Endogenous Coalition Formation with Identical Agents[1]

Davide Fiaschi[2] - Pier Mario Pacini[3]
Università di Pisa
Dipartimento di Scienze Economiche

September 29, 1998

[1] In spite of this paper derives from a joint work of the two authors, the sections 1, 3 and 4 can be attributed to Pier Mario Pacini, while the sections 2 and 5 to Davide Fiaschi.

[2] dfiaschi@ec.unipi.it
[3] pmpacini@ec.unipi.it

1 Introduction

Cooperative behaviour often emerges at a group, rather than social level. In many instances we observe the formation of independent and sometime competing groups, teams, clubs, cooperatives (coalitions for short) each of them persecuting the same goal (in turn provision of commodities, maximization of profits, raising of public funds, standards of behaviour etc.). Examples of this behaviour are numerous both at micro and macro level: scientific research groups, university departments, consumers' associations, firms as organizations, consumption and production cooperatives, industrial districts, international commercial treatises among countries are all instances of volunteer agreements among independent parties that coalesce to obtain a same goal. Once coalitions form, society is partitioned in a coalitional structure. This fact has already been observed and studied in a number of works from different points of view: among them [7] [1] [11] dealt with the problem of coalition formation for the provision of public goods, [8] with the problem of the formation of cooperatives of workers with different working capacity, [6] [2] [3] [4] dealt more directly with the problem of coalitional structures formation in more general settings (see [10] for an excellent review). In these works it has been stressed that the formation of cooperating groups is affected by a moral hazard problem whenever private actions cannot be monitored; this fact may contribute to determine the extent to which cooperation can be sustained as a self-enforcing equilibrium. The quoted works show that the cooperative agreement is self-sustained if the coalition size is under a certain threshold, whereas it would not be incentive compatible in larger coalitions; these conclusions are much in the spirit of [13] (see also [12])

This work takes a similar approach but with a different strategy: first we find a set of conditions under which cooperation can emerge in groups that are subsets of the whole population; then we determine the limits to the sustainability of cooperative behaviour within coalitions and finally we study how a society will form a coalitional structure. We approach the problem in two steps; first we examine agents' behaviour within each of the possible coalitions can form and we investigate the kind of equilibria that we can expect to emerge, coalitions taken as given: we show that the equilibria of the infra-coalitional interaction depend on individual characteristics (outside opportunities) of agents, returns to scale of the available technology for the transformation of cooperative efforts in output and the distributive rule governing the allocation of the obtained output. Once we know how people behave in a coalition, we proceed backward and study what kind of coalitional structures can form in such a way that no one has an incentive to deviate from. In this stage we shall see that the equilibrium coalitional structures depend on the institutional settings ruling the degree of social mobility within population. We examine two extreme institutional arrangements: in the first we assume there to be perfect mobility and capacity to change coalitions, while, in the other, we assume an extremely limited mobility due to the fact that even a single veto can forbid the formation of a coalitional structure. As we shall see, these two extreme cases will give rise to two different sets of equilibrium coalitional structures.

In our setting agents can engage in pre-play non-binding communication among themselves (see [8]). By this they can agree on pareto-dominating, self-enforcing equilibria in any occasion a selection problem is concerned with. However, just because such pre-play talks are unbinding, no one can be forced to perform actions not compatible with individual incentives and, for example, cooperate when only group rationality would call for it. Therefore we adopt a mixed approach in which we suppose agents can deal each other and constraint themselves in a binding manner, but in full respect of the individual freedom non to participate in any agreement which provides individual (and coalitional as well) incentives to deviate from. This approach is not new in the literature and we refer to the extensive work of [9] and [10] for further reference.

The work is clearly limited by some strong assumptions that we make; among them

- the fact that all individuals are identical
- the fact that the distribution rule is fixed
- the fact that we do not examine intermediate institutional settings, in which social mobility may be just partially inhibited.

A throughout work should get rid of these assumptions; however, in this work, we keep them as simplifying devices in order to carry out an analysis capable of manageable results. Section 2 describes the basic characteristics of the economy; Section 3 finds what will be the equilibrium self-enforcing outcomes within any possible coalition; Section 4 analyzes the equilibrium outcomes of the coalitional structure formation process under the two different institutional settings mentioned above; finally Section 5 examines and compares the properties of the two different kinds of equilibrium coalitional structures. Conclusions close the paper with some insight into further work.

2 Description of the model

There is a population $\Im$ of I agents indexed by i; they are supposed identical in all characteristics and endowed with one unit of time and one unit of a physical capital. Capital is used with labour to produce a consumption commodity Y by means of a publicly and freely available technology. There are no alternative uses for capital.

Agents can produce the commodity Y either in isolation or within a group. It takes place in a group when agents in a subset $S \subseteq \Im$ pool their capital units; then agents independently decide the amount of their own time to spend on production. The contribution of the capital endowment is a necessary and sufficient condition to have the right to receive a share of the produced output.

Given a coalition S for the production and distribution of the commodity Y, let X_S denote the cardinality of S. A coalition S is proper if $2 \le X_S \le I$; if $X_S = 1$, S is a singleton. A coalitional structure is a partition of $\Im$ into subsets and is denoted by $\sigma = \{S_1; S_2; \ldots; S_j; \ldots; S_J\}$, where $J \in \mathbb{N}_+$ is the number of non-empty groups in the coalitional structure σ. Any coalitional structure must satisfy the following properties:

$\bigcup_{j=1}^{J} S_j = \Im$

$\forall (h; j)$, $h \neq j$, $S_h \cap S_j = \emptyset$, $S_h; S_j \in \sigma$.

Let Σ be the set of all possible coalitional structures in $\Im$.

2.1 Individual characteristics

Let l_i denote the amount of time that agent i spends on production, $l_i \in [0; 1]$. If an agent chooses to participate into a coalition S he must first contribute his own capital endowment; then he has to decide the amount of time l_i to dedicate to production; hence $(1 - l_i)$ is the leisure time. The more he spends his time on production, the more the total output increases; but one unit of leisure gives him an utility of ω[1]. An agent can always decide not to participate in a coalition; in such a case he keeps his capital endowment to produce in autharchy.

Agents' utility depends on consumption of commodity and on leisure. For simplicity purposes, the utility function is assumed to be linear in all its arguments and takes the form

$$U_i = y_i + (1 - l_i) \cdot \omega$$

where y_i is the quantity agent i receives of commodity Y.

[1] Since agents are supposed identical, the outside opportunity will not be indexed by i.

An agent spending his time on production will be called a cooperator, a defector otherwise. Agents can choose to contribute even a fraction of their working capacity, i.e. $0 \leq l_i \leq 1$, but they are not allowed to play mixed strategies.

Finally we assume that, whereas individual capital contributions can be publicly observed, l_i is a variable known only to agent i and cannot be directly monitored by anyone else.

2.1.1 Technological characteristics

The technology available for the production of Y can be represented by a production function $F(K, L)$, where K is the capital stock and L is the amount of labour; clearly, in any coalition, $0 \leq L \leq K$ since no one will contribute his own working effort in a coalition in which he has no right to receive the produced output. We assume $F(\cdot, \cdot)$ to satisfy the following properties.

Axiom 1 *The production function $F(K, L)$ is such that*

1. $F(K, L) = A \cdot \min\{K^{\alpha}, L^{\alpha}\}$
2. $2 \cdot \omega > A > \omega$
3. $\alpha > 0$

Notice that Axiom 1 together with $0 \leq L \leq K$ yield to $F(K, L) = A \cdot L^{\alpha}$. The particular form of the production function is a simplifying device, allowing for returns to scale to labour invariant to the amount of the working effort put into the production process; the magnitude of the returns to scale depends on the value of α. The fact that the scale parameter A is greater than the outside opportunity serves only to induce the performing of the working effort when an agent acts alone, while the fact that $2 \cdot \omega > A$ serves to exclude that cooperation may arise in proper coalitions for reasons that are not just strategic. It is worth noticing that the production within a coalition of size K has no external effect on the production of other coalitions (see [10]).

2.1.2 Distributive characteristics

Once the total output Y is produced in a given coalition by the working efforts of some of its members, it is then redistributed within the same coalition. The distribution of Y is governed by an equal sharing rule R that we assume to be the same in all coalitions; in other words all members of a coalition that contributed their capital endowments have the right to receive an same share of the total output Y, independently of the respective contribution in terms of working effort. This assumption is motivated by the fact that, l_i being unobservable and the capital assets having equal values, there are no means to discriminate among the participants to a coalition.

To be more precise on the form of R, take a generic coalition S of cardinality X_S and say that Y^S is the total output produced within S by means of the working efforts of its members, i.e. $Y^S = A \cdot \left(\sum_{j \in S} l_j\right)^{\alpha}$; then for all $i \in S$ we have

$$y_i^S = R_i(Y) = \frac{Y^S}{X_S} = \frac{A \cdot \left(\sum_{j \in S} l_j\right)^{\alpha}}{X_S}$$

where $R_i(\cdot)$ denotes the quantity that the distributive rule R allocates to agent i. In view of this, the payoff to an agent i from a coalition S which he participated into is given by

$$\pi_i(l_i|S) = \frac{A \cdot \left(\sum_{j \in S} l_j\right)^{\alpha}}{X_S} + (1 - l_i) \cdot \omega$$

It is clear that, once the level of cooperation $\sum_{j \in S} l_j$ is given, defectors are always better off than cooperators in any coalition. However the essence of the social game we are going to examine in the first part of the paper is whether it is worth or not, from the individual point of view, to increase the level of cooperation considering that the working effort affects the coalitional output and that this returns as a benefit to the subject via the distributive rule, but, at the same time, this reduces leisure.

3 Individual decisions and equilibria within coalitions

Before dealing with the problem of how a society will structure itself in coalitions, we examine how people behave within any possible coalition and which kind of equilibria we can expect in it. We are looking for self-enforcing equilibria, i.e. equilibria no one has an incentive to deviate from. The strategy we follow to prove the existence of such kind of equilibria is the following: first we prove the existence of Nash equilibria (NE), i.e. equilibria robust to unilateral deviations, in the game taking place within a coalition S of X_S agents when they are to decide the level of their time to dedicate to production, given that they have already committed to S. Thereafter we check for individual rationality of NEs; this is a very important step since agents will always be able to improve upon individually not rational solutions by withdrawing from actual coalitions creating new ones (remember that production in autharchy is always a possibility).

3.1 Incentive compatible and individually rational NE within given coalitions

In order to pursue the analysis let us take a generic coalition $S \subset N$ and an agent $i \in S$. Furthermore suppose agent has correct expectations about the level of cooperation supplied by the other agents in S and that this level will not be affected by his own action. Then we can state the following theorem concerning existence of incentive compatible NEs in S; as to the terminology full cooperation, partial cooperation and full defection stand for situations in which $l_i = 1$, $0 < l_i < 1$ and $l_i = 0$ respectively $\forall i \in S$.

Proposition 1 *Suppose the economy satisfies Assumption 1 and that the distributive rule R is an equal sharing rule; then*

1. *in any singleton coalition, $l_i = 1$ is the dominant strategy;*
2. *if $\alpha \leq 1$, partial cooperation is the only NE in any proper coalition;*
3. *if $\alpha > 1$, generalized defection is always a NE in any proper coalition;*
4. *For any $X \geq 2$ there exists a value $\alpha_X < 2$, such that full cooperation is a NE in any coalition with cardinality up to X provided $\alpha \geq \alpha_X$; if $\alpha < \alpha_X$ at least those coalitions with cardinality greater than X admit full defection as the only NE.*
5. *if $\alpha > \alpha_I \equiv \alpha^*$ full cooperation is a NE in any proper coalition.*

The proof of this Proposition is rather long and is postponed to the Appendix. Substantially it establishes that in a proper coalition of cardinality X_S there are either one or two NE depending on the returns to scale of the coalitional production function.

Now we turn to individual rationality of incentive compatible NE, since individually not rational outcomes will not be accepted as equilibrium outcomes. Results are summarized in the following list of remarks, where we always assume the economy to satisfy Assumption 1 and the distributive rule R to be of the equal sharing type, as in Proposition 1.

Remark 1 *If $\alpha \leq 1$, NE are not individually rational in any proper coalition.*

Proof. If returns to scale are at most costant, the only NE in any proper coalition is a partial cooperation one, in which the working effort supplied by any member of a coalition S is

$$\mu = \frac{\left(\frac{\alpha A}{\omega}\right)^{\frac{1}{1-\alpha}}}{X_S^{\frac{2-\alpha}{1-\alpha}}}. \tag{1}$$

A level μ of cooperation is profitable if it grants each agent at least the payoff he could get by himself acting as a singleton, i.e. if

$$\frac{A \cdot \left(\mu \cdot X_S\right)^{\alpha}}{X_S} \geq A,$$

from which

$$\mu \cdot X_S^{\frac{\alpha-1}{\alpha}} \geq 1. \tag{2}$$

Substituting from equation (1) into (2) we get that the NE, when returns to scale are at most constant, is individually rational if

$$\frac{\left(\frac{\alpha A}{\omega}\right)^{\frac{1}{1-\alpha}}}{X_S^{\frac{2-\alpha}{1-\alpha}}} \cdot X_S^{\frac{\alpha-1}{\alpha}} \geq 1 \quad \Rightarrow \quad X_S \leq \left(\frac{\alpha \cdot A}{\omega}\right)^{\alpha} \leq 2,$$

where the last inequality follows from Assumption 1.2. Hence the partial cooperation equilibrium is never individually rational in any coalition with at least two agents. ■

Remark 2 Full defection equilibria are never individually rational.

Proof. The proof follows immediately from equation (2), since it can never be satisfied for $\mu = 0$ and $X_S > 0$. ■

Remark 3 Full cooperation NE (provided it exists) is individually rational.

Proof. From Proposition 1 we know that full cooperation can emerge in proper coalitions only if returns to scale are increasing, i.e. $\alpha > 1$. It can easily be checked that equation (2) is always satisfied for for such values of α and any $X_S \geq 1$. ■

Summarizing, full cooperation is the only individually rational NE, provided it exists. The payoff one gets in any other equilibria (i.e. full defection and sometimes partial cooperation) can be improved upon by withdrawing from the corresponding coalition and producing in autharchy as a singleton.

The substance of the story stylized in the Proposition 1 and Remarks 1, 2 e 3 is simple. If returns to scale are not increasing, there is no incentive to pool together resources, since the allocation rule grants anyone no more than he can do by himself. Incentive to coalesce exist only when there are increasing returns from cooperation. Once this condition is met, cooperation is easier to emerge in groups of limited size (the extent of the maximal coalition capable of sustaining cooperation depends positively on the strength of the returns to scale). The fact that full cooperation can be sustained in coalitions not greater than a certain threshold can be explained in the following way. Suppose all agents in a coalition S of cardinality X_S are cooperating at full extent: their payoff is $A \cdot X_S^{\alpha-1}$. Suppose further that X_S is a continuous variable; if an agent outside the coalition bids for entering and cooperating into S, each of the preexisting X_S agents will receive an increase in payoff equal to the per-capita marginal product of the new agent, i.e. $(\alpha-1) \cdot A \cdot X_S^{\alpha-2}$. The new coalition will continue sustaining full cooperation provided, in the new setting, defection will not be more rewarding than cooperation, i.e. provided $(\alpha-1) \cdot A \cdot X_S^{\alpha-2} \geq \omega$. If $\alpha < 2$, the left-hand side is a decreasing function of the coalitional cardinality, so that there will be a value of X_S constituting an upper bound to the cardinality of the coalition that can sustain cooperation.

3.2 The value of a coalition

In the Proposition 1 section we showed that, when $\alpha > 1$, in some coalitions there can be two pure strategies Nash equilibria: a full cooperation and a full defection one. However a serious coordination problem never arises, since the full defection equilibrium will never be adhered to by agents for at least two reasons: first of all it is not an individually rational equilibrium (see Remark 1), since agents have always the opportunity to leave the coalition and form a singleton granting more to themselves (A rather than ω). Secondly, in such a situation, there is always the possibility for a single or a group of agents to agree to a (joint) deviation forming a cooperating subgroup such that the best reply of everyone in the complementary coalition is to cooperate as well, thus increasing coalitional and individual welfare. Hence, by allowing for a pre-ply communication stage, we can conclude that the only meaningful Nash equilibrium, when a multiplicity is possible, is the full cooperation one that is both individually rational and coalition-proof (as defined in [5]).

Therefore we can say that, given returns to scale as specified by the parameter α, people will accept to cooperate at full extent in all coalitions in which such behaviour is incentive compatible. By this we can define the value to an agent i of a coalition S which he belongs to; in order to do this, let X^1_α be the cardinality of the greatest coalition that can sustain cooperation when returns to scale are given by α[2]. Then we can pose the following:

Definition 1 *The value $V_i(S)$ of a coalition S to an agent i is the payoff that he can get in the NE for that coalition, i.e. $\forall i \in S \in \sigma \in \Sigma$*

$$\alpha \in [0;1] \Rightarrow \quad V_i(S) \leq A$$
$$\alpha \in (1;\alpha^*] \Rightarrow \quad V_i(S) = \begin{cases} A \cdot X_S^{\alpha-1} & \text{if } X_S \leq X^1_\alpha \\ 0 & \text{if } X_S > X^1_\alpha \end{cases}$$
$$\alpha > \alpha^* \Rightarrow \quad V_i(S) = A \cdot X_S^{\alpha-1}$$

In words the value function $V_i(S)$ simply says that all agents recognize and agree that mutual cooperation will be the outcome of their interaction whenever it is possible, where possible means that in playing cooperatively no one, neither in a group nor in isolation, will be incentivated to deviate from the reached agreement. Out of these situations, the impossibility to make binding commitments to cooperate compels agents to recognize that the value of their participation to the coalition is lower than the payoff of the sure alternative that they have always the possibility to choose, i.e. withdraw and act as singletons. For the sake of convention the value of the coalition in this case is set to 0.

4 Equilibrium coalitional structures

Regarding the analysis of coalitional structure formation we confine to the case $1 < \alpha < \alpha^*$. The reason for this is simple; if we allow $\alpha \geq \alpha^*$, there will be no disagreement to agree to mutual cooperation in the grand coalition $\mathcal{N}$. Indeed, by Proposition 1, we know that, when returns to scale are substantial ($\alpha > \alpha^*$), cooperation is an equilibrium outcome in any coalition, $\mathcal{N}$ included; then, in the pre-play communication stage, agents will recognize the advantages of cooperating together since this behaviour maximizes all individual utility and is incentive compatible and individually rational. Therefore the whole society will structure as a sole grand coalition in which everyone cooperates.

The most interesting behaviours can be observed when returns to scale, given with the distributive rule, do not allow for aggregation of the society as a whole. In this case there will be conflicts among the agents about how

[2] In other words, X^1_α is the greatest integer in the set $\{X \mid \alpha_X \leq \alpha\}$.

to partition society and which coalitions to participate into. As we will see, the equilibrium outcomes of the coalitional structure formation game depend crucially on the institutional settings ruling the capacity of agents to form new coalitions and disrupt already existing ones.

Here we take into account two extreme institutional settings. In the first agents are absolutely free to form and disrupt coalitions, in order to maximize individual payoff. In the second, individual mobility is constrained by the vetoes that some agents in a coalition can cast against the transfer of (some of) the other members when this would induce a loss in the payoff levels of the former ones. We shall analyze the two cases separately.

4.1 Perfect mobility

First we examine the case in which agents are completely free to form coalitions and abandon previously established ones, provided they obtain a gain for themselves, independently of what happens to the others. An equilibrium within this institutional setting will be a coalitional structure with respect to which no further rearrangement of coalitions (i.e. transfer of people from a coalition to another) can turn out advantageous to the members of at least one newly formed coalition. In order to make this definition rigorous, let π_i^{σ} be the payoff that agent i receives when he belongs to a coalition in the coalitional structure σ; we say that a coalitional structure σ' is preferred by a coalition $S' \in \sigma'$ to the coalitional structure σ if $\pi_i^{\sigma'} > \pi_i^{\sigma}$ $\forall i \in S'$, i.e. if a coalition in σ' exists such that all its members are better off than in σ. Let us denote by $P(\sigma) = \left\{\sigma' \in \Sigma \mid \exists S' \in \sigma' : \pi_i^{\sigma'} > \pi_i^{\sigma} \; \forall i \in S'\right\}$ the set of coalitional structures that are preferred by at least a coalition to the coalitional structure σ. If perfect mobility is allowed, then a coalitional structure σ can, and indeed will, be disrupted if $P(\sigma) \neq \emptyset$, i.e. if at least a new coalition (and hence a different coalitional structure) can form in which all its members obtain a positive increment in payoff. The disruption of the existing coalitional structure σ takes place independently of what happens to the other agents in $N \setminus S$. These remaining agents have no countermove but a rearrangement among themselves; as a consequence of this rearrangement some of the original deviators could be induced to come back forming a different coalitional structure and so on. When this process admits no further move, i.e. when a position is found such that no group of people can improve upon their payoff, we have reached an equilibrium. This leads to the following definition:

Definition 2 *A coalitional structure $\sigma^* \in \Sigma$ is an equilibrium with perfect mobility (PMCE) if $P(\sigma^*) = \emptyset$.*

In order to clarify this definition examine the following example.

Example 1 *Suppose we have a population of six agents, i.e. $N = \{1, 2, 3, 4, 5, 6\}$. The technological and individual parameters are such that $A = 3$, $\alpha = 1.5$ and $\omega = 2$. Simple computation shows that*

$$V_i(S) = \begin{cases} 3 & \text{if } X_S = 1 \\ 4.2 & \text{if } X_S = 2 \\ 5.2 & \text{if } X_S = 3 \\ 6 & \text{if } X_S = 4 \\ 0 & \text{if } X_S = 5 \\ 0 & \text{if } X_S = 6 \end{cases}$$

Take the coalitional structure $\sigma = \{\{1, 2, 3\}, \{4, 5\}, \{6\}\}$; clearly $P(\sigma) \neq \emptyset$; since any coalitional structure σ' containing a coalition S' with $X_{S'} = 4$ will be preferred to σ by S' itself. Also the coalitional structure $\sigma' = \{\{1, 2, 3, 4\}, \{5\}, \{6\}\}$ can be disrupted, since the coalitional structure $\sigma'' = \{\{1, 2, 3, 4\}, \{5, 6\}\} \in P(\sigma')$; indeed, in the last coalitional structure, both agents 5 and 6 receive 4.2 rather than 3 and hence they increase their own welfare. Finally the coalitional structure $\sigma^* = \{\{1, 2, 3, 4\}, \{5, 6\}\}$ cannot be disrupted by any coalition since no other group can form at mutual advantage of all its participants, so that it is a PMCE.

Note that, in the definition of PMCE, agents are concerned with their own payoff maximization and are free to form any coalition they want, but have no power in blocking the formation of a coalitional structure in which an even negligible group gets an increase in payoff, while all the others suffer a loss. In other words the formation and the acceptance of coalitional structures is subject to group consensus, but there is no veto power.

Before dealing with the main result of this section, let us pose the following notation: $N^1_\alpha = I \bmod X^1_\alpha$ and $Q^1_\alpha = I - N^1_\alpha \cdot X^1_\alpha$. In order to avoid trivial situations let us suppose $Q^1_\alpha > 0$. Then we can introduce the following definition:

Definition 3 *A coalitional structure σ is concentrated when it is formed by exactly $N^1_\alpha + 1$ coalitions such that $Q^1_\alpha \leq X_S \leq X^1_\alpha$; a coalitional structure σ is maximally concentrated when it is concentrated and N^1_α coalitions has the maximal cardinality X^1_α while the last one has cardinality Q^1_α.*

We can now state and prove the following proposition.

Proposition 2 *Suppose that the economy satisfies Assumption 1, $\alpha \in (1; \alpha^*]$ and the distribution is governed by an equal sharing rule R. Then all the maximally concentrated coalitional structures are PMCE for the coalitional structures formation game.*

Proof. First we prove that, in equilibrium, all coalitional structures must be formed by exactly $N^1_\alpha + 1$ coalitions.

1. If a coalitional structure σ were composed by less than $N^1_\alpha + 1$ coalitions, then there would exist at least one coalition, say S, with more than X^1_α agents; in this case all members of S would withdraw from S since, so doing, they get a higher payoff (at least A rather 0). Therefore any coalitional structure σ' containing a coalition $S' \subset S$ such that $X_{S'} \leq X^1_\alpha$ would be preferred to σ by S', so that $P(\sigma) \neq \emptyset$. This shows that in equilibrium there must be at least $N^1_\alpha + 1$ coalitions.

2. If a coalitional structure σ were composed by more than $N^1_\alpha + 1$ coalitions, than at least two of them, say S_1 and S_2, should contain less than X^1_α agents. But then a coalition S' could form such that $S' \subseteq S_1 \cup S_2$ and $X^1_\alpha \geq X_{S'} > \min\{X_{S_1}; X_{S_2}\}$; all the members of S' will get a payoff $V_i(S') > \max\{V_i(S_1); V_i(S_2)\}$. Therefore any coalitional structure $\sigma' \ni S'$ would be preferred to σ, so that $P(\sigma) \neq \emptyset$. This shows that any coalitional structure with more that $N^1_\alpha + 1$ coalitions can be disrupted and hence, in equilibrium, there can be at most $N^1_\alpha + 1$ coalitions.

From the above points, an equilibrium coalitional structure must be concentrated. Now it is immediate to see that in any PMCE there cannot exist a coalition with a cardinality greater than X^1_α, since, would it exist, a new subcoalition could form disrupting the given coalitional structure as in point 1 above. On the other hand, in a PMCE no two coalitions can have fewer than X^1_α members, since, in this case, the same reasoning as in point 2 above could be applied. Combining these facts together, the statement of the Proposition is proved. ■

According to the Proposition, there is a multiplicity of equilibria each of them corresponding to one possible way of combining I agents in $N^1_\alpha + 1$ coalitions, with N^1_α having the maximal cardinality X^1_α. However the structure of all PMCE is always the same (maximally concentrated), only the identity of the members in coalitions change. In Section 5 we shall use this fact to examine some basic properties of this kind of equilibria.

4.2 Veto power

Here we analyze a different institutional setting characterized by the fact that agents are still trying to exploit all opportunities to increase the coalitional outcome, but now they are endowed also with a veto power, i.e. each of them can successfully cast an opposition to the formation of coalitional structures in which he gets a lower payoff. In other words it is as if agents, in forming coalitions, signed contracts with the following clause: anyone of them will be able to leave the coalition he belongs to only if the other members give him their consensus not suffering any loss. Of course, when some agents can get an advantage from the formation of a certain coalitional structure with no other suffering a loss (a Pareto improvement for the society as a whole), then no vetoes will be opposed; however, when the payoff maximizing attempts by someone will induce a loss to someone else, the latter ones will oppose their vetoes and will not allow the formation of the new coalitional structure. When we find a coalitional structure in which any further attempt to increase someone's payoff encounters the veto of someone else, then we have reached an equilibrium; we shall call such a position a vetoing coalitional equilibrium (VCE). As in the case of PMCE, first we have to make this definition rigorous. Suppose that society is arranged in a coalitional structure σ; we say that a coalitional structure $\sigma' \in \Sigma$, $\sigma' \neq \sigma$, is vetoed by someone if there exists at least an agent who suffer a reduction in his payoff passing from σ to σ', i.e. $\exists i \in N$ such that $\pi_i^{\sigma'} < \pi_i^{\sigma}$. Let $W_i(\sigma)$ be the set of coalitional structures that are vetoed if society happens to be structured as in σ, i.e. $W(\sigma) = \left\{\sigma' \in \Sigma \mid \exists i \in N : \pi_i^{\sigma'} < \pi_i^{\sigma}\right\}$. Finally let $P(\sigma)$ have the same meaning as in Section 4.1, i.e. it is the set of coalitional structures in which a coalition could form with a net positive benefit for all its members with respect to what they get in σ. Then we can pose the following definition:

Definition 4 A coalitional structure $\sigma^* \in \Sigma$ is a VCE if $P(\sigma^*) \cap W(\sigma^*) = \emptyset$.

The interpretation of this definition is simple: consider σ^* and take a coalitional structure in $\sigma \in P(\sigma^*)$; this means that in σ there is at least a coalition that was not in σ^* and in which all members get a higher payoff. Since $P(\sigma^*) \cap W(\sigma^*) = \emptyset$, then σ belongs to $W(\sigma^*)$ as well, i.e. there will be some agent suffering from the rearrangement leading to σ; if a veto power is allowed and accepted, the latter people will object to the formation of σ and this coalitional structure will not form. When vetoes are opposed to any coalitional structure that provides individual and group incentive to its formation, then we have reached a position which no one, willingly or unwillingly, moves from, i.e. we are in an equilibrium. As it can be easily checked, the set of VCE corresponds to the set of coalitional structures giving rise to Pareto efficient distributions of payoffs, given the distributive rule R. An example can clarify concepts.

Example 2 Take again the situation described in the Example 1, where we had

$$V_i(S) = \begin{cases} 3 & \text{if } \#S = 1 \\ 4.2 & \text{if } \#S = 2 \\ 5.2 & \text{if } \#S = 3 \\ 6 & \text{if } \#S = 4 \\ 0 & \text{if } \#S = 5 \\ 0 & \text{if } \#S = 6 \end{cases}$$

Take the coalitional structure $\sigma = \{\{1,2,3\},\{4,5\},\{6\}\}$ and the coalitional structure $\sigma' = \{\{1,2,3,4\},\{5,6\}\}$. Clearly $\sigma' \in P(\sigma)$ since the coalition of the first four agents improves with respect to σ; however $\sigma' \notin W(\sigma)$ since no one is worse off in σ' with respect to σ. Therefore σ can and will be disrupted in favour of σ'. Consider now the coalitional structure $\sigma'' = \{\{1,2,3\},\{4,5,6\}\}$. It is easy to check that $\sigma' \in W(\sigma'')$ since agents 5 and 6 will suffer a loss in σ' with respect to σ'' but $\sigma' \in P(\sigma'')$ since the coalition of the first four agent gets an improvement. In this case the coalitional structure σ'' cannot be disrupted.

We now turn to the existence problem and to the description of the characteristics of VCE in terms of coalitions. The main result is stated in the following proposition.

Proposition 3 *Suppose that the economy satisfies Assumption 1, $\alpha \in (1, \alpha^*)$ and the distribution is governed by an equal sharing rule R. Then all the concentrated coalitional structures are VCE for the coalitional structures formation game.*

Proof. First we prove that in any equilibrium coalitional structure there are exactly $N^1_\alpha + 1$ coalitions.

1. Suppose there exists an equilibrium coalitional structure σ with fewer than $N^1_\alpha + 1$ coalitions; then at least one of them, say coalition S, must possess more than X^1_α members so that the value of their coalition is 0. Now construct a coalitional structure σ' obtained by σ simply replacing S by two or more coalitions with cardinality less than or equal to X^1_α, the rest of the coalitional structure remaining the same. The new coalitional structure σ would be such that $\sigma' \in P(\sigma)$ and $\sigma' \notin W(\sigma)$ since the members of new coalitions strictly improve their situations, while the rest of the population is unaffected. From this we get $P(\sigma) \nsubseteq W(\sigma)$, i.e. σ cannot be a VCE.

2. Now suppose that the equilibrium coalitional structure σ possesses more than $N^1_\alpha + 1$ coalitions; suppose further than no one of them has

more than $\bar{X}_\alpha$ members, otherwise the same reasoning as in point 1 above could be applied. Then arrange them in decreasing order of cardinality, obtaining the sequence $S_1, S_2; \ldots; S_{\bar{N}+1}; \ldots; S_{\bar{N}_\alpha+1+v}$, $v > 0$, with $X_{S_1} > \ldots > X_{S_{\bar{N}_\alpha+1+v}}$. Now take the coalitional structure σ' obtained from σ by redistributing the agents in $\bigcup_{j=1}^{v} S_{\bar{N}_\alpha+1+j}$ over the coalitions $S_1; \ldots; S_{\bar{N}_\alpha+1}$ in such a way that $X_{S'} \leq \bar{X}_\alpha$ for any $S' \in \sigma'$. Any of the new coalitions contains at least the same number of agents as in σ so that $\pi_i^{\sigma'} \geq \pi_i^{\sigma}$ $\forall i$ and $\pi_i^{\sigma'} > \pi_i^{\sigma}$ for all the members of at least a coalition $S' \in \sigma'$. This means that $\sigma' \in \mathcal{P}(\sigma)$ and $\sigma' \notin \mathcal{W}(\sigma)$. Again we have $\mathcal{P}(\sigma) \not\subseteq \mathcal{W}(\sigma)$, i.e. σ cannot be a VCE.

Thus far we proved that any VCE must possess exactly $\bar{N}_\alpha + 1$ coalitions. No equilibrium coalitional structure can possess a coalition S with more than $\bar{X}_\alpha$, because such coalitional structure would be dominated by the one in which the members of S withdraw to form singleton coalitions. No equilibrium coalitional structure can possess a coalition with less than $\bar{Q}_\alpha$ agents because, in this case, there would exist another coalition with more than $\bar{X}_\alpha$ agents and the previous reasoning could be applied. Therefore equilibrium coalitional structures must be made up of coalitions with a cardinality included in the interval $[\bar{Q}_\alpha; \bar{X}_\alpha]$. On the other hand all coalitional structures with such characteristics are VCE: indeed take whatever $\sigma^* = \{S_1^*; \ldots; S_{\bar{N}+1}^*\}$ such that $\bar{Q}_\alpha \leq X_{S_j^*} \leq \bar{N}_\alpha$ $\forall j \in [1; \bar{N}_\alpha + 1]$ and let $\pi^{\sigma^*} = \{\pi_1^{\sigma^*}; \ldots; \pi_l^{\sigma^*}\}$ be the corresponding distribution of individual payoffs; only coalitional structures σ^{**} such that $\pi^{\sigma^{**}} \geq \pi^{\sigma^*}$ will not be vetoed. But given that $V_i(S)$ is monotonically increasing in the cardinality of S up to $\bar{X}_\alpha$, the condition $\pi^{\sigma^{**}} \geq \pi^{\sigma^*}$ can be satisfied only if there is an increase in the number of cooperators in at least a non maximal coalition. Since, by construction, cooperation is the observed behaviour in any coalition in σ^*, the above result can be obtained just by an increase in the number of agents forming population =, which contradicts the assumption of a fixed population. This ends the proof of the Proposition. ■

It is worth noting that the set of PMCE is included in the set of VCE; in other words the possibility of an effective veto increases the number of possible self-enforcing agreements.

5 Properties of coalitional equilibria

In this Section we compare the equilibrium outcomes within the institutional settings examined in Sections 4.1 and 4.2.

5.1 Perfect mobility coalitional equilibria

¿From Proposition 2 we know that any PMCE is characterized by a maximally concentrated coalitional structure. First of all we prove the following:

Remark 4 *In any maximally concentrated coalitional structure, and hence in any PMCE, the aggregate production is greater than in any other (not maximally) concentrated coalitional structure.*

Proof. Take a $N^1_\alpha+1$-dimensional vector $\lambda = (\lambda_1; \ldots; \lambda_{N^1_\alpha+1})$; $0 \le \lambda_j \le 1$, such that the first coalition in σ has cardinality $\lambda_1 \cdot X^1_\alpha$, the second $\lambda_2 \cdot X^1_\alpha$, the N^{1}th $\lambda_{N^1} \cdot X^1_\alpha$; for the definition of λ we have that $\sum_{j=1}^{N^1_\alpha+1} \lambda_j \cdot X^1_\alpha = I$, while the aggregate production will be $Y_\sigma = \sum_{j=1}^{N^1_\alpha+1} A\left(\lambda_j \cdot X^1_\alpha\right)$. In order to maximize Y_σ with respect to λ we have to put the maximum number of λ_i equal to 1 (remember that $\frac{\partial Y_\sigma}{\partial \lambda_i} > 0$ and $\frac{\partial^2 Y_\sigma}{\partial \lambda_i^2} > 0$ and $\lambda_i \le 1\ \forall i$). Because $\sum_{j=1}^{N^1_\alpha+1} \lambda_j \cdot X^1_\alpha = I$, then $I \bmod X^1_\alpha = N^1_\alpha$ gives the maximum number of λ_i that we can set equal to 1, while $\lambda_{N^1_\alpha+1} = I - N^1_\alpha \cdot X^1_\alpha$, from which $Q^1_\alpha = \lambda_{N^1_\alpha+1} \cdot X^1_\alpha$. This completes the proof. ■

Therefore a social planner who would maximize only the social output, independently of its distribution, should opt for an institutional setting with perfect mobility, since he is sure that the equilibrium coalition structure is maximally concentrated and hence the aggregate output is maximized.

5.2 Vetoing coalitional equilibria

When VCE are concerned, both maximally and not-maximally concentrated coalitional structures may be the equilibrium outcome of social interaction; however some properties characterizes this kind of equilibria. Then we can state[3]

Remark 5 *The total expected equilibrium outcome in a social setting characterized by a vetoing power is not greater than the total output obtainable in the case of perfect mobility i.e.*

$$E(Y_\sigma) \le Y_{\sigma^*} \quad \sigma^* \in \text{PMCE and } \sigma \in \text{VCE}$$

Proof. The proof of the statement is simple: by Remark 4 we know that total output is maximized in maximally concentrated coalitional structures; therefore the total output obtainable in a VCE is certainly at most as much

[3] We assume that probability of at least a not maximally concentrated VCE is positive.

as that in a PMCE. It follows that the average total output over the VCE set is not greater than the output obtainable in a PMCE. ■

The next result concerns the distribution of payoffs within a VCE. Let $\pi^\sigma = \{\pi_1^\sigma; \ldots; \pi_l^\sigma\}$ be the distribution of payoffs in a coalitional structure σ and let $\underline{\pi}^\sigma = \min \pi^\sigma$. Notice that in any maximally concentrated coalitional structure $\underline{\pi}^\sigma = A \cdot \left(Q^1_\alpha\right)^{\alpha - 1}$; call this value $\underline{\pi}^{\sigma *}$. Then we can state the following[4]:

Remark 6 The expected payoff of the agent in the worst position in a VCE is at least as much as the expected payoff of the same agent in a PMCE i.e.

$$E(\underline{\pi}^\sigma) \geq \underline{\pi}^{\sigma *} \quad \sigma^* \text{ PMCE and } \sigma \text{ VCE}$$

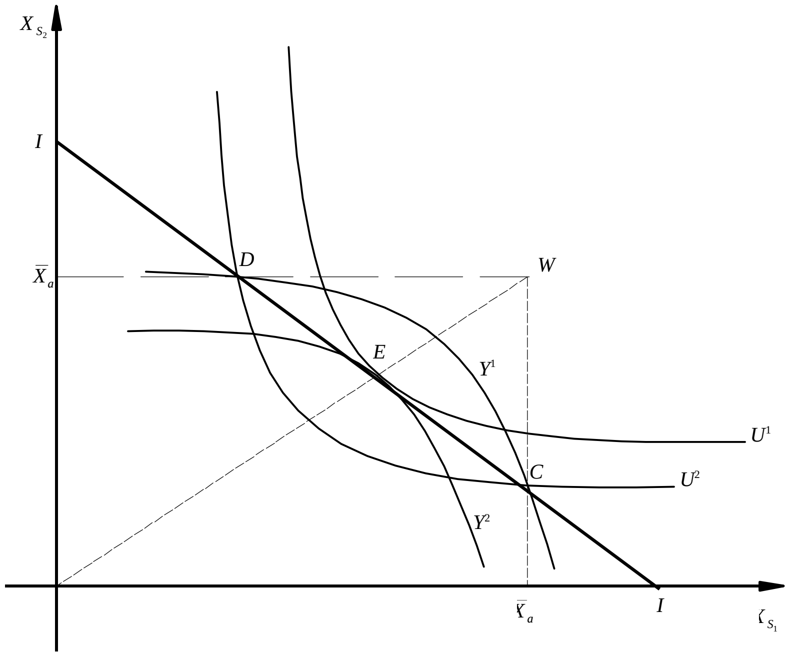
Proof. First we show that the agent in the worst position in a VCE that is not maximally concentrated, gets a higher payoff that an agent in the same position gets in a maximally concentrated coalitional structure, i.e. $\underline{\pi}^\sigma > \underline{\pi}^{\sigma *}$ for any given not-maximally concentrated σ. Clearly the agent with the worst payoff in a maximally concentrated coalition belongs to the residual coalition of cardinality Q^1_α. Another coalitional structure that grants to the agent in the worst position the same payoff must be again maximally concentrated, because there is no other way of arranging the remaining $l - Q^1_\alpha$ agents in N^1_α coalitions preserving individual incentives to cooperation, i.e. remaining within the equilibrium set. Therefore a rearrangement of agents not leading to a maximally concentrated coalitional structure must grant the worst off agent a payoff greater than $\underline{\pi}^{\sigma *}$. From this it follows that the expected payoff of the worst off agent in a VCE must be higher than the payoff the same agent could get in a PMCE since, in the latter case, all coalitional structures are maximally concentrated. ■

>From the Remarks 5 and 6 we have that VCE have lower expected aggregate output than PMCE do, but preserve the worst off agent since his expected payoff is greater in a VCE than in a PMCE. From this it follows also that an egalitarian social planner may prefer (and induce) an institutional setting giving rise to VCE (by allowing veto power) rather than a perfect mobility istitutional setting. So, for example, a Rawlsian social planner would certainly prefer a VCE to a PMCE, but even other social preferences are compatible with the same choice. Suppose, for instance, that the social planner's welfare function is of the form $W(\pi_1; \ldots; \pi_l) = \sum_{i=1}^{l} f(\pi_i)$; then, by the Shorrocks' theorem, we can assert that there always exists a concave function f ($f' > 0$, $f'' < 0$), such that VCE are preferred to PMCE.

[4] See the previous note.

5.3 A graphical illustration

The following example graphically shows the properties regarding the two kinds of equilibria. Suppose that the economy is characterized by values of the individual (!) and technological (A and ®) parameters such that $1 < \frac{\bar{X}_{®}}{I} < 2$, i.e. there will be at most two coalitions in any equilibrium coalitional structure. Therefore we can represent all possible con…gurations in a plane as in the following picture



The horizontal axis shows the cardinality of the …rst coalition, S_1, while the vertical axis shows the cardinality of the second coalition S_2. The line II represents the locus of allocations of the I agents in the two possible coalitions. Every point in the square $\bar{X}_{®} £ \bar{X}_{®}$ corresponds to a coalitional structure in which the cardinality of the largest coalition is compatible with the incentives to mutual cooperation by all its members. For instance C (or D) represents a partition of the whole society into two coalitions; the …rst possesses the maximal cardinality $\bar{X}_{®}$, while the second is the residual coalition. A point as W represents a coalitional structure with two coalitions of cardinality $\bar{X}_{®}$ in which everyone cooperates; however such a point is not a¤ordable because there should be more than I agents.

The two curves labelled Y^1 and Y^2 are aggregate isoproduct curves, i.e. any point on each of them shows how many cooperating agents should be in the two coalitions in order to have the same amount of aggregate output. Since the coalitional production function $F(:,:)$ is convex, and the sum of convex functions is convex as well, the isoproduct curves are bowed out from the origin. Furthermore, from the monotonicity of $F(\cdot;\cdot)$, we have that $Y_1 > Y_2$.

The two curves labelled U^1 and U^2 are the indifference curves of a social planner with convex preferences (as in the case of a concave social welfare function of the type just seen $W(\pi_1; \ldots; \pi_I) = \sum_{i=1}^{I} f(\pi_i)$, $f' > 0$, $f'' < 0$); as it is easily seen, if the social planner preferences are convex enough he would choose the allocation E, that can be obtained only if the institutional setting is the one giving rise to a VCE. If the social planner would maximize the total output, the best result would be obtained on the curve Y_1, intersecting the set of feasible coalitional structures in the points C and D where there is maximal concentration; any other coalitional structure either would not get such result or could not be compatible with individual incentives. In this case, the desired result could be induced by an institutional setting leading only to PMCE, i.e. perfect mobility.

6 Conclusions

In this paper we have analyzed the outcomes of a coalitional structure formation process among identical agents. The problem arises whenever there are increasing returns in the production of the commodity, so that there are incentives to coalesce, but the external effect is not so strong to make cooperation always the dominant action. Notwithstanding that agents can engage in non-binding pre-play communications, the non-observability of individual action induces a moral hazard problem, so that people can engage in cooperative behaviour only when it is individually rational ad self-enforcing. The point we stress is that the outcome of the coalitional structure formation game depends crucially on the institutional settings ruling the economy. If there is perfect social mobility people exploit all opportunities to form coalitions that are advantageous for someone and the resulting equilibrium coalitional structure is maximally concentrated, in the sense that people concentrate as much as possible in the smallest number of coalitions compatible with individual incentives. On the other hand, a more constrained institutional setting, in which even a single agent can cast an effective veto, gives rise to more complex equilibrium coalitional structures that can be not maximally concentrated. In this we see a problem of efficiency against equity: in the equilibria with perfect mobility, the aggregate output is maximized, but the payoff of the agent that is in the worst position is certainly lower than the expected payoff of the worst off agent when social mobility is constrained. The choice of one target (either efficiency or equity) may therefore induce the choice of an institutional setting more appropriate to the obtainment of the former.

One of the main limitations of the present work is that we assume identical agents; heterogeneity may induce further problems, since, in this case, agents have also the opportunity to choose which, and not only how many, agents to coalesce with, provided individual characteristics are common knowledge. In this case also the assumption of a perfectly egalitarian distribution would not be appropriate, since it seems unreasonable that more able agents accept an equal distribution of a product which they eventually contribute more. These is indeed the agenda for our further research.

A Appendix

Proof of Proposition 1

The proof of the first statement is trivial; when an agent is alone his payoff is given by

$$\pi_i(l_i|\{i\}) = A\cdot l_i^{\alpha} + (1-l_i)\cdot\omega, \quad l_i\in[0,1].$$

Since $A>\omega$ by assumption 1.2, the best choice for a singleton is $l_i = 1$.

The proof of the rest of the proposition gets around the properties of the payoff function for different values of α. Denote by $l_{-i} := \sum_{j\in S,\, j\neq i} l_j$ $(0\leq l_{-i}\leq(\Sigma_S-1))$ the level of cooperation supplied by all agents in S except i. Let us start with the case $\alpha\leq 1$. The payoff function, conditional to the fact that i is already in coalition S with a level $l_{-i}\leq\Sigma_S-1$ of expected cooperation by the others, is given by

$$\pi_i(l_i|S,l_{-i}) = \frac{A\cdot[l_i+l_{-i}]^{\alpha}}{\Sigma_S} + (1-l_i)\cdot\omega.$$

By the first order condition we have

$$\alpha\cdot\frac{A\cdot[l_i+l_{-i}]^{\alpha-1}}{\Sigma_S}-\omega=0 \quad\Rightarrow\quad l_i^{*}=\left(\frac{\alpha\cdot A}{\omega\cdot\Sigma_S}\right)^{\frac{1}{1-\alpha}}-l_{-i} \qquad (3)$$

and by the second order condition we get

$$\left.\frac{d^2\pi_i(l_i|S,l_{-i})}{dl_i^2}\right|_{l_i^{*}} = \alpha\cdot(\alpha-1)\cdot\left.\frac{A\cdot[l_i+l_{-i}]^{\alpha-2}}{\Sigma_S}\right|_{l_i^{*}} =$$

$$= \alpha\cdot(\alpha-1)\cdot\frac{A\cdot\left[\left(\frac{\alpha\cdot A}{\omega\cdot\Sigma_S}\right)^{\frac{1}{1-\alpha}}\right]^{\alpha-2}}{\Sigma_S} =$$

$$= \alpha\cdot(\alpha-1)\cdot\frac{A\cdot\left(\frac{\alpha\cdot A}{\omega\cdot\Sigma_S}\right)^{\frac{\alpha-2}{1-\alpha}}}{\Sigma_S} < 0,$$

so that l_i^{*} is indeed a maximum. Clearly such a choice by agent i is meaningful provided $l_i^{*}\geq 0$ otherwise defection would become the dominant strategy; this implies

$$l_{-i}\leq\left(\frac{\alpha\cdot A}{\omega\cdot\Sigma_S}\right)^{\frac{1}{1-\alpha}}$$

so that the quantity $\left(\frac{\alpha\cdot A}{\omega\cdot X_S}\right)^{\frac{1}{1-\alpha}}$ is an upper bound to the number of other cooperators that an agent can expect in order to (partially) cooperate himself. If full cooperation is to be observed as an equilibrium behaviour by all agents, by the above equation it must be true that

$$X_S - 1 \le \left(\frac{\alpha\cdot A}{\omega\cdot X_S}\right)^{\frac{1}{1-\alpha}};$$

i.e.

$$(X_S - 1)^{1-\alpha}\cdot X_S \le \frac{\alpha\cdot A}{\omega}.$$

In view of Assumption 1.2 and $\alpha \le 1$, we have $\frac{\alpha\cdot A}{\omega} < 2$ and the above equation entails $X_S < 2$, i.e. full cooperation can never be observed in coalitions with at least two agents if returns to scale are at most constant.

In any other symmetric equilibrium, the cooperation level supplied by any agent is obtained from equation (3)

$$l = \left(\frac{\alpha\cdot A}{\omega\cdot X_S}\right)^{\frac{1}{1-\alpha}} - l\cdot(X_S - 1);$$

where l is the fraction of his own time anyone devolves to production. Solving for it we get

$$l = \left(\frac{\alpha\cdot A}{\omega}\right)^{\frac{1}{1-\alpha}}\cdot X_S^{\frac{2-\alpha}{\alpha-1}}.$$

Suppose now that returns to scale are increasing, i.e. $\alpha > 1$. In this case the first order condition implies

$$l_i^* = \left(\frac{\omega\cdot X_S}{\alpha\cdot A}\right)^{\frac{1}{\alpha-1}} - l_{-i}. \qquad (4)$$

The second order condition now entails

$$\left.\frac{d^2\pi_i(l_i|S;l_{-i})}{d l_i^2}\right|_{l_i^*} = \alpha\cdot(\alpha-1)\cdot\left.\frac{A\cdot[l_i + l_{-i}]^{\alpha-2}}{X_S}\right|_{l_i^*} =$$

$$= \alpha\cdot(\alpha-1)\cdot\frac{A\cdot\left[\left(\frac{\omega\cdot X_S}{\alpha\cdot A}\right)^{\frac{1}{\alpha-1}}\right]^{\alpha-2}}{X_S} =$$

$$= \alpha\cdot(\alpha-1)\cdot\frac{A\cdot\left(\frac{\omega\cdot X_S}{\alpha\cdot A}\right)^{\frac{\alpha-2}{1-\alpha}}}{X_S} > 0$$

so that the solution l_i^* identifies a minimum of the payoff function. By this, agents' choice will be either to cooperate ($l_i = 1$) whenever

$$(l_{-i}+1)^{\alpha} - l_{-i}^{\alpha} > \frac{\omega \cdot X_S}{A}$$

otherwise they will defect.

Before proceeding, we show that all NE are symmetric, i.e. either all agents defect or cooperate (partial actions are excluded by the previous result). A contrario, suppose that a NE exists in which there are $C < X_S$ cooperators and $X_S - C$ defectors; if this is an equilibrium, any of the C cooperators finds an incentive to cooperate, i.e. for him

$$\frac{A \cdot C^{\alpha}}{X_S} - \frac{A \cdot (C-1)^{\alpha}}{X_S} - \omega > 0,$$

while, for any defector it holds true that

$$\frac{A \cdot (C+1)^{\alpha}}{X_S} - \frac{A \cdot C^{\alpha}}{X_S} - \omega < 0.$$

Combining the two inequalities and multiplying by $\frac{X_S}{A}$ we get

$$C^{\alpha} - (C-1)^{\alpha} > (C+1)^{\alpha} - C^{\alpha},$$

that can never be verified for $\alpha > 1$. It follows that all equilibria must be symmetric, either with full cooperation or defection.

First consider the situation in which everyone defect. It will be immune from deviations if

$$\omega > \frac{A}{X_S},$$

which is always true, in view of Assumption 1.2, in any proper coalition ($X_S \geq 2$). This proves point 3) of Proposition 1.

Next we show that, whenever $\alpha \geq 2$, full cooperation is a NE in any S, the grand coalition included. It is sufficient to prove this statement for the lower bound of the interval, i.e. for $\alpha = 2$; generalized cooperation can be a NE if

$$\frac{A \cdot X_S^2}{X_S} - \frac{A \cdot (X_S - 1)^2}{X_S} \geq \omega,$$

i.e. if

$$2 - \frac{1}{X_S} \geq \frac{\omega}{A}.$$

>From Assumption 1.2 the right-hand side of the above inequality is always lower than 1, while the left-hand side is always greater than 1 in any proper

coalition and increasing with X_S. It follows that the condition for cooperation is satisfied for any X_S.

Finally we prove that for any S with $X_S \leq I$, there is a value α_S such that, for $\alpha > \alpha_S$, full cooperation is a NE in S. In a generic coalition of cardinality X_S full cooperation is a NE if

$$\frac{A \cdot X_S^{\alpha}}{X_S} - \frac{A \cdot (X_S - 1)^{\alpha}}{X_S} \geq \omega .$$

From the previous point, we know that it is certainly satisfied for $\alpha \geq 2$; moreover the left-hand side is an increasing function of α, since

$$\frac{d\left(\frac{A \cdot X_S^{\alpha}}{X_S} - \frac{A \cdot (X_S - 1)^{\alpha}}{X_S}\right)}{d\alpha} = \frac{A}{X_S} \cdot [X_S^{\alpha} \cdot \ln X_S - (X_S - 1)^{\alpha} \cdot \ln(X_S - 1)] > 0 .$$

Therefore it will exist a value α_S such that

$$\frac{A \cdot X_S^{\alpha_S}}{X_S} - \frac{A \cdot (X_S - 1)^{\alpha_S}}{X_S} = \omega .$$

Clearly for all $\alpha > \alpha_I \equiv \alpha^* < 2$ cooperation will be an equilibrium in any coalition. This completes the proof of Proposition 1.

References

[1] Aumann, R. J. and J. H. Dreze (1974). "Cooperative Games with Coalition Structures". International Journal of Game Theory, 217-237.

[2] Bennett, E. (1983). "The Aspiration Approach to Predicting Coalition Formation and Payoff Distribution in Sydepayments Games". International Journal of Game Theory, 12, 1-28.

[3] Bennett, E. (1984). "A New Approach to Predicting Coalition Formation and Payoff Distribution in Characteristic Function Games". In Coalitions and Collective Action. Wuerzburg, W. M.-J. Holler Ed., Germany, and Vienna, Physica, 60-80.

[4] Bennett, E. (1985). "Endogenous vs. Exogenous Coalition Formation". Economie Appliquee, 38 (3/4), 611-35.

[5] Bernheim, B. D., B. Peleg and M. D. Whinston (1987). "Coalition-Proof Nash Equilibria: Concepts". Journal of Economic Theory, 42(1), 1-12.

[6] Chwe, M. S.-Y. (1994). "Farsighted Coalitional Stability". Journal of Economic Theory, 63, 288-325.

[7] Dreze, J. H. and J. Greenberg (1980). "Hedonic Coalitions: Optimality and Stability". Econometrica, 48, 987-1003.

[8] Farrell J. and Scotchmer S. (1988). "Partnerships". Quarterly Journal of Economics, 103, 279-297.

[9] Greenberg, J. (1990). The Theory of Social Institutions. An Alternative Game-Theoretic Approach. Cambridge University Press, Cambridge UK.

[10] Greenberg, J. (1994). "Coalitional Structures". In Handbook of Game Theory. II. R. J. Aumann and S. Hart Eds., Amsterdam, North-Holland, 1305-1337.

[11] Guesnerie, R. and C. Oddou (1988). "Increasing Returns to Size and Their Limits". Scandinavian Journal of Economics, 90, 259-73.

[12] Marwell, G. and Oliver, P. (1993). The Critical Mass in Collective Action. A Microsocial Theory. Cambridge University Press. Cambridge UK.

[13] Olson, M. (1983). The Logic of Collective Action. Cambridege, Mass: Harvard University Press.