Pointwise estimates for a class of non-homogeneous Kolmogorov equations *

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Abstract

We consider a class of ultraparabolic differential equations that satisfy the Hörmander's hypoellipticity condition and we prove that the weak solutions to the equation with measurable coefficients are locally bounded functions. The method extends the Moser's iteration procedure and has previously been employed in the case of operators verifying a further homogeneity assumption. Here we remove that assumption by proving some potential estimates and some ad hoc Sobolev type inequalities for solutions.

Keywords: ultraparabolic equations, measurable coefficients, Moser's iterative method.

1 Introduction

We consider a class of second order partial differential equations of Kolmogorov-Fokker-Planck type with measurable coefficients in the form

$$Lu(x,t) \equiv \sum_{i,j=1}^{m_0} \partial_{x_i}\left(a_{ij}(x,t)\partial_{x_j}u(x,t)\right) + \sum_{i,j=1}^{N} b_{ij}x_i\partial_{x_j}u(x,t) - \partial_t u(x,t) = 0 \tag{1.1}$$

where $(x,t) = (x_1,\dots,x_N,t) = z$ denotes the point in $\mathbb{R}^{N+1}$, and $1 \le m_0 \le N$.

In order to state our assumptions, we define the *principal part of* L as follows

$$K = \Delta_{m_0} + Y, \tag{1.2}$$

where Δ_{m_0} is the Laplace operator in the variables $x_1,\dots,x_{m_0}$ and Y is the first order part of L:

$$Y = \sum_{i,j=1}^{N} b_{ij}x_i\partial_{x_j} - \partial_t. \tag{1.3}$$

Our assumptions are:

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[H.1] *the principal part K of L is hypoelliptic (i.e. every distributional solution of $Ku = 0$ is a C^∞ function);*

[H.2] *the coefficients a_{ij}, $1 \le i,j \le m_0$, are real valued, measurable functions of z. Moreover $a_{ij} = a_{ji}$, $1 \le i,j \le m_0$, and there exists a positive constant μ such that*

$$\mu^{-1}|\xi|^2 \le \sum_{i,j=1}^{m_0} a_{ij}(z)\xi_i\xi_j \le \mu|\xi|^2$$

for every $z \in \mathbb{R}^{N+1}$ and $\xi \in \mathbb{R}^{m_0}$. The matrix $B = (b_{ij})_{i,j=1,\dots,N}$ is constant.

In the sequel, an equation of the form (1.1) satisfying [H.1]-[H.2] will be simply called a *KFP equation.* A well-known criterion for the hypoellipticity of K is the Hörmander's condition [16] which in our setting reads:

$$\text{rank Lie}\left(\partial_{x_1}, \dots, \partial_{x_{m_0}}, Y\right)(z) = N + 1, \qquad \forall z \in \mathbb{R}^{N+1},$$

where $\text{Lie}\big(\partial_{x_1}, \dots, \partial_{x_{m_0}}, Y\big)$ denotes the Lie algebra generated by the first order differential operators (vector fields) $\partial_{x_1}, \dots, \partial_{x_{m_0}}, Y$. Then [H.1] only depends on m_0 and on the first order part of L. We explicitly remark that uniformly parabolic operators (for which $m_0 = N$ and $B \equiv 0$) are KFP operators and the related principal part is the usual heat operator in $\mathbb{R}^{N+1}$. On the other hand, there are also several examples of degenerate KFP operators, i.e. with m_0 strictly lesser than N, from diffusion theory and mathematical finance.

Example 1.1 *Consider the following kinetic equation*

$$\partial_t f - \langle v, \nabla_x f\rangle = \mathcal{Q}(f), \qquad t \ge 0,\ x \in \mathbb{R}^n,\ v \in \mathbb{R}^n \tag{1.4}$$

where $n \ge 1$ and $\mathcal{Q}(f)$ is the so-called "collision operator" which can take either a linear or a non linear form. The solution f corresponds at each time t to the density of particles at the point x with velocity v. If

$$\mathcal{Q}(f) = \text{div}_v\left(\nabla_v f + v\, f\right),$$

then (1.4) *becomes the prototype of the linear Fokker-Planck equation (see, for instance, [8] and [32]) and it can be written in the form* (1.1) *by choosing $m_0 = n$, $N = 2n$ and*

$$B = \begin{pmatrix} I_n & I_n \\ 0 & 0 \end{pmatrix},$$

where I_n is the identity $n \times n$ matrix.

In the Boltzmann-Landau equation (see [20], [6] and [21])

$$\mathcal{Q}(f) = \sum_{i,j=1}^{n} \partial_{v_i}\left(a_{ij}(\cdot, f)\partial_{v_j} f\right),$$

the coefficients a_{ij} depend on the unknown function through some integral expression.

Example 1.2 *Equations of the form* (1.1) *arise in mathematical finance as well. More specifically, the following linear KFP equation*

$$S^2\partial_{SS}V + f(S)\partial_M V - \partial_t V = 0, \qquad S, t > 0,\ M \in \mathbb{R}$$

with either $f(S) = \log S$ *or* $f(S) = S$*, arises in the Black & Scholes theory when considering the problem of the pricing of Asian options (see [3]). Moreover, in the stochastic volatility model by Hobson & Rogers, the price of an European option is given as a solution of the KFP equation*

$$\frac{1}{2}\sigma^2(S-M)\left(\partial_{SS}V - \partial_S V\right) + (S-M)\partial_M V - \partial_t V = 0,$$

for some positive continuous function σ *(see [15] and [9]). In the theory of bonds and interest rates, KFP equations are considered in the study of the possible realization of Heath-Jarrow-Morton [14] models in terms of a finite dimensional Markov diffusion (see, for instance, [33] and [4]). We finally recall that nonlinear KFP equations of the form*

$$\Delta_x u + h(u)\partial_y u - \partial_t u = f(\cdot, u), \qquad (x, y, t) \in \mathbb{R}^{m_0} \times \mathbb{R} \times \mathbb{R}.$$

occur in the theory of stochastic utility theory (see [1], [2], and [7]).

It is well known that the natural geometric setting for the study of KFP operators is the analysis on Lie groups (see for instance Folland [13], Rothschild and Stein [34]).

The theory has been widely developed in the simplest case of *homogeneous* Lie groups. A systematic study of this class of operators, when the coefficents a_{ij} are constant, has been carried out by Kupcov [18], and by Lanconelli and Polidoro [19]. The existence of a fundamental solution has been proved by Weber [36], Il'in [17], Eidelman [12] and Polidoro [30], [29] in the case of Hölder continuous coefficients a_{ij}. Pointwise upper and lower bound for the fundamental solution, mean value formulas and Harnack inequalities are given in [30], [29]; Schauder type estimates have been proved by Satyro [35], Lunardi [22], Manfredini [23].

In the more general case of non-homogeneous groups, the existence of a fundamental solution has been proved in [19] for KFP operators with constant coefficients and by Di Francesco and Pascucci in [10] for Hölder continuous coefficients. We also recall some mean value formulas proved by Morbidelli in [25] and Harnack type inequalities in [25] and in [11].

Concerning the regularity of the weak solutions to (1.1), we recall the papers [5], [24], [31], where the coefficients a_{ij} satisfy a suitable vanishing mean oscillation condition. In [28] we proved some pointwise estimate for the weak solutions to (1.1) by adapting a classical iterative method introduced by Moser [26, 27] to the non Euclidean framework of the Lie groups. In [28] we confined ourselves to the simplest case of homogeneous Lie groups; the aim of this paper is to remove that assumption, in view of the above applications to physics and finance.

We recall that the Moser's method is based on a combination of a Caccioppoli type estimate with the classical embedding Sobolev inequality. Due to the strong degeneracy of the KFP operators, we encountered in [28] a new difficulty: the natural extension of the Caccioppoli estimates gives an $L^2_{\rm loc}$ bound only of the first order derivatives $\partial_{x_1}u, \dots, \partial_{x_{m_0}}u$ of the solution u of (1.1), but it does not give any information on the other spatial directions.

The main idea used in [28] is to prove a Sobolev type inequality only for the solutions to (1.1), by using a representation formula for the solution u in terms of the fundamental solution of the principal part K of L. More specifically, let u be a solution to (1.1), then

$$Ku = (K - L)u = \sum_{i=1}^{m_0} \partial_{x_i} F_i, \tag{1.5}$$

where

$$F_i = \sum_{j=1}^{m_0} (\delta_{ij} - a_{ij})\, \partial_{x_j} u, \qquad i = 1, \dots, m_0.$$

Since the F_i's depend only on the first order derivatives $\partial_{x_j} u$, $j = 1, ..., m_0$, the Caccioppoli inequality yields an $H^{-1}_{\rm loc}$-estimate of the right hand side of (1.5). Thus, by using some potential estimate for the fundamental solution of K, we prove the needed bound for the $L^p_{\rm loc}$ norm of u.

The proof of the Caccioppoli type inequality plainly extends to non-homogeneous groups, whereas the Sobolev inequalities used in [28] heavily rely on the homogeneity of the fundamental solution. The main results of this paper are some L^p potential estimates for the convolution with the non-homogeneous fundamental solution Γ of K and with the derivatives $\partial_{x_1}\Gamma, \dots, \partial_{x_{m_0}}\Gamma$, that are given in Section 3, Theorem 3.1. Section 2 contains some known facts about K and on the related Lie group. Section 4 is devoted to the Moser's iterative procedure.

In order to state our main results we introduce some notations. We denote by $D = (\partial_{x_1}, \dots, \partial_{x_N})$, $\langle \cdot, \cdot \rangle$ respectively the gradient and the inner product in $\mathbb{R}^N$. Besides, D_{m_0} is the gradient with respect to the variables $x_1, \dots, x_{m_0}$. We also write operator L in (1.1) and the vector field Y defined in (1.3) in a more compact form:

$$L = \mathrm{div}(AD) + Y, \qquad Y = \langle x, BD \rangle - \partial_t \tag{1.6}$$

where $A = (a_{ij})_{1 \le i,j \le N}$, $a_{ij} \equiv 0$ if $i > m_0$ or $j > m_0$.

Definition 1.3 *A weak solution of* (1.1) *in a subset* Ω *of* $\mathbb{R}^{N+1}$ *is a function* u *such that* $u, D_{m_0}u, Yu \in L^2_{loc}(\Omega)$ *and*

$$\int_{\Omega} -\langle ADu, D\varphi \rangle + \varphi Y u = 0, \qquad \forall \varphi \in C_0^\infty(\Omega). \tag{1.7}$$

In the sequel we will also consider *weak sub-solutions* of (1.1), namely functions u such that $u, D_{m_0}u, Yu \in L^2_{\rm loc}(\Omega)$ and

$$\int_{\Omega} -\langle ADu, D\varphi \rangle + \varphi Y u \ge 0, \qquad \forall \varphi \in C_0^\infty(\Omega),\ \varphi \ge 0. \tag{1.8}$$

Moreover u is a *weak super-solution* of (1.1) if $-u$ is a sub-solution. Clearly, if u is a sub and super-solution of (1.1), then it is a solution.

As we shall see in Section 2, the natural geometry underlying operator L is determined by a suitable homogeneous Lie group structure on $\mathbb{R}^{N+1}$. Our main results below reflect this non-Euclidean background. Let "$\circ$" denote the Lie product on $\mathbb{R}^{N+1}$ defined in (2.2), and consider the cylinder

$$R_1 = \{(x,t) \in \mathbb{R}^N \times \mathbb{R} \ \mid \ |x| < 1, \ |t| < 1\}.$$

For every $z_0 \in \mathbb{R}^{N+1}$ and $r > 0$, we set

$$R_r(z_0) \equiv z_0 \circ \delta_r(R_1) = \{z \in \mathbb{R}^{N+1} \ \mid \ z = z_0 \circ \delta_r(\zeta), \ \zeta \in R_1\}. \tag{1.9}$$

We also denote $R_r = R_r(0)$. Our main result is the following

Theorem 1.4 *Let u be a non-negative weak solution of* (1.1) *in Ω. Let $z_0 \in \Omega$ and r, ϱ, $0 < \frac{r}{2} \le \varrho < r \le 1$, be such that $\overline{R_r(z_0)} \subseteq \Omega$. Then there exists a positive constant c which depends on μ and on the homogeneous dimension Q (cf.* (2.10)*) such that, for every $p > 0$, it holds*

$$\sup_{R_\varrho(z_0)} u^p \le \frac{c}{(r-\varrho)^{Q+2}} \int_{R_r(z_0)} u^p. \tag{1.10}$$

Estimate (1.10) *also holds for every $p < 0$ such that $u^p \in L^1(R_r(z_0))$.*

Remark 1.5 *Sub and super-solutions also verify estimate* (1.10) *for suitable values of p (see Corollary 4.5). More precisely,* (1.10) *holds for*

(i) $p \ge 1$ or $p < 0$, if u is a non-negative weak sub-solution of (1.1) *such that $u^p \in L^1(R_r(z_0))$;*

(ii) $p \in]0, \frac{1}{2}[$, if u is a non-negative weak super-solution of (1.1)*. In this case, the constant c in* (1.10) *also depends on p.*

A direct consequence of Theorem 1.4 is the local boundedness of weak solutions to (1.1).

Corollary 1.6 *Let u be a weak solution of* (1.1) *in Ω. Let z_0, ϱ, r as in Theorem 1.4. Then, we have*

$$\sup_{R_\varrho(z_0)} |u| \le \left(\frac{c}{(r-\varrho)^{Q+2}} \int_{R_r(z_0)} |u|^p \right)^{\frac{1}{p}}, \qquad \forall p \ge 1, \tag{1.11}$$

where $c = c(Q, \mu)$.

The following result restores the analogy with the classical result by Moser. Denote $R_r^-(x_0,t_0) = R_r(x_0,t_0) \cap \{t < t_0\}$, then

Proposition 1.7 *Let u be a non-negative weak solution of* (1.1) *in Ω. Let $z_0 \in \Omega$ and r, ϱ, $0 < \frac{r}{2} \le \varrho < r \le 1$, be such that $\overline{R_r^-(z_0)} \subseteq \Omega$. Suppose that $u^p \in L^1(R_r^-(z_0))$, for some $p < 0$. Then there exists a positive constant c which depends on μ and on the homogeneous dimension Q such that*

$$\sup_{R_\varrho^-(z_0)} u^p \le \frac{c}{(r-\varrho)^{Q+2}} \int_{R_r^-(z_0)} u^p.$$

2 Preliminaries

In this section we recall some known facts about the principal part K of L, and we give some preliminary results. We first recall that K is invariant with respect to a Lie product in $\mathbb{R}^{N+1}$. More specifically, we let

$$E(s) = \exp(-sB^T), \qquad s \in \mathbb{R}, \tag{2.1}$$

and we denote by ℓ_ζ, $\zeta \in \mathbb{R}^{N+1}$, the left translation $\ell_\zeta(z) = \zeta \circ z$ in the group law

$$(x,t) \circ (\xi,\tau) = (\xi + E(\tau)x, t+\tau), \qquad (x,t),(\xi,\tau) \in \mathbb{R}^{N+1}, \tag{2.2}$$

then we have

$$K \circ \ell_\zeta = \ell_\zeta \circ K.$$

We recall that, by Proposition 2.1 of [19], hypothesis [H.1] is equivalent to assume that for some basis on $\mathbb{R}^N$, the matrix B has the canonical form

$$\begin{pmatrix} * & B_1 & 0 & \cdots & 0 \\ * & * & B_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & B_r \\ * & * & * & \cdots & * \end{pmatrix} \tag{2.3}$$

where B_k is a $m_{k-1} \times m_k$ matrix of rank m_k, $k = 1, 2, \dots, r$ with

$$m_0 \geq m_1 \geq \dots m_r \geq 1, \qquad \text{and} \qquad \sum_{k=0}^{r} m_k = N,$$

and the blocks denoted by "$*$" are arbitrary.

We denote by $\Gamma(\cdot,\zeta)$ the fundamental solution of K in (1.2) with pole in $\zeta \in \mathbb{R}^{N+1}$. An explicit expression of $\Gamma(\cdot,\zeta)$ has been constructed in [16] and [18]:

$$\Gamma(z,\zeta) = \Gamma(\zeta^{-1} \circ z, 0), \qquad \forall z,\zeta \in \mathbb{R}^{N+1},\ z \neq \zeta,$$

where

$$\Gamma((x,t),(0,0)) = \begin{cases} \frac{(4\pi)^{-\frac{N}{2}}}{\sqrt{\det \mathcal{C}(t)}} \exp\left(-\frac{1}{4}\langle \mathcal{C}^{-1}(t)x, x\rangle - t\,\mathrm{tr}(B)\right) & \text{if } t > 0, \\ 0 & \text{if } t \leq 0, \end{cases} \tag{2.4}$$

and

$$\mathcal{C}(t) = \int_0^t E(s) \begin{pmatrix} I_{m_0} & 0 \\ 0 & 0 \end{pmatrix} E^T(s) ds$$

($E(\cdot)$ is the matrix defined in (2.1) and I_{m_0} is the $m_0 \times m_0$ identity matrix). Note that hypothesis [H.1] implies that $\mathcal{C}(t)$ is strictly positive for every positive t (see Proposition A.1

in [19]). If we denote by K^* the formal adjoint of $K : K^* = \Delta_{m_0} + Y^*$, and by Γ^* its fundamental solution, then

$$\Gamma^*(z,\zeta) = \Gamma(\zeta, z), \qquad \text{for every } z,\zeta \in \mathbb{R}^{N+1} : z \neq \zeta. \tag{2.5}$$

Let us explicitly note that, since $Y^* = -Y - \operatorname{tr} B$, we have

$$\int_{\mathbb{R}^N} \Gamma^*(x,t,\xi,0)\, d\xi = e^{t \operatorname{tr} B}, \qquad \text{for every } (x,t) \in \mathbb{R}^N \times \mathbb{R}^+.$$

Let $K_0 = \Delta_{m_0} + Y_0$ be an operator satisfying condition [H.1], where $Y_0 = \langle x, B_0 D\rangle - \partial_t$ and

$$B_0 = \begin{pmatrix} 0 & B_1 & 0 & \cdots & 0 \\ 0 & 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & B_r \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{2.6}$$

Then K_0 is invariant with respect to the dilations defined as

$$\delta_\lambda = \operatorname{diag}(\lambda I_{m_0}, \lambda^3 I_{m_1}, \dots, \lambda^{2r+1} I_{m_r}, \lambda^2) = \operatorname{diag}(D_\lambda, \lambda^2), \qquad \lambda > 0, \tag{2.7}$$

where I_{m_k} denotes the $m_k \times m_k$ identity matrix. More specifically (see Proposition 2.2 of [19]) we have that

$$K_0 \circ \delta_\lambda = \lambda^2 \left(\delta_\lambda \circ K_0\right), \qquad \text{for every } \lambda > 0. \tag{2.8}$$

In (2.8) "$\circ$" denotes the composition law related to K_0. The converse implication is also true: K_0 is invariant with respect to the dilations $(\delta_\lambda)_{\lambda>0}$ if, and only if, the $*$-blocks of B in (2.3) are zero matrices. In that case the corresponding matrices E_0 and $\mathcal{C}_0^{-1}$ satisfy

$$E_0(\lambda^2 s) = D_\lambda E_0(s) D_{\frac{1}{\lambda}}, \qquad \mathcal{C}_0^{-1}(\lambda^{-2} t) = D_\lambda \mathcal{C}_0^{-1}(t) D_\lambda \tag{2.9}$$

for any $s,t \in \mathbb{R}$ and $\lambda > 0$. The fundamental solution Γ_0 of K_0 is a homogeneous function with respect to $(\delta_\lambda)_{\lambda>0}$, namely

$$\Gamma_0(\delta_\lambda(z), 0) = \lambda^{-Q} \Gamma_0(z, 0), \qquad \text{for every } z \in \mathbb{R}^{N+1} \setminus \{0\},\ \lambda > 0,$$

where

$$Q = m_0 + 3m_1 + \cdots + (2r+1)m_r. \tag{2.10}$$

We denote by $\|\cdot\|$ the following norm:

$$\|z\| \equiv \left(\sum_{j=1}^{N} x_j^{\alpha_j} + |t|^{\frac{(2r+1)!}{2}} \right)^{\frac{1}{(2r+1)!}}$$

where $\alpha_j = (2r+1)!$ if $1 \le j \le m_0$ and

$$\alpha_j = \frac{(2r+1)!}{2k+1}, \qquad \text{if } 1 + \sum_{i=0}^{k-1} m_i \le j \le \sum_{i=0}^{k} m_i,\ 1 \le k \le r.$$

Note that $\|\cdot\|$ is δ_λ-homogeneous of degree 1, i.e.

$$\|\delta_\lambda z\| = \lambda \|z\| \qquad \text{for every } \lambda > 0. \tag{2.11}$$

We will denote by

$$\mathcal{B}(\zeta, \varrho) = \left\{ z \in \mathbb{R}^{N+1} : \|\zeta^{-1} \circ z\| < \varrho \right\} \tag{2.12}$$

the ball with center at $\zeta \in \mathbb{R}^{N+1}$ and radius $\varrho > 0$. As we will see in Section 3 (formula (3.17)) we have

$$\operatorname{meas} \mathcal{B}(\zeta, \varrho) = \varrho^{Q+2} \operatorname{meas} \mathcal{B}(0, 1),$$

("meas" denotes the Lebesgue measure) then the natural number $Q + 2$ will be called the *homogeneous dimension of* $\mathbb{R}^{N+1}$ *with respect to* $(\delta_\lambda)_{\lambda>0}$.

It is known that homogeneous operators provide a good approximation of the non-homogeneous ones. In order to be more specific, consider any operator $K = \Delta_{m_0} + \langle x, BD \rangle - \partial_t$ satisfying condition [H.1]. Denote by B_0 the $N \times N$ matrix obtained by annihilating the $*$-blocks of B, define $K_0 = \Delta_{m_0} + \langle x, B_0 D \rangle - \partial_t$ and denote by Γ_0 its fundamental solution of K_0. Then, for every $b > 0$, there exists a positive constant a such that

$$\frac{1}{a} \Gamma_0(z) \leq \Gamma(z) \leq a\, \Gamma_0(z)$$

for every $z \in \mathbb{R}^{N+1}$ such that $\Gamma_0(z) \geq b$ (see [19], Theorem 3.1). The above result says that, in some sense, Γ_0 shares some homogeneity properties with Γ, so that we can use the norm $\|\cdot\|$ also when K is not invariant with respect to $(\delta_\lambda)_{\lambda>0}$. We explicitly note that the dilations $(\delta_\lambda)_{\lambda>0}$ only depend on the matrix B.

For every $\lambda \in]0, 1]$ we set

$$K_\lambda = \lambda^2 \left(\delta_\lambda \circ K \circ \delta_{\frac{1}{\lambda}} \right). \tag{2.13}$$

In order to explicitly write K_λ and its fundamental solution, we note that, if

$$B = \begin{pmatrix} B_{0,0} & B_1 & 0 & \cdots & 0 \\ B_{1,0} & B_{1,1} & B_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B_{r-1,0} & B_{r-1,1} & B_{r-1,2} & \cdots & B_r \\ B_{r,0} & B_{r,1} & B_{r,2} & \cdots & B_{r,r} \end{pmatrix}$$

where $B_{i,j}$ are the $m_i \times m_j$ blocks denoted by "$*$" in (2.3), then $K_\lambda = \Delta_{m_0} + Y_\lambda$, where

$$Y_\lambda := \langle x, B_\lambda D \rangle - \partial_t \tag{2.14}$$

and

$$B_\lambda = \lambda^2 D_\lambda B D_{\frac{1}{\lambda}} = \begin{pmatrix} \lambda^2 B_{0,0} & B_1 & 0 & \cdots & 0 \\ \lambda^4 B_{1,0} & \lambda^2 B_{1,1} & B_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda^{2r} B_{r-1,0} & \lambda^{2r-2} B_{r-1,1} & \lambda^{2r-4} B_{r-1,2} & \cdots & B_r \\ \lambda^{2r+2} B_{r,0} & \lambda^{2r} B_{r,1} & \lambda^{2r-2} B_{r,2} & \cdots & \lambda^2 B_{r,r} \end{pmatrix} \tag{2.15}$$

The fundamental solution Γ_λ of K_λ is given by

$$\Gamma_\lambda\left((x,t),(0,0)\right) = \begin{cases} \frac{(4\pi)^{-\frac{N}{2}}}{\sqrt{\det \mathcal{C}_\lambda(t)}} \exp\left(-\frac{1}{4}\langle \mathcal{C}_\lambda^{-1}(t)x, x\rangle - t\,\mathrm{tr}(B_\lambda)\right) & \text{if } t > 0, \\ 0 & \text{if } t \le 0, \end{cases} \tag{2.16}$$

with

$$E_\lambda(s) = \exp(-sB_\lambda^T), \qquad \mathcal{C}_\lambda(t) = \int_0^t E_\lambda(s) \begin{pmatrix} I_{m_0} & 0 \\ 0 & 0 \end{pmatrix} E_\lambda^T(s)ds. \tag{2.17}$$

Since the translation group related to K_λ depends on λ, it will be denoted by "$\circ_\lambda$":

$$(x,t)\circ_\lambda(\xi,\tau) = (\xi + E_\lambda(\tau)x, t+\tau), \qquad (x,t),(\xi,\tau) \in \mathbb{R}^{N+1}. \tag{2.18}$$

We recall that, for every given $T > 0$, there exists a positive constant c_T such that

$$\begin{aligned} \langle \mathcal{C}_0(t)x, x\rangle \left(1 - c_T\lambda^2 t\right) \le \langle \mathcal{C}_\lambda(t)x, x\rangle \le \langle \mathcal{C}_0(t)x, x\rangle \left(1 + c_T\lambda^2 t\right), \\ \left\langle \mathcal{C}_0^{-1}(t)y, y\right\rangle \left(1 - c_T\lambda^2 t\right) \le \left\langle \mathcal{C}_\lambda^{-1}(t)y, y\right\rangle \le \left\langle \mathcal{C}_0^{-1}(t)y, y\right\rangle \left(1 + c_T\lambda^2 t\right); \end{aligned} \tag{2.19}$$

for every $x, y \in \mathbb{R}^N, t \in]0,T]$ and $\lambda \in [0,1]$ (see [19], formulas (3.23) and (3.24)). In the sequel we will also use the following result

Lemma 2.1 *Let $T > 0$ and c_T as above. Then:*

i) there exists a positive constant c_T' such that

$$\left\| D_{\sqrt{t}}\left(\mathcal{C}_\lambda^{-1}(t) - \mathcal{C}_0^{-1}(t)\right) D_{\sqrt{t}} \right\| \le c_T'\lambda^2 t$$

for every $t \in]0,T]$ and $\lambda \in [0,1]$;

ii) there exist two positive constants c_T'', c_T''' such that

$$c_T'' t^Q (1 - c_T\lambda^2 t) \le \det \mathcal{C}_\lambda(t) \le c_T''' t^Q (1 + c_T\lambda^2 t),$$

for every $(x,t) \in \mathbb{R}^N \times]0,T]$ and $\lambda \in [0,1]$ such that $t < \frac{1}{c_T}$.

PROOF. *i)* Since $\mathcal{C}_\lambda^{-1}(t)$ and $\mathcal{C}_0^{-1}(t)$ are symmetric, we have

$$\begin{aligned} \left\| D_{\sqrt{t}}\left(\mathcal{C}_\lambda^{-1}(t) - \mathcal{C}_0^{-1}(t)\right) D_{\sqrt{t}} \right\| &= \sup_{|y|\le 1} \left\langle \left(\mathcal{C}_\lambda^{-1}(t) - \mathcal{C}_0^{-1}(t)\right) D_{\sqrt{t}}y, D_{\sqrt{t}}y \right\rangle \le \\ & c_T\lambda^2 t \sup_{|y|\le 1} \left\langle \mathcal{C}_0^{-1}(t) D_{\sqrt{t}}y, D_{\sqrt{t}}y \right\rangle = c_T\lambda^2 t \sup_{|y|\le 1} \left\langle \mathcal{C}_0^{-1}(1)y, y\right\rangle, \end{aligned}$$

by the second set of inequalities in (2.19) and the second identity in (2.9). This proves the claim.
ii) Let μ_k be the k-th eigenvalue of $D_{\frac{1}{\sqrt{t}}}\mathcal{C}_\lambda(t)D_{\frac{1}{\sqrt{t}}}$, and let v_k be one of the corresponding eigenvector; it is not restrictive to assume that $|v_k| = 1$. Then (2.19) yields

$$\left\langle \mathcal{C}_0(t)D_{\frac{1}{\sqrt{t}}}v_k, D_{\frac{1}{\sqrt{t}}}v_k \right\rangle (1 - c_T\lambda^2 t) \le \mu_k \le \left\langle \mathcal{C}_0(t)D_{\frac{1}{\sqrt{t}}}v_k, D_{\frac{1}{\sqrt{t}}}v_k \right\rangle (1 + c_T\lambda^2 t);$$

so that, by (2.9),

$$\langle \mathcal{C}_0(1)v_k, v_k\rangle\,(1 - c_T\lambda^2 t) \le \mu_k \le \langle \mathcal{C}_0(1)v_k, v_k\rangle\,(1 + c_T\lambda^2 t).$$

Since $\det\left(D_{\frac{1}{\sqrt{t}}}\mathcal{C}_\lambda(t)D_{\frac{1}{\sqrt{t}}}\right)$ is the product of its eigenvalues, from the above inequality it follows that

$$c''_T(1 - c_T\lambda^2 t)^N \le \det\left(D_{\frac{1}{\sqrt{t}}}\mathcal{C}_\lambda(t)D_{\frac{1}{\sqrt{t}}}\right) \le c'''_T(1 + c_T\lambda^2 t)^N,$$

for suitable positive constants c''_T, c'''_T, provided that t is suitable small. Thus, *(ii)* follows from the fact that $\det D_{\sqrt{t}} = t^{\frac{Q}{2}}$. □

3 Potential estimates

In this section we prove some L^q estimates of the Γ_λ-*potential* of a function $f \in L^p(\mathbb{R}^{N+1})$:

$$\Gamma_\lambda(f)(z) := \int_{\mathbb{R}^{N+1}} \Gamma_\lambda(z,\zeta)f(\zeta)d\zeta, \qquad z \in \mathbb{R}^{N+1}. \tag{3.1}$$

We will also consider the potential $\Gamma_\lambda(D_{m_0}f)$, i.e.

$$\Gamma_\lambda(D_{m_0}f)(z) := -\int_{\mathbb{R}^{N+1}} D^{(\zeta)}_{m_0}\Gamma_\lambda(z,\zeta)f(\zeta)d\zeta \tag{3.2}$$

where $D_{m_0}\Gamma_\lambda(x,t,\xi,\tau)$ is the gradient with respect to the variables $x_1,\dots,x_{m_0}$ and the superscript in $D^{(\zeta)}_{m_0}$ indicates that we are differentiating w.r.t. the variable ζ.

The main result of this section is the following L^p estimate in the domain $S_T = \mathbb{R}^N \times]0,T]$.

Theorem 3.1 *Let $f \in L^2(S_T)$. There exists a positive constant $c = c(T,B)$ such that*

$$\|\Gamma_\lambda(f)\|_{L^{2\widetilde{\kappa}}(S_T)} \le c\|f\|_{L^2(S_T)}, \tag{3.3}$$

$$\|\Gamma_\lambda(D_{m_0}f)\|_{L^{2\kappa}(S_T)} \le c\|f\|_{L^2(S_T)}, \tag{3.4}$$

for every $\lambda \in]0,1]$, where $\widetilde{\kappa} = 1 + \frac{4}{Q-2}$ and $\kappa = 1 + \frac{2}{Q}$.

We first prove an uniform (in λ) pointwise bound for Γ_λ and $D^{(\zeta)}_{m_0}\Gamma_\lambda$.

Proposition 3.2 *For every $T > 0$ there exists a positive constant C_T such that:*

$$\Gamma_\lambda(z,\zeta) \le \frac{C_T}{\|\zeta^{-1}\circ_\lambda z\|^Q}, \tag{3.5}$$

$$\left|D^{(\zeta)}_{m_0}\Gamma_\lambda(z,\zeta)\right| \le \frac{C_T}{\|\zeta^{-1}\circ_\lambda z\|^{Q+1}}, \tag{3.6}$$

for every $z,\zeta \in S_T$ and $\lambda \in]0,1]$.

PROOF. Let $z = (x,t), \zeta = (\xi,\tau) \in S_T$ and $\lambda \in]0,1]$. Denote $w = (y,s) = \zeta^{-1} \circ_\lambda z = (x - E_\lambda(t-\tau)\xi, t-\tau)$. Then, in order to prove (3.5), it is sufficient to show that

$$\|w\|^Q \Gamma_\lambda(w) \le C_T, \qquad \text{for every } w \in S_T \text{ and } \lambda \in]0,1]. \tag{3.7}$$

By (2.16) and Lemma 2.1 we get

$$\Gamma_\lambda(y,s) \le \frac{(4\pi)^{-\frac{N}{2}}}{\sqrt{c''_T s^Q (1 - c_T \lambda^2 s)}} \exp\left(-\frac{1}{4} \langle \mathcal{C}_0^{-1}(s) y, y \rangle (1 - c_T \lambda^2 s) - \lambda^2 s \operatorname{tr}(B) \right)$$

for $s < \frac{1}{c_T}$. On the other hand, (2.9) yields

$$\langle \mathcal{C}_0^{-1}(s) y, y \rangle = \left\langle \mathcal{C}_0^{-1}(1) D_{\frac{1}{\sqrt{s}}} y, D_{\frac{1}{\sqrt{s}}} y \right\rangle \ge \left| D_{\frac{1}{\sqrt{s}}} y \right|^2 \min_{|\eta|=1} \langle \mathcal{C}_0^{-1}(1) \eta, \eta \rangle,$$

and (2.11) gives

$$\|(y,s)\| = \left\| \left(D_{\sqrt{s}} D_{\frac{1}{\sqrt{s}}} y, s \right) \right\| = \sqrt{s} \left\| \left(D_{\frac{1}{\sqrt{s}}} y, 1 \right) \right\| \le \widetilde{c} \sqrt{s} \left(\left| D_{\frac{1}{\sqrt{s}}} y \right| + 1 \right), \tag{3.8}$$

for a constant $\widetilde{c}$ only dependent on the norm. Hence,

$$\|(y,s)\|^Q \Gamma_\lambda(y,s) \le C_0 \left(\left| D_{\frac{1}{\sqrt{s}}} y \right| + 1 \right)^Q \exp\left(-c_0 \left| D_{\frac{1}{\sqrt{s}}} y \right|^2 \right) \tag{3.9}$$

for every $(y,s) \in S_T$ such that $s < \frac{1}{2c_T}$, where the constants C_0, c_0 only depend on T and on the matrix B. This proves the claim (3.7) for $s < \frac{1}{2c_T}$.

If $\frac{1}{2c_T} > T$ the proof is accomplished, otherwise we have to show that (3.7) holds in the set $\mathbb{R}^N \times [\frac{1}{2c_T}, T]$. To that aim, we observe that $\det \mathcal{C}_\lambda(s)$ is a positive function which continuously depend on (s,λ) in the compact set $[\frac{1}{2c_T}, T] \times [0,1]$. The same assertion holds for the positive matrix $D_{\sqrt{s}} \mathcal{C}_\lambda^{-1}(s) D_{\sqrt{s}}$. Then

$$\Gamma_\lambda(y,s) \le C_1 \exp\left(-\frac{c_1}{4} \left| D_{\frac{1}{\sqrt{s}}} y \right|^2 \right),$$

where

$$C_1 = \frac{(4\pi)^{-\frac{N}{2}} e^{T|\operatorname{tr}(B)|}}{\sqrt{\min\limits_{[\frac{1}{2c_T}, T] \times [0,1]} \det \mathcal{C}_\lambda(s)}}, \qquad c_1 = \min_{[\frac{1}{2c_T}, T] \times [0,1]} \min_{|\eta|=1} \left\langle D_{\sqrt{s}} \mathcal{C}_\lambda^{-1}(s) D_{\sqrt{s}} \eta, \eta \right\rangle.$$

Then, by using again (3.8), we get

$$\|(y,s)\|^Q \Gamma_\lambda(y,s) \le C_1 s^{\frac{Q}{2}} \widetilde{c}^Q \left(\left| D_{\frac{1}{\sqrt{s}}} y \right| + 1 \right)^Q \exp\left(-\frac{c_1}{4} \left| D_{\frac{1}{\sqrt{s}}} y \right|^2 \right),$$

for every $(y,s) \in \mathbb{R}^N \times [\frac{1}{2c_T}, T]$. This proves that

$$\|(y,s)\|^Q \Gamma_\lambda(y,s) \le C_2 \left(\left| D_{\frac{1}{\sqrt{s}}} y \right| + 1 \right)^Q \exp\left(-c_2 \left| D_{\frac{1}{\sqrt{s}}} y \right|^2 \right) \tag{3.10}$$

for every $(y,s) \in S_T$, where the constants C_2, c_2 only depend on T and on the matrix B. Since the right hand side of (3.10) is a bounded function, we get the claim (3.7).

In order to simplify the proof of (3.6), we first observe that (2.5) implies

$$D^{(\zeta)}_{m_0}\Gamma_\lambda(z,\zeta) = D_{m_0}\Gamma^*_\lambda(\zeta, z),$$

so that it is sufficient to consider $\partial_{\xi_j}\Gamma^*_\lambda(\xi,\tau,x,t)$ for $j = 1,\dots,m_0$.

As before, we let $(\eta,\sigma) = z^{-1} \circ_\lambda \zeta = (\xi - E_\lambda(\tau - t)x, \tau - t)$ and we note that

$$\partial_{\eta_j}\Gamma^*_\lambda(\eta,\sigma) = -\frac{1}{2}\Gamma^*_\lambda(\eta,\sigma)\left(\mathcal{C}_\lambda^{-1}(-\sigma)\eta\right)_j \quad \text{for } j = 1,\dots,m_0. \tag{3.11}$$

We next claim that

$$\left|\left(\mathcal{C}_\lambda^{-1}(-\sigma)\eta\right)_j\right| \le \frac{C_3}{\sqrt{-\sigma}}\left|D_{\frac{1}{\sqrt{-\sigma}}}\eta\right| \quad \text{for } j = 1,\dots,m_0, \tag{3.12}$$

for every $(\eta,\sigma) \in \mathbb{R}^N \times [-T,0[$, where the constant C_3 only depends on T and on the matrix B, so that, by (3.8), we obtain

$$\|(\eta,\sigma)\| \cdot \left|\left(\mathcal{C}_\lambda^{-1}(-\sigma)\eta\right)_j\right| \le \widetilde{c}\, C_3\left(\left|D_{\frac{1}{\sqrt{-\sigma}}}\eta\right| + 1\right)^2. \tag{3.13}$$

On the other hand, the same argument used in the proof of (3.10) gives the following estimate

$$\|(\eta,\sigma)\|^Q\Gamma^*_\lambda(\eta,\sigma) \le C_2\left(\left|D_{\frac{1}{\sqrt{-\sigma}}}\eta\right| + 1\right)^Q \exp\left(-c_2\left|D_{\frac{1}{\sqrt{-\sigma}}}\eta\right|^2\right)$$

for every $(\eta,\sigma) \in \mathbb{R}^N \times [-T,0[$, and (3.6) follows from (3.13) and (3.11).

We next prove (3.12):

$$\begin{aligned}
\left|\left(\mathcal{C}_\lambda^{-1}(-\sigma)\eta\right)_j\right| \le & \left|\left(\left(\mathcal{C}_\lambda^{-1}(-\sigma) - \mathcal{C}_0^{-1}(-\sigma)\right)\eta\right)_j\right| + \left|\left(\mathcal{C}_0^{-1}(-\sigma)\eta\right)_j\right| = \\
& \frac{1}{\sqrt{-\sigma}}\left|\left(D_{\sqrt{-\sigma}}(\mathcal{C}_\lambda^{-1}(-\sigma) - \mathcal{C}_0^{-1}(-\sigma))D_{\sqrt{-\sigma}}D_{\frac{1}{\sqrt{-\sigma}}}\eta\right)_j\right| + \\
& \frac{1}{\sqrt{-\sigma}}\left|\left(D_{\sqrt{-\sigma}}\mathcal{C}_0^{-1}(-\sigma)D_{\sqrt{-\sigma}}D_{\frac{1}{\sqrt{-\sigma}}}\eta\right)_j\right| \le \\
& \frac{1}{\sqrt{-\sigma}}\left\|D_{\sqrt{-\sigma}}\left(\mathcal{C}_\lambda^{-1}(-\sigma) - \mathcal{C}_0^{-1}(-\sigma)\right)D_{\sqrt{-\sigma}}\right\| \cdot \left|D_{\frac{1}{\sqrt{-\sigma}}}\eta\right| + \\
& \frac{1}{\sqrt{-\sigma}}\left|\mathcal{C}_0^{-1}(1)D_{\frac{1}{\sqrt{-\sigma}}}\eta\right|,
\end{aligned}$$

by (2.9). From Lemma 2.1–*(i)* it follows that

$$\left|\left(\mathcal{C}_\lambda^{-1}(-\sigma)\eta\right)_j\right| \le \frac{1}{\sqrt{-\sigma}}\left(c'_T\lambda^2(-\sigma) + C_4\right)\left|D_{\frac{1}{\sqrt{-\sigma}}}\eta\right|,$$

where C_4 depends on $\mathcal{C}_0^{-1}(1)$. This proves (3.12). □

In view of Proposition 3.2 we define, for $\lambda \in [0,1], \alpha \in]0, Q+2[$ and $p > 1$

$$I^{\alpha}_{\lambda} f(z) = \big(\mathrm{meas}\, \mathcal{B}_{\lambda}(0,1)\big)^{-\frac{\alpha}{Q+2}} \int_{S_T \cap \{\tau < t\}} \frac{f(\zeta)}{\|\zeta^{-1} \circ_{\lambda} z\|^{\alpha}} d\zeta, \qquad z \in \mathbb{R}^{N+1}, \tag{3.14}$$

where $f \in L^p(S_T)$. We next prove a result which is analogous to the classical potential estimates on homogeneous Lie groups (cf., for instance, Folland [13]).

Proposition 3.3 *Let $\alpha \in]0, Q+2[, \lambda \in [0,1]$ and $T > 0$. Consider $f \in L^p(S_T)$ for some $p \in]1, +\infty[$. Then the function $I^{\alpha}_{\lambda} f$ is defined almost everywhere and there exists a constant $c = c(T, B, p, \alpha)$ such that*

$$\|I^{\alpha}_{\lambda} f\|_{L^q(S_T)} \leq c\ \|f\|_{L^p(S_T)}, \tag{3.15}$$

where q is defined by

$$\frac{1}{q} = \frac{1}{p} + \frac{\alpha}{Q+2} - 1.$$

The proof is analogous to that in the framework of homogeneous Lie groups: the main difference occurs in the change of variable of integration, where some extra terms appear.

Remark 3.4 *Let $T \in \mathbb{R}^+, \lambda \in [0,1]$. For any $f \in L^1(S_T)$ we have*

$$\begin{aligned} &\int_{\mathbb{R}^N \times]0,t[} f(\zeta^{-1} \circ_{\lambda} z) d\zeta = \int_{\mathbb{R}^N \times]0,t[} e^{s\lambda^2 \mathrm{tr} B} f(y,s) dy\, ds, \quad \textit{for every } z = (x,t) \in S_T; \\ &\int_{\mathbb{R}^N \times]\tau, T[} f(\zeta^{-1} \circ_{\lambda} z) dz = \int_{\mathbb{R}^N \times]0, T-\tau[} f(y,s) dy\, ds, \quad \textit{for every } \zeta = (\xi, \tau) \in S_T. \end{aligned} \tag{3.16}$$

Indeed, it suffices to perform the change of variable $\Phi(y,s) = (E_{\lambda}(-s)(x-y), t-s)$ in the first integral and note that $\det E_{\lambda}(-s) = e^{s\lambda^2 trB}$, by (2.15) and (2.17). On the other hand, in the second integral we use the change of variable $\Psi(y,s) = (y + E_{\lambda}(s)\xi, \tau + s)$, and note that $\det J_{\Psi}(w) = 1$.

In particular, the second identity in (3.16) yields that for every ball

$$\mathcal{B}_{\lambda}(\zeta, \varrho) = \big\{ z \in \mathbb{R}^{N+1} : \|\zeta^{-1} \circ_{\lambda} z\| < \varrho \big\}$$

it holds

$$\mathrm{meas}\, \mathcal{B}_{\lambda}(\zeta, \varrho) = \varrho^{Q+2}\, \mathrm{meas}\, \mathcal{B}_{\lambda}(0,1). \tag{3.17}$$

We next prove a Young type inequality for the inhomogeneous Lie group related to K_{λ}. Note that the Lie group corresponding to $\lambda = 0$ is homogeneous and Lemma 3.5 restores the standard Young inequality.

Lemma 3.5 *Let $p, q, r \in [1, \infty]$ be three constant such that $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$ and let $\lambda \in [0,1]$. Let $f \in L^p(S_T)$ and $g \in L^q(S_T)$, then the function $f *_{\lambda} g$ defined as*

$$f *_{\lambda} g(z) = \int_{S_T \cap \{\tau < t\}} f(\zeta^{-1} \circ_{\lambda} z) g(\zeta) d\zeta$$

belongs to $L^r(S_T)$ and

$$\|f *_{\lambda} g\|_{L^r(S_T)} \leq e^{\lambda^2 T \ 1 - \frac{1}{q} \ |\mathrm{tr}(B)|} \|f\|_{L^p(S_T)} \|g\|_{L^q(S_T)}.$$

PROOF. We argue as in the proof of the classical Young's inequality, and use Remark 3.4. For any $\alpha, \beta \in [0,1]$ and $p_1, p_2 \geq 0$ such that $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{r} = 1$ we have

$$
\begin{aligned}
|f *_\lambda g(z)| \leq & \int_{S_T \cap \{\tau < t\}} \left| f(\zeta^{-1} \circ_\lambda z) \right|^{(1-\alpha)} |g(\zeta)|^{(1-\beta)} \left| f(\zeta^{-1} \circ_\lambda z) \right|^{\alpha} |g(\zeta)|^{\beta} \, d\zeta \leq \\
& \left(\int_{S_T \cap \{\tau < t\}} \left| f(\zeta^{-1} \circ_\lambda z) \right|^{(1-\alpha) r} |g(\zeta)|^{(1-\beta) r} \, d\zeta \right)^{\frac{1}{r}} \cdot \\
& \left(\int_{S_T \cap \{\tau < t\}} \left| f(\zeta^{-1} \circ_\lambda z) \right|^{\alpha p_1} d\zeta \right)^{\frac{1}{p_1}} \cdot \left(\int_{S_T \cap \{\tau < t\}} |g(\zeta)|^{\beta p_2} \, d\zeta \right)^{\frac{1}{p_2}}
\end{aligned}
$$

by the Hölder inequality. We then change variable in the last but one integral: by Remark 3.4 we have

$$
\int_{S_T \cap \{\tau < t\}} \left| f(\zeta^{-1} \circ_\lambda z) \right|^{\alpha p_1} d\zeta \leq e^{T \lambda^2 |\mathrm{tr} B|} \int_{\mathbb{R}^N \times]0,t[} |f(w)|^{\alpha p_1} \, dw.
$$

Thus, by integrating in the set S_T, we get

$$
\begin{aligned}
\|f *_\lambda g\|_{L^r(S_T)} \leq & e^{\frac{T \lambda^2 |\mathrm{tr} B|}{p_1}} \, \|f\|^{\alpha}_{L^{\alpha p_1}(S_T)} \, \|g\|^{\beta}_{L^{\beta p_2}(S_T)} \cdot \\
& \left(\int_{S_T} |g(\zeta)|^{(1-\beta) r} \left(\int_{S_T \cap \{t > \tau\}} \left| f(\zeta^{-1} \circ_\lambda z) \right|^{(1-\alpha) r} dz \right) d\zeta \right)^{\frac{1}{r}}.
\end{aligned}
$$

We change again the variable in the last integral: in this case Remark 3.4 gives

$$
\|f *_\lambda g\|_{L^r(S_T)} \leq e^{\frac{T \lambda^2 |\mathrm{tr} B|}{p_1}} \, \|f\|^{\alpha}_{L^{\alpha p_1}(S_T)} \, \|f\|^{1-\alpha}_{L^{(1-\alpha) r}(S_T)} \, \|g\|^{\beta}_{L^{\beta p_2}(S_T)} \, \|g\|^{1-\beta}_{L^{(1-\beta) r}(S_T)} . \tag{3.18}
$$

From this point we conclude the proof as in the classical case: for any given $p, q, r \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$ we choose $\alpha = 1 - \frac{p}{r}, \beta = 1 - \frac{q}{r}$ (note that $\alpha, \beta \in [0,1]$), then $p_1 = \frac{p}{\alpha}, p_2 = \frac{q}{\beta}$ (so that $\alpha p_1 = (1-\alpha) r$ and $\beta p_2 = (1-\beta) r$). The proof of the Proposition then follows from (3.18). □

PROOF OF PROPOSITION 3.3. As in the proof of Lemma 3.5, we follow a classical argument and use Remark 3.4 when it is needed.

We first introduce some standard notation. Consider a measurable function $f : \Omega \to \mathbb{R}$ where Ω denotes a measurable subset of $\mathbb{R}^{N+1}$, and let $\beta_f(a) = \mathrm{meas}\{z \in \Omega : |f(z)| > a\}$ denote its *distribution function*. We say that f belongs to the space $L^p_{\mathrm{w}}(\Omega)$ (for $p \geq 1$) if there exists a positive constant C such that $\beta_f(a) \leq \left(\frac{C}{a}\right)^p$, for every positive a. In that case

$$
\|f\|_{L^p_{\mathrm{w}}(\Omega)} = \inf \left\{ C > 0 : \beta_f(a) \leq \left(\tfrac{C}{a}\right)^p \right\},
$$

is the weak-L^p norm of f. We also recall that

$$
\|f\|_{L^p(\Omega)} = \left(p \int_0^\infty a^{p-1} \beta_f(a) da \right)^{\frac{1}{p}}. \tag{3.19}
$$

In order to prove (3.15), we show that, for every $p, q \in]1, \infty[$ satisfying $\frac{1}{q} + 1 = \frac{1}{p} + \frac{\alpha}{Q+2}$, we have

$$\|I_\lambda^\alpha f\|_{L^q_{\rm w}(S_T)} \le \bar{C} \|f\|_{L^p(S_T)}, \tag{3.20}$$

for a positive constant $\bar{C}$ depending on T, α, p and on the matrix B. The thesis follows from the Marcinkiewicz interpolation theorem.

In order to simplify the proof of (3.20), we assume $\|f\|_{L^p(S_T)} = 1$, since it is not restrictive. Moreover we write equation (3.14) as

$$I_\lambda^\alpha f(z) = \int_{S_T \cap \{\tau < t\}} g_\alpha(\zeta^{-1} \circ_\lambda z) f(\zeta) d\zeta \tag{3.21}$$

where

$$g_\alpha(w) = \frac{\left(\text{meas}\, \mathcal{B}_\lambda(0,1)\right)^{-\frac{\alpha}{Q+2}}}{\|w\|^\alpha}.$$

Note that, by (3.17), g_α has norm equal to one in the space $L_{\rm w}^{\frac{Q+2}{\alpha}}(\mathbb{R}^{N+1})$.

For any $a > 0$, we set

$$b = \left(\frac{a}{2} \left(\frac{Q+2}{q\alpha} \right)^{\frac{p-1}{p}} e^{-\frac{p-1}{p}\lambda^2 T |\text{tr}\, B|} \right)^{\frac{q\alpha}{Q+2}}, \tag{3.22}$$

we define

$$g_\alpha^+(w) = \begin{cases} g_\alpha(w), & \text{if } g_\alpha(w) > b, \\ 0, & \text{otherwise,} \end{cases} \qquad g_\alpha^-(w) = g_\alpha(w) - g_\alpha^+(w),$$

and

$$J_\alpha^+ f(z) = \int_{S_T \cap \{\tau < t\}} g_\alpha^+(\zeta^{-1} \circ_\lambda z) f(\zeta) d\zeta, \quad J_\alpha^- f(z) = \int_{S_T \cap \{\tau < t\}} g_\alpha^-(\zeta^{-1} \circ_\lambda z) f(\zeta) d\zeta.$$

To prove (3.20), we recall (3.21) and note that

$$\beta_{I_\lambda^\alpha f}(a) \le \beta_{J_\alpha^+ f}(a/2) + \beta_{J_\alpha^- f}(a/2). \tag{3.23}$$

We first consider the term $\beta_{J_\alpha^- f}$. By the Hölder inequality we get

$$|J_\alpha^- f(z)| \le \left(\int_{S_T \cap \{\tau < t\}} \left| g_\alpha^-(\zeta^{-1} \circ_\lambda z) \right|^{\frac{p}{p-1}} d\zeta \right)^{\frac{p-1}{p}} \|f\|_{L^p(S_T)} \le$$

$$e^{\lambda^2 T \frac{p-1}{p} |\text{tr}\, B|} \left(\int_{S_T \cap \{\tau < t\}} \left| g_\alpha^-(y,s) \right|^{\frac{p}{p-1}} dy\, ds \right)^{\frac{p-1}{p}},$$

by Remark 3.4, since we assume $\|f\|_{L^p(S_T)} = 1$. By using (3.19) and (3.22) we find

$$\left(\int_{S_T \cap \{\tau < t\}} \left| g_\alpha^-(y,s) \right|^{\frac{p}{p-1}} dy\, ds \right)^{\frac{p-1}{p}} \le \frac{a}{2} e^{-\lambda^2 T \frac{p-1}{p} |\text{tr}\, B|},$$

so that $|J_\alpha^- f(z)| \le \frac{a}{2}$ for every $z \in S_T$. Thus

$$\beta_{J_\alpha^- f}(a/2) = 0. \tag{3.24}$$

We next consider the term $\beta_{J_\alpha^+ f}$. By Lemma 3.5 we have

$$\begin{aligned} \|J_\alpha^+ f\|_{L^p(S_T)} \le & e^{\lambda^2 T \;\; 1-\frac{1}{p} \;\; |\mathrm{tr}(B)|} \left\|g_\alpha^+\right\|_{L^1(S_T)} \|f\|_{L^p(S_T)} \le \\ & e^{\lambda^2 T \;\; 1-\frac{1}{p} \;\; |\mathrm{tr}(B)|} \frac{\alpha}{Q+2-\alpha} \, b^{1-\frac{Q+2}{\alpha}}, \end{aligned}$$

since $\|f\|_{L^p(S_T)} = 1$ and

$$\left\|g_\alpha^+\right\|_{L^1(S_T)} \le \frac{\alpha}{Q+2-\alpha} \, b^{1-\frac{Q+2}{\alpha}},$$

by (3.19). Thus, being $\|J_\alpha^+ f\|_{L^p_{\mathrm{w}}(S_T)} \le \|J_\alpha^+ f\|_{L^p(S_T)}$, we get

$$\beta_{J_\alpha^+ f}(a/2) \le a^{-p} \, e^{\lambda^2 T(p-1)|\mathrm{tr}(B)|} \left(\frac{2\alpha}{Q+2-\alpha} \right)^p b^{p\left(1-\frac{Q+2}{\alpha}\right)} = \bar{C} \, a^{-q},$$

where the $\bar{C}$ is a positive constant that depends on T, α, p and on the matrix B (recall our choice (3.22) of b). Then the above inequality and (3.24) give

$$\beta_{I_\lambda^\alpha f}(a) \le \bar{C} \, a^{-q},$$

for any $a > 0$. This proves (3.20) and concludes the proof. □

Proof of Theorem 3.1. By Proposition 3.2 we get

$$\begin{aligned} |\Gamma_\lambda(f)(z)| &\le \bar{C}_T I_\lambda^Q |f(z)|, \\ |\Gamma_\lambda(D_{m_0} f)(z)| &\le \bar{C}_T I_\lambda^{Q+1} |f(z)|, \end{aligned}$$

for every $z \in S_T$. Estimates (3.3) and (3.4) then follow from Proposition 3.3. □

As in the homogeneous case (see [28], Lemma 2.5) we can use the fundamental solution Γ as a test function in the definition of sub and super-solution.

Lemma 3.6 *Let v be a weak sub-solution of $\mathrm{div}(ADv) + Y_\lambda v = 0$ in Ω. For every $\varphi \in C_0^\infty(\Omega)$, $\varphi \ge 0$, and for almost every $z \in \mathbb{R}^{N+1}$, we have*

$$\int_\Omega -\langle ADv, D\left(\Gamma_\lambda(z,\cdot)\varphi\right)\rangle + \Gamma_\lambda(z,\cdot)\varphi Y_\lambda v \ge 0.$$

An analogous result holds for weak super-solutions.

Proof. For every $\varepsilon > 0$, we set

$$\chi_\varepsilon(z,\zeta) = \chi\left(\frac{\|\zeta^{-1} \circ z\|}{\varepsilon} \right), \qquad z, \zeta \in \mathbb{R}^{N+1},$$

where $\chi \in C^1([0,+\infty[,[0,1])$ is such that $\chi(s) = 0$ for $s \in [0,1]$, $\chi(s) = 1$ for $s \geq 2$ and $0 \leq \chi' \leq 2$. By (1.8), for every $\varepsilon > 0$ and $z \in \mathbb{R}^{N+1}$, we have

$$0 \leq \int_\Omega -\langle ADv, D\left(\Gamma_\lambda(z,\cdot)\chi_\varepsilon(z,\cdot)\varphi\right)\rangle + \Gamma_\lambda(z,\cdot)\chi_\varepsilon(z,\cdot)\varphi Y_\lambda v = -I_{1,\varepsilon}(z) + I_{2,\varepsilon}(z) - I_{3,\varepsilon}(z),$$

where

$$I_{1,\varepsilon}(z) = \int_\Omega \langle ADv, D\left(\Gamma_\lambda(z,\cdot)\right)\rangle \chi_\varepsilon(z,\cdot)\varphi,$$

$$I_{2,\varepsilon}(z) = \int_\Omega \Gamma_\lambda(z,\cdot)\chi_\varepsilon(z,\cdot)\left(-\langle ADv, D\varphi\rangle + \varphi Y_\lambda v\right),$$

$$I_{3,\varepsilon}(z) = \int_\Omega \langle ADv, D\chi_\varepsilon(z,\cdot)\rangle \Gamma_\lambda(z,\cdot)\varphi.$$

Consider the first integral. Since $\varphi\chi_\varepsilon(z,\cdot)ADv \to \varphi ADv$ in $L^2(S_T)$, as $\varepsilon \to 0$, (3.4) of Theorem 3.1 gives

$$I_{1,\varepsilon}(z) \to \int_\Omega \langle ADv, D\left(\Gamma_\lambda(z,\cdot)\right)\rangle\varphi,$$

as $\varepsilon \to 0$, for almost every $z \in S_T$. The same argument applies to the second and third integrals, by (3.3) and, since $I_{3,\varepsilon}(z) \to 0$ as $\varepsilon \to 0$, we conclude the proof. □

We next state, without proof, the following

Lemma 3.7 *Let $f \in C^2 \cap \mathrm{Lip}(\mathbb{R})$ be a monotone non-decreasing function. If f is convex (resp. concave) and u is a weak sub-solution (resp. super-solution) of* (1.1)*, then $v = f(u)$ is a weak sub-solution (resp. super-solution) of* (1.1).

4 The Moser method

In this section we prove Theorem 1.4. We first recall that, in the case of homogeneous Lie groups, it is not restrictive to consider the *unit* cylinder R_1 since the transformations of the form

$$\zeta \longmapsto z_0 \circ \delta_r(\zeta), \qquad r > 0,\ z_0 \in \mathbb{R}^{N+1}, \tag{4.1}$$

preserve the class of differential equations considered. In our setting, we rely on the following result.

Lemma 4.1 *The function u is a weak solution of* (1.1) *in the cylinder $R_r(z_0)$ if and only if v defined by*

$$v(\zeta) = u(z_0 \circ \delta_r(\zeta)), \qquad \zeta \in R_1,$$

is a solution to the equation

$$\widetilde{L}_r v = \mathrm{div}(\widetilde{A}Dv) + Y_r v = 0, \qquad \text{in } R_1, \tag{4.2}$$

where $\widetilde{A}(\zeta) = A(z_0 \circ \delta_r(\zeta))$ *satisfies hypothesis [H.2] with the same constant* μ *as* A*, and* Y_r *is defined in* (2.14)*.*

Proof. Since $Y_r \circ \delta_r = r^2 \delta_r \circ Y$ and Y is ℓ_{z_0}-invariant, we have that

$$Y_r \circ \delta_r \circ \ell_{z_0} = r^2 \delta_r \circ \ell_{z_0} \circ Y,$$

or, more explicitly,

$$Y_r v(\zeta) = Y_r^{(\zeta)} u(z_0 \circ \delta_r(\zeta)) = r^2 \big(Y^{(\zeta)}(u \circ \ell_{z_0})\big)(\delta_r(\zeta)) = r^2 \big(Yu\big)(z_0 \circ \delta_r(\zeta)),$$

where the superscript in $Y_r^{(\zeta)}$ indicates that we are differentiating w.r.t. the variable ζ. On the other hand, recalling (2.2) and (2.7), we clearly have

$$D_{m_0} v(\zeta) = D_{m_0}^{(\zeta)} u(z_0 \circ \delta_r(\zeta)) = r(D_{m_0} u)(z_0 \circ \delta_r(\zeta)).$$

Thus we deduce $\widetilde{L}_r v(\zeta) = r^2 (Lu)(z_0 \circ \delta_r(\zeta))$ and the thesis follows. □

Lemma 4.2 *There exists a constant* $\bar{c} \in]0,1[$ *such that*

$$z \circ_r R_{\bar{c}(1-\varrho)} \subseteq R_1, \tag{4.3}$$

for every $r \in [0,1], \varrho \in]0,1[$ *and* $z \in R_\varrho$*.*

Proof. Let $\varrho \in]0,1[$. By the expression (2.7) of the dilations (δ_λ), we see that

$$R_\varrho \subseteq \{(x,t) \in \mathbb{R}^{N+1} \ \mid \ |x| < \varrho, \ |t| < \varrho^2\}.$$

Then the thesis is a consequence of the following inclusion: there exists a positive constant c such that

$$z \circ_r R_\varepsilon \subseteq \{(\xi,\tau) \ \mid \ |x-\xi| < c\varepsilon, \ |t-\tau| < (c\varepsilon)^2\}, \qquad \forall z \in R_\varrho, \ 0 < \varrho, \varepsilon < 1. \tag{4.4}$$

Indeed, if we choose $\varepsilon \le \frac{1-\varrho}{c}$, we get

$$z \circ_r R_\varepsilon \subseteq R_1, \qquad \forall z \in R_\varrho,$$

and this shows (4.3) with $\bar{c} = c^{-1}$.

We are left with the proof of (4.4). If $\zeta = (\xi,\tau) \in z \circ_r R_\varepsilon$ then

$$\zeta = z \circ_r \bar{z} = (\bar{x} + E_r(\bar{t})x, t + \bar{t})$$

for some $\bar{z} \in R_\varepsilon$. Hence

$$|\xi - x| = \big|\bar{x} + \big(E_r(\bar{t}) - E_r(0)\big)x\big| \le |\bar{x}| + |\bar{t}| \max_{|r|,|s| \le 1} \|E_r'(s)\| \le c\,\varepsilon, \qquad |\tau - t| = |\bar{t}| < \varepsilon^2,$$

where $c = 1 + \max_{|r|,|s| \le 1} \|E_r'(s)\|$. □

As a consequence of the above lemmas, we only consider the unit cylinder in the proof of Theorem 1.4 and prove the claim for the operators of the form $\widetilde{L}_r$. We point out that, since its principal part is K_r, we use the group law "$\circ_r$". Theorem 1.4 is a consequence of the following uniform (in r) Caccioppoli and Sobolev type inequalities.

Theorem 4.3 [Caccioppoli type inequalities] *Let u be a non-negative weak solution of (4.2) for a given $r \in [0,1]$. Let $p \in \mathbb{R}, p \neq 0, p \neq \frac{1}{2}$ and let $\varrho, \bar{\varrho}$ be such that $\frac{1}{2} \leq \varrho < \bar{\varrho} \leq 1$. If $u^p \in L^2(R_{\bar{\varrho}})$ then $D_{m_0} u^p \in L^2(R_\varrho)$ and there exists a constant c, only dependent on the homogeneous dimension Q, such that*

$$\|D_{m_0} u^p\|_{L^2(R_\varrho)} \leq \frac{c\sqrt{\mu(\mu+\varepsilon)}}{\varepsilon(\bar{\varrho}-\varrho)} \|u^p\|_{L^2(R_{\bar{\varrho}})}, \quad \textit{where} \quad \varepsilon = \frac{|2p-1|}{4p}. \tag{4.5}$$

Theorem 4.4 [Sobolev type inequalities]. *Let v be a non-negative weak solution of (4.2), for a given $r \in [0,1]$. Then $v \in L^{2\kappa}_{\text{loc}}(R_1)$, $\kappa = 1 + \frac{2}{Q}$, and there exists a constant c, only dependent on Q and μ, such that*

$$\|v\|_{L^{2\kappa}(R_\varrho)} \leq \frac{c}{\bar{\varrho}-\varrho} \left(\|v\|_{L^2(R_{\bar{\varrho}})} + \|D_{m_0} v\|_{L^2(R_{\bar{\varrho}})} \right), \tag{4.6}$$

for every $\varrho, \bar{\varrho}$ with $\frac{1}{2} \leq \varrho < \bar{\varrho} \leq 1$.

The proof of above estimates can be straightforwardly accomplished proceeding as in Theorems 3.1 and 3.3 in [28], by using the potential estimates of the previous section, and therefore is omitted. We are now in position to prove Theorem 1.4.

Proof of Theorem 1.4. Let u be a positive solution to $Lu = 0$ in $R_r(z_0)$. By Lemma 4.1, the function $v(\zeta) = u(z_0 \circ \delta_r(\zeta))$, is a solution to (4.2) in R_1. We first prove that there exists a positive constant c such that

$$\sup_{R_{\varrho/r}} v^p \leq \frac{c}{(1-\varrho/r)^{Q+2}} \int_{R_1} v^p; \tag{4.7}$$

then, by the change of variable (4.1), we obtain (1.10), since

$$\int_{R_1} u^p(z_0 \circ \delta_r(\zeta)) d\zeta = \frac{1}{r^{Q+2}} \int_{R_r} u^p(z_0 \circ w) dw = \frac{1}{r^{Q+2}} \int_{R_r(z_0)} u^p(z) dz, \tag{4.8}$$

by Remark 3.4.

In order to prove (4.7) it is sufficient to set $\theta = \bar{c}(1-\varrho/r)$, where $\bar{c}$ is the constant in Lemma 4.2, and to prove that

$$\sup_{z \circ_r R_{\frac{\theta}{2}}} v^p \leq \frac{c\,\bar{c}^{Q+2}}{\theta^{Q+2}} \int_{z \circ_r R_\theta} v^p \tag{4.9}$$

for every $z \in R_{\varrho/r}$.

By Lemma 4.1, the function $w(\zeta) = v(z \circ_r \delta_\theta(\zeta))$ is a solution of

$$\bar{L}_{r\theta} w = \operatorname{div}(\bar{A} D w) + Y_{r\theta} w = 0, \qquad \text{in } R_1,$$

where $\bar{A}(\zeta) = \widetilde{A}(z \circ_r \delta_\theta(\zeta))$ satisfies hypothesis [H.2] with the same constant μ as A. Hence, by a change of variables analogous to that in (4.8), in order to prove (4.9), it is sufficient to show that there exists a positive constant c_1, only depending on the constant μ in hypothesis [H.2] and on the matrix B, such that

$$\sup_{R_{\frac{1}{2}}} u^p \le c_1 \int_{R_1} u^p \tag{4.10}$$

for every positive solution u of $\bar{L}_\sigma u = 0$: in particular, we emphasize that c_1 does not depend on $\sigma = r\theta \in [0,1]$.

We next prove (4.10). We first consider the case $p > 0$ which is technically more complicated. Combining Theorems 4.3 and 4.4, we obtain the following estimate: if $q, \delta > 0$ verify the condition

$$|q - 1/2| \ge \delta,$$

then there exists a positive constant $c_\delta = c(\delta, Q, \mu)$, such that

$$\|u^q\|_{L^{2\kappa}(R_\varrho)} \le \frac{c_\delta}{(r-\varrho)^2} \|u^q\|_{L^2(R_r)}, \tag{4.11}$$

for every ϱ, r, $\frac{1}{2} \le \varrho < r \le 1$, and $\sigma \in [0,1]$, where $\kappa = 1 + \frac{2}{Q}$.

Fixed a suitable $\delta > 0$ as we shall specify later and $p > 0$, we iterate inequality (4.11) by choosing

$$\varrho_n = \frac{1}{2}\left(1 + \frac{1}{2^n}\right), \quad p_n = \frac{p\kappa^n}{2}, \qquad n \in \mathbb{N} \cup \{0\}.$$

We set $v = u^{\frac{p}{2}}$. If $p > 0$ is such that

$$|p\kappa^n - 1| \ge 2\delta, \qquad \forall n \in \mathbb{N} \cup \{0\}, \tag{4.12}$$

by (4.11), we obtain

$$\|v^{\kappa^n}\|_{L^{2\kappa}(R_{\varrho_{n+1}})} \le \frac{c_\delta}{(\varrho_n - \varrho_{n+1})^2} \|v^{\kappa^n}\|_{L^2(R_{\varrho_n})}, \qquad \forall n \in \mathbb{N} \cup \{0\}. \tag{4.13}$$

Since

$$\|v^{\kappa^n}\|_{L^{2\kappa}} = \left(\|v\|_{L^{2\kappa^{n+1}}}\right)^{\kappa^n} \quad \text{and} \quad \|v^{\kappa^n}\|_{L^2} = \left(\|v\|_{L^{2\kappa^n}}\right)^{\kappa^n},$$

we can rewrite (4.13) in the form

$$\|v\|_{L^{2\kappa^{n+1}}(R_{\varrho_{n+1}})} \le \left(\frac{c_\delta}{(\varrho_n - \varrho_{n+1})^2}\right)^{\frac{1}{\kappa^n}} \|v\|_{L^{2\kappa^n}(R_{\varrho_n})}.$$

Iterating this inequality, we obtain

$$\|v\|_{L^{2\kappa^{n+1}}(R_{\varrho_{n+1}})} \le \prod_{j=0}^{n} \left(\frac{c_\delta}{(\varrho_j - \varrho_{j+1})^2}\right)^{\frac{1}{\kappa^j}} \|v\|_{L^2(R_1)},$$

and letting n go to infinity, we get

$$\sup_{R_{\frac{1}{2}}} v \leq \bar{c}\ \|v\|_{L^2(R_1)},$$

where

$$\bar{c} = \prod_{j=0}^{\infty} \left(\frac{c_\delta}{(\varrho_j - \varrho_{j+1})^2} \right)^{\frac{1}{\kappa^j}},$$

is a finite constant, dependent on δ. Thus, we have proved (4.10) with $c_1 = \bar{c}^2$, for every $p > 0$ which verifies condition (4.12).

We now make a suitable choice of $\delta > 0$, only dependent on the homogeneous dimension Q, in order to show that (4.10) holds for every positive p. We remark that, if p is a number of the form

$$p_m = \frac{\kappa^m(\kappa+1)}{2}, \qquad m \in \mathbb{Z},$$

then (4.12) is satisfied with $\delta = (2Q+4)^{-1}$, for every $m \in \mathbb{Z}$. Therefore (4.10) holds for such a choice of p, with c_1 only dependent on Q, μ. On the other hand, if p is an arbitrary positive number, we consider $m \in \mathbb{Z}$ such that

$$p_m = \frac{\kappa^m(\kappa+1)}{2} \leq p < p_{m+1}. \tag{4.14}$$

Hence, by (4.10), we have

$$\sup_{R_{\frac{1}{2}}} u \leq \left(c_1^2 \int_{R_1} u^{p_m} \right)^{\frac{1}{p_m}} \leq c_1^{\frac{2}{p_m}} \left(\int_{R_1} u^p \right)^{\frac{1}{p}}$$

so that, by (4.14), we obtain

$$\sup_{R_{\frac{1}{2}}} u^p \leq c_1^{\frac{2p}{p_m}} \int_{R_1} u^p \leq c_1^{2\kappa} \int_{R_1} u^p.$$

This concludes the proof of (4.10) for $p > 0$.

We next consider $p < 0$. In this case, assuming that $u \geq u_0$ for some positive constant u_0, estimate (1.10) can be proved as in the case $p > 0$ or even more easily since condition (4.11) is satisfied for every $p < 0$. On the other hand, if u is a non-negative solution, it suffices to apply (1.10) to $u + \frac{1}{n}$, $n \in \mathbb{N}$, and to let n go to infinity, by the monotone convergence theorem. □

Proceeding as in the proof of Theorem 1.4, we obtain the following

Corollary 4.5 *Let u be a non-negative weak sub-solution of* (1.1) *in Ω. Let $z_0 \in \Omega$ and r, ϱ, $\frac{1}{2} \le \varrho < r \le 1$, such that $R_r(z_0) \subseteq \Omega$. Then we have*

$$\sup_{R_\varrho(z_0)} u \le \left(\frac{c}{(r-\varrho)^{Q+2}} \int_{R_r(z_0)} u^p \right)^{\frac{1}{p}}, \qquad \forall p \ge 1, \tag{4.15}$$

$$\inf_{R_\varrho(z_0)} u \ge \left(\frac{c}{(r-\varrho)^{Q+2}} \int_{R_r(z_0)} u^p \right)^{\frac{1}{p}}, \qquad \forall p < 0, \tag{4.16}$$

where $c = c(Q, \mu)$. Estimate (4.16) *is meaningful only when $u^p \in L^1(R_r(z_0))$.*

We close this section by proving the local boundedness of weak solutions to (1.1).

Proof of Corollary 1.6. We consider a sequence $(g_n)_{n\in\mathbb{N}}$ in $C^\infty(\mathbb{R}, [0, +\infty[)$ with the following properties:

$$g_n(s) \downarrow \max(0, s), \qquad s \in \mathbb{R}, \quad \text{as } n \to \infty,$$

and, for every $n \in \mathbb{N}$, g_n is a monotone increasing, convex function which is linear out of a fixed compact set. By Lemma 3.7, $(g_n(u))$ and $(g_n(-u))$ are sequences of non-negative sub-solutions of L, which converge to $u^+ = \max(0, u)$ and $u^- = \max(0, -u)$ respectively. Thus, the thesis follows applying (4.15) of Corollary (4.5) to $g_n(u)$, $g_n(-u)$ and passing at limit as n goes to infinity. □

Proof of Proposition 1.7. As in [28], we follow the lines of the proof of Theorem 1.4, by using the following two estimates:

$$\|D_{m_0} u^p\|_{L^2(R_\varrho^-)} \le \frac{c\sqrt{\mu(\mu+\varepsilon)}}{\varepsilon(r-\varrho)} \|u^p\|_{L^2(R_r^-)}, \quad \text{where} \quad \varepsilon = \frac{|2p-1|}{4p}, \tag{4.17}$$

and

$$\|u^p\|_{L^{2\kappa}(R_\varrho^-)} \le \frac{c}{r-\varrho} \left(\|u^p\|_{L^2(R_r^-)} + \|D_{m_0} u^p\|_{L^2(R_r^-)} \right), \tag{4.18}$$

for every negative p and for any ϱ, r with $\frac{1}{2} \le \varrho < r \le 1$.

The Sobolev type inequality (4.18) can be proved exactly as Theorem 4.4, since the fundamental solution $\Gamma(x, t, \xi, \tau)$ vanishes in the set $\{\tau > t\}$.

In order to prove the Caccioppoli type inequality (4.17) we we follow the method used in the proof of Theorem 4.3, by using $\varphi = u^{2p-1}\psi^2$ as a test function, where $\chi_n(t)$ is defined as

$$\chi_n(s) = \begin{cases} 1, & \text{if } s \le 0, \\ 1 - ns, & \text{if } 0 \le s \le 1/n, \\ 0, & \text{if } s \ge 1/n, \end{cases}$$

for every $n \in \mathbb{N}$. Then, by letting $n \to \infty$, we find

$$\int_{R_1^-} \left(1 - \frac{1}{2p}\right) \psi^2 \langle ADv, Dv \rangle + \psi \langle ADv, D\psi \rangle + \frac{v^2 \psi}{2} Y\psi \leq 0.$$

After that, we follow the same line used in the proof of Theorem 4.3 and we obtain (4.17). We refer to [28] for a more detailed proof of the analogous result in homogeneous Lie groups. □

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