WHICH RPI-NORMS ARE THE SUPREMUM OF STANDARD RPI-NORMS?

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Abstract. We settle some open problems regarding RPI-norms, i.e. norms (on a convenient subspace of C^1 real functions) that are invariant under differentiable reparametrizations. More precisely, we clarify some key points on how RPI-norms can be approximated using *standard* RPI-norms. We first prove that the sum of two standard RPI-norms is the supremum of a set of standard RPI-norms. This allows us to study a class of RPI-norms that can be expressed as the sup of standard ones. Finally, we give a positive answer to an open question on the existence of RPI-norms that are not the sup of standard RPI-norms.

Introduction

Invariant norms on functional spaces are a topic of research that is rapidly developing. In fact, great attention has been given to translation and dilation invariance: for instance, [J] and [MV] present uniqueness results on translation and, respectively, dilation invariant norms.

More recently, an interesting generalization has been introduced in [FL], where the allowed reparametrizations include all orientation-preserving diffeomorphisms of $\mathbb{R}$.

In [FL] the basic definitions regarding these *reparametrization invariant norms* (*RPI-norms*) are given, and some of their main properties are proven.

However, some questions have been left unanswered. An explicit unsolved problem, stated in the concluding remarks of [FL], is the existence of RPI-norms that cannot be written as the sup of *standard* RPI-norms (an important subclass of RPI-norms, see below).

Moreover, some natural questions, not explicitly raised in [FL], need an answer: is the sum of two standard RPI-norms itself a standard RPI-norm? Is it, at least, the supremum of standard RPI-norms? As regards the first question, we will give a counterexample that provides a negative answer. On the other hand, we will prove that the second question has a positive answer.

Besides their intrinsic interest, RPI-norms are an important tool in comparing the shape of manifolds (and in several applications to computer vision).

We point out that there are at least two (independent) approaches to this problem, developed in [DF] and in [MM], and we note that, although they are different, they both put an emphasis on reparametrization invariance. Moreover, they both focus on the one-dimensional case, which is already far from being simple.

From now on, we will closely follow the setting of [DF]. A shape, or a particular aspect of it, can be represented by a *size pair* $(\mathcal{M}, \varphi)$, where $\mathcal{M}$ is a closed manifold and φ a sufficiently regular function on it. The comparison between two size pairs, $(\mathcal{M}_1, \varphi_1)$ and $(\mathcal{M}_2, \varphi_2)$, can be made by means of the *natural pseudodistance*

$$\delta = \delta((\mathcal{M}_1, \varphi_1), (\mathcal{M}_2, \varphi_2)) = \inf_{h \in D} \max_{P \in \mathcal{M}_1} |\varphi_1(P) - \varphi_2(h(P))|,$$

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where D is the set of diffeomorphisms between $\mathcal{M}_1$ and $\mathcal{M}_2$. We can write $\delta = \inf_{h\in D} F(\varphi_1 - \varphi_2 \circ h)$, where $F(\varphi) = \max|\varphi|$, and note that F is a norm (on the space of the functions on $\mathcal{M}_1$) that takes the same value on a function φ and on a differentiable reparametrization $\varphi \circ h$, with $h \in D$. If, in the definition of δ, we choose a different F, provided it is an RPI-norm, we still obtain a pseudodistance which has different features from the natural one.

In [FL] the one-dimensional case is studied: the manifold $\mathcal{M}$ is $\mathbb{R}$. Although $\mathbb{R}$ is not compact, problems caused by non-compactness are avoided by choosing, as admissible functions, *almost-sigmoidal* functions, whose derivatives vanish outside a compact. A family of RPI-norms of particular interest is that of *standard* RPI-norms (Definition 3): they allow us to reconstruct every compactly supported, piecewise monotone function (and possibly every almost-sigmoidal function), up to reparametrization and sign, with arbitrarily small error. Consequently, they determine completely any other RPI-norm.

It was already known that not every RPI-norm is standard. For example, the total variation is not, but it is the supremum of an increasing sequence of standard RPI-norms (see Proposition 6).

In [FL] it is not clear whether there exists an RPI-norm that is *not* even the supremum of standard ones. In fact, the authors suggest that such a pathological norm could be given by $\|\varphi\|_k = \max|\varphi| + k\lim_{t\to+\infty}|\varphi(t)|$, with $k > 0$, but a proof of this claim was missing: we give it in section 4.

The previous sections deal with some remarkable cases in which *it is possible* to express an RPI-norm as the supremum of a family of standard ones. In section 2 we prove that the sum of a finite number of standard RPI-norms is the supremum of standard RPI-norms. As a consequence, in section 3 we show that a large family of RPI-norms, built from standard RPI-norms, is the supremum of standard RPI-norms.

As a final remark, we note that these results can lead to a generalization of the concept of standard RPI-norm, that we can call the *RPI-norm associated with a set of functions.* If $A \subseteq AS^1$ is closed under reparametrization, and we set

$$\|\varphi\|_A = \sup_{\psi\in A}\left|\int_{\mathbb{R}} \varphi(-t)\frac{d\psi}{dt}(t)dt\right|,$$

we obtain an RPI-norm. If A_1 and A_2 are subsets of AS^1, closed with respect to reparametrization and change of sign, then the Minkowski sum $A_1 + A_2 = \{\varphi_1 + \varphi_2 \mid \varphi_1 \in A_1,\, \varphi_2 \in A_2\}$ has the same closure property, and Proposition 8 implies that

$$\|\varphi\|_{A_1} + \|\varphi\|_{A_2} = \|\varphi\|_{A_1+A_2} \quad \text{for every } \varphi \in AS^1.$$

This definition may perhaps lead to future developments.

1. Definitions and main known results

We recall the basic definitions and results from [FL].

The norms we study are defined in the space of real functions with continuous, compactly supported derivatives, vanishing at $-\infty$.

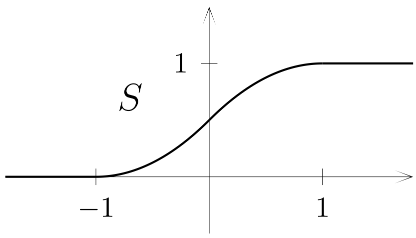
Definition 1. We denote by $AS^1(\mathbb{R})$ (AS^1 for short) the *almost-sigmoidal C^1-functions* on $\mathbb{R}$:

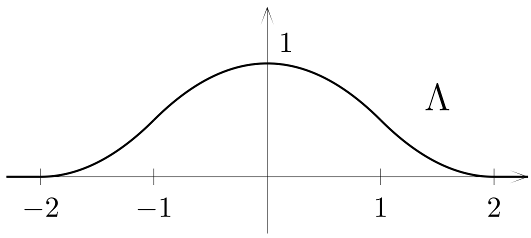
$$AS^1(\mathbb{R}) = \{\varphi \in C^1(\mathbb{R}) \mid \exists\, a, b \in \mathbb{R} \text{ s.t. } \varphi(t) = 0\,\forall\, t < a,\ \varphi(t) = \varphi(b)\,\forall\, t > b\}.$$

Note that AS^1 is a vector space containing $C^1_c(\mathbb{R})$. Moreover, every almost-sigmoidal function φ has a finite total variation $V_\varphi = \int_{\mathbb{R}}\left|\frac{d\varphi}{dt}(s)\right| ds$.

Two useful functions in AS^1 are S and Λ, defined as follows:

$$S(t) = \begin{cases} 0 & \text{if } t < -1, \\ \frac{(t+1)^2}{2} & \text{if } -1 \le t \le 0, \\ 1 - \frac{(t-1)^2}{2} & \text{if } 0 \le t \le 1, \\ 1 & \text{if } t > 1; \end{cases} \qquad \Lambda(t) = S(t+1) - S(t-1).$$





Definition 2. Given $\varphi_1, \varphi_2 \in AS^1$, we say that φ_2 is *obtained from* φ_1 *by reparametrization* (of class C^1), and we write $\varphi_1 \sim \varphi_2$, if there is an orientation-preserving diffeomorphism h of $\mathbb{R}$, such that $\varphi_2 = \varphi_1 \circ h$. The diffeomorphism h is called a *reparametrization*, and "$\sim$" is an equivalence relation. We denote by $[\varphi]$ the equivalence class of $\varphi \in AS^1$.

A norm $\|\cdot\| : AS^1 \to \mathbb{R}$ is *invariant under reparametrization* (or an *RPI-norm*) if it is constant over each equivalence class of $AS^1/\sim$.

The following are examples of RPI-norms (let $\varphi \in AS^1$):

- $\max|\varphi|$, $\max\varphi - \min\varphi$, the total variation V_φ;
- a linear combination, with positive coefficients, of RPI-norms;
- the supremum of a set of RPI-norms, if such sup is always finite;
- the function $\varphi \mapsto \|\varphi\| = \|(\|\varphi\|_1, \ldots, \|\varphi\|_k)\|_*$, where $\|\cdot\|_1, \ldots, \|\cdot\|_k$ are RPI-norms, and $\|\cdot\|_*$ is a norm on $\mathbb{R}^k$, such that if $0 \le x_i \le y_i$ for $1 \le i \le k$, then $\|(x_1, \ldots, x_k)\|_* \le \|(y_1, \ldots, y_k)\|_*$.

The following definition introduces an important class of RPI-norms.

Definition 3. Let $\psi \in AS^1, \psi \not\equiv 0$. The *standard norm* associated with ψ is

$$\|\varphi\|_{[\psi]} = \sup_{\bar\psi \in [\psi]} \left| \int_{\mathbb{R}} \varphi(-t) \frac{d\bar\psi}{dt}(t)dt \right|, \quad \text{for } \varphi \in AS^1. \tag{1}$$

Proposition 4. *Standard RPI-norms satisfy the following properties* ([FL, Th. 3.2]):

i) $\|\varphi\|_{[\psi]} = \|\varphi\|_{[-\psi]}$;
ii) $\|\varphi\|_{[\psi]} = \|\psi\|_{[\varphi]}$ *if* $\varphi \neq 0 \neq \psi$ *(exchange property)*;
iii) $\|\varphi\|_{[\psi]} \le \min\{\max|\varphi| \cdot V_\psi, \max|\psi| \cdot V_\varphi\}$.

Simple examples of standard RPI-norms are given by the L_∞-norm, which is associated with the function S, and by the $(\max - \min)$-norm, associated with the function Λ ([FL, Prop. 3.3, Prop. 3.4]):

$$\|\varphi\|_{L_\infty} = \max|\varphi| = \|\varphi\|_{[S]}\,; \qquad \max\varphi - \min\varphi = \|\varphi\|_{[\Lambda]}\,.$$

Not every RPI-norm is standard. There is a simple necessary condition.

Proposition 5 ([FL, Prop. 3.5])**.** *If* $\|\cdot\|$ *is a finite linear combination, with positive coefficients, of standard RPI-norms, then* $\|S\| \le \|\Lambda\|$.

Proposition 6 ([FL, Prop. 3.8, 3.9])**.** *The total variation is not a finite linear combination of standard RPI-norms. If, for* $n \ge 1$, *we set* $L_n(t) = \sum_{i=0}^{n-1}(-1)^i\Lambda(t-4i)$, *then* $V_\varphi = \sup_{n\ge1}\|\varphi\|_{[L_n]} = \lim_{n\to\infty}\|\varphi\|_{[L_n]}$ *for every* $\varphi \in AS^1$.

2. The sum of standard RPI-norms is the supremum of standard RPI-norms

This section explains how far the sum of two standard RPI-norms is from a standard RPI-norm.

The following proposition shows that this sum is itself not generally standard.

Proposition 7. *The sum of two standard RPI-norms is not necessarily a standard RPI-norm.*

Proof. We work out a counterexample, which can be seen to represent the general situation.

Let $\|\cdot\|$ be the RPI-norm defined as $\|\varphi\| = \max|\varphi| + (\max\varphi - \min\varphi) = \|\varphi\|_{[S]} + \|\varphi\|_{[\Lambda]}$.

We show that there is no $\psi \in AS^1$ such that $\|\varphi\| = \|\varphi\|_{[\psi]}$ for every $\varphi \in AS^1$.

Let us assume that there exists such a ψ.

We claim that the standard RPI-norm associated with ψ must necessarily be

$$\|\varphi\|_{[\psi]} = \sup_{t_0 \le t_1} |2\varphi(t_0) - \varphi(t_1)| \quad \varphi \in AS^1. \tag{2}$$

Let us prove this claim.

The total variation V_ψ of ψ is 3 because $\|\psi\|_{[L_n]} = \|L_n\|_{[\psi]} = 3$ for every $n \ge 2$, and $V_\varphi = \lim_n \|\varphi\|_{[L_n]}$ by Proposition 6. Since $\max|\psi| = \|\psi\|_{[S]} = \|S\|_{[\psi]}$, the maximum of $|\psi|$ is 2. Up to a change of sign, we can assume that $\max\psi = 2$. Therefore, $\lim_{t\to+\infty}\psi(t) \ge 1$.

Let $S_3(t) = S(t) - S(t-2) + S(t-4)$: then $\|\psi\|_{[S_3]} = \sup_{t_1 \le t_2 \le t_3} |\psi(t_1) - \psi(t_2) + \psi(t_3)|$. (If this is not clear, see [FL, Section 4])

If we let $\bar{t} \in \mathbb{R}$ such that $\psi(\bar{t}) = 2$, then $\psi(t)$ must be non-decreasing for $t < \bar{t}$. Otherwise, if $\bar{t}_1 < \bar{t}$ is a local maximum, and $\psi(\bar{t}_1) \neq 0$, there must be some $\bar{t}_2$ in the open interval $(\bar{t}_1, \bar{t})$ such that $\psi(\bar{t}_2) < \psi(\bar{t}_1)$, and therefore $\|\psi\|_{[S_3]} \ge \psi(\bar{t}_1) - \psi(\bar{t}_2) + \psi(\bar{t}) > 2$, contrary to $\|\psi\|_{[S_3]} = \|S_3\|_{[\psi]} = 2$.

Similarly one proves that ψ is non-increasing for $t > \bar{t}$.

As $\psi(t)$ is non-decreasing for $t < \bar{t}$ and non-increasing for $t > \bar{t}$, it follows that $\lim_{t\to+\infty}\psi(t) = 1$, and that equation (2) holds (see again [FL, Section 4]), proving the claim.

Now let us consider $\varphi \in AS^1$, defined by: $\varphi(t) = 2S(t+1) - S(t-1)$ (the function φ increases from 0 to 2, then it decreases to 1, and then remains constant).

Obviously we have:

$$\|\varphi\|_{[\psi]} = \max|\varphi| + (\max\varphi - \min\varphi) = 4. \tag{3}$$

On the other hand, we have from (2) that $\|\varphi\|_{[\psi]} = \sup_{t_0 \le t_1} |2\varphi(t_0) - \varphi(t_1)| = 3$ (this supremum is indeed a maximum, and is reached for $\varphi(t_0) = 2$ and $\varphi(t_1) = 1$). This contradiction shows that $\|\cdot\|$ cannot be a standard RPI-norm. □

The next proposition shows that there is a way to approximate the sum of two standard RPI-norms by means of standard RPI-norms: more precisely, as the supremum of a set of standard RPI-norms. If our original norms are associated with ψ_1 and ψ_2, this set must contain all the standard RPI-norms associated with the sums $\bar{\psi}_1 + \bar{\psi}_2$, where $\bar{\psi}_1$ and $\bar{\psi}_2$ are reparametrizations of ψ_1 and ψ_2 respectively; but this is not enough: we must also allow for the norms associated with the *differences* $\bar{\psi}_1 - \bar{\psi}_2$ (and $\bar{\psi}_1 - \bar{\psi}_2$).

Proposition 8. *Let $\psi_1, \psi_2 \in AS^1$. Let*

$$I_{12} = \{\bar{\psi}_1 + \bar{\psi}_2 \mid \bar{\psi}_1 \in [\psi_1] \cup [-\psi_1], \bar{\psi}_2 \in [\psi_2] \cup [-\psi_2]\}.$$

Then, for every $\varphi \in AS^1$,

$$\|\varphi\|_{[\psi_1]} + \|\varphi\|_{[\psi_2]} = \sup_{\psi \in I_{12}} \|\varphi\|_{[\psi]}.$$

In general, for every $k \geq 2$, for $\{\psi_i\}_{i=1,\dots,k} \subset AS^1$, and real numbers $a_i > 0$,

$$\sum_{i=1}^{k} a_i \|\varphi\|_{[\psi_i]} = \sup_{\psi \in I} \|\varphi\|_{[\psi]}$$

where $I = \{\sum_i \bar{\psi}_i \mid \bar{\psi}_i \in [a_i\psi_i] \cup [-a_i\psi_i], i = 1, \dots, k\}$.

Proof. Let $\varphi \in AS^1$. First, note that, if $\psi = \bar{\psi}_1 + \bar{\psi}_2$, then $\|\varphi\|_{[\psi]} \leq \|\varphi\|_{[\bar{\psi}_1]} + \|\varphi\|_{[\bar{\psi}_2]}$. Indeed, as a consequence of the exchange property, we have:

$$\|\varphi\|_{[\psi]} = \|\varphi\|_{[\bar{\psi}_1+\bar{\psi}_2]} = \left\|\bar{\psi}_1 + \bar{\psi}_2\right\|_{[\varphi]} \leq \left\|\bar{\psi}_1\right\|_{[\varphi]} + \left\|\bar{\psi}_2\right\|_{[\varphi]} = \|\varphi\|_{[\bar{\psi}_1]} + \|\varphi\|_{[\bar{\psi}_2]} .$$

Then we have to prove that, for any $\epsilon > 0$, there are $\bar{\psi}_i \in [\psi_i] \cup [-\psi_i]$ for $i = 1, 2$, such that

$$\|\varphi\|_{[\bar{\psi}_1+\bar{\psi}_2]} \geq \|\varphi\|_{[\psi_1]} + \|\varphi\|_{[\psi_2]} - \epsilon.$$

By definition

$$\|\varphi\|_{[\psi_i]} = \sup_{\widetilde{\psi}_i \in [\psi_i]} \left| \int_{\mathbb{R}} \varphi(-s) \frac{d\widetilde{\psi}_i}{dt}(s) ds \right|.$$

Therefore, for $i = 1, 2$, we can choose $\widetilde{\psi}_i \in [\psi_i]$, such that

$$\left| \int_{\mathbb{R}} \varphi(-s) \frac{d\widetilde{\psi}_i}{dt}(s) ds \right| \geq \|\varphi\|_{[\psi_i]} - \frac{\epsilon}{2}.$$

Now, if necessary, we change the sign of $\widetilde{\psi}_i$. More precisely, we set:

$$\bar{\psi}_i = \begin{cases} \widetilde{\psi}_i & \text{if } \int_{\mathbb{R}} \varphi(-s) \frac{d\widetilde{\psi}_i}{dt}(s) ds \geq 0 \\ -\widetilde{\psi}_i & \text{otherwise.} \end{cases}$$

This allows us to write, *omitting absolute values*,

$$\int_{\mathbb{R}} \varphi(-s) \frac{d\bar{\psi}_i}{dt}(s) ds \geq \|\varphi\|_{[\psi_i]} - \frac{\epsilon}{2} \qquad \text{for } i = 1, 2.$$

Summing up the last couple of inequalities, we have:

$$\begin{aligned} \|\varphi\|_{[\bar{\psi}_1+\bar{\psi}_2]} &\geq \int_{\mathbb{R}} \varphi(-s) \frac{d(\bar{\psi}_1 + \bar{\psi}_2)}{dt}(s) ds \\ &= \int_{\mathbb{R}} \varphi(-s) \frac{d\bar{\psi}_1}{dt}(s) ds + \int_{\mathbb{R}} \varphi(-s) \frac{d\bar{\psi}_2}{dt}(s) ds \\ &\geq \|\varphi\|_{[\psi_1]} + \|\varphi\|_{[\psi_2]} - \epsilon. \end{aligned}$$

This concludes the proof of the first part. The proof of the general case can be made by induction on k, applying the first part (and noting that, if $a > 0$, $a \cdot \|\varphi\|_{[\psi]} = \|\varphi\|_{[a\psi]}$). ☐

3. A class of RPI-norms that are the sup of standard RPI-norms

In this section we examine a class of RPI-norms built starting from standard RPI-norms and expressed as the supremum of standard RPI-norms, although they are not a finite sum of standard RPI-norms.

First we analyze a particular RPI-norm, one of the examples from [FL]. From the definition it is not obvious whether this RPI-norm is standard or not.

Proposition 9. *Let us consider the following RPI-norm:*

$$\|\varphi\| = \sqrt{\max |\varphi|^2 + V_\varphi^2} \qquad \varphi \in AS^1.$$

Then this norm

i) *is not the sum of (finitely many) standard RPI-norms.*
ii) *is the supremum of a family of norms, each of which is the sum of two standard RPI-norms.*
iii) *is the supremum of a family of standard RPI-norms.*

Proof. i) We reason by contradiction.

Assume that there exists a finite set $\{\psi_i \in AS^1\}_{i=1,\ldots,n}$ such that $\|\varphi\| = \sum_{i=1}^n \|\varphi\|_{[\psi_i]}$ for every $\varphi \in AS^1$. From iii) of Proposition 4 we have:

$$\|\varphi\|_{[\psi_i]} \le \max|\varphi| \cdot V_{\psi_i} \qquad \text{for every } i = 1, \ldots, n. \tag{4}$$

We note that, if $\varphi \not\equiv 0$,

$$\|\varphi\| = \max|\varphi| \left(\sqrt{1 + \frac{V_\varphi^2}{\max|\varphi|^2}} \right). \tag{5}$$

From (4) and (5) it follows that

$$\sum_{i=1}^n V_{\psi_i} \ge \sqrt{1 + \frac{V_\varphi^2}{\max|\varphi|^2}} \tag{6}$$

for every nonzero $\varphi \in AS^1$. But the right side of (6) can be arbitrarily large, so $\sum_i V_{\psi_i} = +\infty$. (Indeed, consider the functions $L_n(t) = \sum_{i=0}^{n-1}(-1)^i \Lambda(t-4i)$ used in Proposition 6: the maximum of $|L_n|$ is 1 for every n, while $V_{L_n} = 2n$, so $\lim_n \frac{V_{L_n}}{\max|L_n|} = +\infty$.)

This is absurd: the total variation of each ψ_i must be finite.

ii) Let $\varphi \in AS^1$, $\varphi \not\equiv 0$. Then

$$\|\varphi\| = V_\varphi \sqrt{1 + \left(\frac{\max|\varphi|}{V_\varphi}\right)^2}. \tag{7}$$

We can consider $\|\varphi\|$ as $V_\varphi \cdot f\left(\frac{\max|\varphi|}{V_\varphi}\right)$, where $f(x) = \sqrt{1+x^2}$. Note that $0 < \frac{\max|\varphi|}{V_\varphi}$. Since f is *convex*, the graph of f is the envelope of its tangent lines. If the tangent to the graph of f at the point $(\xi, f(\xi))$, has equation $\{y = a_\xi x + b_\xi\}$ (with $a_\xi = f'(\xi) > 0$ and $b_\xi > 0$), then

$$f(x) = \sup_{\xi\in[0,1]} (a_\xi x + b_\xi) \qquad \text{for } x \in [0,1]. \tag{8}$$

It follows from (8) and (7) that

$$\begin{aligned} \|\varphi\| &= \sup_{\xi\in[0,1]} V_\varphi \left(a_\xi \frac{\max|\varphi|}{V_\varphi} + b_\xi \right) \\ &= \sup_{\xi\in[0,1]} (a_\xi \max|\varphi| + b_\xi V_\varphi). \end{aligned} \tag{9}$$

We can conclude with Proposition 6: total variation is the sup of standard RPI-norms.

iii) is an immediate consequence of *ii)* and Proposition 8. □

The method used in the preceding proposition can be applied to all RPI-norms obtained by means of the third procedure of [FL, Section 2.2].

In the next theorem, x, y, ξ are k-tuples of real numbers, e.g. $x = (x_1, \ldots, x_k)$.

Theorem 10. *Let $g = \|\cdot\|_*$ be a norm on $\mathbb{R}^k$ such that, if $0 \le x_i \le y_i$ for $1 \le i \le k$, then $\|x\|_* \le \|y\|_*$. If $\|\cdot\|_1, \ldots, \|\cdot\|_k$ are k standard RPI-norms, then the function $\|\varphi\| = \|(\|\varphi\|_1, \ldots, \|\varphi\|_k)\|_*$ (which is an RPI-norm) is the supremum of a family of standard RPI-norms.*

Proof. Since g is a norm, it is a *convex* function. Indeed, we have:

$$g(tx+(1-t)y) = \|tx+(1-t)y\|_* \leq t\|x\|_* + (1-t)\|y\|_* = tg(x)+(1-t)g(y),$$

for $x, y \in \mathbb{R}^k$, $t \in [0,1]$. Equivalently, its epigraph $\mathrm{epi}(g) = \{(x,\xi) \in \mathbb{R}^k \times \mathbb{R} \mid \xi \geq g(x)\}$ is a convex set. For $\xi \in \mathbb{R}^k$, denote by $\pi_\xi : x_{k+1} = a_1(\xi)x_1 + \cdots + a_k(\xi)x_k$ a supporting hyperplane of $\mathrm{epi}(g)$, passing through $(\xi, g(\xi))$. (If g is differentiable at ξ, then $a_i(\xi) = \frac{\partial g}{\partial x_i}(\xi)$; if g is not differentiable, π_ξ does not need to be unique: we choose one.) Then, for $x_i \geq 0$, $i = 1, \ldots, k$, we have

$$g(x) = \sup_{\substack{\xi\in\mathbb{R}^k \\ \xi_i \geq 0\,\forall i}} (a_1(\xi)x_1 + \cdots + a_k(\xi)x_k), \tag{10}$$

Note that all the coefficients $a_i(\xi)$ are non negative, provided $\xi_i \geq 0$ for $i = 1, \ldots, k$, for g is increasing in all its arguments, if they are non-negative.

After substituting in (10) the k-tuple $(\|\varphi\|_1, \ldots, \|\varphi\|_k)$ to x, we have:

$$\|\varphi\| = \sup_{\substack{\xi\in\mathbb{R}^k \\ \xi_i \geq 0\,\forall i}} \big(a_1(\xi)\|\varphi\|_1 + \cdots + a_k(\xi)\|\varphi\|_k\big). \tag{11}$$

Since $a_i(\xi)\|\cdot\|_i$, $i = 1, \ldots, k$ are standard RPI-norms, it follows from Proposition 8 that $\|\cdot\|$ is the sup of standard RPI-norms.

□

4. Not every RPI-norm is the sup of standard RPI-norms

The following proposition is the answer to one of the questions raised in the concluding remarks of [FL]: are there RPI-norms which are not the supremum of standard RPI-norms?

Proposition 11. *Let $k \in \mathbb{R}$, $k > 0$. Then the following RPI-norm*

$$\|\varphi\|_k = \max|\varphi| + k \lim_{t\to+\infty}|\varphi(t)| \qquad \varphi \in AS^1$$

is not a standard RPI-norm, nor the supremum (nor the infimum) of standard RPI-norms.

Proof. Since $\|S\|_k = 1 + k > 1 = \|\Lambda\|_k$, the norm $\|\cdot\|_k$ is not standard, because it does not satisfy the necessary condition of Proposition 5.

Now assume that there exists a family of norms $\{\psi_i \in AS^1\}_i$, such that $\|\varphi\|_k = \sup_i \|\varphi\|_{[\psi_i]}$ for every $\varphi \in AS^1$. This assumption leads to a contradiction.

Indeed, let us compute the norm of Λ, and let us apply the exchange property and the equality $\|\varphi\|_\Lambda = \max\varphi - \min\varphi$: then we get:

$$1 = \|\Lambda\|_k = \sup_i \|\Lambda\|_{[\psi_i]} = \sup_i \|\psi_i\|_{[\Lambda]} = \sup_i(\max\psi_i - \min\psi_i) \geq \sup(\max|\psi_i|). \tag{12}$$

On the other hand, taking $\varphi = S$, we have $\|S\|_k = k + 1$. Since $\|\varphi\|_{[S]} = \max|\varphi|$,

$$\sup_i \|S\|_{[\psi_i]} = \sup_i \|\psi_i\|_{[S]} = \sup_i(\max|\psi_i|) \leq 1. \tag{13}$$

From (13) it follows that $k + 1 \leq 1 \Rightarrow k \leq 0$, contradicting the assumption $k > 0$.

Similarly, if we suppose that $\|\varphi\|_k = \inf_i \|\varphi\|_{[\psi_i]}$, and we evaluate the norm at Λ, we have:

$$1 = \|\Lambda\|_k = \inf_i \|\Lambda\|_{[\psi_i]} = \inf_i \|\psi_i\|_{[\Lambda]} = \inf_i(\max\psi_i - \min\psi_i) \Rightarrow \inf(\max|\psi_i|) \leq 1, \tag{14}$$

while, if we evaluate it at S,

$$1 < k + 1 = \|S\|_k = \inf_i \|\varphi\|_{[\psi_i]} = \inf_i \|\psi_i\|_{[S]} = \inf(\max|\psi_i|),$$

implying a contradiction. □

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