GIT quotients of products of projective planes

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Abstract

We study the quotients for the diagonal action of $SL_3(\mathbb{C})$ on the product of n-fold of $\mathbb{P}^2(\mathbb{C})$: we are interested in describing how the quotient changes when we vary the polarization (i.e. the choice of an ample linearized line bundle). We illustrate the different techniques for the construction of a quotient, in particular the numerical criterion for semi-stability and the "elementary transformations" which are resolutions of precisely described singularities (case $n = 6$).

Introduction

Consider a projective algebraic variety X acted on by a reductive algebraic group G. Geometric Invariant Theory (GIT) gives a construction of a G-invariant open subset U of X for which the quotient $U//G$ exists and U is maximal with this property (roughly speaking, U is obtained by X throwing away "bad" orbits). However the open G-invariant subset U depends on the choice of a G-linearized ample line bundle. Given an ample G-linearized line bundle $L \in \mathrm{Pic}^G(X)$ over X, one defines the set of semi-stable points as

$$X^{SS}(L) := \{x \in X \mid \exists n > 0 \text{ and } s \in \Gamma(X, L^{\otimes n})^G \text{ s.t. } s(x) \neq 0\},$$

and the set of stable points as

$$X^S(L) := \{x \in X^{SS}(L) \mid G{\cdot}x \text{ is closed in } X^{SS}(L) \text{ and the stabilizer } G_x \text{ is finite}\}.$$

Then it is possible to introduce a categorical quotient $X^{SS}(L)//G$ in which two points are identified if the closure of their orbits intersect. Moreover as shown in [13], $X^{SS}(L)//G$ exists as a projective variety and contains the *orbit space* $X^S(L)/G$ as a Zariski open subset.

$$\begin{array}{ccc} X & & \\ \cup & & \\ X^{SS}(L) & \xrightarrow{\phi} & X^{SS}(L)//G \\ \cup & & \cup \\ X^S(L) & \xrightarrow{\phi_{|X^S(L)}} & X^S(L)/G \end{array}$$

Question. If one fixes X, G and the action of G on X, but lets the linearized ample line bundle L vary in $\text{Pic}^G(X)$, how do the open set $X^{SS}(L) \subset X$ and the quotient $X^{SS}(L)//G$ change?
Dolgachev-Hu [5] and Thaddeus [19] proved that only a finite number of GIT quotients can be obtained when L varies and gave a general description of the maps relating the various quotients.

In this paper we study the geometry of the GIT quotients for $X = \mathbb{P}^2(\mathbb{C}) \times \ldots \times \mathbb{P}^2(\mathbb{C}) = \mathbb{P}^2(\mathbb{C})^n$. We give examples for $n = 5$ and $n = 6$. The contents of the paper are more precisely as follows.

Section 1 treats the general case $X = \mathbb{P}^2(\mathbb{C})^n$: first of all the numerical criterion of semi-stability is proved (Proposition 1.1). By means of this it is possible to show that only a finite number of quotients $X^{SS}(m)//G$ exists (Subsection 1.2). At the end of the section we introduce the elementary transformations which relate the different quotients.

Section 2 is concerned with the case $n = 5$. Theorem 2.1 contains the main result of Section 2: we show that there are precisely six different quotients.

Section 3 discusses the case $n = 6$: the main results of this Section are concerned with the number of different geometric quotients that may be obtained (it is 38: Table 3.1) and with the singularities that may appear in the quotients. In particular there are only two different types of singularities: in Subsection 3.2 they are described, using the Étale Slice theorem. Theorem 3.2 collects these results. At the end of the Section two examples shows how these singularities are resolved by "crossing the wall".

Acknowledgment

The results of this paper were obtained during my Ph.D. studies at University of Bologna and are also contained in my thesis [7] with the same title. I would like to express deep gratitude to my supervisor prof. Luca Migliorini, whose guidance and support were crucial for the successful completion of this project.

1 The general case $X = \mathbb{P}^2(\mathbb{C})^n$

Let G be the group $SL_3(\mathbb{C})$ acting on the variety $X = \mathbb{P}^2(\mathbb{C})^n$ and let σ be the diagonal action

$$\begin{array}{cccccc} \sigma : & G & \times & \mathbb{P}^2(\mathbb{C})^n & \to & \mathbb{P}^2(\mathbb{C})^n \\ & g & , & (x_1, \ldots, x_n) & \mapsto & (gx_1, \ldots, gx_n) \end{array}$$

A line bundle L over X is determined by $L = L(m) := L(m_1, \ldots, m_n) = \bigotimes_{i=1}^n \pi_i^*(\mathcal{O}_{\mathbb{P}^2(\mathbb{C})}(m_i)),\ m_i \in \mathbb{Z}\ \forall i$, where $\pi_i : X \to \mathbb{P}^2(\mathbb{C})$ is the i-th projection. In particular L is ample iff $m_i > 0$, $\forall i$.
Moreover since each π_i is an G-equivariant morphism, L admits a canonical G-linearization:

$$\text{Pic}^G(X) \cong \mathbb{Z}^n .$$

Thus a *polarization* is completely determined by the line bundle L.

Recall that a point $x \in X$ is said to be *semi-stable* with respect to the polarization m iff there exists a G-invariant section of some positive tensor power of L, $\gamma \in \Gamma(X, L^{\otimes k})^G$, such that $\gamma(x) \neq 0$. A semi-stable point is *stable* if its orbit is closed and has maximal dimension. The *categorical quotient* of the open set of semi-stable points exists and is denoted by $X^{SS}(m)//G$:

$$X^{SS}(m)//G \cong \text{Proj}\left(\bigoplus_{k=0}^{\infty} \Gamma(X, L^{\otimes k})^G\right) .$$

Moreover the open set $X^S(m)/G$ of $X^{SS}(m)//G$ is a *geometric quotient.*
We set $X^{US}(m) = X \backslash X^{SS}(m)$, the closed set of unstable points and $X^{SSS}(m) = X^{SS}(m) \setminus X^S(m)$, the set of strictly semi-stable points.

1.1 Numerical Criterion of semi-stability

Fixed a polarization $L(m)$, we want to describe the set of semi-stable points $X^{SS}(m)$: using the Hilbert-Mumford numerical criterion, we prove the following

Proposition 1.1. *Let $x \in X$. Then we have*

$$x \in X^{SS}(m) \Leftrightarrow \begin{cases} \sum_{k, x_k = y} m_k \leq \frac{|m|}{3} \\ \\ \sum_{j, x_j \in r} m_j \leq 2\frac{|m|}{3} \end{cases} \tag{1}$$

where $|m| := \sum_{i=1}^n m_i$, and y, r are respectively a point and a line in $\mathbb{P}^2(\mathbb{C})$.

Proof. Fixing projective coordinates on the i-th copy of $\mathbb{P}^2(\mathbb{C})$, $[x_{i0} : x_{i1} : x_{i2}]$, a point $x \in X \left(\subset \mathbb{P}(\Gamma(X, L(m))^*) = \mathbb{P}^N(\mathbb{C})\right)$, is described by homogeneous coordinates of this kind:

$$\prod_{i=1}^n x_{i0}^{j_i} x_{i1}^{k_i} x_{i2}^{m_i - (j_i + k_i)}$$

where $0 \leq j_i, k_i \leq m_i$, $j_i + k_i \leq m_i$.

Let $\lambda_{\alpha_0, \alpha_1, \alpha_2}$ a one-parameter subgroup of G; it is defined by $\lambda_{\alpha_0, \alpha_1, \alpha_2}(t) = \text{diag}(t^{\alpha_0}, t^{\alpha_1}, t^{\alpha_2})$ where $\alpha_0 + \alpha_1 + \alpha_2 = 0$; we can assume $\alpha_0 \geq \alpha_1 \geq \alpha_2$.
The subgroup $\lambda_{\alpha_0, \alpha_1, \alpha_2}$ acts on every component of $\mathbb{C}^{N+1}$, multiplying by

$$t^{\alpha_0 \sum_i j_i + \alpha_1 \sum_i k_i + \alpha_2 \sum_i (m_i - (j_i + k_i))} .$$

By the definition of the numerical function of Hilbert-Mumford $\mu_L(x, \lambda)$, we are interested in determining the minimum value of

$$\alpha_0 \sum_{i=1}^n j_i + \alpha_1 \sum_{i=1}^n k_i + \alpha_2 \sum_{i=1}^n (m_i - (j_i + k_i)) .$$

This should be obtained when $j_i = k_i = 0,\ \forall i$; but if there are some $x_{i2} = 0$, then the minimum value becomes:

$$\alpha_2 \sum_{i, x_{i2} \neq 0} m_i + \alpha_1 \sum_{j, x_{j2}=0, x_{j1} \neq 0} m_j + \alpha_0 \sum_{k, x_{k2}=x_{k1}=0} m_k\,. \qquad (2)$$

Thus $x \in X$ is semi-stable if and only if expression (2) is less or equal than zero. Let

$$\alpha_0 = \beta_0 + \beta_1\,, \quad \alpha_1 = -\beta_0\,, \quad \alpha_2 = -\beta_1\,;$$

it follows that $\beta_1 \geq -2\beta_0$, $\beta_1 \geq \beta_0$ e $\beta_1 \geq 0$.
The expression (2) can be rewritten and the minimum value is

$$\beta_0 \left(\sum_{k, x_{k2}=x_{k1}=0} m_k - \sum_{j, x_{j2}=0, x_{j1} \neq 0} m_j \right) + \beta_1 \left(\sum_{k, x_{k2}=x_{k1}=0} m_k - \sum_{i, x_{i2} \neq 0} m_i \right) \leq 0 \qquad (3)$$

The figure 1 shows that every couple (β_0, β_1) that satisfies (3) is a positive

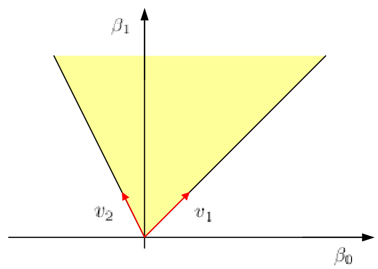


Figure 1: Plane β_0, β_1

linear combination of $v_1 = (1,1)$ e $v_2 = (-1,2)$. Thus the relation (3) must be verified in the two cases $\beta_0 = \beta_1 = 1$ e $\beta_0 = -1, \beta_1 = 2$. After few calculations we obtain

$$\begin{cases} \sum_{h, x_h = y} m_h \leq |m|/3\,, & y \in \mathbb{P}^2(\mathbb{C})\,; \\ \sum_{l, x_l \in r} m_l \leq 2|m|/3\,, & r \subset \mathbb{P}^2(\mathbb{C})\,. \end{cases}$$

■

Remark 1.2. *$x \in X^S(m)$ iff the numerical criterion (1) is verified with strict inequalities.*

The numerical criterion can be restated as follows: if K, J are subset of $[n] := \{1, \ldots, n\}$, then we can associate them with the numbers:

$$\gamma_K^C(m) = |m| - 3 \sum_{k \in K} m_k\,, \qquad \gamma_J^L(m) = 2|m| - 3 \sum_{j \in J} m_j\,.$$

In particular we have: $\gamma_J^C(m) = -\gamma_{J'}^L(m)$ where $J' = [n] \setminus J$.

Now for every subset $K \subseteq [n]$, we consider the set of configurations $(x_1, \ldots, x_n)$ where the points indexed by K are coincident, while the others are all distinct:

$$U_K^C = \{x \in X \,|\, x_{k_1} = \ldots = x_{k_{|K|}} \neq x_i,\ x_j \neq x_l\ \forall i, j, l \notin K\}\,;$$

if $U_K^C \subset X^{SS}(m)$, then $\gamma_K^C(m) \geq 0$.
In the same way if r is a fixed line of $\mathbb{P}^2(\mathbb{C})$, let

$$U_J^L = \{x \in X \mid x_{j_1}, .., x_{j_{|J|}} \in r, x_i \notin r, x_i, x_k, x_l \text{ not collinear}\,,\ \forall i, k, l \notin J\},$$

the set of configurations $(x_1, \ldots, x_n)$ where the points indexed by J are collinear, while the others are not; if $U_J^L \subset X^{SS}(m)$, then $\gamma_J^L(m) \geq 0$.

1.2 Quotients

Proposition 1.3. *Let*

$$U^{GEN} := \{x \in X | \ x_1, \ldots, x_n \ \textit{ in general position}\} \subset X\,,$$

(i.e. every four points among $\{x_1, \ldots, x_n\}$ are a projective system of $\mathbb{P}^2(\mathbb{C})$). Then:

1. $X^{SS}(m) \neq \emptyset \Leftrightarrow U^{GEN} \subset X^{SS}(m)$*;*
2. $X^S(m) \neq \emptyset \Leftrightarrow U^{GEN} \subset X^S(m) \Leftrightarrow \dim(X^{SS}(m)//G) = 2(n-4)$.

We know that the quotient $X^{SS}(m)//G$ depends on the choice of the polarization $L(m)$: moreover Dolgachev-Hu [5] and Thaddeus [19] have proved that when $L(m)$ varies, then there exists only a *finite* number of different quotients.

Now we give a proof of the same result in our case.
If $X^{SS}(m) \neq \emptyset$, then by the previous Proposition we have $U^{\mathrm{GEN}} \subset X^{SS}(m)$. Moreover sets U_K^C and U_J^L are in a finite number since they consist in particular combinations of $x_1, \ldots, x_n$.
Fixed a polarization m, $X^{SS}(m)$ can be described as

$$X^{SS}(m) = U^{\mathrm{GEN}} \cup \mathcal{U}^{SS}(m)\,,$$

where $\mathcal{U}^{SS}(m) := \left\{ U_K^C, U_J^L \,|\, U_K^C, U_J^L \subset X^{SS}(m) \right\}$. In particular we can construct only a finite number of different sets $\mathcal{U}^{SS}(m)$ and as a consequence there exists a finite number of different open sets $X^{SS}(m)$; in conclusion only a finite number of quotients $X^{SS}(m)//G$ exists.

1.3 Elementary transformations

Let m be a polarization such that 3 divides $|m|$ and $X^S(m) \neq \emptyset, X^S(m) \subsetneq X^{SS}(m)$; let us consider "variations" of m as follows:

$$\widehat{m} = m \pm (0, \ldots, 0, \underbrace{1}_{i}, 0, \ldots, 0)\,.$$

We can have two different kind of variations, depending on the value $|\widehat{m}|$:

1. $\widehat{m} \xrightarrow{+1_i} m \quad (\text{i.e. } |\widehat{m}| \equiv 2 \mod 3)\,;$

2. $\widehat{m} \xrightarrow{-1_i} m \quad (\text{i.e. } |\widehat{m}| \equiv 1 \mod 3)\,.$

In both cases we have $X^S(\widehat{m}) = X^{SS}(\widehat{m})$; studying the relations between values $\gamma_J^C(\widehat{m}), \gamma_K^L(\widehat{m})$ and values $\gamma_J^C(m), \gamma_K^L(m)$, we observe that

1. $\widehat{m} \xrightarrow{+1_i} m$

$$X^S(\widehat{m}) \subset X^{SS}(m)\,, \quad X^S(\widehat{m}) = X^{SS}(m) \setminus \bigcup_{i \notin J, \gamma_J^C(m)=0 \,\vee\, \gamma_J^L(m)=0} U_J^*\,;$$

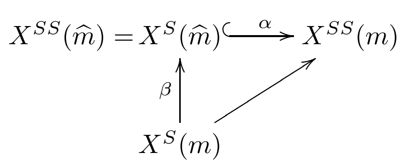
$$X^S(m) \subset X^S(\widehat{m})\,, \quad X^S(m) = X^S(\widehat{m}) \setminus \bigcup_{i \in H, \gamma_H^C(\widehat{m})=2 \,\vee\, \gamma_H^L(\widehat{m})=1} U_H^*\,.$$

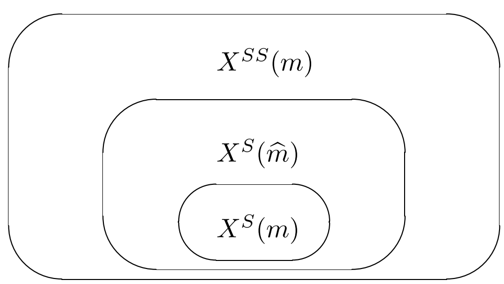
2. $\widehat{m} \xrightarrow{-1_i} m$

$$X^S(\widehat{m}) \subset X^{SS}(m)\,, \quad X^S(\widehat{m}) = X^{SS}(m) \setminus \bigcup_{i \in J, \gamma_J^C(m)=0 \,\vee\, \gamma_J^L(m)=0} U_J^*\,;$$

$$X^S(m) \subset X^S(\widehat{m})\,, \quad X^S(m) = X^S(\widehat{m}) \setminus \bigcup_{i \notin H, \gamma_H^C(\widehat{m})=1 \,\vee\, \gamma_H^L(\widehat{m})=2} U_H^*\,.$$

At the end, we can illustrate the inclusions of the open sets of stable and semi-stable points, with the following diagrams:





The inclusions $X^S(m) \subset X^S(\widehat{m}) \subset X^{SS}(m)$ induce a morphism

$$\theta : X^S(\widehat{m})/G \longrightarrow X^{SS}(m)//G\,, \tag{4}$$

which is an isomorphism over $X^S(m)/G$, while over $\big(X^{SS}(m)//G\big) \setminus \big(X^S(m)/G\big)$ is a contraction of subvarieties.

In fact, let us consider a point $\xi \in \big(X^{SS}(m)//G\big) \setminus \big(X^S(m)/G\big)$: this is

the image in $X^{SS}(m)//G$ of different open, strictly semi-stable orbits, that all have in their closure a closed, minimal orbit Gx, for a certain configuration $x = (x_1, \dots, x_n) \in X^{SSS}(m)$. In particular this configuration x has $|J|$ coincident points, and the others $n - |J|$ collinear; by the numerical criterion, we get $\gamma_J^C(m) = 0$ and $\gamma_{J'}^L(m) = 0$, where J indicates the coincident points, while $J' = [n] \setminus J$ indicates the collinear ones.
For the sake of simplicity, we can assume x as

$$\begin{pmatrix} 1 & \dots & 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 1 & 0 & 1 & 1 & \dots & 1 \\ 0 & \dots & 0 & 0 & 1 & \beta_1 & \beta_2 & \dots & \beta_{n-|J|-2} \end{pmatrix}, \quad \beta_k \in \mathbb{C}^*, \ \forall k\,.$$

The open orbits O that contain Gx in their closure, are characterized by $\gamma_J^C(m) = 0$ or $\gamma_{J'}^L(m) = 0$; there are two different cases:

1. $\gamma_J^C(m) = 0$: orbits look as

$$O_1 = \begin{pmatrix} 1 & \dots & 1 & 0 & 0 & \alpha_1 & \alpha_2 & \dots & \alpha_{n-|J|-2} \\ 0 & \dots & 0 & 1 & 0 & 1 & 1 & \dots & 1 \\ 0 & \dots & 0 & 0 & 1 & \rho\beta_1 & \rho\beta_2 & \dots & \rho\beta_{n-|J|-2} \end{pmatrix}, \rho \in \mathbb{C}^*, \alpha_k \in \mathbb{C}.$$

2. $\gamma_{J'}^L(m) = 0$: orbits look as

$$O_2 = \begin{pmatrix} 1 & 1 & \dots & 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & \delta_1 & \dots & \delta_{|J|-1} & 1 & 0 & 1 & 1 & \dots & 1 \\ 0 & \epsilon_1 & \dots & \epsilon_{|J|-1} & 0 & 1 & \beta_1 & \beta_2 & \dots & \beta_{n-|J|-2} \end{pmatrix}, \delta_k, \epsilon_k \in \mathbb{C}.$$

Now, calculating $\theta^{-1}(\xi)$, it follows:

$$\theta^{-1}(\xi) = \theta^{-1}\left(\phi(\overline{U_J^C} \cup \overline{U_{J'}^L})\right)\ ;$$

by the numerical criterion, only one between U_J^C and $U_{J'}^L$ is included in $X^S(\widehat{m})$.

Dealing with an elementary transformation of the first type ($\widehat{m} \xrightarrow{+1_i} m$), then

- if $i \in J \Rightarrow \theta^{-1}(\xi) = \theta^{-1}\left(\phi(\overline{U_J^C} \cup \overline{U_{J'}^L})\right) = \widehat{\phi}\left(\overline{U_J^C} \cap X^S(\widehat{m})\right)$.
  When $n \geq 5$, this has dimension:

$$d = n - |J| - 3\,. \tag{5}$$

  In fact, let us consider the minimal closed orbit Gx: all the orbits that contain Gx in their closure and are stable in $X^S(\widehat{m})$, are characterized by the coincidence of $|J|$ points (O_1 orbits).

- if $i \in J' \Rightarrow \theta^{-1}(\xi) = \theta^{-1}\left(\phi(\overline{U_J^C} \cup \overline{U_{J'}^L})\right) = \widehat{\phi}\left(\overline{U_{J'}^L} \cap X^S(\widehat{m})\right)$.
  Now the dimension d of $\theta^{-1}(\xi)$ is

$$d = 2\,(n - |J'| - 1) - 1\,. \tag{6}$$

Dealing with an elementary transformation of the second type ($\widehat{m} \xrightarrow{-1_i} m$), then

$$i \in J \Rightarrow d = 2\,(n - |J'| - 1) - 1\,; \qquad i \in J' \Rightarrow d = n - |J| - 3\,. \tag{7}$$

2 $X = \mathbb{P}^2(\mathbb{C})^5$

2.1 Number of quotients

Let us study the case $n = 5$: $X = \mathbb{P}^2(\mathbb{C})^5$. First of all let us determine how many different quotients we may get when the polarization varies.

Let us examine the number of *Geometric* quotients; let $L(m)$ be a polarization such that $X^S(m) \neq \emptyset$: by the Proposition 1.3 it follows that $U^{\text{GEN}} \subset X^S(m)$. In particular

$$X^S(m) = U^{\text{GEN}} \cup \mathcal{U}^S(m)\,,$$

where $\mathcal{U}^S(m) := \left\{ U_K^C, U_J^L \,|\, U_K^C, U_J^L \subset X^S(m) \right\}$. Obviously there is only a finite number of sets $\mathcal{U}^S(m)$: we want to describe their structure.

Let $m = (m_1, \ldots, m_5)$ be a polarization such that $X^S(m) = X^{SS}(m) \neq \emptyset$; we can assume $m_i \in \mathbb{Q}$ and

$$0 < m_i < \frac{1}{3}\,, \quad m_i \geq m_{i+1}\,, \quad |m| = 1\,.$$

As a consequence only strictly inequalities are allowed in the numerical criterion:

$$x \in X^S(m) \Leftrightarrow \sum_{k, x_k = y, k \in K} m_k < \frac{1}{3}\,, \sum_{j, x_j \in r, j \in J} m_j < \frac{2}{3} \Leftrightarrow \gamma_K^C(m) > 0\,, \gamma_J^L(m) > 0$$

In particular sets K that indicate coincident points, can have only two elements (otherwise it would be possible to find a weight m_i greater than 1/3), and in the same way non trivial sets J that indicate collinear points, have only three elements.

Moreover by the numerical criterion, only some sets U_K^C, U_J^L may be included in $X^S(m)$:

$$\begin{array}{ccccc} U_{15}^C\,, & U_{25}^C\,, & U_{34}^C\,, & U_{35}^C\,, & U_{45}^C\,, \\ U_{234}^L\,, & U_{134}^L\,, & U_{125}^L\,, & U_{124}^L\,, & U_{123}^L\,. \end{array} \tag{8}$$

They can be examined in couple, because $\gamma_K^C(m) = -\gamma_{K'}^L(m)\,, K' = [5] \setminus K$ and then only one between U_K^C and $U_{K'}^L$ may be included in $X^S(m)$.

The number of geometric quotient is *six*.

In fact

0. in $\mathcal{U}^S(m)$ there may be only sets as U_J^L: an example is the polarization $m = (1/5, 1/5, 1/5, 1/5, 1/5)$;

1. if in $\mathcal{U}^S(m)$ there is one set as U_K^C, it is U_{45}^C: in fact if it were U_{34}^C, then it follows

   $$m_3 + m_4 < 1/3 \text{ and } m_4 + m_5 > 1/3 \Rightarrow m_3 < m_5 \Rightarrow \text{Impossible.}$$

   Example: $m = (1/4, 1/4, 1/4, 1/8, 1/8)$;

2. if in $\mathcal{U}^{\mathcal{S}}(m)$ there are two sets as U^C_K, they are U^C_{45} and U^C_{35}: the argument is similar to the previous one.
   Example: $m = (3/11, 3/11, 2/11, 2/11, 1/11)$;

3. if in $\mathcal{U}^{\mathcal{S}}(m)$ there are three sets as U^C_K, we can have two cases:

   (a) U^C_{45}, U^C_{35} and U^C_{25} , example $m = (3/10, 1/5, 1/5, 1/5, 1/10)$;

   (b) U^C_{45}, U^C_{35} and U^C_{34} , example $m = (3/10, 3/10, 1/5, 1/10, 1/10)$.

4. if in $\mathcal{U}^{\mathcal{S}}(m)$ there are four sets as U^C_K, they are $U^C_{45}, U^C_{35}, U^C_{25}$ and U^C_{15}. Example: $m = (1/4, 1/4, 1/4, 2/9, 1/36)$;

5. the case of all U^C_K sets in $\mathcal{U}^{\mathcal{S}}(m)$ is impossible, because U^C_{45}, U^C_{35}, U^C_{34}, U^C_{25} are incompatible.

We have found six cases:

$$\begin{array}{ll}
0. & \mathcal{U}^{\mathcal{S}}(m) = \{U^L_{234}, U^L_{134}, U^L_{124}, U^L_{123}, U^L_{125}\} \\
1. & \mathcal{U}^{\mathcal{S}}(m) = \{U^L_{234}, U^L_{134}, U^L_{124}, U^L_{125}, U^C_{45}\}\,, \\
2. & \mathcal{U}^{\mathcal{S}}(m) = \{U^L_{234}, U^L_{134}, U^L_{125}, U^C_{35}, U^C_{45}\}\,, \\
3a. & \mathcal{U}^{\mathcal{S}}(m) = \{U^L_{234}, U^L_{125}, U^C_{25}, U^C_{35}, U^C_{45}\}\,, \\
3b. & \mathcal{U}^{\mathcal{S}}(m) = \{U^L_{234}, U^L_{134}, U^C_{34}, U^C_{35}, U^C_{45}\}\,, \\
4. & \mathcal{U}^{\mathcal{S}}(m) = \{U^L_{125}, U^C_{15}, U^C_{25}, U^C_{35}, U^C_{45}\}\,.
\end{array}$$

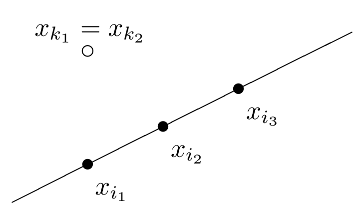
Then there are only six different open sets of stable points and thus six geometric quotients.

Now let us examine the number of *Categorical* quotients. First of all let us observe that sets $U^C_K, U^L_{K'}$ that may be included in $X^{SS}(m)$ are the same of (8). What is different from the previous case is that now two sets U^C_K and $U^L_{K'}$ may be *both* included in $X^{SS}(m)$ (if $\gamma^C_K(m) = \gamma^L_{K'}(m) = 0$); this means that in $X^{SS}(m)$ there are two distinct strictly semi-stable orbits:

- an orbit O_1 with $x_{k_1} = x_{k_2}$, $K = \{k_1, k_2\}$;

- orbits O_2 with $x_{i_1}, x_{i_2}, x_{i_3}$ collinear, $i_1, i_2, i_3 \in K'$.

Orbit O_1 and all orbits O_2 contain in their closure a closed, minimal, strictly semi-stable orbit O_{12}, that is characterized by $x_{k_1} = x_{k_2}$ *and* $x_{i_1}, x_{i_2}, x_{i_3}$ collinear:



In the categorical quotient $X^{SS}(m)//G$, orbits O_1 and O_2 determine the *same* point; in fact $O_{12} \subset (\overline{O_1} \cap \overline{O_2})$.

Let us examine the stable case more accurately: we know that only one between O_1 and O_2 is included in $X^S(m)$; when O_1 is included, it determines a point of the geometric quotient. In fact if for example $U_{45}^C \subset X^S(m)$, then $\phi(U_{45}^C)$ may regarded as $\mathbb{P}^2(\mathbb{C})^4(m_1, m_2, m_3, m_4 + m_5)/SL_3(\mathbb{C})$ and the we have a point. When orbits O_2 are included in $X^S(m)$, they determine a $\mathbb{P}^1(\mathbb{C})$ in $X^S(m)/G$. In fact if for example $U_{123}^L \subset X^S(m)$, then we can assume

$$O_2 = \left(\begin{array}{ccccc} 1 & 0 & 1 & 0 & \alpha \\ 0 & 1 & 1 & 0 & \beta \\ 0 & 0 & 0 & 1 & 1 \end{array} \right), (\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0,0)\}.$$

Applying to O_2 a projectivity G_λ of $\mathbb{P}^2(\mathbb{C})$ that fixes the line that contains x_1, x_2, x_3 ($G_\lambda \cong \mathrm{diag}(\lambda, \lambda, \lambda^{-2})$, with $\lambda \in \mathbb{C}^*$), , it follows:

$$G_\lambda \cdot x = \left(\begin{array}{ccccc} 1 & 0 & 1 & 0 & \lambda^3\alpha \\ 0 & 1 & 1 & 0 & \lambda^3\beta \\ 0 & 0 & 0 & 1 & 1 \end{array} \right).$$

If $\alpha \neq 0$, then we can assume $\lambda^3 = \alpha^{-1}$; thus we obtain $x_5 = [1 : \alpha^{-1}\beta : 1]$; in the same way if $\beta \neq 0$, then $x_5 = [\alpha\beta^{-1} : 1 : 1]$.
Then it is clear that $\phi(O_2) \cong \mathbb{P}^1(\mathbb{C})$.

In the semi-stable case when $U_K^C, U_{K'}^L \subset X^{SS}(m)$, we know that $\overline{U_K^C} \cap \overline{U_{K'}^L} \neq \emptyset$ and they determine a non-singular point of $X^{SS}(m)//G$, just as in the stable case when $U_K^C \subset X^S(m)$. In this way it follows that every categorical quotient $X^{SS}(m)//G$, where

$$X^{SS}(m) = U^{\mathrm{GEN}} \cup \{ \underbrace{U_J^C, U_I^L, \ldots,}_{stablesets} \underbrace{U_K^C, U_{K'}^L, \ldots, U_H^C, U_{H'}^L}_{semi-stablesets} \},$$

is isomorphic to a geometric one $X^S(m')/G$, whose open set of stable points is

$$X^S(m') = U^{\mathrm{GEN}} \cup \{U_J^C, U_I^L, \ldots, U_K^C, U_{\ldots}^C, U_H^C, \}.$$

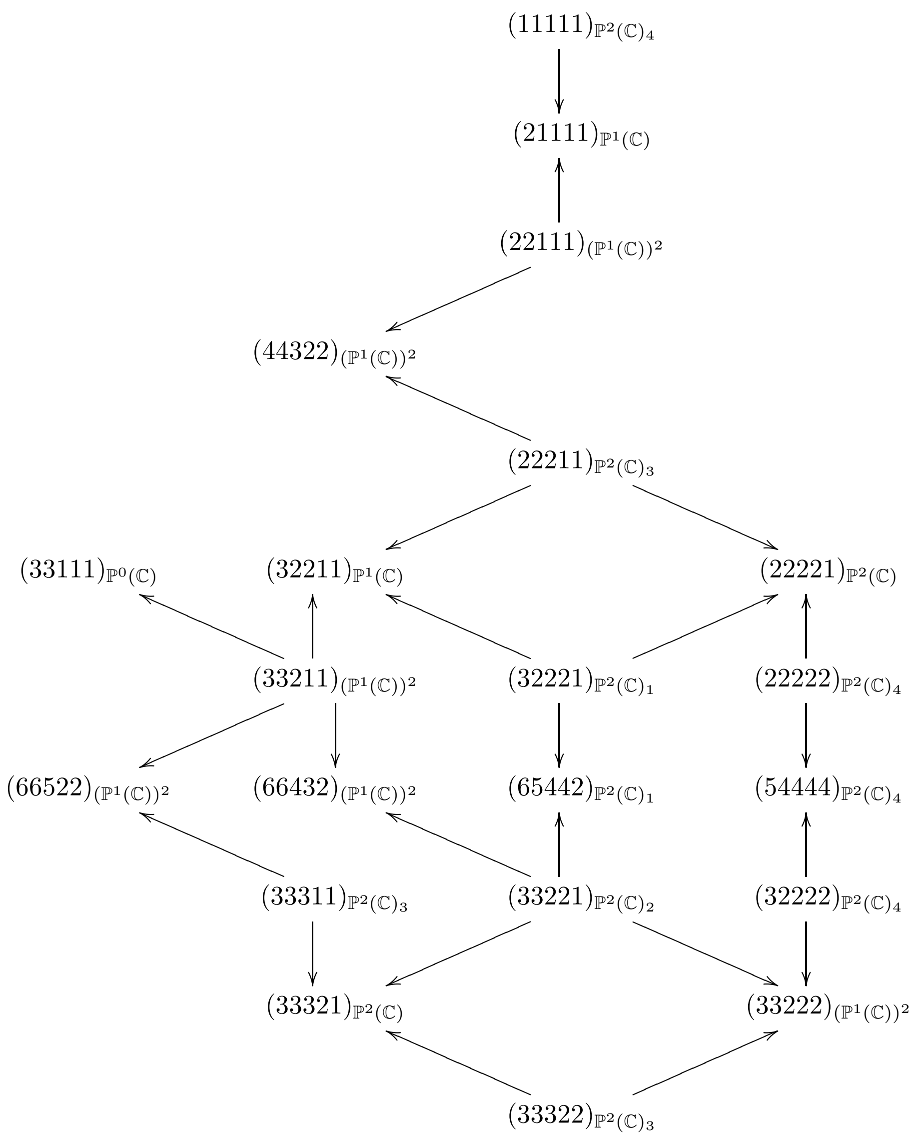
In conclusion:

Theorem 2.1. *Let $X = \mathbb{P}^2(\mathbb{C})^5$: then there are* six *non trivial quotients. Moreover a quotient $X^{SS}(m)//G$ is isomorphic to one of the following:*

$\mathbb{P}^2(\mathbb{C})$	
$\mathbb{P}^2(\mathbb{C})$ *with a point blown up*	$(\mathbb{P}^2(\mathbb{C})_1)$
$\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$	$(\mathbb{P}^1(\mathbb{C})^2)$;
$\mathbb{P}^2(\mathbb{C})$ *with two points blown up*	$(\mathbb{P}^2(\mathbb{C})_2)$
$\mathbb{P}^2(\mathbb{C})$ *with three points blown up*	$(\mathbb{P}^2(\mathbb{C})_3)$
$\mathbb{P}^2(\mathbb{C})$ *with four points blown up*	$(\mathbb{P}^2(\mathbb{C})_4)$

2.2 Quotients $\mathbb{P}^2(\mathbb{C})^5//G$

The following diagram shows the relations between some polarizations that realize the quotients; for example if $m = (22211)$, then $X^S(m) = \mathbb{P}^2(\mathbb{C})_3$ and there is a morphism $\theta : X^S(22211)/G = \mathbb{P}^2(\mathbb{C})_3 \to X^{SS}(44322)//G = \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$.



3 $X = \mathbb{P}^2(\mathbb{C})^6$

3.1 Number of quotients

Now we study the case $n = 6$: $X = \mathbb{P}^2(\mathbb{C})^6$; as in the previous case we first determine how many different *Geometric* quotient we can get when the polarization varies.

For a polarization $m = (m_1, \ldots, m_6)$ such that $X^S(m) \neq \emptyset$, then

$$X^S(m) = U^{\mathrm{GEN}} \cup \mathcal{U}^S(m)\,.$$

We want to describe the structure of the sets $\mathcal{U}^S(m)$; assume that $0 < m_i < \frac{1}{3}$, $m_i \geq m_{i+1}$, $|m| = 1$.
We are interested in those sets U_K^C that are included in $X^S(m)$: some are *always* included in $X^S(m)$:

$$U_{36}^C\,,\quad U_{46}^C\,,\quad U_{56}^C\,,$$

and others *may* be included in $X^S(m)$:

$$\begin{array}{lllllllll} U_{15}^C\,, & U_{16}^C\,, & U_{23}^C\,, & U_{24}^C\,, & U_{25}^C\,, & U_{26}^C\,, & U_{34}^C\,, & U_{35}^C\,, & U_{45}^C\,, \\ U_{156}^C\,, & U_{256}^C\,, & U_{345}^C\,, & U_{346}^C\,, & U_{356}^C & U_{456}^C\,. \end{array}$$

The number of different sets $\mathcal{U}^S(m)$ is 38.

First of all the minimum number of sets U_K^C with $|K| = 2$, included in $X^S(m)$ is five: in fact for example consider only the sets $U_{36}^C, U_{46}^C, U_{56}^C$ that are always included in $X^S(m)$, then obviously

$$m_1 + m_6 > \frac{1}{3}\,,\; m_2 + m_5 > \frac{1}{3}\,,\; m_3 + m_4 > \frac{1}{3} \;\Rightarrow\; \sum_{i=1}^{6} m_i > 1 \;:\; \text{impossible.}$$

In a similar way it is impossible to have only four sets U_K^C ($|K| = 2$) in $X^S(m)$.

Then for five sets U_K^C, we have $U_{16}^C, U_{26}^C, U_{36}^C, U_{46}^C, U_{56}^C$: in fact with another 5-tuple (for example $U_{45}^C, U_{26}^C, U_{36}^C, U_{46}^C, U_{56}^C$), it gets $|m| > 1$, that is impossible. Moreover with these combinations, it is impossible to obtain a set as U_K^C with $|K| = 3$.

Going on with the calculations, we are able to construct the following table, that shows all the possible cases (in the "admissible" cells we exhibit an example of a polarization that realize the geometric quotient). In particular it is not possible to have more than ten sets U_K^C ($|K| = 2$) in $X^S(m)$: we would obtain $|m| < 1$.

Table 3.1.

U_K^C, $\lvert K\rvert=2$	*No* U_K^C, $\lvert K\rvert=3$	1 *set* U_K^C, $\lvert K\rvert=3$	2 *sets* U_K^C, $\lvert K\rvert=3$	3 *sets* U_K^C, $\lvert K\rvert=3$	4 *sets* U_K^C, $\lvert K\rvert=3$
$U_{16}^C, U_{26}^C, U_{36}^C, U_{46}^C, U_{56}^C$	✓ $\frac{1}{11}(222221)$	*No*$^{(*)}$	*No*	*No*	*No*
$U_{16}^C, U_{26}^C, U_{36}^C, U_{45}^C, U_{46}^C, U_{56}^C$	✓ $\frac{1}{14}(333221)$	U_{456}^C $\frac{1}{17}(444221)$	*No*$^{(*)}$	*No*	*No*
$U_{34}^C, U_{35}^C, U_{36}^C, U_{45}^C, U_{46}^C, U_{56}^C$	✓ $\frac{1}{8}(221111)$	U_{456}^C $\frac{1}{11}(332111)$	U_{456}^C, U_{356}^C $\frac{1}{14}(442211)$	$U_{456}^C, U_{356}^C, U_{346}^C$ $\frac{1}{17}(552221)$	$U_{456}^C, U_{356}^C, U_{346}^C, U_{345}^C$ $\frac{1}{10}(331111)$
$U_{25}^C, U_{26}^C, U_{35}^C, U_{36}^C, U_{45}^C, U_{46}^C, U_{56}^C$	✓ $\frac{1}{11}(322211)$	U_{456}^C $\frac{1}{14}(433211)$	U_{456}^C, U_{356}^C $\frac{1}{17}(543311)$	$U_{456}^C, U_{356}^C, U_{256}^C$ $\frac{1}{19}(644311)$	*No*$^{(*)}$
$U_{26}^C, U_{34}^C, U_{35}^C, U_{36}^C, U_{45}^C, U_{46}^C, U_{56}^C$	✓ $\frac{1}{14}(432221)$	U_{456}^C $\frac{1}{17}(543221)$	U_{456}^C, U_{356}^C $\frac{1}{26}(875321)$	$U_{456}^C, U_{356}^C, U_{346}^C$ $\frac{1}{16}(542221)$	*No*$^{(**)}$
$U_{16}^C, U_{26}^C, U_{35}^C, U_{36}^C, U_{45}^C, U_{46}^C, U_{56}^C$	✓ $\frac{1}{17}(443321)$	U_{456}^C $\frac{1}{20}(554321)$	U_{456}^C, U_{356}^C $\frac{1}{26}(775421)$	*No*$^{(*)}$	*No*
$U_{16}^C, U_{26}^C, U_{34}^C, U_{35}^C, U_{36}^C, U_{45}^C, U_{46}^C, U_{56}^C$	✓ $\frac{1}{13}(332221)$	U_{456}^C $\frac{1}{16}(443221)$	U_{456}^C, U_{356}^C $\frac{1}{19}(553321)$	$U_{456}^C, U_{356}^C, U_{346}^C$ $\frac{1}{25}(774331)$	*No*$^{(**)}$
$U_{16}^C, U_{25}^C, U_{26}^C, U_{35}^C, U_{36}^C, U_{45}^C, U_{46}^C, U_{56}^C$	✓ $\frac{1}{16}(433321)$	U_{456}^C $\frac{1}{26}(766421)$	U_{456}^C, U_{356}^C $\frac{1}{26}(765521)$	$U_{456}^C, U_{356}^C, U_{256}^C$ $\frac{1}{25}(755521)$	*No*$^{(*)}$
$U_{25}^C, U_{26}^C, U_{34}^C, U_{35}^C, U_{36}^C, U_{45}^C, U_{46}^C, U_{56}^C$	✓ $\frac{1}{31}(965542)$	U_{456}^C $\frac{1}{26}(865322)$	U_{456}^C, U_{356}^C $\frac{1}{13}(432211)$	*No*$^{(\dagger)}$	*No*

U^C_K, $\lvert K\rvert=2$	*No* U^C_K, $\lvert K\rvert=3$	*1 set* U^C_K, $\lvert K\rvert=3$	*2 sets* U^C_K, $\lvert K\rvert=3$	*3 sets* U^C_K, $\lvert K\rvert=3$	*4 sets* U^C_K, $\lvert K\rvert=3$
$U^C_{15}, U^C_{16}, U^C_{25}, U^C_{26}, U^C_{35}, U^C_{36}, U^C_{45}, U^C_{46}, U^C_{56}$	✓ $\frac{1}{10}(222211)$	U^C_{456} $\frac{1}{13}(333211)$	U^C_{456}, U^C_{356} $\frac{1}{16}(443311)$	$U^C_{456}, U^C_{356}, U^C_{256}$ $\frac{1}{25}(766411)$	$U^C_{456}, U^C_{356}, U^C_{256}, U^C_{156}$ $\frac{1}{22}(555511)$
$U^C_{24}, U^C_{25}, U^C_{26}, U^C_{34}, U^C_{35}, U^C_{36}, U^C_{45}, U^C_{46}, U^C_{56}$	✓ $\frac{1}{17}(533222)$	U^C_{456} $\frac{1}{10}(322111)$	*No*$^{(\dagger\dagger)}$	*No*	*No*
$U^C_{23}, U^C_{24}, U^C_{25}, U^C_{26}, U^C_{34}, U^C_{35}, U^C_{36}, U^C_{45}, U^C_{46}, U^C_{56}$	✓ $\frac{1}{7}(211111)$	*No*$^{(\dagger\dagger\dagger)}$	*No*	*No*	*No*

$^{(*)}$ This case is not possible, because there is not any available tern;

$^{(**)}$ U^C_{345} is not included in $X^S(m)$, because otherwise $m_3+m_4+m_5<\frac{1}{3}$, $m_2+m_6<\frac{1}{3} \Rightarrow m_1>\frac{1}{3}$, that is impossible;

$^{(\dagger)}$ $U^C_{256}, U^C_{345}, U^C_{346} \not\subseteq X^S(m)$;

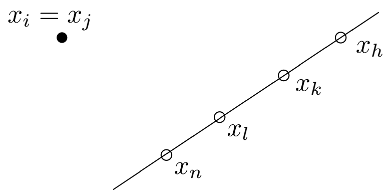
$^{(\dagger\dagger)}$ $U^C_{246}, U^C_{256}, U^C_{345}, U^C_{346}, U^C_{356} \not\subseteq X^S(m)$;

$^{(\dagger\dagger\dagger)}$ $U^C_{236}, U^C_{246}, U^C_{256}, U^C_{345}, U^C_{346}, U^C_{356}, U^C_{456} \not\subseteq X^S(m)$.

3.2 Singularities

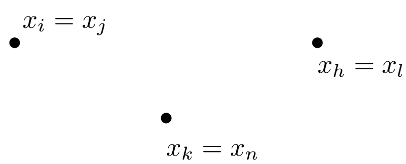
In this section we study the singularities which appear in the categorical quotients.
Suppose that $|m|$ is divisible by 3, and that there exist strictly semi-stable orbits (included in $X^{SSS}(m)$); then we can have different cases depending on some "partitions" of the polarization $m \in \mathbb{Z}^6_{>0}$:

1. there are two distinct indexes i, j such that $m_i+m_j=|m|/3$; as a consequence, for the other indexes it holds $m_h+m_k+m_l+m_n=2|m|/3$ (i.e. minimal closed orbits have $x_i=x_j$ and x_h, x_k, x_l, x_n collinear).

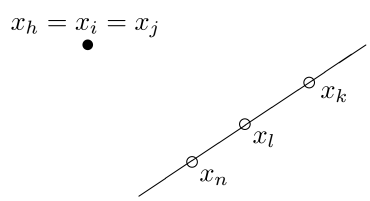


In $X^{SS}(m)//G$ these orbits determine a curve $C_{ij} \cong \mathbb{P}^1(\mathbb{C})$.

1.1 particular case: $m_i + m_j = m_h + m_l = m_k + m_n = |m|/3$ for distinct indexes (i.e. there is a “special” minimal, closed orbit other than the orbits previously seen, characterized by $x_i = x_j$, $x_h = x_l$, $x_k = x_n$).



2. there are three distinct indexes h, i, j such that $m_h + m_i + m_j = |m|/3$; as a consequence for the other indexes it holds $m_k + m_l + m_n = 2|m|/3$ (i.e. there is a minimal, closed orbit such that $x_h = x_i = x_j$, and x_k, x_l, x_n collinear).



Let us study minimal, closed orbits and what they determine in $X^{SS}(m)//G$.

3.2.1 $x_i = x_j$ and x_h, x_k, x_l, x_n collinear

Consider a polarization $m = (m_1, \ldots, m_6)$ as previously indicated and an orbit Gx such that $x_i = x_j \, (m_i + m_j = |m|/3)$, and the other four points x_h, x_k, x_l, x_n collinear $(m_h + m_k + m_l + m_n = 2|m|/3)$.
Gx is a minimal, closed, strictly semi-stable orbit and its image in $X^{SS}(m)//G$ is a point $\xi \in C_{ij}$. For the sake of generality, suppose that x_h, x_k, x_l, x_n are collinear, but distinct; for example assume x as:

$$x = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & a & b \end{pmatrix}, \quad a, b \in \mathbb{C}^*, a \neq b \, .$$

Now let us apply the Luna Étale Slice Theorem, to make a *local* study of ξ: in fact it states that if Gx is a closed semi-stable orbit and ξ is the corresponding point of $X^{SS}(m)//G$, then the pointed varieties $(X^{SS}(m)//G, \xi)$ and $(N_x//G_x, 0)$ are locally isomorphic in the étale topology, where $N_x = N_{Gx/X,x}$ is the fiber over x of the normal bundle of Gx in X (for more details about the Étale Slice Theorem, see [12], [20] and [8]).

In our case the dimension of the stabilizer G_x is equal to one and $G_x \cong \{\mathrm{diag}(\lambda^{-2}, \lambda, \lambda), \lambda \in \mathbb{C}^*\} \cong \mathbb{C}^*$. Moreover the orbit Gx is a 7-dimensional regular variety in $\mathbb{C}^{12}$ and the space $T_x\mathbb{C}^{12} = \mathbb{C}^{12}$ can be decomposed G_x-invariantly as the direct sum $T_xGx \oplus N_x$.
So we study the action of the torus $\mathbb{C}^*$ on N_x: it is induced by the diagonal action of $SL_3(\mathbb{C})$ on $\mathbb{P}^2(\mathbb{C})^6(m)$ and it can be written as

$$v_1 \mapsto \lambda^3 v_1; \quad v_2 \mapsto \lambda^3 v_2; \quad v_3 \mapsto \lambda^{-3} v_3; \quad v_4 \mapsto \lambda^{-3} v_4; \quad v_5 \mapsto v_5$$

where $(v_1, \ldots, v_5)$ is a basis of $N_x \cong \mathbb{C}^5$.
In this way a local model of $(X^{SS}(m)//G, \xi)$ is given by $(\mathbb{C}^5//\mathbb{C}^*, 0)$ with "weights" $(3, 3, -3, -3, 0)$ that is the 4-dimensional toric variety

$$Y := \mathbb{C}[T_1, \ldots, T_5]/(T_1T_4 - T_2T_3)\,.$$

In conclusion, the variety $(X^{SS}(m)//G, \xi)$, where ξ is a point of the curve $C_{ij} \cong \mathbb{P}^1(\mathbb{C})$, is locally isomorphic to the toric variety Y: it is singular and there are different ways to resolve it ([10], [2]).

3.3 $x_i = x_j$, $x_h = x_l$, $x_k = x_n$

This study is analogous to the previous one.
Consider a polarization m such that it is possible to "subdivide" it as $m_i + m_j = m_h + m_l = m_k + m_n$ (for different indexes); we are examining the configuration x, with $x_i = x_j\,, x_h = x_l\,, x_k = x_n$ (this configuration is a particular case of the previous one).
In the quotient $X^{SS}(m)//G$ the image of the orbit Gx is a point $O_{ij,hl,kn}$ that lies on the three singular curves C_{ij}, C_{hl}, C_{kn}.

The orbit Gx is minimal, closed and strictly semi-stable: assume x equal to

$$x = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

Let us apply the Étale Slice Theorem: the stabilizer G_x is isomorphic to a 2-dimensional torus $G_x \cong \{\mathrm{diag}(\lambda, \mu, \lambda^{-1}\mu^{-1}), \lambda, \mu \in \mathbb{C}^*\}$ which implies that $\dim Gx = 6$. By the Étale Slice Theorem, let us study the action of G_x on N_x: on the basis $\{v_1, \ldots, v_6\}$ of N_x it gives

$$\begin{array}{lll} v_1 \mapsto \lambda^{-1}\mu \cdot v_1\,; & v_2 \mapsto \lambda^{-2}\mu^{-1} \cdot v_2\,; & v_3 \mapsto \lambda\mu^{-1} \cdot v_3\,; \\ v_4 \mapsto \lambda^{-1}\mu^{-2} \cdot v_4\,; & v_5 \mapsto \lambda^2\mu \cdot v_5\,; & v_6 \mapsto \lambda\mu^2 \cdot v_6\,. \end{array}$$

It follows that a local model for $(X^{SS}(m)//G, O_{ij,hl,kn})$ is given by $Y := (\mathbb{C}^6//(\mathbb{C}^*)^2, 0)$, where the action of $(\mathbb{C}^*)^2$ can be written (in the coordinates $(z_1, \ldots, z_6)$ of $N_x \cong \mathbb{C}^6$) as

$$(\lambda, \mu)(z_1, \ldots, z_6) \to (\lambda^{-1}\mu z_1, \lambda^{-2}\mu^{-1} z_2, \lambda\mu^{-1} z_3, \lambda^{-1}\mu^{-2} z_4, \lambda^2\mu z_5, \lambda\mu^2 z_6). \quad (9)$$

Thus we obtain a 4-dimensional toric variety:

$$Y = \mathbb{C}[T_1, \ldots, T_5]/(T_1T_2T_3 - T_4T_5)\,. \quad (10)$$

Its singular locus is given by three lines $s_1 = \{(t, 0, 0, 0, 0), t \in \mathbb{C}\}$, $s_2 = \{(0, t, 0, 0, 0), t \in \mathbb{C}\}$ and $s_3 = \{(0, 0, t, 0, 0), t \in \mathbb{C}\}$ that have a common point, the origin. These lines correspond to the curves C_{ij}, C_{hl}, C_{kn}.

A toric representation of Y is determined by a rational, polyhedral cone $\sigma \subset \mathbb{R}^4$, such that $Spec(\sigma^\vee \cap \mathbb{Z}^4) \cong Y$. The generators of the semi-group $\sigma^\vee \cap \mathbb{Z}^4$ are $w_1, \ldots, w_5 \in \mathbb{Z}^4$ and satisfy $w_1 + w_2 + w_3 = w_4 + w_5$. Assume

$$w_1 = (1, 0, 0, 0)\,, \quad w_2 = (0, 1, 0, 0)\,, \quad w_3 = (0, 0, 1, 0)\,,$$
$$w_4 = (0, 0, 0, 1)\,, \quad w_5 = (1, 1, 1, -1)\,.$$

The primitive elements of σ are:

$$\mathbf{n}_1 = (0, 0, 1, 1)\,, \quad \mathbf{n}_2 = (1, 0, 0, 0)\,, \quad \mathbf{n}_3 = (0, 0, 1, 0)\,,$$
$$\mathbf{n}_4 = (0, 1, 0, 1)\,, \quad \mathbf{n}_5 = (1, 0, 0, 1)\,, \quad \mathbf{n}_6 = (0, 1, 0, 0)\,.$$

It is clear that the cone σ is singular.

Let us intesect σ with a transversal hyperplane π of $\mathbb{R}^4$ and then consider the projection on π. With $\pi : y_1 + y_2 + y_3 + y_4 = 2$ we get the polytope Π of $\mathbb{R}^3$, with verteces

$$u_1 = (0, 0, 1)\,, \quad u_2 = (2, 0, 0)\,, \quad u_3 = (0, 0, 2)\,,$$
$$u_4 = (0, 1, 0)\,, \quad u_5 = (1, 0, 0)\,, \quad u_6 = (0, 2, 0)\,.$$

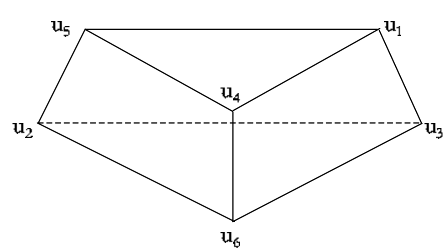


Figure 2: Polytope Π

In conclusion the pointed variety $(X^{SS}(m)//G, O_{ij,hl,kn})$ is isomorphic to the toric variety $\mathbb{C}[T_1, \ldots, T_5]/(T_1T_2T_3 - T_4T_5)$, where the action has weights

$$\begin{pmatrix} -1 & -2 & 1 & -1 & 2 & 1 \\ 1 & -1 & -1 & -2 & 1 & 2 \end{pmatrix}.$$

3.4 $x_h = x_i = x_j$ and x_k, x_l, x_n collinear

Consider a polarization m such that $m_h + m_i + m_j = |m|/3$ and $m_k + m_l + m_n = 2|m|/3$ (for different indexes); then let us study the configuration x where: $x_h = x_i = x_j$ and x_k, x_l, x_n collinear.
The orbit Gx is minimal, closed, strictly semi-stable and its image in $X^{SS}(m)//G$ is a point O_{hij}. In particular x_k, x_l, x_nhave to be all distinct.
As in the previous cases, by the Étale Slice Theorem, we obtain a local model for $(X^{SS}(m)//G, O_{hij})$: this is determined by $Y := (\mathbb{C}^5//\mathbb{C}^*, 0)$, where the action of $\mathbb{C}^*$ over $\mathbb{C}^5$ with coordinate $(z_1, \ldots, z_5)$ has weights $(3,\ 3,\ 3,\ 3,\ -3)$. Y is a 4-dimensional toric variety that corresponds to the smooth affine variety

$$Y = \mathbb{C}[T_1, \ldots, T_4] \cong \mathbb{C}^4 \,.$$

In conclusion the corresponding point O_{hij} in $X^{SS}(m)//G$ is nonsingular.

We have classified the different singularities of $X^{SS}(m)//G$:

Theorem 3.2. *Let $X = \mathbb{P}^2(\mathbb{C})^6$ and $m \in \mathbb{Z}^6_{>0}$ a polarization:*

1. *m s.t.*
   - $3 \nmid |m|$,
   - $m_i < |m|/3 \, \forall i$,

   then the quotient is geometric;
2. *m s.t.*
   - $3 \mid |m|$,
   - $m_i < |m|/3 \, \forall i$,
   - *for all couples and triples of indexes we have $m_i + m_j \neq |m|/3$ or $m_h + m_i + m_j \neq |m|/3$,*

   then the quotient is geometric;
3. *m s.t.*
   - $3 \mid |m|$,
   - *there exists an index i s.t. $m_i = |m|/3$, while for the other indexes $j \neq i$, $m_j < |m|/3$,*

   then the quotient is $(\mathbb{P}^1(\mathbb{C}))^5(m')//SL_2(\mathbb{C})$; its dimension is equal to two, and the polarization $m' \in \mathbb{Z}^5_{>0}$ is obtained from m by eliminating m_i;
4. *m s.t.*
   - $3 \mid |m|$,

- *there exist two different indexes* i, j *s.t.* $m_i = m_j = |m|/3$*, while for the others* $h \neq i, j$, $m_h < |m|/3$,

then the quotient is $(\mathbb{P}^1(\mathbb{C}))^4(m'')//SL_2(\mathbb{C}) \cong \mathbb{P}^1(\mathbb{C})$*; the polarization* $m'' \in \mathbb{Z}^4_{>0}$ *is obtained from* m *by eliminating* m_i *and* m_j*;*

5. m *s.t.*

- $3 \mid |m|$,
- $m_i < |m|/3 \, \forall i$,
- *there are two different indexes* i, j *s.t.* $m_i + m_j = |m|/3$,

then the quotient is categorical; moreover it includes a curve $C_{ij} \cong \mathbb{P}^1(\mathbb{C})$*, that corresponds to strictly semi-stable orbits s.t.* $x_i = x_j$ *or* x_h, x_k, x_l, x_n *collinear. In particular points* ξ *of* C_{ij} *are singular: locally, the variety* $(X^{SS}(m)//G, \xi)$ *is isomorphic to the toric variety*

$$\mathbb{C}[T_1, T_2, T_3, T_4, T_5]/(T_1T_4 - T_2T_3)\,.$$

6. m *s.t.*

- $3 \mid |m|$,
- $m_i < |m|/3 \, \forall i$,
- *there is a "partition" of* m *such that* $m_i + m_j = m_h + m_l = m_k + m_n$,

then the quotient is categorical; moreover it includes three curves C_{ij}, C_{hl}, $C_{kn} \cong \mathbb{P}^1(\mathbb{C})$*, that have a common point* $O_{ij,hl,kn}$*.*
In particular $O_{ij,hl,kn}$ *is singular: locally the variety* $(X^{SS}(m)//G, O_{ij,hl,kn})$ *is isomorphic to the toric variety*

$$\mathbb{C}[T_1, T_2, T_3, T_4, T_5]/(T_1T_2T_3 - T_4T_5)\,.$$

7. m *s.t.*

- $3 \mid |m|$,
- $m_i < |m|/3 \, \forall i$,
- *there are three indexes* h, i, j *s.t.* $m_h + m_i + m_j = |m|/3$,

then the quotient is categorical; moreover it includes a point O_{hij} *that correspond to the minimal, closed, strictly semi-stable orbit* Gx *such that* $x_h = x_i = x_j$ *and* x_k, x_l, x_n *are collinear. The point* O_{hij} *is non singular.*

3.5 Examples

Now we provide two examples that illustrate how to get explicitly a quotient, via its coordinates ring, or via an elementary transformation.

3.6 $\mathbb{P}^2(\mathbb{C})^6(222111)$

$|m| = 9$; by the numerical criterion: $\sum_{k, x_k = y} m_k \leq 3$, $\sum_{j, x_j \in r} m_j \leq 6$. Then $X^S(m) \subset X^{SS}(m)$.
Moreover it is easy to verify that there are nine C_{ij} curves, six $O_{ij,hl,kn}$ points and one O_{hij} point.

Let us study the graded algebra of G-invariant functions $R_2^6(m)^G$. A standard tableau τ of degree k associated to the polarization m looks like

$$\tau = \left[\begin{array}{ccc} a_1^1 & a_2^2 & a_3^3 \\ a_2^1 & a_3^2 & a_4^3 \\ a_3^1 & a_4^2 & a_5^3 \\ a_4^1 & a_5^2 & a_6^3 \end{array}\right] \left.\vphantom{\begin{array}{c} a \\ a \\ a \\ a \end{array}}\right\} 3k \qquad (11)$$

where

$$\begin{array}{lll} |a_1^1| = 2k, & |a_6^3| = k, & |a_2^1| + |a_2^2| = 2k, \\ |a_3^1| + |a_3^2| + |a_3^3| = 2k, & |a_4^1| + |a_4^2| + |a_4^3| = k, & |a_5^2| + |a_5^3| = k, \\ \sum_{i=2}^4 |a_i^1| = k, & \sum_{i=2}^5 |a_i^2| = 3k, & \sum_{i=3}^5 |a_i^3| = 2k\,. \end{array}$$

Let $\alpha_3 := |a_3^1|\,, \alpha_4 := |a_4^1|\,, \beta_3 := |a_3^3|\,, \beta_4 := |a_4^3|$. Then it follows:

$$\begin{array}{lll} |a_1^1| = 2k, & |a_2^2| = k + \alpha_3 + \alpha_4, & |a_3^3| = \beta_3, \\ |a_2^1| = k - (\alpha_3 + \alpha_4), & |a_3^2| = 2k - (\alpha_3 + \beta_3), & |a_4^3| = \beta_4, \\ |a_3^1| = \alpha_3, & |a_4^2| = k - (\alpha_4 + \beta_4), & |a_5^3| = 2k - (\beta_3 + \beta_4), \\ |a_4^1| = \alpha_4, & |a_5^2| = \beta_3 + \beta_4 - k, & |a_6^3| = k. \end{array}$$

Moreover $\alpha_3, \alpha_4, \beta_3, \beta_4$ must satisfy the inequalities:

$$\begin{array}{lll} 0 \leq \alpha_3, \alpha_4, \beta_3, \beta_4 \leq 2k, & \alpha_3 + 2\alpha_4 \leq \beta_3, & \alpha_3 + \alpha_4 \leq k, \\ k + \alpha_4 \leq \beta_3 + \beta_4 \leq 2k, & \beta_3 \leq k + \alpha_3 + \alpha_4, & 2\beta_3 + \beta_4 \leq 3k + \alpha_4. \end{array}$$

Assume

$$x := \alpha_4, \quad y := \alpha_3 + \alpha_4, \quad z := \beta_3, \quad w := \beta_3 + \beta_4\,;$$

the standard tableau τ (11) is completely determined by the vector (x, y, z, w) that satisfy:

$$\begin{array}{llll} 0 \leq x \leq y \leq k, & 0 \leq z \leq w \leq 2k, & 0 \leq y + z - x \leq 2k, & \\ x + y \leq z \leq y + k, & z \leq w \leq k + z, & 0 \leq w + x - z \leq k, & w \geq x + k\,. \end{array}$$

After few calculations we find out that for any k, there are

$$\frac{1}{8}(k^4 + 6k^3 + 15k^2 + 18k) + 1 \,(= \dim(R_2^6(m)_k^G)\,)$$

standard tableaux. Thus the Hilbert function of the graded ring $R_2^6(m)^G$ is equal to

$$\sum_{k=0}^{\infty} \left(\frac{1}{8}(k^4 + 6k^3 + 15k^2 + 18k) + 1\right) t^k = \frac{1 - t^3}{(1 - t)^6}\,.$$

This suggests that the quotient $X^{SS}(m)//G$ is isomorphic to a cubic hypersurface in $\mathbb{P}^5(\mathbb{C})$.

First of all we have the following generators of $R_2^6(m)^G$:

$$\begin{array}{lll} t_0 = [124][135][236], & t_1 = [123][135][246], & t_2 = [123][134][256], \\ t_3 = [123][125][346], & t_4 = [123][124][356], & t_5 = [123][123][456]. \end{array}$$

For every $(i,j) \neq (2,3),(3,2)$, the product $t_i t_j$ is a standard tableau function from $R_2^6(m)_2^G$. Applying the straightening algorithm (that allows to write any tableau function as a linear combination of tableau standard functions), we obtain:

$$t_2 t_3 = t_1 t_4 - u + t_5(-t_0 + t_1 - t_2 - t_3 + t_4 - t_5). \tag{12}$$

So the standard monomial $u = [123][123][123][145][246][356]$ can be expressed as polynomials of degree two in the t_i.

In we take a tableau function $\mu_{(x,y,z,w,k)}$ corresponding to a standard tableau τ (11), we can write it as

$$\mu_{(x,y,z,w,k)} = \begin{cases} t_0^{k+x-z} t_1^{k+z-x-w} t_2^{w-y-k} t_4^{y-x} t_5^x, & z \leq x+k,\ w \leq k+z-x; \\ t_0^{k+x-z} t_1^{z-x-y} t_3^{k+y-w} t_4^{w-x-k} t_5^x, & z \leq x+k,\ y \leq z-x; \\ t_1^{3k+x-w-z} t_2^{w-y-k} t_4^{k+y-z} t_5^x u^{z-x-k}, & z \geq x+k,\ w \leq 3k+x-z; \\ t_1^{2k+x-y-z} t_3^{k+y-w} t_4^{w-z} t_5^x u^{z-x-k}, & z \geq x+k,\ y \leq 2k+x-z. \end{cases}$$

Applying the straightening algorithm to the non-standard product $t_0 u$, we have:

$$t_0 u \ = \ t_1 t_4 (t_1 - t_2 - t_3 + t_4 - t_5)\,.$$

Then by relation (12), it follows

$$t_0\big(t_1t_4 - t_2t_3 + t_5(-t_0 + t_1 - t_2 - t_3 + t_4 - t_5)\big) = t_1t_4(t_1 - t_2 - t_3 + t_4 - t_5) \Rightarrow$$

$$t_0\big(-t_2t_3 + t_5(-t_0 + t_1 - t_2 - t_3 + t_4 - t_5)\big) = t_1t_4(-t_0 + t_1 - t_2 - t_3 + t_4 - t_5) \Rightarrow$$

$$(-t_0 + t_1 - t_2 - t_3 + t_4 - t_5)(t_0t_5 - t_1t_4) - t_0t_2t_3 = 0$$

Let

$$F_3 = (-T_0 + T_1 - T_2 - T_3 + T_4 - T_5)(T_0T_5 - T_1T_4) - T_0T_2T_3\,, \tag{13}$$

there is a surjective homomorphism of the graded algebras

$$\mathbb{C}[T_0,T_1,T_2,T_3,T_4,T_5]/(F_3(T_0,T_1,T_2,T_3,T_4,T_5)) \longrightarrow R_2^6(m)^G\,.$$

Thus the quotient $X^{SS}(m)//G$ is isomorphic to the cubic hypersurface $F_3(T_0,T_1,T_2,T_3,T_4,T_5) = 0$.

3.7 $\mathbb{P}^2(\mathbb{C})^6(221111)$

$|\widehat{m}| = 8$; by the numerical criterion $\sum_{k,x_k=y} \widehat{m}_k \leq 8/3$, $\sum_{j,x_j\in r} \widehat{m}_j \leq 16/3$ and thus $X^S(\widehat{m}) = X^{SS}(\widehat{m})$.

In order to determine this geometric quotient, we have to introduce the elementary transformation $\widehat{m} = (221111) \xrightarrow{+1_3} (222111) = m$, and consequently

$$\widehat{\theta} : X^S(\widehat{m})/G \longrightarrow X^{SS}(m)//G\,.$$

First of all let us study $\widehat{\theta}^{-1}(O_{456})$: by relation (6) its dimension is equal to $d = 3$; the semi-stable orbits of $X^{SS}(m)$ that determine O_{456} in the quotient $X^{SS}(m)//G$ and are included in $X^S(\widehat{m})$, are characterized by x_1, x_2, x_3 collinear. Applying a projectivity of $\mathbb{P}^2(\mathbb{C})$ such that it fixes the line that contains x_1, x_2, x_3, we have $\widehat{\theta}^{-1}(O_{456}) \cong \mathbb{P}^3(\mathbb{C})$.

Then $\widehat{\theta}^{-1}(\xi), \xi \in C_{ij}$; studying how semi-stable orbits change going from $X^{SS}(m)$ to $X^S(\widehat{m})$, there can be two different cases: coincidence or collinearity.

1. Consider the curve C_{14}: by the numerical criterion for $X^S(\widehat{m})$, orbits which have x_2, x_3, x_5, x_6 collinear are stable. In particular by relation (6), the dimension of $\widehat{\theta}^{-1}(\xi_1)$, $\xi_1 \in C_{14}$ is equal to $d = 1$: in fact

$$\widehat{\theta}^{-1}(\xi_1) \cong \mathbb{P}^1(\mathbb{C})\,. \tag{14}$$

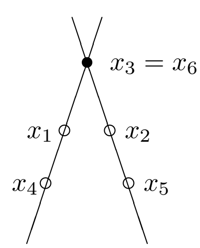
2. Consider the curve C_{36}: by the numerical criterion for $X^S(\widehat{m})$ orbits which have $x_3 = x_6$ are stable. In particular by relation (5), the dimension of $\widehat{\theta}^{-1}(\xi_2)$, $\xi_2 \in C_{36}$ is equal to $d = 1$; in fact

$$\widehat{\theta}^{-1}(\xi_2) \cong \mathbb{P}^1(\mathbb{C})\,. \tag{15}$$

Let us study $\widehat{\theta}^{-1}(O_{ij,hl,kn})$; consider $O_{14,25,36}$. Strictly semi-stable orbits that contain the orbit Gx ($x_1 = x_4, x_2 = x_5, x_3 = x_6$) in their closure, are characterized by one of the following properties:

1. $x_1 = x_4$ and x_1, x_2, x_5 collinear; 2. $x_1 = x_4$ and x_1, x_3, x_6 collinear;
3. $x_2 = x_5$ and x_1, x_2, x_4 collinear; 4. $x_2 = x_5$ and x_2, x_3, x_6 collinear;
5. $x_3 = x_6$ and x_1, x_3, x_4 collinear; 6. $x_3 = x_6$ and x_2, x_3, x_5 collinear.

In particular configurations $1, 2, 3, 4$ are unstable for the polarization $\widehat{m}$, while 5 and 6 are included in $X^S(\widehat{m})$; moreover these sets have a common configuration: ($x_3 = x_6$, x_1, x_3, x_4 collinear, x_2, x_3, x_5 collinear):



Every one of these two sets of stable configurations determine a copy of $\mathbb{P}^1(\mathbb{C})$ in the quotient $X^S(\widehat{m})/G$: thus these two copies of $\mathbb{P}^1(\mathbb{C})$ have a common point.

$$\widehat{\theta}^{-1}(O_{ij,hl,kn}) \cong \mathbb{P}^1(\mathbb{C}) \cup \mathbb{P}^1(\mathbb{C}) \text{ with a common point}\,.$$

We can get this result in a different way, by constructing a subdivision of the polytope Π (figure 2).

Since $X^{US}(m) \subset X^{US}(\widehat{m})$ and $\big(X^{US}(\widehat{m}) \setminus X^{US}(m)\big) \subset X^{SSS}(m)$, we determine (locally in N_x), which strictly semi-stable orbits for the polarization m are unstable for $\widehat{m}$. By the machinery of the theory of homogeneous coordinates for a toric variety ([1],[2], [4]), the local resolution of $(X^{SS}(m)//G, O_{14,25,36}) \cong (\mathbb{C}^6/(\mathbb{C}^*)^2, 0)$ in the quotient $X^S(\widehat{m})/G$ is determined by $(\mathbb{C}^6 \setminus Z)//H$, where $\mathbb{C}^6 \setminus Z = \mathbb{C}^6 \setminus \{z \in \mathbb{C}^6 \,| z_1z_4 = 0, z_2z_3 = 0, z_2z_4 = 0\}$, and H is the 2-dimensional torus $H = \{(\lambda_1, \lambda_2, \lambda_1^{-1}, \lambda_1^{-1}\lambda_2, \lambda_2^{-1}, \lambda_1\lambda_2^{-1}), \lambda_1, \lambda_2 \in \mathbb{C}^*\}$.

The set $\mathbb{C}^6 \setminus Z$ describes a particular resolution of Π.

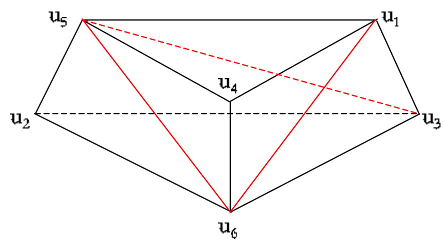


Figure 3: Subdivision of type (221111) of Π

We can find three simplicial polytopes: figure 4.

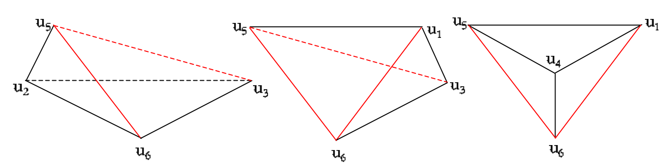


Figure 4: The three polytopes of the subdivision (221111) of Π

The toric representation of Y, described by the polytope Π, is determined by the cone σ: to solve its singularities let us construct a fan Σ, refinement of σ. By the theory of toric varieties, there exists a proper, birational morphism φ

$$X_\Sigma \cong (\mathbb{C}^6 \setminus Z)//H \cong (\mathbb{C}^6 \setminus Z)//(\mathbb{C}^*)^2 \xrightarrow{\varphi} (\mathbb{C}^6//(\mathbb{C}^*)^2) \cong (N_x//G_x) \cong X_\sigma,$$

induced by the identity over the lattice $\mathbb{R}^4$: this application allows us to specify the map $\widehat{\theta}$:

$$\widehat{\theta} : X^S(\widehat{m})/G \longrightarrow X^{SS}(m)//G\,.$$

First of all let us take a cover of $(\mathbb{C}^6 \setminus Z)$: for example the three open sets U_1, U_2, U_3:

$$U_1 = \mathbb{C}^6 \setminus \{z \in \mathbb{C}^6 \mid z_1 z_4 = 0\}; \qquad U_2 = \mathbb{C}^6 \setminus \{z \in \mathbb{C}^6 \mid z_2 z_3 = 0\};$$
$$U_3 = \mathbb{C}^6 \setminus \{z \in \mathbb{C}^6 \mid z_2 z_4 = 0\}.$$

Now let us consider the action of $H \cong (\mathbb{C}^*)^2$ on these three open sets and construct the three quotients: in the first case, the quotient $\widetilde{U_1} = U_1//H$ is the smooth variety $\mathbb{C}[X_1, X_2, X_3, X_4, X_6]/(X_2 - X_4 X_6)$.
In the same way $\widetilde{U}_2 = U_2//H = \mathbb{C}[Y_1, Y_2, Y_3, Y_5, Y_7]/(Y_3 - Y_5 Y_7)$ and $\widetilde{U}_3 = U_3//H = \mathbb{C}[Z_1, Z_2, Z_3, Z_8, Z_9]/(Z_1 - Z_8 Z_9)$.

How do these quotients $\widetilde{U}_i (i = 1, 2, 3)$ fit together? We have the following "gluing"

$$\begin{array}{lll} X_1 = Y_1 = Z_8 Z_9 & Y_1 = X_1 = Z_8 Z_9 & Z_2 = X_4 X_6 = Y_2 \\ X_3 = Y_5 Y_7 = Z_3 & Y_2 = X_4 X_6 = Z_2 & Z_3 = X_3 = Y_5 Y_7 \\ X_4 = Y_1 Y_2 Y_7 = Z_2 Z_8 & Y_5 = X_1 X_3 X_6 = Z_3 Z_8 & Z_8 = X_6^{-1} = Y_1 Y_7 \\ X_6 = (Y_1 Y_7)^{-1} = Z_8^{-1} & Y_7 = (X_1 X_6)^{-1} = Z_9^{-1} & Z_9 = X_1 X_6 = Y_7^{-1} \end{array} \tag{16}$$

The birational maps $\widehat{\theta}_i : \widetilde{U}_i \to Y$ that resolve the singularities of Y are described by the pull back of the generators of the ring of G_x-invariant functions $(T_1, T_2, T_3, T_4, T_5)$:

$$\begin{array}{lll} \widehat{\theta}_1^*(T_1) = X_1, & \widehat{\theta}_2^*(T_1) = Y_1, & \widehat{\theta}_3^*(T_1) = Z_8 Z_9, \\ \widehat{\theta}_1^*(T_2) = X_4 X_6, & \widehat{\theta}_2^*(T_2) = Y_2, & \widehat{\theta}_3^*(T_2) = Z_2, \\ \widehat{\theta}_1^*(T_3) = X_3, & \widehat{\theta}_2^*(T_3) = Y_5 Y_7, & \widehat{\theta}_3^*(T_3) = Z_3, \\ \widehat{\theta}_1^*(T_4) = X_4, & \widehat{\theta}_2^*(T_4) = Y_1 Y_2 Y_7, & \widehat{\theta}_3^*(T_4) = Z_2 Z_8, \\ \widehat{\theta}_1^*(T_5) = X_1 X_3 X_6, & \widehat{\theta}_2^*(T_5) = Y_5, & \widehat{\theta}_3^*(T_5) = Z_3 Z_9. \end{array}$$

The point $O_{14,25,36}$ corresponds to the origin in Y: let us study $\widehat{\theta}_i^{-1}(0)$

$$\widehat{\theta}_1^{-1}(0) = (0, 0, 0, t_1) \cong \mathbb{C}, \quad \widehat{\theta}_2^{-1}(0) = (0, 0, 0, u_1) \cong \mathbb{C},$$
$$\widehat{\theta}_3^{-1}(0) = (0, 0, t_2, u_2) \cong \mathbb{C} \cup \mathbb{C}$$

where $t_1, u_1, t_2, u_2 \in \mathbb{C}$ and $t_2 u_2 = 0$.
In particular the fiber $\widehat{\theta}_3^{-1}(0)$ is isomorphic to the union of two copies of $\mathbb{C}$ that have a common point $(0, 0, 0, 0) \in \widetilde{U}_3$. Moreover by the gluing (16), $t_1, t_2 \in \mathbb{C}$ give a cover of $\mathbb{P}^1(\mathbb{C})$, just like $u_1, u_2 \in \mathbb{C}$.

In conclusion the resolution of $O_{14,25,36}$ in $X^S(221111)/G$ is determined by the union of two copies of $\mathbb{P}^1(\mathbb{C})$ that have a common point

$$\widehat{\theta}^{-1}(O_{14,25,36}) \cong \mathbb{P}^1(\mathbb{C}) \cup \mathbb{P}^1(\mathbb{C}) \quad \text{with a common point.}$$

Let us calculate the resolutions of the three singular curves C_{14}, C_{25}, C_{36} that meet in $O_{14,25,36}$: we know that there is a correspondence between C_{ij}, C_{hl}, C_{kn}

and the three lines $s_3 = \{(0,0,t,0,0)\}, s_2 = \{(0,t,0,0,0)\}, s_1 = \{(t,0,0,0,0)\}$ of Y. Now let us calculate the fiber of a "generic" point of each line s_j, for the maps $\widehat{\theta}_i$.
Let $\xi_3 \in C_{14}$: $\widehat{\theta}_1^{-1}(\xi_3) = (0,t,0,\tau)$, $\widehat{\theta}_2^{-1}(\xi_3) =$ Imposs., $\widehat{\theta}_3^{-1}(\xi_3) = (0,t,\tau^{-1},0)$; thus

$$\widehat{\theta}^{-1}(\xi_3) \cong \mathbb{P}^1(\mathbb{C}), \quad \forall \xi_3 \in C_{14} \quad \xi_3 \neq O_{ij,hl,kn}.$$

In the same way for $\xi_2 \in C_{25}$ and $\xi_1 \in C_{36}$, $\xi_1, \xi_2 \neq O_{ij,hl,kn}$ we obtain:

$$\widehat{\theta}^{-1}(\xi_2) \cong \mathbb{P}^1(\mathbb{C}), \quad \widehat{\theta}^{-1}(\xi_1) \cong \mathbb{P}^1(\mathbb{C}).$$

In conclusion the map

$$\widehat{\theta} : X^S(\widehat{m})/G = (\mathbb{P}^2)^6(221111)/G \longrightarrow (\mathbb{P}^2)^6(222111)//G = X^{SS}(m)//G$$

determines the quotient $X^S(\widehat{m})/G$: in fact $\widehat{\theta}$ is an isomorphism over

$$X^S(\widehat{m})/G \setminus \left(\bigcup_{\xi \in S} \widehat{\theta}^{-1}(\xi) \right) \xrightarrow{\sim} X^S(m)/G,$$

where $S = \{\xi \in X^{SSS}(m)//G\}$.
Then the map $\widehat{\theta}$ is a contraction of subvarieties over $\bigcup_{\xi \in S} \widehat{\theta}^{-1}(\xi)$:

- if $\xi \in C_{ij}$, then $\widehat{\theta}^{-1}(\xi) = \mathbb{P}^1(\mathbb{C})$;
- if $\xi = O_{ij,hl,kn}$, then $\widehat{\theta}^{-1}(\xi) = \mathbb{P}^1(\mathbb{C}) \cup \mathbb{P}^1(\mathbb{C})$, with a common point;
- if $\xi = O_{456}$, then $\widehat{\theta}^{-1}(\xi) = \mathbb{P}^3(\mathbb{C})$.

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