ON THE X-RANK WITH RESPECT TO LINEARLY NORMAL CURVES.

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ABSTRACT: *In this paper we study the X-rank of points with respect to smooth linearly normal curves $X \subset \mathbb{P}^n$ of genus g and degree $n+g$.*
We prove that, for such a curve X, under certain circumstances, the X-rank of a general point of X-border rank equal to s is less or equal than $n+1-s$.
In the particular case of $g=2$ we give a complete description of the X-rank if $n=3,4$; while if $n \geq 5$ we study the X-rank of points belonging to the tangential variety of X.

Introduction

Let $X \subset \mathbb{P}^n$ be an integral, smooth, non degenerate curve defined over an algebraically closed field K of characteristic 0.

The X-*rank* of a point $P \in \mathbb{P}^n$, that we will denote with $r_X(P)$, is the minimum positive integer $s \in \mathbb{N}$ of points $P_1, \ldots, P_s \in X$ such that

$$P \in \langle P_1, \ldots, P_s \rangle \subset \mathbb{P}^n, \tag{1}$$

where $\langle \ \rangle$ denote the linear span.

The knowledge of the X-rank of an element $P \in \mathbb{P}^n$ with respect to a variety $X \subset \mathbb{P}^n$ is a theme of great interest both in mathematics and in recent applications. In particular in the literature a large space is devoted to computation of the X-rank of points $P \in \mathbb{P}^n$ with respect to projective varieties X that parameterize certain classes of homogeneous polynomials and also particular kind of tensors (see e.g. [3], [4], [5], [6], [7], [8], [9] [11], [12], [13], [14], [17], [19], [22], [23]).

Actually, from a pure mathematical point of view, the notion of X-rank of a point is preceded by the notion of X-border rank and that one of secant varieties.
The *s-th secant variety* $\sigma_s(X) \subseteq \mathbb{P}^n$ of a projective variety $X \subset \mathbb{P}^n$ is defined as follows:

$$\sigma_s(X) := \overline{\bigcup_{P_1, \ldots, P_s \in X} \langle P_1, \ldots, P_s \rangle} \subseteq \mathbb{P}^n, \tag{2}$$

where the closure is in terms of Zariski topology.
Observe that $\sigma_1(X) = X$ and also that

$$X \subset \sigma_2(X) \subset \cdots \subset \sigma_{s-1}(X) \subset \sigma_s(X) \subseteq \mathbb{P}^n.$$

If $P \in \sigma_s(X) \setminus \sigma_{s-1}(X)$ is said to be of X-*border rank* equal to s. Obviously the X-border rank of a point $P \in \mathbb{P}^n$ is less or equal than its X-rank.
Since the set

$$\sigma_s^0(X) := \{P \in \mathbb{P}^n \mid r_X(P) \leq s\} \tag{3}$$

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is not a closed variety (except obviously when $s=1$ and when $\sigma_s(X)=\mathbb{P}^n$), it turns out that, in Algebraic Geometry, the notion of X-border rank is more natural than that one of X-rank, because it is not possible to find ideals and equations for $\sigma_s^0(X)$, while there is a wide open research area interested in a description of ideals for $\sigma_s(X)\subsetneq\mathbb{P}^n$ (see e.g. [2], [10], [15], [16], [18], [20]).

If $X\subset\mathbb{P}^n$ is a rational normal curve of degree n, the knowledge of X-rank of a point coincides both with that one of "symmetric rank" of a two variables n-dimensional symmetric tensor, and with the knowledge of the so called "Waring rank" of a two variables homogeneous polynomial of degree n (see [6], [11], [12] [23]). In this case a complete description of the X-rank is given for any point $P\in\sigma_s(X)\setminus\sigma_{s-1}(X)$ and for any positive integer s. In particular the first description of this result is due to Sylvester ([23]), then in [11] there is a reformulation of it in more modern terms. Recently [12] and [6] have given explicit algorithms for the computation of the X-rank with respect to a rational normal curve. What is proved in all those papers is that, with respect to a rational normal curve $X\subset\mathbb{P}^n$, the X-rank of a point $P\in\sigma_s(X)\setminus\sigma_{s-1}(X)$, for $2\le s\le\lceil\frac{n+1}{2}\rceil$, can only be either s or $n-s+2$.

When one looks to the X-rank with respect to projective curves $X\subset\mathbb{P}^n$ of higher genus, the informations becomes immediately isolated. For example, if $X\subset\mathbb{P}^n$ is a genus 1 curve of degree $n+1$, the only result we are aware of, is about points belonging to tangent lines to X (see [6], Theorem 3.13).

We introduce here the definition of the *tangential variety* $\tau(X)\subset\mathbb{P}^n$ of a projective variety $X\subset\mathbb{P}^n$ as follows:

$$\tau(X):=\overline{\bigcup_{Q\in X_{\mathrm{reg}}}T_QX}. \tag{4}$$

where T_QX is the tangent space to X at Q, and the closure is in terms of the Zariski topology. Observe that if $X\subset\mathbb{P}^n$ is smooth then $\tau(X)=\bigcup_{Q\in X_{\mathrm{reg}}}T_QX$.

If $X\subset\mathbb{P}^n$ is a smooth elliptic curve of degree $n+1$, the X-rank of the points $P\in\tau(X)$ is described in Theorem 3.13 of [6]. The authors proved that, for all $Q\in X$, if $n=3$ then the elements $P\in T_QX\backslash\sigma_2^0(X)$ are such that $r_X(P)=3$, while if $n\ge4$ then any $P\in T_QX\backslash\sigma_2^0(X)$ is such that $r_X(P)=n-1$.
Clearly this result runs out the case of the X-rank of all points $P\in\sigma_2(X)$ when $X\subset\mathbb{P}^n$ is an elliptic curve of degree $n+1$, because $\sigma_2(X)=\sigma_2^0(X)\cup\tau(X)$, and any point $P\in\sigma_2^0(X)$ can only be of X-rank equal either to 1 or to 2 by definition (3). Anyway, on our knowledge, nothing is known on $r_X(P)$ with respect to elliptic curves X if $P\notin\sigma_2(X)$.

What we do in this paper is to treat the case of smooth and linearly normal curves $X\subset\mathbb{P}^n$ of genus g and degree $n+g$ with a particular attention to the case of genus 2 curves.

Definition 1. A non-degenerate projective curve $X\subset\mathbb{P}^n$ is called *linearly normal* if $H^1(\mathbb{P}^n,\mathcal{I}_X(1))=0$.

From now on $X\subset\mathbb{P}^n$ will be a linearly normal non-degenerate projective curve of genus g and degree $n+g$.

If a point $P\in\mathbb{P}^n$ belongs to a $\sigma_s^0(X)$, for certain value of $s\in\mathbb{N}$, then, by definition (3), there exists at least an effective reduced divisor $Z\subset X$ of degree less or equal than s such that $P\in\langle Z\rangle$.
Otherwise if there exists an integer s such that $P\in\sigma_s(X)\setminus\sigma_s^0(X)$, then (by Proposition 2.8 in [6]) there exists an effective non-reduced divisor $Z\subset X$ of degree s such that $P\in\langle Z\rangle$, no other effective divisor of X of degree strictly less then s can contain the point P in its span,

and the smallest degree of a reduced effective divisor $Z' \subset X$ such that $P \in \langle Z' \rangle$ has to be bigger than s.
A first way of investigation in order to compute the X-rank of a point $P \in \mathbb{P}^n$ for whom only its X-border rank is known, is to study the X-rank of points belonging to some projective subspace $\langle Z \rangle \subset \mathbb{P}^n$ where $Z \subset X$ is an effective non-reduced divisor of X and try to understand if there is a relation between the X-rank of $P \in \langle Z \rangle$ and the structure of $Z \subset X$.
The result that we can give for this general case is Theorem 1 stated below (it will be proved in Section 1). That theorem shows that if the X-border rank s of a point $P \in \sigma_s(X) \subset \mathbb{P}^n$ does not exceed $\lceil \frac{n-2}{2} \rceil$ and P belongs to the span of an effective non reduced divisor $Z \subset X$ such that $\deg(\langle Z \rangle \cap X) \leq \deg(X) - 2p_a(X)$ then the X-rank of P cannot be greater than $n+1-s$ (only one very particular embedding of X in $\mathbb{P}^n$ is excluded form that result).

Theorem 1. *Let $X \subset \mathbb{P}^n$ be an integral non-degenerate and linearly normal curve. Let $Z \subset X_{reg}$ be a 0-dimensional scheme such that $\dim(\langle Z \rangle) = s$ and $n \geq 2s+2$. Let $Z' \subset X$ be the Cartier divisor obtained by the schematic intersection $Z' := X \cap \langle Z \rangle$. Assume $h^1(\mathbb{P}^n, \mathcal{I}_{Z'}(1)) = 0$ and $\deg(Z') \leq \deg(X) - 2p_a(X)$. If $\deg(X) = 2p_a(X) + \deg(Z')$ and X admits a degree 2 morphism $\phi : X \to \mathbb{P}^1$, then assume $\mathcal{O}_X(1)(-Z') \neq \phi^*(\mathcal{O}_{\mathbb{P}^1}(p_a(X)))$. Then for a general $P \in \langle Z \rangle$ the X-rank of P is:*

$$r_X(P) \leq n+1-s.$$

Section 1 is almost entirely devoted to the proof of that theorem and to another result (Corollary 1) on linearly normal curves $X \subset \mathbb{P}^n$ of genus $g \leq n-1$ where we give an immediate lower bound for the X-rank of points belonging to $\tau(X) \setminus \sigma_2^0(X)$ that will be useful in the sequel.

In Section 2 we will focus on non-degenerate linearly normal curves $X \subset \mathbb{P}^n$ of genus 2 and degree $n+2$. We will first treat the cases $n = 3, 4$ (in subsections 2.1 and 2.2 respectively). If $n = 3$ then $\sigma_2(X) = \mathbb{P}^3$ (see [1]), hence, the only meaningful case to study is that one of points in $\tau(X) \setminus \sigma_2^0(X)$ (if such a set is not empty). R. Piene in [21] shown that it is possible to find a linearly normal embedding of X in $\mathbb{P}^3$ for which there exists $P \in \mathbb{P}^3$ such that $r_X(P) = 3$. When such point exists it has to belong to $\tau(X) \setminus \sigma_2^0(X)$. In Proposition 1 we will give a geometric description of those points.
If $n = 4$ then $\sigma_3(X) = \mathbb{P}^4$ (see [1]). We will actually prove in Proposition 2 that $\sigma_3^0(X) = \mathbb{P}^4$. This will be proved by showing both that if $P \notin \sigma_2(X)$ and also if $P \in \tau(X) \setminus \sigma_2^0(X)$ than $r_X(P) = 3$. We will also give in Proposition 3 a geometric description of the points $P \in \mathbb{P}^4$ of X-rank equal to 3.

Finally, in Subsection 2.3, we will treat the case of a linearly normal genus 2 curve in $\mathbb{P}^n$ of degree $n+2$ for $n \geq 5$ and we will prove the following theorem.

Theorem 2. *Fix an integer $n \geq 8$. and let $X \subset \mathbb{P}^n$ be a linearly normal smooth curve of genus 2 and degree $n+2$. Then the X-rank $r_X(P)$ of a point $P \in T_Q X \setminus X$, for any $Q \in X$, is $r_X(P) = n-2$.*

If $n = 5, 6, 7$ we can actually show that the set of points $\{P \in \tau(X) \mid r_X(P) = n-2\}$ is not empty, but we can only prove that the X-rank of $P \in \tau(X)$ can be at most $n-1$ (see Proposition 4).

From all these results we end up in Section 3 with some natural but open questions concerning the highest realization of the X-rank with respect to a linearly normal smooth genus g curve $X \subset \mathbb{P}^n$ of degree $n+g$. More precisely, we expect that the maximum possible X-rank with respect to such a curve can be reached, at least for big values of n, by points on $\tau(X)$ (see questions 2 and 4). We also expect that, when $n \gg s$, the X-rank equal to s cannot be realized out of $\sigma_{n-s}(X)$ (Question 3).

1. The X-rank with respect to a linearly normal curve

In this section we study the X-rank of projective points with respect to a smooth and linearly normal curve $X \subset \mathbb{P}^n$ of genus g and degree $n+g$.
First of all we give in Corollary 1 a lower bound for the X-rank of points belonging to $\tau(X)\backslash X$ if X is embedded in a $\mathbb{P}^n$ with $n \geq g+1$.

Lemma 1. *Let $X \subset \mathbb{P}^n$ be an integral and linearly normal curve and let $Z \subset X$ be a zero-dimensional subscheme such that* $\deg(X) - \deg(Z) > 2p_a(X) - 2$. *Then* $h^1(\mathbb{P}^n, \mathcal{I}_Z(1)) = 0$.

Proof. The degree of the canonical sheaf ω_X is $2p_a(X) - 2$, even if X is not locally free. Hence for degree reasons we have $h^1(X, \mathcal{I}_{Z,X}(1)) = 0$ Since X is linearly normal, then $h^1(\mathbb{P}^n, \mathcal{I}_Z(1)) = 0$. □

Corollary 1. *Let $X \subset \mathbb{P}^n$ be an integral, non degenerate, linearly normal and smooth curve of genus g and degree $n+g$ with $n \geq g+1$. Then, for any regular point $Q \in X$, the X-rank of a point $P \in T_QX \setminus X$ is:*

$$r_X(P) \geq n - g.$$

Proof. By Lemma 1 we have that $h^1(\mathbb{P}^n, \mathcal{I}_{\langle 2Q \rangle}(1)) = 0$ hence $h^0(\mathbb{P}^n, \mathcal{I}_{\langle 2Q \rangle}(1)) = n-1$, therefore any hyperplane H containing T_QX cuts on X a divisor D_H of degree n having $2Q$ as a fixed part, i.e. for any hyperplane H containing T_QX there exists a divisor D'_H on X of degree $n-g$ such that $D_H = 2Q + D'_H$. Now the D'_H's give a linear serie g^{n-g+1}_{n-g} on X. This implies the existence of a hyperplane $\tilde{H}$ such that $D_{\tilde{H}} = 2Q + D'_{\tilde{H}}$ and the divisor $D'_{\tilde{H}}$ belonging to g^{n-g+1}_{n-g} spans a $\mathbb{P}^{n-g+1}$ containing P. Therefore $r_X(P) \geq n-g$. □

Before proving Theorem 1 stated in the Introduction, we need the following Lemma.

Lemma 2. *Let $X \subset \mathbb{P}^n$ be an integral non-degenerate and linearly normal curve. Let $Z \subset X_{reg}$ be a 0-dimensional scheme such that* $\dim(\langle Z \rangle) = s$ *and* $n \geq 2s+2$. *Let $Z' \subset X$ be the Cartier divisor obtained by the schematic intersection $Z' := X \cap \langle Z \rangle$. If* $h^1(\mathbb{P}^n, \mathcal{I}_{Z'}(1)) = 0$ *and* $\deg(Z') \leq \deg(X) - 2p_a(X)$, *then $Z' = Z$.*

Proof. The hypothesis on the degree of $X \subset \mathbb{P}^n$, i.e. $\deg(X) \geq 2p_a(X) - 1$, implies the vanishing of $h^1(X, \mathcal{O}_X(1))$. Now X is linearly normal, then $\deg(X) = n + p_a(X)$. Moreover $h^1(\mathbb{P}^n, \mathcal{I}_{Z'}(1)) = 0$ and $\langle Z' \rangle$ is a projective subspace of dimension s, then $\deg(Z') = s+1$. Now, since $\langle Z \rangle = \langle Z' \rangle$ by hypothesis, then $Z' = Z$. □

We are now ready to give the proof of Theorem 1 that will allow to study the X-rank for points $P \in \sigma_s(X) \setminus \sigma_s^0(X)$ if the dimension of the ambient space is greater or equal than $2s-2$ and if $P \in \langle Z \rangle$ where $Z \subset X$ is an effective divisor such that $h^1(\mathbb{P}^n, \mathcal{I}_{\langle Z \rangle \cap X}) = 0$ and $\deg(\langle Z \rangle \cap X) \leq \deg(X) - 2p_a(X)$ (only one linearly normal embedding of X in $\mathbb{P}^n$ is excluded from this theorem). We stress that the two conditions required for the divisor $Z' = \langle Z \rangle \cap X$ are used to ensure that $Z' = Z$ as it is shown in Lemma 2 (i.e. that $\langle Z \rangle$ does not intersect X in other points than those cut by Z itself). We notice moreover that if $n \leq 2s-1$ then $\sigma_s(X) = \mathbb{P}^n$ (in fact, by [1], the dimension of the s-th secant variety to a smooth, reduced, non degenerate curve $X \subset \mathbb{P}^n$ is $2s-1$), hence the hypothesis $n \geq 2s-2$ excludes only the case of $\sigma_s(X) = \mathbb{P}^n$.

Proof of Theorem 1.

By Lemma 2, the scheme Z' coincides with Z. Now Z' is the base locus of the linear system induced on X by the set of all hyperplanes containing Z. By hypothesis, Z is also a Cartier divisor and $\deg(X) \geq 2p_a(X) + \deg(Z)$, then the line bundle

$$R := \mathcal{I}_X(1)(-Z) \tag{5}$$

is spanned. Therefore, if with $R_0 \in |R|$ we denote the general zero-locus of R, then R_0 is reduced and contained in $X_{reg} \backslash (Z)_{red}$. Since $h^1(\mathcal{I}_Z(1)) = 0$, then the restriction map $H^0(\mathbb{P}^n, \mathcal{I}_Z(1)) \to H^0(X, \mathcal{O}_X(1)(-Z))$ is surjective. Hence $R_0 \cup Z$ is the scheme-theoretic intersection of X with a hyperplane $H \subset \mathbb{P}^n$ and containing $\langle Z \rangle$.
Let $\varphi_{|R|} : X \to \mathbb{P}^{n-s-1}$ be the morphism induced by the complete linear system $|R|$. Clearly, by the definition (5), the degree of R is $\deg(R) = \deg(\varphi_{|R|}) \cdot \deg(\varphi_{|R|}(X))$. Moreover the image of X via $\varphi_{|R|}$ spans $\mathbb{P}^{n-s-1}$, then $\deg(R) \geq \deg(\varphi_{|R|}) \cdot (n-s-1)$ and equality holds if and only if $\varphi_{|R|}(X) \subset \mathbb{P}^{n-s-1}$ is a rational normal curve. Hence our numerical assumptions give that either $\deg(\varphi_{|R|}) = 1$ or $\deg(X) = 2p_a(X) + \deg(Z)$, $\deg(\varphi_{|R|}) = 2$, $\varphi_{|R|}(X) \subset \mathbb{P}^{n-s-1}$ is a rational normal curve and $R \cong \psi^*(\mathcal{O}_{\mathbb{P}^1}(p_a(X)))$. Now, the latter case is excluded by hypothesis, therefore $\varphi_{|R|}$ is birational onto its image.
Consider a subset Ψ defined as follows:

$$\Psi := \{B \in |R| \mid \varphi_{|R|}|B \text{ is } 1{-}1, \forall\, S_B \subset \varphi_{|R|}(B) \text{ with } \sharp(S_B) \leq n{-}s{-}1, \langle S_B \rangle = \mathbb{P}^{n-s-2}\}. \tag{6}$$

By a monodromy argument such a $\Psi \subset |R|$ exists, is non-empty and open.
For any $B \in \Psi$ and for any set of points $S_B \subset \varphi_{|R|}(B)$, we have that $\varphi_{|R|}(S_B) \subset \mathbb{P}^{n-s-1}$ itself is not linearly independent, but any proper subset of it is linearly independent, then, for any $B \in \Psi$, we have that $\langle Z \rangle \cap \langle S_B \rangle$ is made by only one point and we denote it with P_{B,S_B}:

$$P_{B,S_B} := \langle Z \rangle \cap \langle S_B \rangle. \tag{7}$$

Obviously $r_X(P_{B,S_B}) \leq n - s + 1$ because $P_{B,S_B} \in \langle S_B \rangle$ and $S_B \subset \varphi_{|R|}(B)$ for some $B \in \Psi$. Now to conclude the proof it is sufficient to show that, varying $B \in \Psi$ and $S_B \subset \varphi_{|R|}(B)$, the set of all points P_{B,S_B} obtained as in (7) covers a non-empty open subset of $\langle Z \rangle$.

Let $G(n-s, n)$ denote the Grassmannian of all $(n-s)$-dimensional projective linear subspaces of $\mathbb{P}^n$. Now the set Ψ defined in (6) can be viewed as an irreducible component of maximal dimension of the constructible subset of $G(n-s, n)$ which parametrizes all linear spaces $\langle S_B \rangle$ with (B, S_B) as in (6) (whit an abuse of notation we will write $(B, S_B) \in \Psi$ when we think $\Psi \subset G(n-s, n)$).
With $\overline{\Psi}$ we denote the closure of Ψ in $G(n-s, n)$.
Since $h^0(X, R) = n - s \geq s + 3$ and $g(X) > 0$, there is an element in $|R|$ that contains $s+1$ general points of X (remind that $(n-s-2)$ general points on an integral curve $Y \subset \mathbb{P}^{n-s-1}$ are contained in a linear space that has $(n-s-1)$ on Y, if and only if Y is not the rational normal curve; in our case $g(Y) > 0$ and $|R|$ induces a birational map). Hence the closure Γ in $\mathbb{P}^n$ of the union of all $\langle B \rangle$ with $B \in \overline{\Psi}$ contains $\sigma_{s+1}(X)$. Therefore such a Γ clearly contains $\langle Z \rangle$. Then for every $P \in \langle Z \rangle$ there is $B \in \overline{\Psi}$ such that $P \in \langle B \rangle$. To prove the theorem it is sufficient to prove that for a general P in $\langle Z \rangle$ we may take $B \in \Psi$ with $P \in \langle B \rangle$. Since Ψ contains a non-empty open subset of $\overline{\Psi}$, it is sufficient to find at least one $P \in \langle Z \rangle$ that actually belongs to an element $B \in \Psi$. It is just sufficient to take, for such required P, the element $P_{B,S_B} \in \langle Z \rangle \cap \langle S_B \rangle$ defined in (7) where $(B, S_B) \in \Psi \subset G(n-s, n)$, in fact we saw that for every $\langle S_B \rangle \in \Psi$ with $(B, S_B) \in \Psi \subset G(n-s, n)$ as above $\langle Z \rangle \cap \langle S_B \rangle$ is a unique point P_{B,S_B} of $\langle Z \rangle$. □

Remark 1. Notice that the proof of Theorem 1 works also if X is smooth, of genus $g \geq 2$, embedded in $\mathbb{P}^n$ by a degree $n+g$ line bundle and if $Z \subset X$ is an effective non-reduced degree s divisor such that n is greater or equal both than $2s+2$ and than $2g+s+1$.

Corollary 2. *Let $X \subset \mathbb{P}^n$ be a smooth linearly normal curve of genus $g \geq 2$ and degree $n+g$. Let $s \in \mathbb{N}$ be such that $n \geq 2s+2, 2g+s+1$ and $P \in \sigma_s(X)$. Then $r_X(P) \leq n+1-s$.*

Proof. If $P \in \sigma_s^0(X)$, then, by definition (3), $r_X(P) \leq s$, and, by hypothesis on n, $s < n+1-s$. If $P \in \sigma_s(X) \setminus \sigma_s^0(X)$, then (by [6], Proposition 2.8) there exists an effective non-reduced divisor $Z \subset X$ of degree s such that $P \in \langle Z \rangle$. Such divisor Z satisfies the hypothesis of Remark 1, and then those of Theorem 1, therefore $r_X(P) \leq n+1-s$. □

2. The X-rank with respect to a linearly normal curve of genus two.

In this section we restrict our attention to the case of smooth genus 2 curves embedded linearly normal in $\mathbb{P}^n$ and of degree $n+2$ for $n \geq 3$.

In this case the Corollary 1 together with the Theorem 1 (when applicable) will assure that the X-rank $r_X(P)$ of a point $P \in \tau(X) \setminus X$ can only be

$$n-2 \leq r_X(P) \leq n-1.$$

If $n = 3, 4$ this tells that, if there is some point $P \in \tau(X)$ that does not belong to $\sigma_2^0(X)$, then the elements of the set $\tau(X) \setminus \sigma_2^0(X)$ can only have X-rank equal to 3. What we will do in subsections 2.1 and 2.2 will be to study that set in the cases $n = 3, 4$ respectively. First we will give examples in which such a set is not empty, then we will relate the choice of the tangent line to X with its number of points P with $r_X(P) = 3$.

2.1. The case of a smooth linearly normal curve of degree 5 in $\mathbb{P}^3$.

For all this subsection $X \subset \mathbb{P}^3$ will be linearly normal curve of degree 5 and genus 2.

In this case only the X-rank on the tangential variety of X is not know.

Remark 2. Let $X \subset \mathbb{P}^3$ be a smooth non degenerate curve. Since $r_X(P) \leq 3$ for all $P \in \mathbb{P}^3$ (see Proposition 5.1 in [17]) and since $\sigma_2(X) = \mathbb{P}^3$ (see [1]), then we have that

$$r_X(P) = 3 \Leftrightarrow P \in \tau(X) \setminus (\sigma_2^0(X)).$$

Clearly this does not prove the actual existence of a point $P \in \mathbb{P}^3$ such that $r_X(P) = 3$. But, R. Piene proved the existence of a smooth genus 2 linearly normal curve $X \subset \mathbb{P}^3$ and $P \in \mathbb{P}^3$ whose X-rank is greater or equal than 3 ([21], Example 4, pag. 110). This shows that there exists at least one case in which $\tau(X) \setminus (\sigma_2^0(X)) \neq \emptyset$.

The Proposition 1 below shows that there are infinitely many manners to embed X in $\mathbb{P}^3$ in such a way that $\tau(X) \setminus (\sigma_2^0(X)) \neq \emptyset$ and, moreover, that for any such embedding there exists at least one tangent line to X on which there are exactly 6 points of X-rank equal to 3.

Before proving Proposition 1 we need to recall standard facts on Weierstrass points that we will need in the sequel.

Definition 2. A point P on an algebraic curve C of genus g is a *Weierstrass point* if there exists a non-constant rational function on C which has at P a pole of order less or equal than g and which has no singularities at other points of C.

Remark 3. If the algebraic curve C has genus $g \geq 2$ then there always exist at least $2g+2$ Weierstrass points, and only hyper-elliptic curves of genus g have exactly $2g+2$ Weierstrass points.

The presence of a Weierstrass point on an algebraic curve C of genus $g \geq 2$ ensures the existence of a morphism of degree less or equal than g from the curve C onto the projective line $\mathbb{P}^1$.

We can now prove the following proposition.

Proposition 1. *Let C be a smooth curve of genus 2. Fix $O \in C$ such that there is no $U \in C$ such that $\mathcal{O}_C(3O) \cong \omega_C(U)$ (this condition is satisfied by a general $O \in C$). Set*

$$L := \omega_C(3O).$$

Let $\varphi_{|L|} : C \to \mathbb{P}^3$ be the degree 5 linearly normal embedding of C induced by the complete linear system $|L|$. Set $X := \varphi_{|L|}(C)$ and $Q := \varphi_{|L|}(O)$. Then there are exactly 6 points of T_QX with X-rank equal to 3.

Proof. Since $\deg(L) = 5 = 2p_a(C) + 1$, then L is very ample and $h^0(C, L) = 4$ (as implicitly claimed in the statement). By hypothesis $\mathcal{O}_C(3O) \nsim \omega_C(O)$, then we have also that $\mathcal{O}_C(2O) \nsim \omega_C$, that means that O is not a Weierstrass point of C (see Definition 2).
Now $\mathcal{O}_X(1) \cong \omega_X(3Q)$, then, by Riemann-Roch theorem, the point $Q = \varphi_{|L|}(O) \in X$ defined in the statement, is the unique base point of $\mathcal{O}_X(1)(-2Q)$. Therefore:

$$T_QX \cap X = 3Q$$

where the intersection is scheme-theoretic and $3Q \subset X$ is an effective Cartier divisor of X. Since the genus of X is $g = 2$ we have that, by Remark 3, the number of Weierstrass points of X, in characteristic different from 2, is exactly 6. Moreover the the canonical morphism $u : X \to \mathbb{P}^1$ recalled in Remark 3 is induced by the linear projection from $T_QX \subset \mathbb{P}^3$, and the ramification points of such a morphism u are, by definition, the Weierstrass points of X. Let $B \in X$ be one of these Weierstrass points (by assumption $B \neq Q$). Since $\mathcal{O}_X(3Q+2B) \cong \mathcal{O}_X(1)$, $B \neq Q$, and X is linearly normal, then $\langle T_QX \cup T_BX \rangle$ is a plane, and let

$$P = T_QX \cap T_BP.$$

Now $\deg(X) = 5$ and $P \notin \{Q, B\}$, then we have that $P \notin X$, i.e.

$$r_X(P) \geq 2.$$

We claim that $r_X(P) \geq 3$ and hence $r_X(P) = 3$ ([17], Proposition 5.1). We also check that $2B$ and $2Q$ are the only degree 2 effective divisors Z on X such that $P \in \langle Z \rangle$.
Assume that there is a degree 2 divisor $Z \subset X$ such that $P \in \langle Z \rangle$ but $Z \neq 2Q, 2B$. Since $\langle Z \rangle \cap T_QX = \{P\}$, then $\langle T_QX \cup \langle Z \rangle \rangle$ is a plane. Since the effective Cartier divisor $3Q$ of X is the scheme-theoretic intersection of X and T_QX, we get that $Z + 3Q \in |\mathcal{O}_X(1)|$, i.e. $Z \in |\omega_X|$. Analogously $\{P\} = \langle Z \rangle \cap T_BX$, then again $\langle T_BX \cup \langle Z \rangle \rangle$ is a plane. Thus there is a point $A \in X$ such that $2B + Z + A \in |\mathcal{O}_X(1)|$, i.e. $Z + A \in |3Q|$. Let U be the only point of C such that $\varphi_{|L|}(U) = A$. Since $Z \in |\omega_X|$, we get $\omega_C(U) \cong \mathcal{O}_C(3O)$, contradicting our assumption on $O \in C$. Now, varying B among the 6 Weierstrass points, we get 6 points of T_QX with X-rank 3. All other points of $T_QX \backslash \{Q\}$ have X-rank 2, because they are in the linear span of a reduced divisor $Z \in |\omega_X|$. □

2.2. **The case of a smooth linearly normal curve of degree 6 in $\mathbb{P}^4$.** For all this subsection $X \subset \mathbb{P}^4$ will be linearly normal curve of degree 6 and genus 2.

In order to completely describe the X-rank of points in $\mathbb{P}^4$ for such a X, we need to recall that from [1] we have that $\mathbb{P}^4 = \sigma_3(X)$. Clearly $r_X(P) \leq 3$ for all $P \in \sigma_3^0(X)$, but this does not give any information neither on the points $P \in \tau(X) \setminus \sigma_2^0(X)$ nore on $\mathbb{P}^4 \setminus \sigma_3^0(X)$. In the next proposition we show that:

$$r_X(P) \leq 3$$

for all $P \in \mathbb{P}^4$.

Proposition 2. *Let $X \subset \mathbb{P}^4$ be a smooth and linearly normal curve of degree 6 and genus 2. Then every $P \in \mathbb{P}^4 \backslash \sigma_2^0(X)$ has X-rank 3.*

Proof. Since, by hypothesis, $P \notin \sigma_2^0(X)$, then obviously $r_X(P) \geq 3$. It is sufficient to prove the reverse inequality.

We start proving the statement for $P \notin \sigma_2(X)$, i.e. P neither in $\sigma_2^0(X)$, nor in $\tau(X)$. Let $\ell_P : \mathbb{P}^4 \backslash P \to \mathbb{P}^3$ be the linear projection from P. Since $P \notin \sigma_2(X)$, the image $\ell_P(X) \subset \mathbb{P}^3$ is a smooth and degree 6 curve isomorphic to X. Since $\binom{6}{2} = 15$ and $\mathcal{O}_X(2)$ is not special, while $h^0(X, \mathcal{O}_X(2)) = 12 + 1 - 2 = 11$, we get that $h^0(\mathcal{I}_X(2)) \geq 4$. Now $2^3 > 6$ and Bezout's theorem imply that X is contained in a minimal degree surface $S \subset \mathbb{P}^4$ and that it is a complete intersection of S with a quadric hypersurface. The surface S is either a cone over a rational normal curve or a degree 3 smooth surface isomorphic to the Hirzebruch surface. In both cases the adjunction formula shows that a smooth curve, which is scheme-theoretically the intersection with S and a quadric hypersurface, is a genus 2 linearly normal curve. Since X is cut out by quadrics, there is no line $L \subset \mathbb{P}^4$ such that $\mathrm{length}(L \cap X) \geq 3$. For every smooth and non-degenerate space curve Y (except the rational normal curves and the degree 4 curves with arithmetic genus 1) there are infinitely many lines $M \subset \mathbb{P}^3$ such that $\mathrm{length}(M \cap Y) \geq 3$ (it is sufficient to consider a projection of Y into a plane from a general point of Y). In characteristic zero only finitely many tangent lines $T_O Y$, with $O \in Y_{reg}$, have order of contact ≥ 3 with Y at O. Hence if $Y \subset \mathbb{P}^3$ is the smooth curve $\ell_O(X)$, there are infinitely many lines $M \subset \mathbb{P}^3$ such that $\sharp((M \cap \ell_O(X))_{red}) \geq 3$. Any such line M is the image via ℓ_P, of a plane $\Pi \subset \mathbb{P}^4$ containing P and at least 3 points of X. Since X has no trisecant (o multisecant) lines, the plane Π must be spanned by the points of X contained in it.

To complete the picture, it only remains to show that if $P \in \tau(X)$ then $r_X(P) \leq 3$. Clearly if $P \in X \subset \tau(X)$ then $r_X(P) = 1$, and if there exists a bisecant line L to X such that $L \cap \tau(X) = P$, then $r_X(P) = 2$. We actually have to prove that the points $P \in \tau(X)$ such that $P \notin \sigma_2^0(X)$ have X-rank 3.

Let $\ell_O : \mathbb{P}^4 \setminus \{O\} \to \mathbb{P}^3$ be the linear projection from $O \in X \subset \mathbb{P}^4$, and let $C \subset \mathbb{P}^3$ be the closure of $\ell_O(X \backslash \{O\})$ in $\mathbb{P}^3$. Since $\deg(X) = 4 + 2p_a(X) - 2$, we have $h^1(X, \mathcal{O}_X(1)(-Z)) = 0$ for every effective divisor $Z \subset X$ such that $\deg(Z) \leq 3$. Hence the scheme $T_O X \cap X$ has length 2. Thus $C \subset \mathbb{P}^3$ is a degree 6 space curve birational to X, with arithmetic genus 3 and an ordinary cusp at $\ell_O(P)$ as its unique singular point. Fix a general $A \in C$. Since A is general, it is not contained in the tangent plane to C at $\ell_O(P)$. Moreover, by the same reason, there is no $B \in C_{reg} \backslash \{A\}$ such that $A \in T_B C$. Thus, if $\ell_A : \mathbb{P}^3 \setminus \{A\} \to \mathbb{P}^2$ is the linear projection from $A \in \mathbb{P}^3$, the closure of $\ell_A(C \backslash \{A\})$ in $\mathbb{P}^2$ is a degree 5 plane curve with an ordinary cusp and at least one non-unibranch point with multiplicity ≥ 2. Hence there is a line $M \subset \mathbb{P}^3$ such that $\sharp(M \cap C) \geq 3$ and $A \in M$. Hence C has a one-dimensional family Γ of lines L such that $\sharp(L \cap C) = 3$. Fix any $L \in \Gamma$ and let $\Pi \subset \mathbb{P}^4$ be the only plane such that $P \in \Pi$ and $\ell_P(\Pi \backslash \{P\}) = L$. Since $\ell_P|X : X \to C$ is injective, then $\sharp(\Pi \cap X) = 3$. Since any length 3 subscheme of X is linearly independent, $\Pi = \langle \Pi \cap X \rangle$. Since $P \in \Pi$, we get $r_X(P) \leq 3$. □

Corollary 3. *Let $X \subset \mathbb{P}^4$ be a smooth and linearly normal curve of degree 6 and genus 2. Then $\overline{\sigma_3^0(X)} = \sigma_3^0(X) = \mathbb{P}^4$*

Proof. By the definition of secant variety that we gave in (2) we have that $\sigma_3(X) = \overline{\sigma_3^0(X)}$ and for any $X \subset \mathbb{P}^4$ smooth and linearly normal curve $\sigma_3(X) = \mathbb{P}^4$. By Proposition 2, $r_X(P) \leq 3$ for all $P \in \mathbb{P}^4$, hence if $P \in \mathbb{P}^4$ is such that there exists a non reduced scheme $Z \subset X$ of length 3 for which $P \in \langle Z \rangle$, there always exists another reduced scheme $Z' \subset X$ of length at most 3 such that $P \in \langle Z' \rangle$. Hence in the Zariski closure of $\sigma_3^0(X)$ the X-rank doesn't increase. □

The Proposition 2 gives a complete stratification of the X-rank of the points in $\mathbb{P}^4$ with respect to a genus 2 curve $X \subset \mathbb{P}^4$ of degree 6 embedded linearly normal. We implicitly proved that if $P \in \mathbb{P}^4$ is such that $r_X(P) = 3$ then $P \in \tau(X) \cup \sigma_3^0(X)$. Clearly if $P \in \sigma_3^0(X) \subset \mathbb{P}^4$ then $r_X(P) = 3$. We can actually be more precise about the points belonging to $\tau(X) \setminus X$ for $X \subset \mathbb{P}^4$ as above. Are all of them of X-rank 3 or is the intersection between $\tau(X)$ and $\sigma_2^0(X)$ not empty? Moreover, which is the cardinality of $\tau(X) \cap \sigma_2^0(X)$? We describe it in the following proposition.

Proposition 3. *Let $X \subset \mathbb{P}^4$ be a smooth and linearly normal curve of degree 6 and genus 2. Fix $O \in X$. The linear projection from $T_O X$ does not induce a birational morphism onto a degree 4 plane curve if and only if $\mathcal{O}_X(1) \cong \omega_X^{\otimes 2}(2O)$. The space $T_O X$ contains only*

- 1 *point of X-rank equal to 2 if and only if $T_O X$ induces a birational morphism from X to a plane curve;*
- 5 *points of X-rank equal to 2 if and only if $T_O X$ doesn't induce a birational morphism from X to a plane curve and $O \in X$ is a Weiestrass point of X,*
- 6 *points of X-rank equal to 2 if and only if $T_O X$ doesn't induce a birational morphism from X to a plane curve and $O \in X$ is not a Weiestrass point of X.*

All the other points in $T_O(X)$ have X-rank equal to 3.

Proof. Since $\text{length}(T_O X \cap X) = 2$, $\deg(X) = 6$, and X is smooth, then, by Lemma 1, the morphism $\ell_O|X \backslash \{O\}$ extends to a morphism $v_O : X \to \mathbb{P}^2$ such that $\deg(v_0) \cdot \deg(v_O(X)) = 4$. Since $v_O(X)$ spans $\mathbb{P}^2$, $\deg(v_O(X)) \geq 2$. Hence either v_O is birational or $\deg(v_O) = 2$ and $v_O(X)$ is a smooth conic. The latter case occurs if and only if $\mathcal{O}_X(1) \cong \omega_X^{\otimes 2}(2O)$.

The last sentence of the statement is a direct consequence of Proposition 2. Let us prove the previous part. Since X has no trisecant lines, then $r_X(P) = 2$ if and only if there is a line L such that $P \in L$ and $\sharp(D \cap X) = 2$.

First assume that the linear projection from $T_O X$ induces a birational morphism from X onto a plane curve. Since $\text{length}(T_O X \cap X) = 2$, the linear projection from $T_O X$ and the genus formula for degree 4 plane curves show the existence of exactly one $P' \in T_O X$ contained in another tangent or secant line; both cases may occur for some pairs (X, O).

Now assume that the linear projection from $T_O X$ does not induce a birational morphism of X onto a plane curve. Since $\text{length}(T_O X \cap X) = 2$, it induces a degree 2 morphism $\phi : X \to E$, with $E \subset \mathbb{P}^2$ a smooth conic. Hence ϕ is the hyperelliptic pencil. Therefore $\mathcal{O}_X(1) \cong \omega_X^{\otimes 2}(2O)$. Then, for a fixed abstract curve of genus 2, there is a one-dimensional family of linearly normal embeddings having such tangent lines, while a general element of $\text{Pic}^6(X)$ has no such tangent line. For that tangent line $T_O X$, the morphism ϕ has 6 ramification points (by Riemman-Hurwitz formula) and O may be one of them (it is one of them if and only if O is one of the 6 Weierstrass points of X). Hence all except 5 or 6 points of $T_O X \backslash \{O\}$ have X-rank 2. □

Remark 4. Let X be an abstract smooth curve of genus 2. Every element of $\text{Pic}^6(X)$ is very ample. The algebraic set $\text{Pic}^6(X)$ is isomorphic to a 2-dimensional abelian variety $\text{Pic}^0(X)$. A one-dimensional closed subset of it (isomorphic to X) parametrizes the set Σ all line bundles of the form $\omega_X^{\otimes 2}(2O)$ for some $O \in X$. Fix $L \in \Sigma$. Since the 2-torsion of $\text{Pic}^0(X)$ is formed by 2^4 points, there are exactly 2^4 points $O \in X$ such that $L \cong \omega_X^{\otimes 2}(2O)$.

2.3. **The case of a smooth linearly normal curve of degree $n + 2$ in $\mathbb{P}^n$ for $n \geq 5$.** For all this subsection $X \subset \mathbb{P}^n$ will be linearly normal curve of degree $n + 2$ and genus 2 and $n \geq 5$.

We treated the cases of $n = 3, 4$ separately from the others because for small values of n's the behaviour of the X-rank for points in $\tau(X)$ is not consistent to the general case. In fact if $n = 3, 4$ then $r_X(P) = 3$ for all $P \in \tau(X) \setminus \sigma_2^0(X)$ as proved in propositions 1 and 2. The Theorem 2 that we stated in the Introduction (and that we will prove in this section) shows that, if $n \geq 8$, then the X-rank of $P \in \tau(X) \setminus \sigma_2^0(X)$ is $r_X(P) = n - 2$. If $n = 5, 6, 7$ the behaviour of the X-rank of points in $\tau(X) \setminus \sigma_2^0(X)$ is not inconsistent whit that one of the general case in fact in Proposition 4 we show that if $n \geq 5$ the X-rank of a point $P \in \tau(X)$ is $r_X(P) \leq n - 1$ and there are points $P \in \tau(X)$ such that $r_X(P) = n - 2$.

Proposition 4. *Let $X \subset \mathbb{P}^n$ be a smooth and linearly normal curve of genus 2 and let $n \geq 5$. Fix $Q \in X$. If $n = 5$, then assume $\mathcal{O}_X(1) \neq \omega_X^{\otimes 2}(3Q)$. Then:*

(1) *there is $P \in T_QX \backslash X$ such that $r_X(P) = n - 2$.*
(2) *$r_X(P) \leq n - 1$ for all $P \in T_QX$.*

Proof. From Corollary 1 we immediately get that for all $P \in T_QX \backslash X$ the X-rank of P is at least $n - 2$. Hence to prove part (1) it is sufficient to find a point $P \in T_QX \backslash X$ such that $r_X(P) \leq n - 2$. Set

$$R := \mathcal{O}_X(1)(-2Q)$$

and

$$M := R \otimes \omega_X^*.$$

Since $\deg(R) = n \geq 2p_a(X) + 1$, then R is very ample and $h^0(X, R) = n - 1$.
Let $\varphi_R : X \hookrightarrow \mathbb{P}^{n-2}$ be the embedding induced by $|R|$. Notice that $\varphi_R(X)$ is obtained projecting X from the line T_QX. Since $\deg(M) = n - 2 \geq p_a(X) + 1$, we have $h^0(X, M) \geq 2$.

Since $\deg(M) = n - 2$, then M is spanned if $n - 2 \geq 2p_a(X)$, then if $n \geq 6$. We distinguish two cases: M spanned and M not spanned.

(i) First assume that M is spanned, and hence that $n \geq 6$. Obviously $|M|$ contains at least a reduced element $A \in |M|$. Now, since $h^0(X, R(-A)) = h^0(X, \omega_X) = 2$, then $\dim(\langle \varphi_R(A) \rangle) = n - 4$. By definition of $\varphi_{|R|}$ the curve $\varphi_R(X)$ is the linear projection of X from T_QX, then $\dim(\langle T_QX \cup A \rangle) \leq n - 2$. Since $\deg(\mathcal{O}_X(1)(-A)) = 4 > \deg(\omega_X)$, the set $A \in |M|$ is linearly independent in $\mathbb{P}^n$, i.e. $\dim(\langle A \rangle) = n - 3$. Since $\dim(\langle T_QX \cup \langle A \rangle) \leq n - 2$, we get $T_QX \cap \langle A \rangle \neq \emptyset$. If $T_QX \subset \langle A \rangle$, then $r_X(P) \leq n - 2$ for all $P \in T_QX$. Hence we may assume that $T_QX \cap \langle A \rangle$ is a unique point:

$$T_QX \cap \langle A \rangle = P'.$$

If $P' \neq Q$, then $r_X(P') = n - 2$ as required in part (1) of the statement.
If $P' = Q$ (i.e. if $Q \in \langle A \rangle$), then Q has actually to belong to A itself, in fact $h^1(X, \mathcal{O}_X(1)(-Z)) = 0$ for every zero-dimensional scheme of X with degree $\leq n - 1$. However, since R is assumed to have no base points, we may always take $A \in |M|$ such that $Q \notin A$.

(ii) Now assume that M is not spanned, hence $n = 5$ and $R = \omega_X(B)$ for some $B \in X$. Since ω_X is spanned, there is a reduced $A' \in |\omega_X|$ such that $Q \notin A'$ and $B \notin A'$. Therefore $A := B + A'$ is a reduced element of $|M|$. We may use step (i) to prove the part (1) of the statement even in this case, unless $B = Q$ (but this is exactly the case excluded).

We can now prove part (2). Take any $P \in T_QX \backslash X$. By part (1) there are $P_1 \in T_QX \backslash X$ and $S_1 \subset X$ such that $\sharp(S_1) = n - 2$ and $P_1 \in \langle S_1 \rangle$. Since $P \in T_QX = \langle \{P_1, Q\} \rangle$, we have $P \in \langle \{Q\} \cup S_1 \rangle$. Hence $r_X(P) \leq n - 1$. □

We can state the analogous of Remark 4.

Remark 5. Let X be an abstract smooth curve of genus 2. Every element of $\mathrm{Pic}^7(X)$ is very ample. The algebraic set $\mathrm{Pic}^7(X)$ is isomorphic to a 2-dimensional abelian variety $\mathrm{Pic}^0(X)$. A one-dimensional closed subset of it (isomorphic to X) parametrizes the set Σ all line bundles of the form $\omega_X^{\otimes 2}(3O)$ for some $O \in X$. Fix $L \in \Sigma$. Since the 3-torsion of $\mathrm{Pic}^0(X)$ is formed by 3^4 points, there are exactly 3^4 points $O \in X$ such that $L \cong \omega_X^{\otimes 2}(3O)$.

We prove here the Theorem 2 stated in the Introduction that gives the precise value $r_X(P) = n-2$ for points $P \in \tau(X) \setminus X$ if $n \geq 8$.

Proof of Theorem 2.

By Corollary 1, the X-rank of P is $r_X(P) \geq n-2$. We prove the reverse inequality. Consider the linear projection $\ell_P : \mathbb{P}^n \setminus P \to \mathbb{P}^{n-1}$ of $\mathbb{P}^n$ to $\mathbb{P}^{n-1}$ from $P \in \mathbb{P}^n$ and set:

$$Y := \ell_P(X) \subset \mathbb{P}^{n-1}$$

and

$$O := \ell_P(Q) \in \mathbb{P}^{n-1}$$

to be the linear projections via ℓ_P of $X \subset \mathbb{P}^n$ and $Q \in \mathbb{P}^n$ respectively. For every 0-dimensional subscheme $Z \subset X$ of length at most 4 we have that $\dim(\langle Z \rangle) = \mathrm{length}(Z) - 1$, hence $\ell_P|X\setminus\{Q\}$ is an embedding, the curve Y is singular only in O where there is a cusp and the embedding $Y \subset \mathbb{P}^{n-1}$ is linearly normal. Set:

$$R := \mathcal{O}_Y(1) \otimes \omega_Y^*.$$

Since $p_a(Y) = 3$ and $\deg(R) = n-2 \geq 6$, then R is spanned. Hence a general divisor $B \in |R|$ is reduced and does not contain O. Therefore there is a unique set of points $S \subset X$ such that $\sharp(S) = \sharp(B)$ and $\ell_P(S) = B$. Since $h^1(Y, \mathcal{O}_Y(1)(-B)) = h^1(Y, \omega_Y) = 1$, we have $\dim(\langle B \rangle) = \sharp(B) - 2$. Hence $\dim(\langle \{P\} \cup S \rangle) = n-3$. In order to get $P \in \langle S \rangle$, and hence $r_X(P) \leq n-2$, it is sufficient to prove that S is linearly independent. This is true, because X is linearly normal and $\deg(\mathcal{O}_X(1)(-S)) = 4 > 2 = \deg(\omega_X)$. □

For the next proposition we need to recall the definition of X-rank of subspaces.

Definition 3. Let $V \subset \mathbb{P}^n$ be a non-empty linear subspace. The X-rank $r_X(V)$ of V is the minimal cardinality of a finite set $S \subset X$ such that $V \subseteqq < S >$.

Proposition 5. *Fix an integer $n \geq 4$. Let $X \subset \mathbb{P}^n$ be a non-degenerate, smooth and linearly normal curve of genus 2 and degree $n+2$. Fix $Q \in X$ and let Δ_Q be the set of all $S \subset X$ such that $T_Q X \subset \langle S \rangle$ and $\sharp(S) = r_X(T_Q X)$. Then:*

(i) *$r_X(T_Q X) = n-1$;*

(ii) *every $S \in \Delta_Q$ contains Q and $\{S\setminus\{Q\}\}_{S \in \Delta_Q}$ is the non-empty open subset of the projective space $|\mathcal{O}_X(1)(-2Q) \otimes \omega_X^*|$ parameterizing the reduced divisors not containing Q.*

Proof. By item (1) in Proposition 4, there is $P \in T_Q X\setminus X$ such that $r_X(P) = n-2$. Take $S_1 \subset X$ computing $r_X(P)$. Since $T_Q X \subset \langle \{Q\} \cup S_1 \rangle$, we get $r_X(T_Q X) \leq n-1$. Hence to prove (i) it is sufficient to prove the reverse inequality.

Fix a finite subset of points $S \subset X$ computing $r_X(T_Q X)$, i.e. $T_Q X \subset \langle S \rangle$ and it does not exist any $\mathbb{P}^t$ with $t < \dim(\langle S \rangle)$ containing $T_Q X$. Here we prove $\sharp(S) \geq n-1$ and that if $\sharp(S) = n-1$, then $Q \in S$.

Assume either $\sharp(S) \leq n-2$ or $\sharp(S) = n-1$ and $Q \notin S$. Hence in both cases it is possible to find a projective linear subspace $M \subset \mathbb{P}^n$ such that $\dim(M) \leq n-2$ and $\mathrm{length}(X \cap M) \geq \dim(M) + 3$. So if $\sharp(S) \leq n-2$ or $\sharp(S) = n-1$ and $Q \notin S$ we are able to find a scheme $X \cap M \subset X$ of length greater than n that is linearly independent; but this is not possible, in fact, since $\deg(\mathcal{O}_X(1)) = n + \deg(\omega_X)$ and $X \subset \mathbb{P}^n$ is linearly normal, a zero-dimensional

subscheme $Z \subset X$ is linearly independent if either length$(Z) \le n-1$ or length$(Z) = n$ and $Z \notin |\mathcal{O}_X(1) \otimes \omega_X^*|$ (if $Z \in |\mathcal{O}_X(1) \otimes \omega_X^*|$, then $\dim(\langle Z \rangle) = \text{length}(Z) - 2$, because $h^1(X, \omega_X) = 1$).
Hence we get that $r_X(T_Q X) = n-1$ that proves the first part of the statement. Moreover this also shows that every $S \backslash \{Q\}$ is a reduced element of $|\mathcal{O}_X(1)(-2Q) \otimes \omega_X^*|$. Conversely, fix a reduced $B \in |\mathcal{O}_X(1)(-2Q) \otimes \omega_X^*|$ not containing Q and set $E := B \cup \{Q\}$. Notice that $\omega_X(Q)$ has Q as its base-points and that $\mathcal{O}_X(1)(-E) \cong \omega_X(Q)$. Hence $\langle E \rangle \cap X$ contains Q with multiplicity at least 2. Thus $T_Q X \subset \langle E \rangle$, that concludes the proof of the second part of the statement. □

Remark 6. Observe that the space Δ_Q of Proposition 5 has dimension $n-3$ if $n \ge 5$.

3. Questions

We end the paper with a number of progressive questions that should give a line for further investigations on the X-rank of points in $\mathbb{P}^n$ with respect to linearly normal curves of genus g and degree $n+g$.

A first question is on the possible sharpness of the bound given in Theorem 2 for the dimension n of the ambient space. Clearly Theorem 2 cannot hold for any $n \ge 3$ because we know that it is false for $n = 3, 4$ (by propositions 1 and 2), but it can maybe be extended to $n \ge 7$.

Question 1. Let $X \subset \mathbb{P}^n$ be a genus 2 linearly normal curve of degree $n+2$. Is it possible to prove that if $n \ge 7$ then the X-rank of any point $P \in \mathbb{P}^n$ is at most $n-2$?

Next question comes up from the fact that, in all the examples that we have studied in this paper, the X-rank with respect to a smooth genus 2 linearly normal curve $X \subset \mathbb{P}^n$, the highest value of the X-rank is realized on points belonging to the tangential variety to X.

Question 2. Let $X \subset \mathbb{P}^n$ be a genus 2 linearly normal curve of degree $n+2$. Does it exist a positive integer $n_0 \in \mathbb{N}$ such that for any $n \ge n_0$ every point $P \in \mathbb{P}^n \setminus \tau(X)$ have X-rank less or equal than $n-3$?

Actually Question 2 can be generalized to any s-th secant variety $\sigma_s(X) \subset \mathbb{P}^n$ for the same X linearly normal genus 2 curve.

Question 3. Let $X \subset \mathbb{P}^n$ be a genus 2 linearly normal curve of degree $n+2$. Is the maximal X-rank s of a point $P \in \mathbb{P}^n$ realized on $\sigma_{n-s}(X)$ when $n \gg s$?

All the above questions can be formulated in an analogous way for any projective, smooth and genus g linearly normal curve.

Question 4. Fix an integer $g \ge 0$. Are there integers $n_g, m_g \ge 2g+3$ such that for every integer $n \ge n_g$ (resp. $n \ge m_g$), every smooth genus g curve Y and every linearly normal embedding $j : Y \hookrightarrow \mathbb{P}^n$, we have $r_{j(Y)}(P) \le n-g$ for all $P \in \mathbb{P}^n$ (resp. $r_{j(Y)}(P) \le n-g-1$ for all $P \in \mathbb{P}^n \setminus Tj(Y)$)?

In the set-up of Question 4 we have $r_X(P) \ge n-g$ for every $P \in TX \backslash X$ by Corollary 1.

Question 5. Take the set-up of Question 4, but assume $g \ge 3$. Is it possible to find integers n'_g and m'_g as in Question 4 (but drastically lower) such that the same statements holds for $n \ge n'_g$ and $n \ge m'_g$ if we make the further assumption that Y has general moduli?

Hint: in the set-up of Question 5 in the first non-trivial case $g = 3$ perhaps it is sufficient to distinguish between hyperelliptic curves and non-hyperelliptic curves.

References

[1] B. Ådlansdvik. Joins and higher secant varieties. Math. Scand. 61, 213–222, (1987).

[2] S. Allman, J. Rhodes. Phylogenetic ideals and varieties for the general Markov model. Advances in Applied Mathematics, 40, no. 2, 127–148, (2008).

[3] E. Ballico, A. Bernardi. On the X-rank with respect to linear projections of projective varieties. Preprint: http://arxiv.org/abs/0912.4834, (2009).

[4] J. M. F. ten Berge, J. Castaing, P. Comon, L. De Lathauwer. Generic and typical ranks of multi-way arrays. Linear Algebra Appl. 430, no. 11–12, 2997–3007, (2009).

[5] A. Bernardi. Ideals of varieties parameterizing certain symmetric tensors. Journ. of P. and A. Algebra, 212, no. 6, 1542–1559, (2008).

[6] A. Bernardi, A. Gimigliano and M. Idà. Computing symmetric rank for symmetric tensors. Preprint: http://arxiv.org/abs/0908.1651, (2009).

[7] J. Buczynski, J. M. Landsberg. Ranks of tensors and a generalization of secant varieties. Preprint: http://arxiv.org/abs/0909.4262, (2009).

[8] M. V. Catalisano, A. V. Geramita, A. Gimigliano. On the rank of tensors, via secant varieties and fat points, Zero-Dimensional Schemes and Applications (Naples, 2000). Queen's Papers in Pure and Appl. Math., vol. 123, Queen's Univ., Kingston, ON, 2002, 133–147.

[9] M. V. Catalisano, A. V. Geramita, A. Gimigliano. Ranks of tensors, secant varieties of Segre varieties and fat points. Linear Algebra Appl. 355, 263–285 (2002).

[10] M.V. Catalisano, A.V. Geramita, and A. Gimigliano. On the ideals of secant varieties to certain rational varieties. Journal of Algebra 319, 1913–1931, (2008).

[11] G. Comas, M. Seiguer. On the rank of a binary form. Preprint: http://arxiv.org/abs/math/0112311, (2001).

[12] P. Comon, G. Golub, L.-H. Lim and B. Mourrain. Symmetric tensors and symmetric tensor rank. SIAM Journal on Matrix Analysis Appl., 30, 1254–1279, (2008).

[13] P. Comon, G. Ottaviani. On the typical rank of real binary forms. Preprint http://arxiv.org/abs/math/0909.4865, (2009).

[14] S. Friedland. On the generic rank of 3-tensors. Preprint: http://arxiv.org/abs/math/0805.3777, (2008).

[15] J.M. Landsberg, L. Manivel. On the ideals of secant varieties of Segre varieties. Found. Comput. Math. 4, no. 4, 397–422, (2004).

[16] J.M. Landsberg and L. Manivel. Generalizations of Strassen's equations for Secant varieties of Segre varieties. Communications in Algebra 36, 1–18, (2008).

[17] J. M. Landsberg, Z. Teiler. On the ranks and border ranks of symmetric tensors. Preprint: http://arxiv.org/abs/0901.0487v3, (2009).

[18] J.M. Landsberg, J. Weyman. On the ideals and singularities of secant varieties of Segre varieties. Bull. London Math. Soc. 39, no. 4, 685–697, (2007).

[19] L. H. Lim, V. de Silva. Tensor rank and the ill-posedness of the best low-rank approximation problem. SIAM J. Matrix Anal. Appl. 30, no. 3, 1084–1127, (2008).

[20] L. Manivel. On Spinor Varieties and Their Secants. SIGMA 5, 78-100, Contribution to the Special Issue lie Cartan and Differential Geometry, (2009).

[21] R. Piene. Cuspidal projections of space curves. Math. Ann. 256, no. 1, 95–119, (1981).

[22] V. Strassen. Rank and optimal computation of generic tensors. Linear Algebra Appl. 52/53 (1983), 645–685.

[23] J. Sylvester. Sur une extension d'un théorème de Clebsh relatif aux courbes du quatrième degré. Comptes Rendus, Math. Acad. Sci. Paris, 102, 1532–1534, (1886).

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