Stratification of the fourth secant variety of Veronese variety via the symmetric rank

Edoardo Ballico

Dept. of Mathematics, University of Trento, 38123 Povo (TN), Italy

Alessandra Bernardi*

CIRM -FBK, 38123 Povo (TN), Italy

Abstract

If $X \subset \mathbb{P}^n$ is a projective non degenerate variety, the X-rank of a point $P \in \mathbb{P}^n$ is defined to be the minimum integer r such that P belongs to the span of r points of X. We describe the complete stratification of the fourth secant variety of any Veronese variety X via the X-rank. This result has an equivalent translation in terms both of symmetric tensors and homogeneous polynomials. It allows to classify all the possible integers r that can occur in the minimal decomposition of either a symmetric tensor or a homogeneous polynomials of X-border rank 4 (see Not. 1) as a linear combination of either completely decomposable tensors or powers of linear forms respectively.

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Introduction

Fix integers $m \geq 2$ and $d \geq 2$ and set $n_{m,d} := m + dm - 1$. All along this paper the number field K over which all the projective spaces and all the vector spaces will be defined is algebraically closed and of caracteristic 0. Let $\nu_{m,d} : \mathbb{P}^m \hookrightarrow \mathbb{P}^{n_{m,d}}$ be the order d Veronese embedding of $\mathbb{P}^m$ defined by the sections of the sheef $\mathcal{O}(d)$. Set:

$$X_{m,d} := \nu_{m,d}(\mathbb{P}^m). \tag{1}$$

We often set $X := X_{m,d}$ and $n := n_{m,d}$. The Veronese variety can be regarded both as the variety that parameterizes projective classes of homogeneous polynomials of degree d in $m + 1$ variables that can be written as d-th powers of linear forms, and as

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*Corresponding author.
Email addresses: ballico@science.unitn.it (Edoardo Ballico), bernardi@fbk.eu (Alessandra Bernardi)

the variety that parameterizes projective classes of symmetric tensors $T \in V^{\otimes d}$ where V is a vector space of dimension $m+1$ and $T = v^{\otimes d}$ for certain $v \in V$ (symmetric tensors of the form $v^{\otimes d}$ are often called "completely decomposable tensors"). Hence if we indicate with $K[x_0, \dots, x_m]_d$ the vector space of homogeneous polynomials of degree d in $m+1$ variables, and with $S^d V$ the subspace of symmetric tensors in $V^{\otimes d}$, then the Veronese variety $X_{m,d} \subset \mathbb{P}^{n_{m,d}}$ can be described both as $\{[F] \in \mathbb{P}(K[x_0, \dots, x_m]_d) \mid \exists\, L \in K[x_0, \dots, x_m]_1 \text{ s.t. } F = L^d\}$ and as $\{[T] \in \mathbb{P}(S^d V) \mid \exists\, v \in V \text{ s.t. } T = v^{\otimes d}\}$.

A very classical problem coming from a number theory problem known as the Big Waring Problem (see [22]) is the knowledge of the minimum integer s for which a generic form $F \in K[x_0, \dots, x_m]_d$ can be written as the sum of s d-th powers of linear forms $L_1, \dots, L_s \in K[x_0, \dots, x_m]_1$:

$$F = L_1^d + \cdots + L_s^d. \tag{2}$$

The same s gives the minimum integer for which the generic symmetric tensor $T \in S^d V$ can be written as a sum of s completely decomposable tensors $v_1^{\otimes d}, \dots, v_s^{\otimes d} \in S^d V$:

$$T = v_1^{\otimes d} + \cdots + v_s^{\otimes d}. \tag{3}$$

This problem was solved by J. Alexander and A. Hirshowithz in [2] (see also [5] for a modern proof).

A natural question arising form the applications (see for example [1], [13], [8], [12], [19], [10], [15]) is:

Question 1. *Given a symmetric tensor $T \in S^d V$ (or a homogeneous polynomial $F \in K[x_0, \dots, x_m]_d$), which is the minimum integer r for wich we can write it as a linear combination of r completely decomposable tensors, i.e. as in (3) with $r = s$ (or as a linear combination of r d-th powers of linear forms, i.e. as in (2) with $r = s$)?*

A first useful definition in order to formalize the problem in terms of linear algebra is the following:

Definition 1. Let $X \subset \mathbb{P}^n \simeq \mathbb{P}(S^d V)$ be a Veronese variety. The X-rank of a point $P \in \mathbb{P}^n$ is the minimum integer r such that $P \in \langle P_1, \dots, P_r \rangle$ with $P_1, \dots, P_r \in X$. Such an integer is often called "the symmetric rank of P". We denote it by $r_X(P)$.

With this definition it turns out to be obvious that answering to Question 1 is equivalent to find the X-rank of a given point $P \in \mathbb{P}^{n_{m,d}}$ since $\mathbb{P}^{n_{m,d}}$ can be both the projective space of homogeneous polynomials of degree d in $m+1$ variables and the projective space of symmetric tensors of order d over an $m+1$ dimensional vector space.

The answer to Question 1, in the particular case of $m = 2$, is known from [21] and [9]. In that case the Veronese variety coincides with a rational normal curve. In more recent papers [7] and [11] one can find an algebraic theoretical algorithm for the general case with $m \geq 2$.

Both the Big Waring Problem and Question 1 have a very interesting reformulation in Algebraic Geometry by using Linear Algebra tools. The authors of [6] give some effective algorithms for the computation of the X-rank of certain kind of symmetric tensors by using this algebraic geometric interpretation (about the results appeared in [6], we will be more precisely in the sequel). The advantage of those last algorithms is that they are effective and that they arise from an algebraic geometric perspective that gives the idea on how one can proceed in the study of the X-rank either of a form or of a symmetric tensor. Let us go into the details of that geometric description.

Let $\sigma_s(X) \subset \mathbb{P}^n$ be the so called "higher s-th secant variety of X" (for brevity we will quote it only as "s-th secant variety of X"):

$$\sigma_s(X) := \overline{\bigcup_{P_1,\dots,P_s \in X} \langle P_1, \dots, P_s \rangle}. \tag{4}$$

From this definition it turns out that generic element of $\sigma_s(X)$ has X-rank equal to s, but obviously not all the elements of $\sigma_s(X)$ have X-rank equal to s.

Notation 1. For any $P \in \mathbb{P}^n$ let $b_X(P)$ denote the "X-border rank of P", i.e. the first integer $s > 0$ such that $P \in \sigma_s(X)$. Sometimes $b_X(P)$ is called either "the secant X-rank of P" or the "symmetric border rank of P".

Remark 1. Obviously $b_X(P) \leq r_X(P)$ for all $P \in \mathbb{P}^n$.

First of all the definition (4) of the secant varieties of the Veronese variety implies the following chain of containments:

$$X = \sigma_1(X) \subseteq \sigma_2(X) \subseteq \cdots \subseteq \sigma_{k-1}(X) \subseteq \sigma_k(X) = \mathbb{P}^n \tag{5}$$

for certain natural number k. Therefore $\sigma_s(X)$ contains all the elements of X-rank less or equal than s.
Moreover the set

$$\sigma_s^0(X) := \bigcup_{P_1,\dots,P_s \in X} \langle P_1, \dots, P_s \rangle \tag{6}$$

is contained in $\sigma_s(X)$ and it is made by the elements $P \in \mathbb{P}^n$ whose X-rank is less or equal than s, hence the elements of $\sigma_s(X) \setminus (\sigma_{s-1}(X) \cup \sigma_s^0(X))$ have X-rank bigger than s.

What is done in [6] is to start giving a stratification of $\sigma_s(X) \setminus \sigma_{s-1}(X)$ via the X-rank: in that paper the cases of $\sigma_2(X_{m,d})$ and $\sigma_3(X_{m,d})$ for any $m, d \geq 2$ are completey classified (among others). The authors gives algorithms that produce the X-rank of an element of $\sigma_2(X_{m,d})$ and $\sigma_3(X_{m,d})$.
If we indicate

$$\sigma_{s,r}(X) := \{P \in \sigma_s(X) \,|\, r_X(P) = r\} \subset \sigma_s(X) \subset \mathbb{P}^n, \tag{7}$$

then we can write the stratifications quoted above as follows:

- $\sigma_2(X_{m,d}) \setminus X_{m,d} = \sigma_{2,2}(X_{m,d}) \cup \sigma_{2,d}(X_{m,d})$, for $m \geq 1$ and $d \geq 2$ (cfr. [21], [9], [6], [11], [7]);
- $\sigma_3(X_{1,d}) \setminus \sigma_2(X_{1,d}) = \sigma_{3,3}(X_{1,d}) \cup \sigma_{3,d-1}(X_{1,d})$, for $d \geq 4$ (cfr. [21], [9], [6], [11], [7]);
- $\sigma_3(X_{m,3}) \setminus \sigma_2(X_{m,3}) = \sigma_{3,3}(X_{m,3}) \cup \sigma_{3,4}(X_{m,3}) \cup \sigma_{3,5}(X_{m,3})$, for $m \geq 2$ (see [6]);
- $\sigma_3(X_{m,d}) \setminus \sigma_2(X_{m,d}) = \sigma_{3,3}(X_{m,d}) \cup \sigma_{3,d-1}(X_{m,d}) \cup \sigma_{3,d+1}(X_{m,d}) \cup \sigma_{3,2d-1}(X_{m,d})$, for $m \geq 2$ and $d \geq 4$ (see [6]).

What we want to do in this paper is to give the analogous stratification for $\sigma_4(X_{m,d})$ for any $m, d \geq 2$. We will prove the following:

Theorem 1. *The stratification of $\sigma_4(X_{m,d}) \setminus \sigma_3(X_{m,d})$ via the $X_{m,d}$-rank is the following:*

1. $\sigma_4(X_{1,d}) \setminus \sigma_3(X_{1,d}) = \sigma_{4,4}(X_{1,d}) \cup \sigma_{4,d-2}(X_{1,d})$, *if $d \geq 6$;*
2. $\sigma_4(X_{2,3}) \setminus \sigma_3(X_{2,3}) = \mathbb{P}^9 \setminus \sigma_3(X_{2,3}) = \sigma_{4,4}(X_{2,3})$;
3. $\sigma_4(X_{2,4}) \setminus \sigma_3(X_{2,4}) = \sigma_{4,4}(X_{2,4}) \cup \sigma_{4,6}(X_{2,4}) \cup \sigma_{4,7}(X_{2,4})$;
4. $\sigma_4(X_{2,5}) \setminus \sigma_3(X_{2,5}) = \sigma_{4,4}(X_{2,5}) \cup \sigma_{4,5}(X_{2,5}) \cup \sigma_{4,7}(X_{2,5}) \cup \sigma_{4,9}(X_{2,d})$;
5. $\sigma_4(X_{2,d}) \setminus \sigma_3(X_{2,d}) = \sigma_{4,4}(X_{2,d}) \cup \sigma_{4,d-2}(X_{2,d}) \cup \sigma_{4,d}(X_{2,d}) \cup \sigma_{4,d+2}(X_{2,d}) \cup \sigma_{4,2d-2}(X_{2,d}) \cup \sigma_{4,2d-1}(X_{2,d})$, *if $d \geq 6$;*
6. $\sigma_4(X_{m,4}) \setminus \sigma_3(X_{m,4}) = \sigma_{4,4}(X_{m,4}) \cup \sigma_{4,6}(X_{m,4}) \cup \sigma_{4,7}(X_{m,4}) \cup \sigma_{4,8}(X_{m,4}) \sigma_{4,10}(X_{m,4})$, *if $m \geq 3$;*
7. $\sigma_4(X_{m,5}) \setminus \sigma_3(X_{m,5}) = \sigma_{4,4}(X_{m,5}) \cup \sigma_{4,5}(X_{m,5}) \cup \sigma_{4,7}(X_{m,5}) \cup \sigma_{4,9}(X_{m,5}) \cup \sigma_{4,10}(X_{m,5}) \sigma_{4,13}(X_{m,5})$, *if $m \geq 3$;*
8. $\sigma_4(X_{m,d}) \setminus \sigma_3(X_{m,d}) = \sigma_{4,4}(X_{m,d}) \cup \sigma_{4,d-2}(X_{m,d}) \cup \sigma_{4,d}(X_{m,d}) \cup \sigma_{4,d+2}(X_{m,d}) \cup \sigma_{4,2d-2}(X_{m,d}) \cup \sigma_{4,2d-1}(X_{m,d}) \cup \sigma_{4,2d}(X_{m,d}) \sigma_{4,3d-2}(X_{m,d})$, *if $m \geq 3$ and $d \geq 6$.*

Moreover all listed $\sigma_{s,r}(X_{n,d})$ are non-empty.

The case of the rational normal curve of item (1) is done in [21], [9], [11], [7] and [6]. The cases of the Veronese surfaces in degrees 3 and 4 (i.e. item (2) and item (3)) are done in [6].
We complete the case of Veronese surface (item (5)) in the Subsection 4.1 of the present paper.
In the Subsection 4.2 of this paper we will give the stratification of $\sigma_4(X_{3,d})$ that will be the same stratification for any $m \geq 3$ (items (6), (7) and (8)).

Before going into the details of the proof we need some preliminary and auxiliary sections. In Section 1 we present the construction that will allow to associate two different zero-dimensional schemes of $\mathbb{P}^m$ to two zero-dimensional sub-schemes of $X_{m,d}$ realizing the $X_{m,d}$-border rank and the $X_{m,d}$-rank, respectively, of a point $P \in \mathbb{P}^{n_{m,d}}$. Section 2 and Section 3 will be crucial and useful for the proof of Theorem 1. In fact in Section 2 we give bounds for the Y-rank (see Definition 3) of a point with respect to some particular projective curves $Y \subset \mathbb{P}^t$ that will be used in the proof of the Theorem 1. Section 3 is made by preliminary lemmas on the linear dependence of the pre-image via the Veronese map $\nu_{m,d}$ of the zero-dimensional schemes realizing the X-rank and the X-border rank of a point $P \in \mathbb{P}^n$. Finally in Section 4 we collect all the previous results into the proof of Theorem 1.

Moreover we will describe case by case how to find the scheme that realizes the X-rank of a point P (modulo the scheme that realizes the X-border rank). This allows to explicitly describe the subset $\sigma_{s,r}(X) \subset \sigma_s(X)$ defined in (7).

We like to stress here that the defining ideals of $\sigma_2(X_{1,d})$ and $\sigma_3(X_{2,d})$ are known (see [16] and [20] respectively) and this allows the authors of [6] to give algorithms for the X-rank of points in $\sigma_s(X)$ with $s = 2, 3$. Given an element $P \in \mathbb{P}^n$ they can firstly check if its X-border rank is actually either 2 or 3, and then they can produce the algorithm for the computation of the X-rank of P. Unfortunately equations defining $\sigma_4(X_{3,d})$ at least set-theoretically are not known yet (at least on our knowledge), therefore we could write algorithms for the X-rank of an element $P \in \sigma_4(X)$ but only if we already know by other reason that $b_X(P) = 4$.

1. Construction

In this paper we want to study the X-rank of the points P belonging to the fourth secant variety of the Veronese variety X, i.e. $P \in \sigma_4(X)$. By the chain of containments (5) we have that $\sigma_3(X) \subseteq \sigma_4(X)$, then, since the stratification of $\sigma_3(X)$ via the X-rank is already known by [6], it is sufficient to understand the X-rank of points $P \in \sigma_4(X) \setminus \sigma_3(X)$.

Moreover the definition (6) of $\sigma_s^0(X)$ implies that if $P \in \sigma_4^0(X)$ then $r_X(P) \leq 4$, hence, for the purpose of this paper, it is sufficient to study the X-rank of points belonging to

$$\boxed{\sigma_4(X) \setminus (\sigma_3(X) \cup \sigma_4^0(X))}\,.$$

Start our construction by taking $P \in \sigma_4(X)$.
For such a point P, by the definition (4) of $\sigma_4(X)$, it exists at least one zero-dimensional scheme $Z \subset X$ of degree 4 such that $P \in \langle Z \rangle$.

Definition 2. We say that a zero-dimensional scheme $Z \subset X$ such that $P \in \langle Z \rangle$ and $P \notin \langle Z' \rangle$ for any $Z' \subset X$ with $\deg(Z') < \deg(Z)$ "computes the X-border rank of P".

Since we want to take $P \in \sigma_4(X) \setminus \sigma_3(X)$ we can assume the existence of such a degree 4 zero-dimensional scheme $Z \subset X_{m,d}$ such that $P \in \langle Z \rangle$ and moreover that $P \notin \langle Z' \rangle$ for all $Z' \subsetneq Z$ (cfr. [6], Proposition 2.8). More precisely we can assume that $P \notin \langle Z' \rangle$ for all zero-dimensional schemes $Z' \subset X_{m,d}$ with $\deg(Z') \leq 3$.

We also want that $P \notin \sigma_4^0(X)$ hence we can assume that the above zero-dimensional scheme Z of degree 4 that computes the X-border rank of P is unreduced (otherwise $r_X(P) = 4$ and then $P \in \sigma_4^0(X)$).

We fix the following notation.

Notation 2. If $P \in \sigma_4(X) \setminus (\sigma_3(X) \cup \sigma_4^0(X))$, we fix $Z \subset X$ to be one of the degree 4 unreduced zero-dimensional schemes that computes the X-border rank of P, i.e. $P \in \langle Z \rangle$ and $P \notin \langle Z' \rangle$ for all zero-dimensional schemes $Z' \subset X$ of degree less or equal than 3 such that $P \in \langle Z' \rangle$.

In order to study the stratification of $\sigma_4(X_{m,d})$ it is therefore necessary to understand the $X_{m,d}$-rank of the points belonging to the span of a non reduced zero-dimensional subscheme $Z \subset X_{m,d}$ of degree 4. Clearly, for such a degree 4 scheme Z we have that $\dim(\langle Z \rangle) \leq 3$. By [18], Proposition 3.1, or [17], Subsection 3.2, [6], Remark 4.2, it is sufficient to do the cases $m = 2, 3$. In fact the stratification of $\sigma_4(X_{1,d})$ is already known by [21], [9] and [6]. Hence it remains to study the stratification of $\sigma_4(X_{2,d})$ (in fact [6] gives it for the cases $d = 3, 4$), i.e. when $m = 2$, and the stratification of $\sigma_4(X_{m,d})$ for $m \geq 3$. What the already quoted results in [6], [17] and [18] allow to do is that, once we will have the stratification of $\sigma_4(X_{3,d})$ then we will straightforwardly have that the same stratification will hold for $\sigma_4(X_{m,d})$ for the same d and $m \geq 3$.

Notation 3. Let $P \in \sigma_4(X_{m,d}) \setminus (\sigma_3(X_{m,d}) \cup \sigma_4^0(X_{m,d}))$ and let $Z \subset X_{m,d}$ be, as in Notation 2, a scheme computing the $X_{m,d}$-border rank of P. Take $A \subset \mathbb{P}^m$ to be an unreduced zero-dimensional scheme of degree 4 such that $\nu_{m,d}(A) = Z$.

By the discussion above we may assume that such a scheme A just defined in Notation 3 is not contained in a 2-dimensional projective subspace of $\mathbb{P}^m$. In fact, $\deg(A) = 4$ then $\langle A \rangle \subseteq \mathbb{P}^3$, but if $\langle A \rangle \subseteq \mathbb{P}^2$ then there exist a zero-dimensional scheme $B \subset \mathbb{P}^2$ of degree 3 such that $\langle A \rangle \subseteq \langle B \rangle = \mathbb{P}^2$. This would imply that any point $P \in \langle \nu_{m,d}(A) \rangle$ belongs to $\langle \nu_{m,d}(B) \rangle$ for some zero-dimensional scheme $B \subset \mathbb{P}^m$ of degree 3. Now, since $\deg(B) = 3$ then $\langle \nu_{m,d}(B) \rangle \subset \sigma_3(X)$. Therefore if $\langle A \rangle \subset \mathbb{P}^2$ we get that, if $Z = \nu_{m,d}(A)$, any point $P \in \langle Z \rangle$ belongs to $\sigma_3(X)$, but we want to study the X-rank of the points $P \in \sigma_4(X) \setminus \sigma_3(X)$. Therefore we assume that the scheme $A \subset \mathbb{P}^m$ defined in Notation 3 spans a projective subspace of dimension 3.

Notation 4. Let $P \in \sigma_4(X_{m,d}) \setminus (\sigma_3(X_{m,d}) \cup \sigma_4^0(X_{m,d}))$. We fix $S \subset X_{m,d}$ to be a reduced zero-dimensional scheme that computes the $X_{m,d}$-rank of P. I.e. $S \subset X_{m,d}$ is a reduced zero-dimensional scheme such that $P \in \langle S \rangle$ and $P \notin \langle S' \rangle$ for any reduced $S' \subset X_{m,d}$ with $\deg(S') < \deg(S)$.

Notation 5. Let $P \in \sigma_4(X_{m,d}) \setminus (\sigma_3(X_{m,d}) \cup \sigma_4^0(X_{m,d}))$. Let also $S \subset X_{m,d}$ be a reduced zero-dimensional scheme that computes the $X_{m,d}$-rank of P as in Notation 4. Take $B \subset \mathbb{P}^m$ be a reduced zero-dimensional scheme of degree $\deg(B) = \deg(S)$ such that $\nu_{m,d}(B) = S$.

We introduced this notation because we will often use it in the sequel. More precisely it allows to use many results on the reduced and unreduced zero-dimensional schemes in $\mathbb{P}^m$ and translate them in informations on the zero-dimensional sub-schemes of $X_{m,d}$ (see Section 3 and in particular Lemma 3).

2. Useful reducible curves

First of all let us recall the generic notion of Y-rank of a point P with respect to any non-degenerate projective curve $Y \subset \mathbb{P}^t$.

Definition 3. Let $Y \subset \mathbb{P}^t$ be an embedded non-degenerate projective curve, and let $P \in \mathbb{P}^t$. The Y-rank of P with respect to Y is the minimum number of points belonging to Y whose span contains P:

$$r_Y(P) := \min\{r \in \mathbb{N} \mid P \in \langle P_1, \ldots, P_r \rangle, \text{ with } P_1, \ldots, P_r \in Y\}.$$

Definition 4. Let $Y \subset \mathbb{P}^t$ be an embedded non-degenerate projective curve, and let $P \in \mathbb{P}^t$. The Y-border rank of P is the minimum positive integer s such that $P \in \sigma_s(Y) = \overline{\bigcup_{P_1,\ldots,P_s \in Y} \langle P_1, \ldots, P_s \rangle}$.

We prove here four propositions on the Y-rank of points belonging to $\langle Z \rangle$ where Z is a degree 4 non-reduced zero-dimensional sub-scheme a projective non-degenerate reduced curve Y obtained by the union of two rational normal curves Y_1, Y_2. We will not study here neither all the possible configurations of the scheme Z nor all the possible reciprocal

positions of the curves Y_1, Y_2, but only those that will be needed in the proof of the Theorem 1.
We like to stress here that in almost all the following propositions of this section (except in Proposition 3) we can prove only that the Y-rank of certain points is less or equal than a value. Nevertheless in Section 4 all these inequalities will be proved to be equalities (cfr. Corollary 1, Corollary 2 and Corollary 3).

Proposition 1. *Fix an integer $d \geq 3$. Let $Y \subset \mathbb{P}^{2d}$ be a reduced and connected curve union of two smooth degree d curves Y_1, Y_2, each of them a rational normal curve in its linear span, with a unique common point point, Q, and with $\langle Y \rangle = \mathbb{P}^{2d}$. Let $Z \subset Y$ be a length 4 zero-dimensional scheme such that $Z_{red} = \{Q\}$, Z is a Cartier divisor of Y and $\deg(Z \cap Y_i) \geq 2$ for $i = 1, 2$. Fix $P \in \langle Z \rangle$ such that $P \notin \langle Z' \rangle$ for any $Z' \subsetneq Z$. Then:*

$$r_Y(P) \leq 2d - 2 \tag{8}$$

and there is a reduced zero-dimensional sub-scheme $S \subset Y$ computing $r_Y(P)$ such that $Q \notin S$ and $\sharp(S \cap Y_i) = d - 1$ for $i = 1, 2$. We may find S as above and not intersecting any finite prescribed subset of Y.
If $d \geq 4$, then for a general pair of sets of points $(A_1, A_2) \subset Y_1 \times Y_2$ such that $\sharp(A_1) = \sharp(A_2) = d - 3$ there is S as above with the additional property that $A_1 \cup A_2 \subset S$.

Proof. Fix a finite set of points $M \subset Y$.
In Steps (a) and (b) of this proof we will show the existence of a reduced zero-dimensional sub-scheme $S \subset Y \setminus M$ of length $2d - 2$ that intersects both Y_1 and Y_2 in degree $d - 1$ and such that $P \in \langle S \rangle$, but the point $Q = \{Y_1 \cap Y_2\} \notin S$, and moreover, if $d \geq 4$, then the set $A_1 \cup A_2$ is contained in S.

Step (a) Here we assume $d = 3$. Let $\ell_P : \mathbb{P}^6 \setminus \{P\} \to \mathbb{P}^5$ be the linear projection from P. Since $P \notin Y$, then the map $\ell_P|Y$ is obviously a morphism. Since $P \notin \langle Y_i \rangle$, each curve $C_i := \ell_P(Y_i)$, $i = 1, 2$, is a rational normal curve in its 3-dimensional linear span. Now the zero-dimensional sub-scheme $Z \subset Y$ is, by hypothesis, such that $P \in \langle Z \rangle$ and $P \notin \langle Z' \rangle$ for any proper sub-scheme $Z' \subset Z$. If $P \in \langle T_Q Y_1 \cup T_Q Y_2 \rangle$ we would have that P belongs to the span of a proper sub-scheme of Z of degree 3 (in fact $\deg(T_Q Y_1 \cup T_Q Y_2) = 3$ and $T_Q Y_1 \cup T_Q Y_2 \subset Z$) that contradicts the hypothesis. Then $P \in \langle Z \rangle \setminus \langle T_Q Y_1 \cup T_Q Y_2 \rangle$, hence $D := \langle C_1 \rangle \cap \langle C_2 \rangle$ is a line not tangent either to C_1 or to C_2, but intersecting each C_i only at their common point $\ell_P(Q)$. Hence the linear projection from D induces a degree 2 morphism $\psi_i : C_i \to \mathbb{P}^1$. Thus, for a general $O \in D$, there are two sets of points $B_i \subset C_i$ such that $\sharp(B_i) = 2$ and $O \in \langle B_i \rangle$, for $i = 1, 2$. Let $S_i \subset Y_i$ be the only set of points such that $\ell_P(S_i) = B_i$ for $i = 1, 2$. Since $\dim(\langle \ell_P(S_1 \cup S_2) \rangle) = 2$, we have $\dim(\langle \{P\} \cup S_1 \cup S_2 \rangle) = 3$. We easily find $O \in D$ such that $\dim(\langle S_1 \cup S_2 \rangle) = 3$ and $Q \notin S_1 \cup S_2$. Hence $P \in \langle S_1 \cup S_2 \rangle$. We can then take $S := S_1 \cup S_2$ as a solution for $d = 3$.

Step (b) Here we assume $d \geq 4$. Take a general pair of sets of points $(A_1, A_2) \subset Y_1 \times Y_2$ such that $\sharp(A_1) = \sharp(A_2) = d - 3$. Let $\ell : \mathbb{P}^{2d} \setminus \langle A_1 \cup A_2 \rangle \to \mathbb{P}^6$ denote the linear projection from $\langle A_1 \cup A_2 \rangle$. Apply Step (a), i.e. the case $d = 3$, to the curve $Y' \subset \mathbb{P}^6$ which is the closure of $\ell(Y \setminus Y \cap \langle A_1 \cup A_2 \rangle)$. Let $S_1 \cup S_2$ be a solution for Y' with respect to the point $\ell(P)$. For general $O \in D$ (as in Step (a)) we may find $S_1 \cup S_2$ not through the finitely many points of Y' which are in $Y' \setminus \ell(Y \setminus (A_1 \cup A_2))$. Hence there are unique $B_i \subset Y_i$ such that $\ell(B_i) = S_i$ for $i = 1, 2$. Set $S := B_1 \cup B_2 \cup A_1 \cup A_2$. □

Remark 2. We notice that the triple (Y_1, Y_2, Q) described in the statement of Proposition 1 is unique, up to a projective transformation, and that $p_a(Y) = 0$.

In Corollary 2 we will show that (8) is an equality.

The following result can be proved in a similar way as we proved Proposition 1.

Proposition 2. *Fix an integer $d \geq 3$. Let $Y \subset \mathbb{P}^{2d}$ be a reduced and connected curve union of two smooth degree d curves Y_1, Y_2, each of them a rational normal curve in its linear span, with a unique common point point, Q, and with $\langle Y \rangle = \mathbb{P}^{2d}$. Fix $P_i \in Y_i \setminus \{Q\}$ for $i = 1, 2$. Let $Z_i \subset Y_i$, $i = 1, 2$, be the degree 2 effective divisor of Y_i supported on P_i. Set $Z := Z_1 \cup Z_2$. Fix $P \in \langle Z \rangle$ such that $P \notin \langle Z' \rangle$ for any $Z' \subset Y$ with $\deg(Z') < \deg(Z)$. Then:*

$$r_Y(P) \leq 2d - 2 \tag{9}$$

and there is reduce zero-dimensional sub-scheme $S \subset Y$ computing $r_Y(P)$ such that $Q \notin S$ and $\sharp(S \cap Y_i) = d - 1$ for $i = 1, 2$. We may find S as above and not intersecting any finite prescribed subset of Y. If $d \geq 4$, then for a general pair of sets of points $(A_1, A_2) \subset Y_1 \times Y_2$ such that $\sharp(A_1) = \sharp(A_2) = d - 3$ there is S as above with the addition property that $A_1 \cup A_2 \subset S$.

We do not write the proof of Proposition 2 because it is completely analogous to the proof of the above Proposition 1.

Corollary 3 will prove that (9) is an equality.

Proposition 3. *Fix an integer $d \geq 3$. Let $Y \subset \mathbb{P}^{2d+1}$ be a reduced curve union of two smooth degree d curves Y_1, Y_2, each of them a rational normal curve in its linear span and such that $\langle Y_1 \rangle \cap \langle Y_2 \rangle = \emptyset$. Fix $P_i \in Y_i$ for $i = 1, 2$. Let $Z_i \subset Y_i$ be the degree 2 effective Cartier divisor $2P_i$ of Y_i, $i = 1, 2$. Set $Z := Z_1 \cup Z_2$. Fix $P \in \langle Z \rangle$ such that $P \notin \langle Z' \rangle$ for all $Z' \subsetneq Z$. Then $b_Y(P) = 4$, Z is the only subscheme of Y computing $b_Y(P)$,*

$$r_Y(P) = 2d$$

and $\sharp(S \cap Y_1) = \sharp(S \cap Y_2) = d$ for all reduced zero-dimensional sub-schemes $S \subset Y$ computing $r_Y(P)$.

Proof. Since $\deg(Z) = 4$ and since Y is a smooth curve, we have $b_Y(P) \leq 4$.

Obviously $\langle Z \rangle \cap \langle Y_i \rangle \supset \langle Z_i \rangle$. Let's see the other containment. We show that $\langle Z \rangle \cap \langle Y_1 \rangle \subset \langle Z_1 \rangle$ (the same proof holds for $\langle Z \rangle \cap \langle Y_2 \rangle \subset \langle Z_2 \rangle$). If $\langle Z \rangle \cap \langle Y_1 \rangle$ is not contained in $\langle Z_1 \rangle$ then there exists a point $Q \in \langle Z \rangle \cap \langle Y_1 \rangle$ such that $Q \notin \langle Z_1 \rangle$. Since such a $Q \in \langle Z \rangle$, we have that $\dim(\langle Z_1, Q \rangle) = 2$ and $\langle Z_1, Q \rangle := \Pi \subset \langle Z \rangle$. Now Π is spanned by a zero-dimensional scheme of degree 3 that is contained in $\langle Y_1 \rangle$, by hypothesis $\langle Y_1 \rangle \cap \langle Y_2 \rangle = \emptyset$, then Π cannot intersect $\langle Z_2 \rangle$ which is entirely contained in $\langle Y_2 \rangle$. Now boht Π and $\langle Z_2 \rangle$ are contained in $\langle Z \rangle$ which has projective dimension 3. Therefore if such a Q exists, we would have a projective plane Π and a line $\langle Z_2 \rangle$ that are contained in a $\mathbb{P}^3$ without intersecting each other, but this is impossible. Then $\langle Z \rangle \cap \langle Y_i \rangle \subset \langle Z_i \rangle$ for $i = 1, 2$.

Since $P \notin \langle Z' \rangle$ for all $Z' \subsetneqq Z$, we get $P \notin \langle Y_1 \rangle$ and $P \notin \langle Y_2 \rangle$. Since for $i = 1, 2$ Y_i is a rational normal curve, then Z_i is the only sub-scheme of Y_i computing $b_{Y_i}(Q)$ for all $Q \in T_{P_i}Y_i \setminus \{P_i\}$. We immediately get that Z is the only sub-scheme of Y with length at most 4 whose linear span contains P. Hence we have proved that Z is the unique zero-dimensional scheme that computes the Y-border rank of P and that $P \in \sigma_4(Y) \setminus \sigma_3(Y)$.

Now we compute $r_Y(P)$.

Let $\ell_P : \mathbb{P}^{2d+1} \setminus \{P\} \to \mathbb{P}^{2d}$ denote the linear projection from P. Set $C := \ell_P(Y)$ and $C_i := \ell_P(Y_i)$. Since $P \notin \sigma_2(Y)$, then $\ell_P|Y$ is an embedding. Hence $C_1 \cap C_2 = \emptyset$. Since $P \notin \langle Y_1 \rangle \cup \langle Y_2 \rangle$, then each C_i is a degree d rational normal curve in its linear span. Thus $r_Y(P)$ is the minimal cardinality of a set of points $A := A_1 \cup A_2$ such that $A_1 \subset C_1$, $A_2 \subset C_2$ and $A_1 \cup A_2$ is linearly dependent. Notice that $\langle C_1 \rangle \cap \langle C_2 \rangle$ is a unique point $O \notin C$. Set $Q_i := \ell_P(P_i)$ with $P_i = (Z_i)_{red}$, $W_i := \ell_P(Z_i)$ and $W := W_1 \cup W_2$. Hence W_i is the degree 2 effective divisor $2Q_i$ of C_i. Since $P \in \langle Z \rangle$, then $\langle W \rangle$ is a plane. Thus the two lines $T_{Q_i}C_i$, $i = 1, 2$, meets each other. Since $\{O\} = \langle C_1 \rangle \cap \langle C_2 \rangle$, then O is their unique common point. Since $O \in T_{Q_i}C_i$, we have $r_{C_i}(O) = d$ (see [9] or [17], Theorem 4.1). Hence $\sharp(A \cap C_1) \geq d$ and $\sharp(A \cap C_2) \geq d$. Since $\langle C_1 \rangle \cap \langle C_2 \rangle = \{O\}$ and any $d+1$ points of C_i are linearly independent, $\sharp(A \cap C_1) = d$ and $\sharp(A \cap C_2) = d$ for every linearly dependent $A \subset C$ such that $\sharp(A \cap C_i) \leq d$ for all i. Then $r_Y(P) \geq 2d$, but $r_{Y_i}(R_i) = d$ for all $R_i \in T_{P_i}Y_i \setminus Y_i$ and $i = 1, 2$, hence $r_Y(P) \leq 2d$ and therefore $r_Y(P) = 2d$. □

Proposition 4. *Fix an integer $d \geq 3$. Let $Y \subset \mathbb{P}^{2d}$ be a reduced and connected curve union of two smooth degree d curves Y_1, Y_2, each of them a rational normal curve in its linear span, with a unique common point, Q, and with $\langle Y \rangle = \mathbb{P}^{2d}$. Fix $P_1 \in Y_1 \setminus \{Q\}$ and let $Z_1 \subset Y_1$ the degree 2 effective divisor with P_1 as its reduction. Let $Z_2 \subset Y_2$ be the degree 2 effective divisor of Y_2 with Q as its reduction. Set $Z := Z_1 \cup Z_2$. Z is a degree 4 Weil divisor of Y, but not a Cartier divisor of Y. Fix $P \in \langle Z \rangle$ such that $P \notin \langle Z' \rangle$ for any $Z' \subsetneqq Z$. Then:*

$$r_Y(P) \leq 2d - 1 \qquad (10)$$

and there is a reduced zero-dimensional sub-scheme $S \subset Y$ computing $r_Y(P)$ such that $\sharp(S \cap Y_1) = d - 1$ and $\sharp(Y_2 \cap S) = d$. Moreover $\langle Z \rangle \cap \langle S \rangle$ is a line intersecting both $\langle Y_1 \rangle$ and $\langle Y_2 \rangle$.

Proof. First of all observe that, by construction, $P \in \langle T_Q Y_2, T_{P_1} Y_1 \rangle$. Since $Q = Y_1 \cap Y_2$, we may also observe that $P \in \langle \sigma_3(Y_2), T_Q Y_1 \rangle$. Hence there exists a zero-dimensional scheme $Z' \subset Y_1$ such that $P \in \langle Z', T_Q Y_1 \rangle$. The construction of the scheme Z allows us to be more precise: $Z' = Z_1 \cup Q$. Therefore the point P can be written as a linear combination of a point $P_1' \in \langle Z' \rangle$ and $P_2' \in T_Q Y_1$ where $Z' = Z_1 \cup Q$. Now $r_{Y_2}(P_2') = d$ because $P_2' \in T_Q Y_1$ and, by [21], [6], [9], [11] and [7], the points belonging to the tangent line to a rational normal curve of degree d have symmetric rank equal to d. Moreover $r_{Y_1}(P_1') = d - 1$ because $P_1' \in \langle Z_1 \rangle$ that is not reduced and the points belonging to the span of a degree 3 non-reduced zero-dimensional sub-scheme of a rational normal curve belongs to $\sigma_3(Y_2) \setminus \sigma_3^0(Y_2)$ and then, by [21], [6], [9], [11] and [7], they have symmetric rank $d - 1$. Hence $P = P_1' + P_2'$, then $r_Y(P) \leq 2d - 1$. □

In Corollary 1 of Section 4.1.1 we will prove that (10) is actually an equality.

3. Lemmas

In Notation 3 and in Notation 5 we defined two zero-dimensional schemes $A, B \subset \mathbb{P}^m$ such that $\nu_{m,d}(A) = Z$ and $\nu_{m,d}(B) = S$ respectively and two different zero-dimensional schemes $Z, S \subset X$ realizing the X-border rank and the X-rank respectively of a point $P \in \sigma_4(X) \setminus (\sigma_4^0(X) \cup \sigma_3(X))$. Here, but only for this Section 3, we do not care about the fact that $P \in \langle \nu_{m,d}(A)\rangle \cap \langle \nu_{m,d}(B)\rangle$ is a point of $\sigma_4(X) \setminus (\sigma_4^0(X) \cup \sigma_3(X))$: for this section $A, B \subset \mathbb{P}^m$ are zero-dimensional schemes whose images via $\nu_{m,d}$ still realize the X-border rank and the X-rank respectively of a point $P \in \mathbb{P}^n$, but here we do not give any restriction on the minimum secant variety $\sigma_s(X)$ such that $P \in \sigma_s(X)$. This is summarized in the following notation.

Notation 6. In this section, and only in this section, we only require that:

- $A \subset \mathbb{P}^m$ is a non-reduced zero-dimensional scheme such that $\nu_{m,d}(A) = Z \subset \mathbb{P}^n$ realizes the X-border rank of $P \in \mathbb{P}^n$,
- $B \subset \mathbb{P}^m$ is a reduced zero-dimensional scheme such that $\nu_{m,d}(B) = S \subset \mathbb{P}^n$ realizes the X-rank of $P \in \mathbb{P}^n$,
- $\deg(A) < \deg(B)$.

More assumptions on the degrees of A and B will be explained in each Lemma.

We can now give the following lemmata. It will be crucial in the proof of Theorem 1.

Lemma 1. *Fix $P \in \mathbb{P}^{n_{m,d}}$. Let $A, B \in \mathbb{P}^m$ be two zero-dimensional schemes as in Notation 6. We have that*

$$h^1(\mathbb{P}^m, \mathcal{I}_{A\cup B}(d)) > 0.$$

Proof. The statement is equivalent to the fact that the zero-dimensional scheme $\nu_{m,d}(A\cup B)$ is linearly dependent in $\mathbb{P}^{n_{m,d}}$, i.e. $\dim(\langle \nu_{m,d}(A\cup B)\rangle) \leq \deg(A\cup B) - 2$. The latter inequality is true even if $A \cap B \neq \emptyset$, because $P \in \langle \nu_{m,d}(A)\rangle \cap \langle \nu_{m,d}(B)\rangle$, but $P \notin \langle \nu_{m,d}(A')\rangle$ for any $A' \subsetneq A$ and $P \notin \langle \nu_{m,d}(B')\rangle$ for any $B' \subsetneq B$. □

We introduce here a tool that we will use in the proofs of the next lemmata.

Notation 7. Let $E \subset \mathbb{P}^m$ be a zero-dimensional scheme and let $H \subset \mathbb{P}^m$ be a hyperplane, then the sequence that defines the residual scheme $\mathrm{Res}_H(E)$ of E with respect to H is the following:

$$0 \to \mathcal{I}_{\mathrm{Res}_H(E)}(t-1) \to \mathcal{I}_E(t) \to \mathcal{I}_{E\cap H, H}(t) \to 0 \tag{11}$$

Lemma 2. *Fix an integer $d \geq 2$ and a zero-dimensional and curvilinear subscheme E of $\mathbb{P}^2$ such that $\deg(E) = 2d+2$ and $h^1(\mathcal{I}_E(d)) > 0$.*

(i) If E is in linearly general position, then $h^1(\mathcal{I}_E(d)) = 1$ and there is a smooth conic C such that $E \subset C$.

(ii) Assume that E is not in linearly general position. Then either there is a line $L \subset \mathbb{P}^2$ such that $\deg(L \cap E) \geq d+2$ and $h^1(\mathcal{I}_E(d)) = \deg(L\cap E) - d - 1$ or there are two lines L_1, L_2 such that $E \subset L_1 \cup L_2$ and $\sharp(E \cap L_1) = \sharp(E \cap L_2) = d+1$.

Proof. First assume that E is in linearly general position. Let $C \subset \mathbb{P}^2$ be a reduced conic such that $y := \deg(E \cap C)$ is maximal. Since $\deg(E) \geq 5$ and $52 = 6$, we have $y \geq 5$. Since $\mathrm{Res}_C(E) \subset E$, the scheme $\mathrm{Res}_C(E)$ is in linearly general position. Since $\deg(\mathrm{Res}_C(E)) = 2d + 2 - y \leq 2(d-2) + 1$, we have $h^1(\mathcal{I}_{\mathrm{Res}_C(E)}(d-2)) = 0$ ([14], Theorem 3.2). Thus the exact sequence

$$0 \to \mathcal{I}_{\mathrm{Res}_C(E)}(d-2) \to \mathcal{I}_E(d) \to \mathcal{I}_{C \cap E}(d) \to 0$$

gives $h^1(C, \mathcal{I}_{C \cap E}(d)) > 0$. Thus $\deg(E \cap C) \geq 2d + 2$. Since $\deg(E) = 2d + 2$, we get $E \subset C$, concluding the proof of (i).

Now assume that E is not in linearly general position. Take a line $L \subset \mathbb{P}^2$ such that $x := \deg(L \cap E)$ is maximal. By assumption we have $x \geq 3$. First assume $x \geq d + 2$. Since $\mathrm{Res}_L(E)$ has degree $2d + 2 - x$, we have $h^1(\mathcal{I}_{\mathrm{Res}_L(E)}(d-1)) = 0$ ([6], Lemma 4.6). From the exact sequence (11) we get the result in this case. Now assume $x \leq d + 1$. If $h^1(\mathcal{I}_{\mathrm{Res}_L(E)}(d-1) = 0$, then (11) gives $h^1(\mathcal{I}_E(d)) = 0$ that is a contradiction. Thus $h^1(\mathcal{I}_{\mathrm{Res}_L(E)}(d-1)) > 0$. Since $2d + 2 - x \leq 2(d-1) + 1$ ([6], Lemma 4.6) gives the existence of a line R such that $z := \deg(R \cap \mathrm{Res}_L(E)) \geq d + 1$. The maximality property of x and the inclusion $\mathrm{Res}_L(E) \subseteq E$ gives $x \geq d + 1$. Since $z \leq 2d + 2 - x$, we get $z = x = d + 1$. □

We remind here part of the Theorem 1 proved in [4], and we write the part that will be useful in our paper applied in the particular case of $\deg(Z) = 4$.

Lemma 3. (Theorem 1 in [4]) *Assume $m \geq 2$ and let $A, B \subset \mathbb{P}^m$ as in Notation 6. Assume also that $\langle A \rangle = \langle B \rangle = \mathbb{P}^m$ and $\deg(A \cup B) \leq 2d + 1$. Then there are a line $L \subset \mathbb{P}^m$ and a finite set of points $F_2 \subset \mathbb{P}^m \setminus L$ such that:*

$$\deg(L \cap (A \cup B)) \geq d + 2,$$

$A \cap L \neq \emptyset$, $(B \setminus A_{red}) \cap L \neq \emptyset$, $\sharp(F_2) \geq m - 1$, $B = F_2 \sqcup (B \cap L)$ and $A = F_2 \sqcup (A \cap L)$ (as schemes).

Remark 3. We like to observe that the proof of the above Lemma (ref. proof of Theorem 1 in [4]) holds even if either $A \subset \mathbb{P}^m$ is reduced or if $B \subset \mathbb{P}^m$ does not compute $r_{X_{m,d}}(P)$. We do not give the details of the proof of this fact because it is sufficient to retrace the proof of Theorem 1 in [4] without the assumptions on the non-reducibility of A and on the fact that B computes the $X_{m,d}$-rank of P. In fact that proof only requires the existence of $P \in \langle \nu_{m,d}(A) \rangle \cap \langle \nu_{m,d}(B) \rangle$ such that $P \notin \langle \nu_{m,d}(A') \rangle$ for any $A' \subsetneq A$ and $P \notin \langle \nu_{m,d}(B') \rangle$ for any $A' \subsetneq B$.

Because of the above remark we will often use, especially in Section 4.2, the Lemma 3 in the following form.

Lemma 4. *Assume $m \geq 2$ and let $A, B \subset \mathbb{P}^m$ two zero-dimensional schemes such that there exists a point $P \in \langle \nu_{m,d}(A) \rangle \cap \langle \nu_{m,d}(B) \rangle$ such that $P \notin \langle \nu_{m,d}(A') \rangle$ for any $A' \subsetneq A$ and $P \notin \langle \nu_{m,d}(B') \rangle$ for any $A' \subsetneq B$. Assume also that $\langle A \rangle = \langle B \rangle = \mathbb{P}^m$ and $\deg(A \cup B) \leq 2d + 1$. Then there are a line $L \subset \mathbb{P}^m$ and a finite set of points $F_2 \subset \mathbb{P}^m \setminus L$ such that:*

$$\deg(L \cap (A \cup B)) \geq d + 2,$$

$A \cap L \neq \emptyset$, $(B \setminus A_{red}) \cap L \neq \emptyset$, $\sharp(F_2) \geq m-1$, $B = F_2 \sqcup (B \cap L)$ and $A = F_2 \sqcup (A \cap L)$ (as schemes).

Lemma 5. *Assume the existence of a hyperplane $M \subset \mathbb{P}^m$ such that $h^1(\mathbb{P}^m, \mathcal{I}_{Res_M(A \cup B)}(d-1)) = 0$. Let $A, B \subset \mathbb{P}^m$ as in Notation 6. Set $F_3 := B \setminus B \cap M$. Then $F_3 \subset A_{red}$. If either $\langle A \rangle = \mathbb{P}^m$ or $B \nsubseteq M$, then $F_3 \neq \emptyset$.*

Proof. Set $G := F_3 \cap A$ and $S_3 := F_3 \setminus G$. Notice that S_3 (if it is not empty) is a union of reduced connected components of $A \cup B$. From the exact sequence (11) we get $h^1(M, \mathcal{I}_{(A\cup B)\cap M}(d)) = h^1(\mathbb{P}^3, \mathcal{I}_{A \cup B}(d))$ and $h^1(M, \mathcal{I}_{(A\cup B)\cap M}(d)) > 0$. Thus $\langle \nu_{m,d}(A \cap M)\rangle \cap \langle \nu_{m,d}(B \cap M)\rangle \neq \emptyset$. Since $\nu_{m,d}$ is a complete embedding, then we have just proved that $h^1(\mathcal{I}_{(Z \cup S)\cap \nu_{m,d}(M)}(1)) = h^1(\mathcal{I}_{Z\cup S}(1))$. We have $\dim(\langle Z \cup S\rangle) = \deg(Z \cup S) - 1 - h^1(\mathbb{P}^3, \mathcal{I}_{A\cup B}(d))$ and $\dim(\langle \nu_{m,d}((A \cup B) \cap M)\rangle = \deg((A\cup B)\cap M) - 1 - h^1(M, \mathcal{I}_{(A \cup B)\cap M}(d))$. We also have $\dim(\langle E \rangle) = \deg(E) - 1$ for every $E \in \{A, B, A \cap M, B \cap M\}$. Hence Grassmann's formula gives $\dim(\langle Z \rangle \cap \langle S \rangle) = \sharp(G) + \dim(\langle \nu_{m,d}(A \cap M)\rangle \cap \langle \nu_{m,d}(B \cap M)\rangle)$ and $\dim(\langle Z \cup S \rangle) = \dim(\langle \nu_{m,d}(A \cup B \setminus S_3)\rangle) + \sharp(S_3)$ (i.e. $\dim(\langle Z \cup S)\rangle) = \sharp(S_3) + \dim(\langle \nu_{m,d}(A \cup B) \setminus S_3 \rangle)$.

Now we prove $S_3 = \emptyset$, i.e. $F_3 \subset A_{red}$. Since $S \cap ((A \cup B) \setminus S_3) = S_1 \cup S_2$, $A \cup B = ((A \cup B) \setminus S_3) \sqcup S_3$, $W = W' \sqcup S_3$ and $Z \cup (S \cap M) \cup \nu_{m,d}(G) = (A \cup B) \setminus S_3$, we get $\langle (Z \cup S) \setminus \nu_{m,d}(S_3)\rangle \cap \langle S \rangle = \langle Z \rangle \cap \langle (S \cap M) \cup \nu_{m,d}(G)\rangle$. Since $P \in \langle Z \cup (S \cap M) \cup \nu_{m,d}(G)\rangle \cap \langle S \rangle$, we get $P \in \langle (S \cap M) \cup Z \cup (S \cap M) \cup \nu_{m,d}(G)\rangle$. The minimality of S gives $S_3 = \emptyset$.

To conclude the proof it is sufficient to show that $F_3 \neq \emptyset$. Assume $F_3 = \emptyset$. Then $B \subset M$. Hence $\langle S \rangle \subset \langle \nu_{m,d}(M)\rangle$. Hence P may be defined using a smaller number of homogeneous variables, contradicting [18], Proposition 3.1, or [17], Subsection 3.2, and the assumption that m is minimal for P. □

4. Preliminaries of the proof of the main theorem

This section is essentially the core of the proof of Theorem 1 but it is not the proof yet. That proof will be done in the next section. Here we give only all the preliminaries, they will reduce the proof of Theorem 1 to its structure: it will be sufficient to show the frame of the proof and all the details will be already proved in this section.

For all this section we will use the notation given in the Introduction that we remind here for the reader.

- Let $Z \subset X_{m,d}$, as in Notation 2, be a degree 4 zero-dimensional sub-scheme of $X_{m,d}$ that computes the $X_{m,d}$-border rank of a point $P \in \sigma_4(X_{m,d}) \setminus (\sigma_4^0(X_{m,d}) \cup \sigma_3(X_{m,d}))$;
- Let $A \subset \mathbb{P}^m$, as in Notation 3, be the pre-image of $Z \subset X_{m,d}$ as above via the Veronese map $\nu_{m,d}$;
- Let $S \subset X_{m,d}$, as in Notation 4, be a reduced zero-dimensional sub-scheme of $X_{m,d}$ that computes the $X_{m,d}$-rank of a point $P \in \sigma_4(X_{m,d}) \setminus (\sigma_4^0(X_{m,d}) \cup \sigma_3(X_{m,d}))$;
- Let $B \subset \mathbb{P}^m$, as in Notation 5, be the pre-image of $S \subset X_{m,d}$ as avobe via $\nu_{m,d}$.

Here we give two auxiliary lemmas using these assumptions.

Lemma 6. *Fix a line $L \subset \mathbb{P}^m$ and assume $h^1(\mathcal{I}_{A\cup B}(d)) = \deg(L \cap (A \cup B)) - d - 1$. Then $\langle Z \rangle \cap \langle S \rangle \subseteq \langle D \rangle$ with $D = \nu_{m,d}(L)$ where $A, B \subset \mathbb{P}^m$ are as in Notation 3 and Notation 5 respectively and $S, Z \subset X_{m,d}$ are as in Notation 4 and Notation 2 respectively.*

Proof. The assumption $h^1(\mathcal{I}_{A\cup B}(d)) = \deg(L \cap (A \cup B)) - d - 1$ implies $\dim(\langle A \cup B \rangle) = \dim(\langle D \rangle) + \deg(A \cup B) - \deg((A \cup B) \cap D)$. $\square$

Lemma 7. *Let $M \subset \mathbb{P}^m$ be a hyperplane such that $h^1(\mathbb{P}^m, \mathcal{I}_{Res_M(A\cup B)}(d-1)) = 0$. Then $h^1(\mathbb{P}^m, \mathcal{I}_{A\cup B}(d)) = h^1(M, \mathcal{I}_{(A\cup B)\cap M}(d))$ and $\langle Z \rangle \cap \langle S \rangle \subseteq \langle \nu_{m,d}(M) \rangle$ where $A, B \subset \mathbb{P}^m$ are as in Notation 3 and Notation 5 respectively and $S, Z \subset X_{m,d}$ are as in Notation 4 and Notation 2 respectively.*

Proof. Since $h^2(Y_m, \mathcal{I}_{A\cup B}(d-1)) = 0$, the first equality follows from the residual sequence (11) . Thus $\dim(\langle A \cup B \rangle) - \dim(\langle \nu_{m,d}((A \cup B) \cap D) \rangle) = \deg(A \cup B) - \deg((A \cup B) \cap M)$. $\square$

Now we split the section in two subsections where we study the $X_{m,d}$-rank of a point $P \in \langle \nu_{m,d}(A) \rangle$ for particular configurations of the scheme $A \subset \mathbb{P}^m$ with $m = 2, 3$ respectively (if $A \subset \mathbb{P}^1$ we refer to the Sylvester algorithm in [21], [9], [6], [11] and [7] for the computation of the $X_{1,d}$-rank of a point $P \in \langle \nu_{1,d}(\mathbb{P}^1) \rangle$).

4.1. Two dimensional case

Here we study the $X_{2,d}$-rank of a point $P \in \sigma_4(X_{2,d}) \setminus (\sigma_4^0(X_{2,d}) \cup \sigma_3(X_{2,d}))$ with $X_{2,d}$ the Veronese surface $\nu_{2,d}(\mathbb{P}^2) \subset \mathbb{P}^{n_{2,d}}$. Moreover we assume in this sub-section that the scheme $A \subset \mathbb{P}^2$ such that $Z = \nu_{2,d}(A)$ computes the $X_{2,d}$-border rank of P is not contained in a line, that is to say that $m = 2$ is the minimum integer that contains A where A is defined as in Notation 3. Since A is not contained in a line we have that $\langle A \rangle = \mathbb{P}^2$ and that

$$\boxed{h^0(\mathbb{P}^2, \mathcal{I}_A(2)) = 2.} \tag{12}$$

4.1.1. Here assume the existence of a line $L \subset \mathbb{P}^2$ such that the schematic intersection between A and L has degree at least 3

Since we are assuming that there exists a line $L \subset \mathbb{P}^2$ such that $\deg(A \cap L) \geq 3$ and since, by (12), $\langle A \rangle = \mathbb{P}^2$, we have necessary that:

$$\boxed{\deg(A \cap L) = 3.}$$

Hence, in this case, the scheme $\mathrm{Res}_L(A)$ has degree 1, i.e.

$$\mathrm{Res}_L(A) = O \tag{13}$$

is a point with its reduced structure.
Notice that every point P' of $\langle \nu_{2,d}(A \cap L) \rangle \setminus \sigma_2(\nu_{2,d}(L))$ has rank $d-1$ ([9] or [17], Theorem 4.1), unless $A \cap L$ is reduced. In the latter case any such a point has rank 3.

In Proposition 5 we study the case of $O \notin L$, while the case of $O \in L$ is done in Proposition 6, Proposition 7 and Proposition 8.

Proposition 5. *Let $A \subset \mathbb{P}^2$ a degree 4 zero-dimensional non-reduced scheme with $h^0(\mathbb{P}^2, \mathcal{I}_A(2)) = 2$. Suppose that it exists a line $L \subset \mathbb{P}^2$ such that $deg(L \cap A) \geq 3$ and that $\mathrm{Res}_L(A) =: \{O\} \notin L$. Let $Z \subset X_{2,d}$, as in Notation 2, be $Z = \nu_{2,d}(A)$. Then the $X_{2,d}$-rank of a point $P \in \langle Z \rangle \setminus (\sigma_4^0(X_{2,d}) \cup \sigma_3(X_{2,d}))$ is*

$$r_{X_{2,d}}(P) = d.$$

PROOF. Since A is not reduced, then $O \notin L$, $\{O\} = \mathrm{Res}_L(A)$ and the scheme $A \cap L$ cannot be reduced. Moreover let $P' \in \langle \nu_{2,d}(L \cap A) \rangle \setminus \sigma_2(\nu_{2,d}(L \cap A))$, then $P' \in \sigma_3(\nu_{2,d}(L)) \setminus (\sigma_3^0(\nu_{2,d}(L)) \cup \sigma_2(\nu_{2,d}(L)))$, hence $r_{\nu_{2,d}(L)}(P') = d-1$ (in fact $\nu_{2,d}(L)$ is a rational normal curve of degree d, hence we can apply the Sylvester algorithm [21], [9], [6], [11], [7]). Now, by [9] or [17], Theorem 4.1, we have that also $r_{X_{2,d}}(P') = d-1$.
Obviously a point $P \in \langle Z \rangle$ is a linear combination of a point $P' \in \langle \nu_{2,d}(L \cap A) \rangle \setminus \sigma_2(\nu_{2,d}(L \cap A))$ and the point O, hence $r_X(P) \leq r_X(P') + 1 = d$.
Assume $r_X(P) < d$, i.e. $\sharp(B) \leq d-1$ where $B \subset \mathbb{P}^2$ is defined as in Notation 5 to be the pre-image via $\nu_{2,d}$ of a scheme $S \subset X$ that computes the X-rank of P. Hence $\deg(A \cup B) \leq d+3 \leq 2d+1$. Apply Lemma 3, calling R the line such that $\deg(R \cap (A \cup B)) \geq d+2$. Since $\deg(A \cup B) \leq d+3$ and $\deg(A \cap L) = 3$, we have $R = L$ and $B \subset R$. Since $P \notin \langle \nu_{2,d}(L) \rangle$, we have $P \notin \langle S \rangle$, that is a contradiction. □

In the next three propositions we will do the cases in which the point $O = \mathrm{Res}_L(A)$ defined in (13) is contained in L. Observe that the definition of the residual scheme shows that the connected component A_O of A containing O is not reduced. We will distinguish the three propositions below by the cardinality of the support of A.

Proposition 6. *Let $A \subset \mathbb{P}^2$ be a degree 4 zero-dimensional scheme with $h^0(\mathbb{P}^2, \mathcal{I}_A(2)) = 2$ with support on a point O. Suppose that it exists a line $L \subset \mathbb{P}^2$ such that $\deg(A \cap L) = 3$. Let $Z \subset X_{2,d}$, as in Notation 2, be $Z = \nu_{2,d}(A)$. Then the $X_{2,d}$-rank of a point $P \in \langle Z \rangle \setminus (\sigma_4^0(X_{2,d}) \cup \sigma_3(X_{2,d}))$ is*

$$r_{X_{2,d}}(P) = 2d-2.$$

PROOF. Since $\deg(A) = 4$ and $42 = 6$, there is a reduced conic $T \supset L$, say $T = L \cup L_1$ with L_1 a line and $L_1 \neq L$, such that $A \subset T$. We are in the set-up of Proposition 1 taking $Y := \nu_{2,d}(T)$, $Y_1 = \nu_{2,d}(L_1)$ and $Y_2 = \nu_{2,d}(L)$. In this case we have that $\deg(Y_i \cap Z) = 2$ for $i = 1, 2$. Remember that in the proof of Proposition 1 we proved the inequality $r_Y(P) \leq 2d-2$. Since $Y \subset X_{2,d}$, this inequality gives $r_X(P) \leq 2d-2$. Assume that $r_{X_{2,d}}(P) \leq 2d-3$. This implies that $\deg(A \cup B) \leq 2d+1$. Hence we may apply Lemma 3. Since Z_{red} is a single point, we must have $F_2 = \emptyset$, contradicting the inequality $\sharp(F_2) \geq m-1 = 1$ of the statement of Lemma 3. □

Proposition 7. *Let $A_1, A_2 \subset \mathbb{P}^2$ be two degree 2 non-reduced zero-dimensional schemes such that if $A = A_1 \cup A_2$ then $h^0(\mathbb{P}^2, \mathcal{I}_A(2)) = 2$. Let also $A \subset \mathbb{P}^2$ be such that there exists a line $L \subset \mathbb{P}^2$ with the property that $\deg(L \cap A) = 3$. Let $Z \subset X_{2,d}$, as in Notation 2, be $Z = \nu_{2,d}(A)$. Then the $X_{2,d}$-rank of a point $P \in \langle Z \rangle \setminus (\sigma_4^0(X_{2,d}) \cup \sigma_3(X_{2,d}))$ is*

$$r_{X_{2,d}}(P) = 2d-1.$$

PROOF. The inequality $r_X(P) \leq 2d-1$ follows from the inequality $r_Y(P) \leq 2d-1$ proved in Proposition 4. Hence to prove that $r_X(P) = 2d-1$ it is sufficient to prove $r_X(P) \geq 2d-1$. Let $B \subset \mathbb{P}^2$ be as in Notation 5 such that $\nu_2(B) = S$ and $S \subset X_{2,d}$ be a sub-scheme computing the $X_{2,d}$-rank of P.
Assume $r_X(P) \leq 2d-2$, hence $\deg(A \cup B) \leq 2d+2$. First assume $\deg(A \cup B) \leq 2d+1$. Since no component of Z is reduced, Lemma 3 gives both $F_2 = \emptyset$ and $\sharp(F_2) \geq 1$ that is a contradiction. Now assume $\deg(A \cup B) = 2d+2$. Since $h^1(\mathbb{P}^2, \mathcal{I}_{A\cup B}(d)) > 0$ (Lemma 1), we may apply Lemma 2 to $E := A \cup B$. Since A is not in linearly general position, Lemma 2 gives $\deg((A \cup B) \cap L) \geq d+1$ and the existence of a line $R \neq L$ such that $A \cup B \subset L \cup R$. If $R \cap L \notin B$, then we may apply the proven part of Proposition 4. Since $\deg(A) + \deg(B) \leq 2d+2$, if $R \cap L \in B$, then $\deg(A \cup B) \leq 2d+1$ that is a contradiction.
$\square$

This proposition allows to prove that the inequality (10) of Proposition 4 is actually an equality.

Remark 4. Let Y be a projective curve contained in the variety $X_{m,d}$, and let $P \in \mathbb{P}^{n_{m,d}}$. Then obviously

$$r_Y(P) \geq r_{X_{m,d}}(P).$$

Corollary 1. *Fix an integer $d \geq 3$. Let $Y \subset \mathbb{P}^{2d}$ be a reduced and connected curve union of two smooth degree d curves Y_1, Y_2, each of them a rational normal curve in its linear span, with a unique common point, Q, and with $\langle Y \rangle = \mathbb{P}^{2d}$. Fix $P_1 \in Y_1 \setminus \{Q\}$ and let $Z_1 \subset Y_1$ the degree 2 effective divisor with P_1 as its reduction. Let $Z_2 \subset Y_2$ be the degree 2 effective divisor of Y_2 with Q as its reduction. Set $Z := Z_1 \cup Z_2$. Fix $P \in \langle Z \rangle$ such that $P \notin \langle Z' \rangle$ for any $Z' \subsetneq Z$. Then*

$$r_Y(P) = 2d-1.$$

PROOF. The inequality $r_Y(P) \leq 2d-1$ is proved in Proposition 4.
In the proof of Proposition 7 we showed that $r_{X_{2,d}}(P) \geq 2d-1$.
Since $Y \subseteq X_{2,d}$ then, as Remark 4 shows, also $r_Y(P)$ as to be bigger or equal than $2d-1$.
$\square$

Remark 5. The case proved in Proposition 7 is a very interesting case, because $h^1(\mathbb{P}^3, \mathcal{I}_{A\cup B}(d)) = 2$ (i.e. $\dim(\langle Z \rangle \cap \langle S \rangle) = 1$.

Proposition 8. *Let $A = A_O \sqcup O_1 \sqcup O_2 \subset \mathbb{P}^2$ with $O_1 \neq O_2$ be two simple points of $\mathbb{P}^2$ and $A_O \subset \mathbb{P}^2$ be a degree 2 non-reduced zero-dimensional scheme with support on a point $O \in L := \langle O_1, O_2 \rangle$ but $O \notin \{O_1, O_2\}$ and $\deg(A_O \cap L) = 1$. Let $Z \subset X_{2,d}$, as in Notation 2, be $Z = \nu_{2,d}(A)$. Then, if $d \geq 4$, the $X_{2,d}$-rank of a point $P \in \langle Z \rangle \setminus (\sigma_4^0(X_{2,d}) \cup \sigma_3(X_{2,d}))$ is*

$$r_{X_{2,d}}(P) = d+2.$$

PROOF. Define $Z_O := \nu_{2,d}(A_O) \subset X_{2,d}$. Every point $P' \in \langle \nu_{2,d}(Z_O) \rangle \setminus X_{2,d}$ has $X_{2,d}$-rank equal to d (see [6], Theorem 4.3). Thus $r_X(P) \leq d+2$ in this case. Assume $r_X(P) \leq d+1$. Since $d+5 \leq 2d+1$ (here we are using the hypothesis $d \geq 4$), we may apply Lemma 3. We get the existence of a line $R \subset Y_2$ and of a set of points $F_2 \subset Y_2 \setminus R$

such that $\deg((A\cap B)\cap R)\geq d+2$, $\sharp(F_2)\geq 1$, $B=((B\setminus B\cap A_{red})\cap R)\sqcup F_2$, $A\cap R\neq\emptyset$, $B\cap R\neq\emptyset$, $B=(B\cap R)\sqcup F_2$ and $A=(A\cap R)\sqcup F_2$ where B is as in Notation 5. First assume $R=L$. Since $A_{red}\subset L$, we get $F_2=\emptyset$, contradicting the inequality $\sharp(F_2)\geq 1$. Now assume $R\neq L$. Thus $\{O\}=R\cap L$, A_O is the degree 2 effective divisor of R supported by O and $F_2=\{O_1,O_2\}$. Since $P\notin\langle\nu_{2,d}(O_1),\nu_{2,d}(O_2),\nu_{2,d}(O)\rangle$ (in fact we have assumed that $Z=\nu_{2,d}(A)$ computes the X-border rank of P and $\deg(Z)=4$), we have $\langle\nu_{2,d}(A\cap L)\rangle\cap\langle\nu_{2,d}(A_O\rangle\cap\langle(B\cap R)\setminus\{O\}\rangle$. Since $r_{\nu_{2,d}(R)}(U)=d$ for all $U\in\langle\nu_{2,d}(A_O)\rangle\setminus\{O\}$ (see [9]), we get $\sharp((B\cap R)\setminus\{O\})\geq d$. Thus $\sharp(B)\geq d+2$ that is a contradiction. □

Remark 6. Take $m\geq 2$ and $A=A_O\sqcup O_1\sqcup O_2\subset\mathbb{P}^m$ with $A_O\subset\mathbb{P}^m$ connected and $\deg(A_O)=2$ and $O_1,O_2\in\mathbb{P}^m$. Notice that if $m>2$ we are not assuming that A is contained in a plane. As in Proposition 8 if $P\in\langle\nu_{m,d}(A_O\sqcup O_1\sqcup O_2)\rangle\setminus(\sigma_4^0(X_{m,d})\cup\sigma_3(X_{m,d}))$ we have that:

$$r_{X_{m,d}}(P)=d+2.$$

Let $L\subset\mathbb{P}^m$ be the line spanned by A_O. Set $\{O\}:=(A_O)_{red}$. Let T be the tangent line to the degree d rational normal curve $\nu_{m,d}(L)$ at $\nu_{m,d}(O)$. The plane $\langle\{\nu_{m,d}(O_1),\nu_{m,d}(O_2),P\}\rangle$ intersects T at a unique point P_1 and $P_1\neq\nu_{m,d}(O)$. Hence $r_{\nu_{m,d}(L)}(P_1)=d$. Using Sylvester's algorithm (see [6], §3) to find a set $S_1\subset\nu_{m,d}(L)$ computing $r_{\nu_{m,d}(L)}(P_1)$. The set $S_1\cup\{\nu_{m,d}(O_1),\nu_{m,d}(O_2)\}$ computes $r_{X_{m,d}}(P)$.

This concludes our considerations on the Subsection 4.1.1 in which we were assuming the existence of a line $L\subset\mathbb{P}^2$ such that $\deg(A\cap L)\geq 3$.

4.1.2. Here assume that the schematic intersection of A with a line $L\subset\mathbb{P}^2$ has degree less or equal than 2 for every line L

First of all observe that the assumption

$$\boxed{\deg(L\cap A)\leq 2}$$

for all lines $L\subset\mathbb{P}^2$ is equivalent to the spannedness of the sheaf $\mathcal{I}_A(2)$.

Notation 8. Fix a general $E\in|\mathcal{I}_A(2)|$ for $A\subset\mathbb{P}^2$ a non-reduced zero-dimensional scheme of degree 4 such that $\mathcal{I}_A(2)$ is spanned.

Remark 7. If the degree 4 non-reduced zero-dimensional scheme $A\subset\mathbb{P}^2$ is such that $\mathcal{I}_A(2)$ is spanned then A is the complete intersection of the conic $E\subset\mathbb{P}^2$ fixed in Notation 8 with another conic (perhaps a double line). Thus A is a Cartier divisor of E.

Let's do first the case in which the generic conic $E\in|\mathcal{I}_A(2)|$ of Notation 8 is smooth.

Proposition 9. *Let $A\subset\mathbb{P}^2$ be a non-reduced zero dimensional scheme of degree 4 such that $\mathcal{I}_A(2)$ is spanned. Suppose that the general conic $E\in|\mathcal{I}_A(2)|$ is smooth. Let $Z\subset X_{2,d}$ be equal to $\nu_{2,d}(A)$ as in Notation 5. Then the $X_{2,d}$-rank of a point $P\in\langle Z\rangle\setminus(\sigma_4^0(X_{2,d})\cup\sigma_3(X_{2,d}))$ for $d\geq 4$ is*

$$r_{X_{2,d}(P)}=2d-2.$$

PROOF. Notice that $Y := \nu_{2,d}(E)$ is a degree $2d$ rational normal curve in its linear span. Let $B \subset \mathbb{P}^2$ be defined as in Notation 5. Since $A \cup B \subset E$, we have $P \in \langle Y \rangle$. Since $r_Y(P) = 2d - 2$ (see [9] or [17], Theorem 4.1) we have that $r_{X_{2,d}}(P) \le 2d - 2$. Assume $r_{X_{2,d}}(P) \le 2d - 3$. Thus $\deg(A \cup B) \le 2d + 1$. Take L and F_2 as in the statement of Lemma 3. Since $\deg(L \cap E) \le 2$, we have $\sharp(F_2) \ge 2$. Since S is in linearly general position, we have $\sharp(L \cap B) \le d+1$. Thus $r_{X_{2,d}}(P) = \sharp(B) \le d+3$ that is a contradiction. □

Remark 8. Assume $m \ge 2$ and that the scheme $A \subset \mathbb{P}^m$ of Proposition 9 is contained in a smooth conic $E \subset \mathbb{P}^m$. Set $Y := \nu_{m,d}(E)$. In Proposition 9 we proved that $r_Y(P) = 2d - 2$. Since one can use Sylvester's algorithm (see [6], §3) to compute a set of points $S \subset C$ that computes $r_Y(P)$, then one can use the same S in order to compute $r_{X_{m,d}}(P)$, too.

Suppose now that the generic conic $E \in |\mathcal{I}_A(2)|$ of Notation 8 is reduced, but not smooth.

Remark 9. If the generic conic $E \in |\mathcal{I}_A(2)|$ of Notation 8 is reduced, but not smooth then $E = L_1 \cup L_2$ with L_1 and L_2 lines and $L_1 \ne L_2$. Notice that $Y := \nu_{2,d}(E)$ is a degree $2d$ reduced and connected curve (it is a nodal union of 2 smooth degree d rational normal curves meeting in one point). Hence $\dim(\langle Y \rangle) = 2d$. Since we are not in the set-up of § 4.1.1, $\deg(A \cap L_i) \le 2$ for all i. Since $\deg(A) = 4$, we get that:

$$\deg(A \cap L_i) = 2$$

for all i and that A is a Cartier divisor of E. Since A is not reduced, then $1 \le \sharp(A_{red}) \le 3$.

In the next propositions and in the following remarks we will study the cases $\sharp(A_{red}) = 1, 2, 3$.

Proposition 10. *Let $A \subset \mathbb{P}^2$ be a non-reduced zero dimensional scheme of degree 4 such that $\mathcal{I}_A(2)$ is spanned. Moreover suppose that the generic conic $E \in |\mathcal{I}_A(2)|$ of Notation 8 is reduced, but not smooth: $E = L_1 \cup L_2$ with L_1 and L_2 lines and $L_1 \ne L_2$, then $\deg(A \cap L_i) = 2$ for $i = 1, 2$. Assume $\sharp(A_{red}) = 1$. Let $Z \subset X_{2,d}$ be equal to $\nu_{2,d}(A)$ as in Notation 5. Then the $X_{2,d}$-rank of a point $P \in \langle Z \rangle \setminus (\sigma_4^0(X_{2,d}) \cup \sigma_3(X_{2,d}))$ is*

$$r_{X_{2,d(P)}} = 2d - 2.$$

PROOF. Since $\deg(A \cap L_i) > 0$ for all i, we have $A_{red} = L_1 \cap L_2$. Since A is a Cartier divisor of E, we may apply Proposition 1. Thus $r_Y(P) = 2d - 2$ for $Y = \nu_{2,d}(E)$. Hence $r_{X_{2,d}}(P) \le 2d - 2$. Thus it is sufficient to prove $r_{X_{2,d}}(P) \ge 2d - 2$. Assume $r_{X_{2,d}}(P) \le 2d-3$. Hence $\deg(A \cup B) \le 2d+1$ for B as in Notation 5. Thus we may apply Lemma 3. Since A is connected, $F_2 = \emptyset$, contradicting the inequality $\sharp(F_2) \ge m - 1 = 1$. □

We can now prove that the reverse inequality of (8) appeared in Proposition 1 is an equality.

Corollary 2. *Fix an integer $d \geq 3$. Let $Y \subset \mathbb{P}^{2d}$ be a reduced and connected curve union of two smooth degree d curves Y_1, Y_2, each of them a rational normal curve in its linear span, with a unique common point point, Q, and with $\langle Y \rangle = \mathbb{P}^{2d}$. Let $Z \subset Y$ be a length 4 zero-dimensional scheme such that $Z_{red} = \{Q\}$, Z is a Cartier divisor of Y and $\deg(Z \cap Y_i) \geq 2$ for $i = 1, 2$. Fix $P \in \langle Z \rangle$ such that $P \notin \langle Z' \rangle$ for any $Z' \subsetneqq Z$. Then:*

$$r_Y(P) = 2d - 2.$$

Proof. The inequality $r_Y(P) \leq 2d - 2$ is proved in Proposition 1.
In the proof of Proposition 6 we showed that if $\deg(A \cap L_i) = 3$ for one $i \in \{1, 2\}$ (i.e. if $\deg(Z \cap Y_i) \geq 2$ for $i = 1, 2$) then $r_{X_{2,d}}(P) \geq 2d - 2$. Since $Y \subseteq X_{2,d}$ then, as Remark 4 shows, also $r_Y(P)$ as to be bigger or equal than $2d - 2$.
Finally in Proposition 10 we showed that if $\deg(A \cap L_i) = 2$ for $i = 1, 2$ then $r_{X_{2,d}}(P) \geq 2d - 2$. Therefore. with the same argument above, we get that $r_Y(P) = 2d - 2$. □

Proposition 11. *Let $A \subset \mathbb{P}^2$ be a non-reduced zero dimensional scheme of degree 4 such that $\mathcal{I}_A(2)$ is spanned. Moreover suppose that the conic $E \in |\mathcal{I}_A(2)|$ of Notation 8 is reduced, but not smooth: $E = L_1 \cup L_2$ with L_1 and L_2 lines and $L_1 \neq L_2$. Assume $\sharp(A_{red}) = 2$ and that A has two unreduced connected components, say A_1 and A_2 with $A_1 \cap L_1 \neq \emptyset$. Let $Z \subset X_{2,d}$ be equal to $\nu_{2,d}(A)$ as in Notation 5. Then the $X_{2,d}$-rank of a point $P \in \langle Z \rangle \setminus (\sigma_4^0(X_{2,d}) \cup \sigma_3(X_{2,d}))$ is*

$$r_{X_{2,d}(P)} = 2d - 2.$$

Proof. Since $\deg(A \cap L_i) > 0$ for all i and $\deg(L_i \cap A) < 3$ for all i, we have $A_i \subset L_i \setminus (L_1 \cap L_2)$ for all i. Proposition 2 gives $r_Y(P) \leq 2d - 2$ with $Y = \nu_{2,d}(E)$. Hence $r_{X_{2,d}}(P) \leq 2d - 2$. Assume $r_{X_{2,d}}(P) \leq 2d - 3$. Thus $\deg(A \cup B) \leq 2d + 1$. We may apply Lemma 3. Since A has no reduced component, we get $F_2 = \emptyset$, contradicting the inequality $\sharp(F_2) \geq m - 1 = 1$. □

In the following corollary we show that one can substitute the inequality (9) of the Proposition 2 with an equality.

Corollary 3. *Fix an integer $d \geq 3$. Let $Y \subset \mathbb{P}^{2d}$ be a reduced and connected curve union of two smooth degree d curves Y_1, Y_2, each of them a rational normal curve in its linear span, with a unique common point point, Q, and with $\langle Y \rangle = \mathbb{P}^{2d}$. Fix $P_i \in Y_i \setminus \{Q\}$ for $i = 1, 2$. Let $Z_i \subset Y_i$, $i = 1, 2$, be the degree 2 effective divisor of Y_i supported by P_i. Set $Z := Z_1 \cup Z_2$. Fix $P \in \langle Z \rangle$ such that $P \notin \langle Z' \rangle$ for any $Z' \subset Y$ with $\deg(Z') < \deg(Z)$. Then*

$$r_Y(P) = 2d - 2.$$

Proof. In Proposition 2 it is proved that $r_Y(P) \leq 2d - 2$.
In the above Proposition 11 we proved that $r_{X_{2,d}}(P) = 2d - 2$, then, by Remark 9, we have that also $r_Y(P) = 2d - 2$. □

Remark 10. Here we cover the cases described in Proposition 10 and in Proposition 11. Fix an integer $m \geq 2$ and two lines $L_1, L_2 \subset \mathbb{P}^m$ such that $L_1 \neq L_2$ and $L_1 \cap L_2 \neq \emptyset$. Set $\{O\} := L_1 \cup L_2$. Assume that no connected component of A is reduced, $A \subset L_1 \cup L_2$ and $\deg(L_1 \cap A) = \deg(L_2 \cap A) = 2$. Set $A_i := A \cap L_i$ and $Z_i := Z \cap L_i$. By [18], Proposition

3.1, or [17], Subsection 3.2, and Proposition 10 or Proposition 11 we have $r_{X_{2,d}}(P) = 2d-2$. Set $Y_i := \nu_{m,d}(L_i)$. Since $d \geq 3$, we have $\langle \nu_{m,d}(L_1)\rangle \cap \langle \nu_{m,d}(L_2)\rangle = \{\nu_{m,d}(O)\}$. Set $D_1 := \langle \{P\} \cup Y_2\rangle \cap \langle Y_1\rangle$ and $D_2 := \langle \{P\} \cup Y_1\rangle \cap \langle Y_2\rangle$. The schemes D_1 and D_2 are lines and $D_1 \cap D_2 = \{\nu_{m,d}(O)\}$. Fix $P_i \in D_i \setminus \{\nu_{m,d}(O)\}$. Take any $S_i \subset Y_i$ computing $r_{Y_i}(P_i)$ and set $S := S_1 \cup S_2$.

Remark 11. Let $A \subset \mathbb{P}^2$ be a non-reduced zero-dimensional scheme of degree 4 such that $\mathcal{I}_A(2)$ is spanned. Moreover suppose that the conic $E \in |\mathcal{I}_A(2)|$ of Notation 8 is reduced, but not smooth: $E = L_1 \cup L_2$ with L_1 and L_2 lines and $L_1 \neq L_2$. Assume $\sharp(A_{red}) = 2$ and that A is the disjoint union of a connected component, A_O, with degree 3 and a point O', say $O' \in L_1$. Since no line intersects A in a scheme of degree ≥ 3, A_O is neither the first infinitesimal neighborhood of $L_1 \cap L_2$ in Y_2 (the line $\langle O', L_1 \cap L_2\rangle$ would give a contradiction) nor a degree 3 effective divisor either of L_1 or of L_2. Hence $O' \neq L_1 \cap L_2$ and $(A_O)_{red} = L_1 \cap L_2$. Let E' another reducible conic such that $A = E \cap E'$. Let R be the line of E' containing O'. Since $E \cap R \subseteq A$, then R is the line of $L_1 \cup L_2$ containing O'. Call it L_1. Since $h^0(Y_2, \mathcal{I}_A(2)) = 2$, we have infinitely many such conics. Thus L_1 is in the base locus of $|\mathcal{I}_A(2)|$. Hence $\deg(L_1 \cap A) \geq 3$ that is in contradiction with our hypothesis.

Proposition 12. *Let $A \subset \mathbb{P}^2$ be a non-reduced zero-dimensional scheme of degree 4 such that $\mathcal{I}_A(2)$ is spanned. Moreover suppose that the conic $E \in |\mathcal{I}_A(2)|$ of Notation 8 is reduced, but not smooth: $E = L_1 \cup L_2$ with L_1 and L_2 lines and $L_1 \neq L_2$. Assume $\sharp(A_{red}) = 3$. Let A_O be the unreduced connected component of A and O_1, O_2 the reduced ones. Let $Z \subset X_{2,d}$ be equal to $\nu_{2,d}(A)$ as in Notation 5. Then the $X_{2,d}$-rank of a point $P \in \langle Z\rangle \setminus (\sigma_4^0(X_{2,d}) \cup \sigma_3(X_{2,d}))$ is*

$$r_{X_{2,d}(P)} = d+2.$$

Proof. Since $\nu_{2,d}(A_O)$ is a tangent vector of $X_{2,d}$, $r_{X_{2,d}}(P') = d$ for all $P' \in \langle \nu_{2,d}(A_O)\rangle \setminus (A_O)_{red}$ ([6], Theorem 4.3). Thus $r_{X_{2,d}}(P) \leq d+2$. Using Lemma 3 we easily get $r_{X_{2,d}}(P) \geq d+2$ (see Proposition 8 for a similar case). Thus $r_{X_{2,d}}(P) = d+2$. □

Remark 12. Let $A \subset \mathbb{P}^2$ be a non-reduced zero-dimensional scheme of degree 4 such that $\mathcal{I}_A(2)$ is spanned. Moreover suppose that the conic $E \in |\mathcal{I}_A(2)|$ of Notation 8 is reduced, but not smooth: $E = L_1 \cup L_2$ with L_1 and L_2 lines and $L_1 \neq L_2$. Aassume that E is not reduced. Hence E is a double line. Bertini's theorem and the generality of E gives that E is smooth outside the base locus of $\mathcal{I}_U(2)$. Thus $L := E_{red}$ is in the base locus of $\mathcal{I}_A(2)$. Hence $\deg(A \cap L) \geq 3$. Thus we are in the set-up of § 4.1.1.

4.2. Three dimensional case

Here we assume that $m = 3$ and that the degree 4 non-reduced zero-dimensional scheme $A \subset \mathbb{P}^3$ introduced in Notation 3 and such that $\nu_{3,d}(A) = Z$ computes the $X_{3,d}$-border rank of a point $P \in \sigma_4(X_{3,d}) \setminus (\sigma_4^0(X_{3,d}) \cup \sigma_3(X_{3,d}))$ is not contained in any plane of $\mathbb{P}^3$, that is to say:

$$\boxed{\dim(\langle A\rangle) = 3}$$

Remark 13. If $A \subset \mathbb{P}^3$ is the first infinitesimal neighborhood $2Q$ of some point $Q \in \mathbb{P}^3$ then, if $Z \subset X_{3,d}$ is as in Notation 2, the linear span $\langle Z \rangle$ is actually the tangent space $T_{\nu_{3,d}(Q)}X_{3,d}$ of $X_{3,d}$ at $\nu_{3,d}(Q)$. Therefore, by [6], Theorem 4.3, we have $r_{X_{3,d}}(P) = d$, but also that $P \in \sigma_2(X_{3,d})$.

Proposition 13. *Let $U_1, U_2 \subset \mathbb{P}^3$ be two disjoint non-reduced zero-dimensional schemes of degree 2 such that $A = U_1 \sqcup U_2$ spans $\mathbb{P}^3$. Let $Z_i = \nu_{3,d}(U_i) \subset X_{3,d}$ for $i = 1, 2$ and $Z = \nu_{3,d}(A)$ as in Notation 2. Then, if $d \geq 4$,*

$$r_{X_{3,d}}(P) = 2d$$

for every $P \in \langle Z \rangle \setminus (\sigma_4^0(X_{3,d}) \cup \sigma_3(X_{3,d}))$.

Proof. Since no proper linear subspace of $\mathbb{P}^3$ contains A, the lines $\langle U_1 \rangle$ and $\langle U_2 \rangle$ of $\mathbb{P}^3$ are disjoint. Set $E := Z \cup \{P\}$. Since $P \notin \langle Z' \rangle$ for any $Z' \subsetneq Z$ and Z is linearly independent (in fact Z as in Notation 2 computes the $X_{3,d}$-border rank of P), the degree 5 scheme E is in linearly general position in its 3-dimensional linear span $\langle Z \rangle$. Any two such schemes E are projectively equivalent (this can be seen by using [14], Corollary 2, and the fact that $\sharp(E_{red}) = 3$).
Proposition 3 gives $r_{X_{3,d}}(P) \leq 2d$. Here we will prove the reverse inequality and hence that $r_{X_{3,d}}(P) = 2d$ for $d \geq 4$.
Assume $r_{X_{3,d}}(P) \leq 2d-1$ and fix $S \subset X_{3,d}$ computing $r_{X_{3,d}}(P)$. We will show that this is actually a contradiction.
Now if $r_{X_{3,d}}(P) \leq 2d-1$ and if B is defined as in Notation 5, we have that $\deg(A \cup B) \leq 2d+3$. Lemma 1 gives $h^1(\mathbb{P}^3, \mathcal{I}_{A\cup B}(d)) > 0$. Since $\deg(A \cup B) \leq 3d+1$ and $h^1(\mathbb{P}^3, \mathcal{I}_{A\cup B}(d)) > 0$, $A \cup B$ is not in linearly general position ([14], Theorem 3.2). Thus there is a plane $M \subset \mathbb{P}^3$ such that $\deg(M \cap (A \cup B)) \geq 4$. Among all such planes we take one, say M, such that $\deg(M \cap (A \cup B))$ is maximal. The residual scheme $\mathrm{Res}_M(A \cup B)$ of $A \cup B$ with respect to the effective Cartier divisor M of $\mathbb{P}^3$ is the closed subscheme of $\mathbb{P}^3$ with $(\mathcal{I}_{A\cup B} : \mathcal{I}_M)$ as its ideal sheaf. We have $\mathrm{Res}_M(A \cup B) \subseteq A \cup B$ and $\deg(A \cup B) = \deg((A \cup B) \cap M) + \deg(\mathrm{Res}_M(A \cup B))$.
Set $W := A \cup B$, $M_0 := M$ and $W_1 := \mathrm{Res}_M(W)$.
Define inductively the planes $M_i \subset \mathbb{P}^3$, $i \geq 1$, and the schemes W_{i+1}, $i \geq 1$, by the condition that M_i is one of the planes such that $\deg(M_i \cap W_i)$ is maximal and then set $W_{i+1} := \mathrm{Res}_{M_i}(W_i)$.
We have $W_{i+1} \subseteq W_i$ for all $i \geq 1$ and $W_i = \emptyset$ for all $i \gg 0$. For all integers t and $i \geq 1$ there is the residual exact sequence

$$0 \to \mathcal{I}_{W_{i+1}}(t-1) \to \mathcal{I}_{W_i}(t) \to \mathcal{I}_{W_i \cap M_i, M_i}(t) \to 0. \tag{14}$$

In what follows we distinguish the case $h^1(M, \mathcal{I}_{W_1 \cap M, M}(d)) = 0$ (see item (1) in the proof) from the case $h^1(M, \mathcal{I}_{W_1 \cap M, M}(d)) > 0$ (see item (2) and item (3) in the proof) and we will show that in all possible cases we will get that the assumption $r_{X_{3,d}}(P) \leq 2d-1$ leads to a contradiction, then we will be allowed to conclude that $r_{X_{3,d}}(P) = 2d-1$.

(1) First assume:

$$h^1(M, \mathcal{I}_{W_1 \cap M, M}(d)) = 0.$$

Since by Lemma 1 we have $h^1(\mathcal{I}_{W_1}(d)) > 0$, then (14) gives $h^1(\mathbb{P}^3, \mathcal{I}_{W_2}(d-1)) > 0$. Since $\deg(W_1 \cap M_1) \geq 4$, we have $\deg(W_2) \leq 2(d-1)+1$. Hence [6], Lemma 4.6,

gives the existence of a line $L \subset \mathbb{P}^3$ such that $\deg(L \cap W_2) \geq (d-1)+2$.
Since $W_2 \subset W_1$, we get $\deg(L \cap W_1) \geq d+1$. Since W_1 is non-degenerate, $\deg(M \cap W_1) \geq 1 + \deg(L \cap W_1) \geq d+2$. Thus $\deg(W_1) \geq d+2+\deg(W_2) \geq 2d+3$. Thus we must have equalities everywhere. Hence $\deg(W_2) = d+1$ and $\deg(W_1) = 2d+3$. Since $\deg(W_2) = \deg(W_2 \cap L)$, we also get $W_2 \subset L$. Since $L \subset M_1$ no reduced connected component of W_1 supported by a point of L survives to a component of W_2. Since B is reduced and $\deg(A) = 4$, we get $d+1 \leq 2$ that is a contradiction.

(2) Now, by the previous item (1) we are allowed to assume here (and in the next item (3)) that
$$h^1(M, \mathcal{I}_{W_1 \cap M, M}(d)) > 0.$$
Actually here in this item (2) we start assuming also that
$$\deg(W_1 \cap M) \geq 2d+2.$$
Since W_1 spans $\mathbb{P}^3$ and $\deg(W_1) \leq 2d+3$, we get $\deg(W_1 \cap M) = 2d+2$ and that W_2 is a reduced point, say Q. Since $P \in \langle Z \rangle \cap \langle S \rangle$, to compute $r_{X_{3,d}}(P)$ we cannot use a smaller number of variables. Thus $Q \in A_{red} \cap B_{red}$. Thus $\deg(A \cup B) \leq \deg(A) + \deg(B \setminus \{Q\}) \leq 2d+2$. We also get $\deg(M \cap (A \cup B)) \leq 2d+1$, even if $Q \in M$. Hence we may apply Remark 3 to M. Take the line $L \subset M$ such that $\deg((A \cup B) \cap L) \geq d+2$. Since S is linearly independent, we have $\sharp(B \cap L) \leq d+1$. Since $\sharp(A_{red}) \leq 2$, Remark 3 gives $\sharp(B \cap M) \leq d+3$. Hence $r_{X_{3,d}}(P) := \sharp(B) \leq d+4$ that is a contradiction.

(3) Finally we still assume (as in item (2)) that
$$h^1(M, \mathcal{I}_{W_1 \cap M, M}(d)) > 0$$
but also that
$$\deg(W_1 \cap M) \leq 2d+1.$$
Since $h^1(M, \mathcal{I}_{W_1 \cap M, M}(d)) > 0$, Remark 3 gives the existence of a line L such that $\sharp((A \cup B) \cap L) \geq d+2$ and a non-empty set F_2 such that $A \cap M = (A \cap L) \sqcup F_2$ and $B \cap M = (B \cap L) \sqcup F_2$. Since $\deg(A \cup B) \leq 2d+3-\sharp(F_2)$ and no connected component of A is reduced, Lemma 3 gives $\sharp(F_2) = 1$.
First assume $h^1(\mathbb{P}^3, \mathcal{I}_{\mathrm{Res}_M(A \cup B)}(d-1)) = 0$.
Lemma 6 gives the existence of a set F_3 such that $F_3 \subset A_{red}$ and $B = (B \cap M) \sqcup F_3$. Since B spans $\mathbb{P}^3$, $F_3 \neq \emptyset$. Thus $\deg(A \cup B) \leq 2d+3-\sharp(F_2)-\sharp(F_3) \leq 2d+1$. Since no connected component of A is reduced, Lemma 3 gives a contradiction.
Now assume $h^1(\mathbb{P}^3, \mathcal{I}_{\mathrm{Res}_M(A \cup B)}(d-1)) > 0$.
We get $\deg(\mathrm{Res}_M(A \cup B) \geq d+1$. Since $\deg(A \cup B) \leq 2d+2$ and $\deg((A \cup B) \cap M) \geq d+2$, we obtained a contradiction.

All the cases above lead to a contradiction, then $r_{X_{3,d}}(P) = 2d-1$ and this ends the proof. □

Remark 14. Assume, for $m > 2$, that the zero-dimensional scheme $A \subset \mathbb{P}^m$ of Notation 3 has two connected components, $A_1, A_2 \subset \mathbb{P}^m$, both of degree 2 and that the lines $L_i := \langle A_i \rangle$, $i = 1, 2$, are disjoint. Thus $\dim(\langle L_1 \cup L_2 \rangle) = 3$. Set $Y_i := \nu_{m,d}(L_i)$, $i = 1, 2$, and $Y := Y_1 \cup Y_2$. Notice that $Y_1 \cap Y_2 = \emptyset$. Now let $Z \subset X_{m,d}$ be defined as in Notation 2 as a scheme that computes the $X_{m,d}$-border rank of a point $P \in \langle Z\rangle \setminus (\sigma_4^0(X_{m,d}) \cup \sigma_3(X_{m,d}))$. By [18], Proposition 3.1, or [17], Subsection 3.2, $r(X_{m,d})(P) = r_{Y_1\cup Y_2}(P)$. We proved in Proposition 3 that $r_{X_{m,d}}(P) = 2d$ and that it may be computed by a set $S \subset Y$ such that $\sharp(S \cap Y_i) = d$, $i = 1, 2$. The set S may be found in the following way (here we just translate the proof of Proposition 3):

Step 1. Set $P_2 := \langle \{P\} \cup Y_1\rangle \cap \langle Y_2\rangle$ and $P_1 := \langle \{P\} \cup Y_2\rangle \cap \langle Y_1\rangle$.
Step 2. Find $S_i \subset Y_i$ computing the Y_i-rank of P_i (e.g. use Sylvester's algorithm [21], [9], [6], [11] and [7]).
Step 3. Set $S := S_1 \cup S_2$.

Proposition 14. *Let $A \subset \mathbb{P}^3$ be a degree 4 curvilinear zero-dimensional scheme with support on only one point and such that $\langle A\rangle = \mathbb{P}^3$. Let $Z \subset X_{3,d}$ be defined as in Notation 2 as $\nu_{3,d}(A)$. If $d \geq 4$ then the $X_{3,d}$-rank of a point $P \in \langle Z\rangle \setminus (\sigma_4^0(X_{m,d}) \cup \sigma_3(X_{3,d}))$ is*

$$r_{X_{3,d}}(P) = 3d - 2.$$

Proof. Since A spans $\mathbb{P}^3$, it is projectively equivalent to a connected degree 4 divisor of a smooth rational curve of $\mathbb{P}^3$. Thus $r_{X_{3,d}}(P) \leq 3d-2$. We want to prove $r_{X_{3,d}}(P) = 3d-2$. Assume $r_{X_{3,d}}(P) \leq 3d - 3$ and get a contradiction.
Take $S \subset X_{3,d}$ computing $r_{X_{3,d}}(P)$. We have $\deg(A \cup B) = 4 + r_{X_{3,d}}(P) - \deg(S \cap Z) \leq 3d + 1$ where $B \subset \mathbb{P}^3$ is as in Notation 5 the pre-image of S via $\nu_{3,d}$. Lemma 1 gives $h^1(\mathbb{P}^m, \mathcal{I}_{A\cup B}(d)) > 0$. Hence $A \cup B$ is not in linearly general position (see [14], Theorem 3.2). Thus there is a plane $M \subset \mathbb{P}^3$ such that $\deg(M \cap (A \cup B)) \geq 4$. Among all such planes we take one, say M_1, such that $x_1 := \deg(M_1 \cap (A \cup B))$ is maximal. Set $E_1 := A \cup B$ and $E_2 := \mathrm{Res}_{M_1}(E_1)$. Notice that $\deg(E_2) = \deg(E_1) - x_1$. Define inductively the planes $M_i \subset \mathbb{P}^3$, $i \geq 2$, the schemes E_{i+1}, $i \geq 2$, and the integers x_i, $i \geq 2$, by the condition that M_i is one of the planes such that $x_i := \deg(M_i \cap E_i)$ is maximal and then set $E_{i+1} := \mathrm{Res}_{M_i}(E_i)$. We have $E_{i+1} \subseteq E_i$ (with strict inclusion if $E_i \neq \emptyset$) for all $i \geq 1$ and $E_i = \emptyset$ for all $i \gg 0$. For all integers t and $i \geq 1$ there is the residual exact sequence

$$0 \to \mathcal{I}_{E_{i+1}}(t-1) \to \mathcal{I}_{E_i}(t) \to \mathcal{I}_{E_i\cap M_i,M_i}(t) \to 0 \tag{15}$$

Let z be the minimal integer i such that $1 \leq i \leq d+1$ and $h^1(M_i, \mathcal{I}_{M_i\cap E_i}(d+1-i)) > 0$. Use at most $d+1$ times the exact sequences (15) to prove the existence of such an integer z. We now study the different possibilities that we have for the integer z just defined.

$z = 1$ Since A_{red} is a single point, Lemma 7 gives $h^1(\mathbb{P}^3, \mathcal{I}_{E_2}(d-1)) > 0$. Hence $x_2 \geq d + 1$. Since by hypothesis $d \geq 4$, $x_2 \leq x_1$ and $x_1 + x_2 \leq 3d + 1$, we have $x_2 \leq 2d - 1$. Hence Lemma 4 applied for the integer $d - 1$ gives the existence of a line $R \subset \mathbb{P}^3$ such that $\deg(E_2 \cap R) \geq d + 1$. Since A_{red} is a single point, we also get that either $A_{red} \in B \setminus (R \cap B)$ and $B \cap E_2 = B \cap R \sqcup \{A_{red}\}$ or $E_2 \subset R$. In the latter case we have $x_2 = d + 1$ and $x_3 = 0$. Since $h^1(\mathbb{P}^3, \mathcal{I}_{E_2}(d-1)) > 0$, we get $h^1(\mathbb{P}^3, \mathcal{I}_{A\cup B}(d)) \geq 2$. Since $x_2 \geq d + 1$, we have $x_1 \leq 2d$. Since $h^1(M_1, \mathcal{I}_{(A\cup B)\cap M_1}(d)) = 0$, $h^1(M_1, \mathcal{I}_{B\cap M_1}(d)) = 0$, and A is connected, Lemma 4 gives the existence of a line L such that $B \cap M_1 \subset L$ and $(\deg((A\cup B)\cap L)) \geq d+2$. Since $\deg(A\cap M_1) \leq 3$, we have $x_1 \leq d+4$. Since no point of $B \cap M \setminus A_{red}$ survives

in E_2, we have $L \neq R$.
First assume $L \cap R \neq \emptyset$. We have $\deg((A \cup B) \cap (\langle L \cup R \rangle) \geq 2d+1$, contradicting the inequality $x_1 \leq 2d$.
Now assume $L \cap R = \emptyset$. Since $h^1(\mathcal{I}_B(d)) = 0$, we have $\sharp(B \cap L) \leq d+1$. Hence $A_{red} \in L$. Since we assumed $L \cap R = \emptyset$, we have $A_{red} \neq R$, $B \subset L \cup R$ and either $B \cap E_2 = B \cap R \sqcup \{A_{red}\}$ or $A_{red} \notin B$ and $B \cap E_2 = B \cap R$. Since $A_2 \nsubseteq E_2$, we have $M_1 = \langle A_3 \rangle$. Let H be a plane spanned by R and a point, O, of $B \cap (L \setminus A_{red})$. We have $\deg((A \cup B) \cap H) = \deg(B \cap H) = d+2$ and $(A \cup B) \cap H = B \cap H = \{O\} \cup (B \cap R)$. Since $h^1(L \cup R, \mathcal{I}_{B \cap R}(d)) = 0$, Horace lemma (see [**?**]) applied in H with respect to L gives $h^1(H, \mathcal{I}_{A \cup B}(d)) = 0$. Since $\mathrm{Res}_H(A \cup B) = A \cup (B \cap L \setminus \{O\})$, we have $h^1(M_1, \mathcal{I}_{M_1 \cap \mathrm{Res}_H(A \cup B)}(d-1)) = 0$. Since $\mathrm{Res}_{M_1}(\mathrm{Res}_H(A \cup B)) = A_3$ and $h^1(\mathbb{P}^3, \mathcal{I}_{A_3}(d-2)) = 0$, Horace lemma with respect to M_1 gives $h^1(\mathbb{P}^3, \mathcal{I}_{\mathrm{Res}_H(A \cup B)}(d-1)) = 1$. Hence $h^1(\mathbb{P}^3, \mathcal{I}_{A \cup B}(d)) = 1$, contradiction.

$z > 1$ Since $h^1(M_z, \mathcal{I}_{M_z \cap E_z}(d+1-z)) > 0$, we have $x_z := \deg(M_z \cap E_z) \geq d+3-z$. Since the function $z \mapsto x_z$ is non-decreasing, we get $x_i \geq d+3-z$ for all $i \in \{1, \ldots, z+1\}$. Since $\deg(A \cup B) \geq z(d+3-z)$, we get $3d+1 \geq z(d+3-z)$. Hence either $z \in \{2,3\}$ or $z \geq d$ (here we are assuming $d \geq 4$).

$z = d$ The condition $h^1(\mathcal{I}_{M_d \cap E_d}(1)) > 0$ says that either $M_d \cap E_d$ contains a scheme of length ≥ 3 contained in a line R or $x_d \geq 4$. If $x_d \geq 4$, then we get $x_1 + \cdots + x_d \geq 4d$, that is a contradiction. Hence we may assume $x_1 = 4$, $x_i = 3$ for $2 \leq i \leq d$ and $x_{d+1} = 0$. Since $x_2 = 3$, the maximality of the integer x_2 gives that E_2 is in linearly general position. Since $\deg(E_2) = \deg(E_1) - x_4 \leq 3(d-1)+1$ and E_2 is in linearly general position, then $h^1(\mathcal{I}_{E_2}(d-1)) = 0$. Since $z > 1$. $h^1(M_1, \mathcal{I}_{E_1 \cap M_1}(d)) = 0$. Hence (15) with $i = 1$ and $t = d$ gives a contradiction.

$z = d+1$ The condition $h^1(M_i, \mathcal{I}_{M_z \cap E_z}) > 0$ only says $x_{d+1} \geq 2$. Taking the first integer $y \leq d$ such that $x_y \leq 3$ and E_y is not collinear, we get a contradiction as above.

$z = 2$ Since $3d+1 \geq x_1 + x_2 \geq 2x_2$, we get $x_2 \leq 2(d-1)+1$. Since A_{red} is a single point, the contradiction comes applying Lemma 4, unless $m = 1$, i.e. unless $E_2 \cap M_2$ is contained in a line R. Since B is reduced and $\sharp(B \cap R) \geq d-2 > 0$ then R is not contained in M_1. Since $E_2 \subset R$, while $\mathbb{P}^3$ is the Zariski tangent space to A at P, then M_1 must contain Z_1. Since $R \nsubseteq M_1$, we also get $M_1 = \langle Z_2 \rangle$. Hence we get that either $\sharp(B \cap R) = d$ (case $A_{red} \notin B$) or $\sharp(R \cap B) = d+1$ (case $A_{red} \in B$). Let H be any plane containing R and such that $\deg(H \cap (A \cup B)) \geq d+2$; for instance we may take the plane spanned by R and a point of $B \setminus R \cap B$ or the plane spanned by R and the line $\langle A_2 \rangle$. First assume $h^1(H, \mathcal{I}_{(A \cup B) \cap H}(d)) > 0$. Since $\deg((A \cup B) \cap H) \leq x_1 \leq 3d+1-x_2 \leq 2d$, we may apply Lemma 4. We get a line $L \subset H$ such that $\deg((A \cup B) \cap L) \geq d+2$. First assume $L \neq R$. Hence at least one of the lines R, L is not the line $\langle A_2 \rangle$. Hence $\sharp(B \cap (R \cup L)) \geq 2d$. Since $A_{red} \in L \cap R$, $\langle R \cup L \rangle$ is a plane. Hence $\deg((A \cup B) \cap \langle R \cup L \rangle \leq x_1 \leq 2d$. Since $A_{red} \in L$, we get a contradiction (notice that $\deg(A \cup B) \leq 3d$ and hence

$x_1 \le 2d-1$ if $A_{red} \in B$). Now assume $L = R$. Since $h^1(\mathbb{P}^3, \mathcal{I}_B(d)) = 0$ (the linear independence of S) we get $A_{red} \in L$. Since A_{red} is connected, Lemma 4 gives $F_2 = \emptyset$, i.e. $H \cap B = L \cap R$. Since we could choose H containing a point of $B \setminus R \cap B$, we got a contradiction.

$z = 3$ Since $x_1 \ge x_2 \ge x_3 \ge d$, we get $x_2 = x_3 = d$, $x_1 \le d+1$ and the existence of a line $R \subset M_3$ such that $E_3 \cap M_3 \subset R$. Since M_3 is a plane for which $\deg(E_3 \cap M_3)$ is maximal, while there is a pencil of planes containing R, we have $E_3 \subset M_3$ and $E_4 = \emptyset$. Now instead of M_1 we take a plane M_1' containing R and at least another point of B. As above we get schemes E_i', planes M_i' and a non-decreasing sequence of integers x_i', $i \ge 1$, such that $x_i' \ge d+1$. Since $x_1' \le x_1$, we get $x_1' = x_1 = d+1$. Hence the definitions of M_2' and M_2 gives that we may assume $x_2' = d$. Then we may assume $z = 3$ for this new sequence of data E_i', M_i'. We get the existence of a line R' such that $E_3' \subset R$ and $\deg(E_3') = d$. Since B is reduced and $R \subset M_1'$, we have $R' \ne R$. Lemma 4 also gives $A \cap R \ne \emptyset$, i.e. $A_{red} \in R$. Lemma 4 also gives $A \cap R' \ne \emptyset$, i.e. $A_{red} \in R'$. Thus $\deg((A \cup B) \cap T) \ge 2d-2$, where T is the plane $\langle R \cup R' \rangle$. Thus $x_1 \ge 2d-2$, contradiction.

Therefore we may conclude that $r_{X_{3,d}}(P) = 3d-2$. □

Remark 15. Fix $P \in \sigma_4(X_{m,d}) \setminus \sigma_3(X_{m,d})$, $m \ge 3$ and $d \ge 4$, for which $A \subset \mathbb{P}^m$ and hence $Z \subset X_{m,d}$ are as in Proposition 14. Here we want to describe and produce algorithmically several sets of points $S \subset X_{m,d}$ computing $r_{X_{m,d}}(P)$.
Fix any 3-dimensional linear subspace M of $\mathbb{P}^m$ containing A and any smooth rational normal curve T of M such that $A \subset T$. Set $Y := \nu_{m,d}(T)$. Thus Y is a degree $3d$ rational normal curve in its linear span. Since $Z \subset Y$, we have $P \in \langle Y \rangle$. Since $\deg(Z) = 4$ and Z is contained in a rational normal curve, we have $r_Y(P) = 3d-2$ (see [9] or [17], Theorem 4.1). Hence $r_Y(P) = r_{X_{m,d}}(P)$. Hence any $S \subset Y$ computing $r_Y(P)$ computes $r_{X_{m,d}}(P)$. Sylvester's algorithm produces one such set S (see [21], [9], [6], [11], [7]).

Remark 16. Observe that it cannot happen that the degree 4 zero-dimensional scheme $A \subset \mathbb{P}^3$ defined in Notation 3 and such that $\langle A \rangle = \mathbb{P}^3$ is simultaneously supported on one point and it has a 2-dimensional Zariski tangent space. Indeed, since A has a 2-dimensional Zariski tangent space at A_{red}, then $Z \cong A$ is not curvilinear. Since $\dim \langle \nu_{3,d}(A) \rangle = 3$, Z is linearly independent. Since Z is not curvilinear and $\deg(Z) = \dim(\langle Z \rangle) + 1$, Z is not as in case III of [14] (Vol 1.1, Theorem 1.3), it must be the first infinitesimal neighborhood of Z_{red} in its linear span (case II of [14], Vol 1.1 Theorem 1.3). Hence Z has 3-dimensional Zariski tangent space at Z_{red}, contradicting our assumption.

Proposition 15. *Let $A_1 \subset \mathbb{P}^3$ be a degree 3 non-reduced zero-dimensional scheme contained in a smooth conic. Let $A = A_1 \sqcup \{O\}$ with $O \in \mathbb{P}^3$ a simple point such that $\langle A \rangle = \mathbb{P}^3$. Then if $Z \subset X_{3,d}$ is as in Notation 2 and if $P \in \langle Z \rangle \setminus (\sigma_4^0(X_{3,d}) \cup \sigma_3(X_{3,d}))$, then*

$$r_{X_{3,d}}(P) = 2d.$$

Proof. Since $P \in \langle \nu_{3,d}(A_1) \cup \nu_{3,d}(O) \rangle$ we have that P is a linear combination of a point $P_1 \in \langle \nu_{3,d}(A_1) \rangle$ and O itself. Since A_1 is contained in a smooth conic $C \subset \mathbb{P}^3$, then $\nu_{3,d}(A_1)$ is a degree 3 non-reduced zero-dimensional scheme contained in a degree

$2d$ rational normal curve, then $r_{\nu_{3,d}(C)}(P') = 2d-1$ then $r_{X_{3,d}}(P) \le 2d$. Suppose that $r_{X_{3,d}}(P) \le 2d-1$ and take $S \subset X_{3,d}$ computing $r_{X_{3,d}}(P)$. Let $B \subset \mathbb{P}^3$ be as in Notation 5 such that $\nu_{3,d}(B) = S$. We have $\deg(A \cup B) \le 4 + 2d - 1 = 2d + 3$. Lemma 1 gives $h^1(\mathbb{P}^3, \mathcal{I}_{A\cup B}(d)) > 0$. Hence $A \cup B$ is not in linearly general position ([14], Theorem 3.2). Thus there is a plane $M \subset \mathbb{P}^3$ such that $\deg(M \cap (A \cup B)) \ge 4$. Among all such planes we take one, say M_1, such that $x_1 := \deg(M_1 \cap (A \cup B))$ is maximal. Set $E_1 := A \cup B$ and $E_2 := \mathrm{Res}_{M_1}(E_1)$. Notice that $\deg(E_2) = \deg(E_1) - x_1$. Define inductively the planes $M_i \subset \mathbb{P}^3$, $i \ge 2$, the schemes E_{i+1}, $i \ge 2$, and the integers x_i, $i \ge 2$, by the condition that M_i is one of the planes such that $x_i := \deg(M_i \cap E_i)$ is maximal and then set $E_{i+1} := \mathrm{Res}_{M_i}(E_i)$. We have $E_{i+1} \subseteq E_i$ (with strict inclusion if $E_i \ne \emptyset$) for all $i \ge 1$ and $E_i = \emptyset$ for all $i \gg 0$. We have again the exact sequence (15). Use at most $d+1$ times the exact sequences (15) to prove the existence of such an integer z. We study now the different possibilities we have for the integer z.

$z = 1$ Since A_{red} is a single point, Lemma 7 gives $h^1(\mathbb{P}^3, \mathcal{I}_{E_2}(d-1)) > 0$. Hence $x_2 \ge d+1$. Since $x_2 \le x_1$ and $x_1 + x_2 \le 2d+3$, we have $x_2 \le d+1$. Hence Lemma 4 applied for the integer $d-1$ gives the existence of a line $R \subset \mathbb{P}^3$ such that $E_2 \subset R$. We have $\deg(R \cap A) \le 2$. Thus $\sharp(B \cap R) \ge d-1$. Now one has either that $x_1 = d+2$ and $x_2 = d+1$ or $x_1 = x_2 = d+1$ and $x_3 = 0$. Since $h^1(M_1, \mathcal{I}_{(A\cup B)\cap M_1}(d)) > 0$ by assumption, there is a line $L \subset M_1$ such that $\deg((A \cup B) \cap R) \ge d+2$. Hence $x_1 = d+2$ and $(A\cup B)\cap M_1 \subset L$. Now we have that $L \ne R$ because B is a reduced scheme and there is no point in $B \cap M$ that still belongs to E_2. In order to get a contradiction it is sufficient to observe that it exists a plane N that contains L with $\deg((A\cup B)\cap N) > \deg((A\cup B)\cap L) = x_1$, and this contradicts the definition of x_1.

$z \ge 2$ Since $h^1(M_z, \mathcal{I}_{M_z \cap E_z}(d+1-z)) > 0$, we have $x_z := \deg(M_z \cap E_z) \ge d+3-z$. Since the function $z \mapsto x_z$ is non-decreasing, we get $x_i \ge d+3-z$ for all $i \in \{1, \dots, z+1\}$. Since $\deg(A \cup B) \ge z(d+3-z)$, we get $2d+3 \ge z(d+3-z)$. Since $2 \le z \le d+1$, we get a contradiction.

Therefore it was absurd to suppose that $r_{X_{3,d}}(P) \le 2d-1$, then $r_{X_{3,d}}(P)$ has to be equal to $2d-1$. □

5. Proof of the main theorem

We state here the last remark that will allow to simplify the proof of the main theorem.

Remark 17. Let $Z \subset X_{m,d}$ be as in Notation 2 a zero-dimensional scheme that computes the $X_{m,d}$-border rank of a point $P \in \mathbb{P}^{n_{m,d}}$. Assume the existence of an integer s such that $1 \le s \le m-1$ and of an s-dimensional linear subspace M_s of $\mathbb{P}^n$ containing Z. Let also $A \subset \mathbb{P}^m$ be as in Notation 3 such that $\nu_{m,d}(A) = Z$. If $Z \subset M_s$ then $\dim(\langle A \rangle) \le s$; actually if $\langle Z \rangle \simeq \mathbb{P}^s$ then also $\langle A \rangle \simeq \mathbb{P}^s$ and $\nu_{m,d}(\langle A \rangle) \simeq X_{s,d} \subset \mathbb{P}^{n_{s,d}} \subset \mathbb{P}^{n_{m,d}}$. Now, by [18], Proposition 3.1, or [17], Subsection 3.2, the integer $r_{X_{m,d}}(P)$ is the $(\nu_{s,d}(\langle A \rangle))$-rank of the same point P seen as a point in the linear span $\langle \nu_{m,d}(\langle A \rangle) \rangle \cong \mathbb{P}^{n_{s,d}}$ of $\nu_{s,d}(\langle A \rangle)$. In our case $\deg(Z) = 4$, hence $Z \subseteq M_3$ with $\dim(M_3) = 3$ and $\langle Z \rangle = M_3$. Hence if $P \in \langle Z \rangle \setminus (\sigma_4^0(X_{m,d}) \cup \sigma_3(X_{m,d}))$ then

$$r_{X_{m,d}}(P) = r_{X_{3,d}}(P)$$

where $X_{3,d} = \nu_{m,d}(\langle A\rangle) \simeq \nu_{3,d}(\langle A\rangle)$. Therefore, in order to prove Theorem 1, it is sufficient to prove it in all cases with $m \le 3$.

We are now ready to give the proof of Theorem 1 stated in the Introduction.

PROOF OF THEOREM 1. We want to classify the X-rank of a point $P \in \sigma_4(X) \setminus \sigma_3(X)$ where X is the Veronese embedding of $\mathbb{P}^m$ into $\mathbb{P}^n$ with $n = \binom{m+d}{d}$.

Now $\sigma_4(X) \setminus \sigma_3(X)$ can be split into two components:

$$\sigma_4^0(X) \setminus \sigma_3(X) = \{P \in \sigma_4(X) \,|\, r_X(P) = 4\}$$

(the set $\sigma_4^0(X)$ is defined in (6)) and

$$\sigma_4(X) \setminus (\sigma_4^0(X) \cup \sigma_3(X)) = \{P \in \sigma_4(X) \,|\, r_X(P) > 4\}.$$

Obviously there is nothing to say on the X-rank of points belonging to $\sigma_4^0(X) \setminus \sigma_3(X)$, except that it is equal to 4. Therefore it remains to study only the X-rank of points $P \in \sigma_4(X) \setminus (\sigma_4^0(X) \cup \sigma_3(X))$. In order to do that, as already showed in Section 1, we have to study the X-rank of points belonging to the span of a degree 4 zero-dimensional non-reduced sub-scheme $Z \subset X$ computing the X-border rank of such a point $P \in \sigma_4(X) \setminus (\sigma_3(X) \cup \sigma_4^0(X))$ (as in Notation 2).

By Remark 17 we may restrict our attention to the case $m \le 3$. Therefore we study separately the $X_{m,d}$-rank of a point $P \in \langle Z\rangle \setminus (\sigma_4^0(X_{m,d}) \cup \sigma_3(X_{m,d}))$ for $Z \in X_{m,d}$ non-reduced and of degree 4 and for $m = 1$, $m = 2$ and $m = 3$.

$m = 1$. In this case $Z = \nu_{m,d}(A)$ for A contained in a line $L \subset \mathbb{P}^m$, hence $r_{X_{m,d}}(P) = r_{\nu_{m,d}(L)}(P) = d - 2$ (for [21], [9], [6], [11], [7] or [17], Theorem 4.1).

$m = 2$. The scheme A now is a degree 4 zero-dimensional scheme that is contained in a plane but not in a line, hence it can intersect at least one line in degree 3 or it does not exist any line that intersects A in degree 3.

- If $\deg(A \cup L) = 3$ for at least one line $L \subset \mathbb{P}^m$ then we distinguish the following cases:
  - If $Res_L(A) \notin L$ then $r_{X_{m,d}}(P) = d$ for Proposition 5.
  - If $Res_L(A) \in L$ then we study the cardinality of the support of the scheme A.
    - If $\sharp(Supp(A)) = 1$, then $r_{X_{m,d}}(P) = 2d - 2$ by Proposition 6.
    - If $\sharp(Supp(A)) = 2$, then either A is the union of two unreduced zero-dimensional schemes both of degree 2 or A is the union of a simple point O and a first infinitesimal neighborhood of another point $Q \in \mathbb{P}^2$. In the first case $r_{X_{m,d}}(P) = 2d-1$ by Proposition 7, in the second case we have that $P \in \langle O, T_{\nu_{2,d}(Q)}X\rangle$, but since $T_Q X \subset \sigma_2(X)$, then $P \in \sigma_3(X_{m,d})$.
    - If $\sharp(Supp(A)) = 3$, then $r_{X_{m,d}}(P) = d + 2$ by Proposition 8.
- Now assume that $\deg(A \cup L) < 3$ for all lines L's contained in $\mathbb{P}^m$ and $\dim(\langle A\rangle) = 2$. In this case there exists a one parameter system of conics containing A. We can have two cases: either the generic conic through A is smooth or it is a union of two distinct lines.

* If the generic conic through A is smooth, then, by Proposition 9, $r_{X_{m,d}}(P) = 2d-1$.
* If the generic conic through A is the union of two distinct lines L_1, L_2, then $r_{X_{m,d}}(P) = 2d-2$ in all possible configurations of A except if $A = A_1 \cup O_1 \cup O_2$ where $A_1 \subset L_1$ is a non-reduced scheme of degree 2, the points $O_1, O_2 \in L_2$ and $\{L_1 \cap L_2\} \notin \{Supp(A_1), O_1, O_2\}$, in this last case we have that $r_{X_{m,d}}(P) = d+2$ (see Proposition 10, Proposition 11 and Proposition 12).

$m=3$. Let us study the cardinality of the support of A with A such that $\langle A \rangle \simeq \mathbb{P}^3$.

- If $\sharp(Supp(A)) = 1$ we may assume that A is not the first infinitesimal neighborhood of a point $Q \in \mathbb{P}^3$, otherwise $P \in \sigma_2(\nu_{m,d}(\langle A \rangle)) \subset \sigma_2(X_{m,d})$. Moreover, by Remark 16 we can also suppose that it has not a 2-dimensional tangent space. Hence A can only be a curvilinear scheme; in this case by Proposition 14 we have that $r_{Xm,d}(P) = 3d-2$.
- If $\sharp(Supp(A)) = 2$ we may have the following cases.
  * The scheme A is the union of a simple point O and a degree 3 zero-dimensional scheme A' supported on a point $Q \subset \mathbb{P}^3$ such that $\dim(\langle A' \rangle) = 2$ and $\langle \nu_{3,d}(A') \rangle \subset T_{\nu_{3,d}(Q)}X$. Therefore $P \in \langle O, T_{\nu_{3,d}(Q)}X \rangle \subset \langle O, \sigma_2(X) \rangle \subset \sigma_3(X)$.
  * The scheme A is the union of two unreduced zero dimensional schemes both of degree 2. Since $\langle A \rangle = \mathbb{P}^3$ we are in the case of Proposition 14 where we get that $r_{X_{m,d}}(P) = 2d$.
  * The scheme A is the union of a simple point and of a degree 3 curvilinear zero-dimensional scheme supported on one point. Proposition 15 gives us that $r_{X_{m,d}}(P) = 2d$.
- If $\sharp(Supp(A)) = 3$ then A can only be the union of two simple points and a degree 2 unreduced scheme. By Proposition 6 we have that $r_{X_{m,d}}(P) = d+2$.

All this cases prove the statement of the Theorem 1. □

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