UNIQUE DECOMPOSITION FOR A POLYNOMIAL OF LOW RANK

EDOARDO BALLICO, ALESSANDRA BERNARDI

Abstract

Let F be a homogeneous polynomial of degree d in $m+1$ variables defined over an algebraically closed field of characteristic 0 and suppose that F belongs to the s-th secant variety of the d-uple Veronese embedding of $\mathbb{P}^m$ into $\mathbb{P}^{\binom{m+d}{d}-1}$ but that its minimal decomposition as a sum of d-th powers of linear forms requires more than s addenda. We show that if $s \leq d$ then F can be uniquely written as $F = M_1^d + \cdots + M_t^d + Q$, where $M_1, \ldots, M_t$ are linear forms with $t \leq (d-1)/2$, and Q a binary form such that $Q = \sum_{i=1}^{q} l_i^{d-d_i} m_i$ with l_i's linear forms and m_i's forms of degree d_i such that $\sum(d_i+1) = s-t$.

Introduction

In this paper we will always work with an algebraically closed field K of characteristic 0. Let $X_{m,d} \subset \mathbb{P}^N$, with $m \geq 1$, $d \geq 2$ and $N := \binom{m+d}{m} - 1$, be the classical Veronese variety obtained as the image of the d-uple Veronese embedding $\nu_d : \mathbb{P}^m \to \mathbb{P}^N$. The *s-th secant variety* $\sigma_s(X_{m,d})$ of the Veronese variety $X_{m,d}$ is the Zariski closure in $\mathbb{P}^N$ of the union of all linear spans $\langle P_1, \ldots, P_s \rangle$ with $P_1, \ldots, P_s \in X_{m,d}$. For any point $P \in \mathbb{P}^N$, we indicate with $\mathrm{sbr}(P) = s$ the minimum integer s such that $P \in \sigma_s(X_{m,d})$. This integer is called the *symmetric border rank* of P. Since $\mathbb{P}^m \simeq \mathbb{P}(K[x_0, \ldots, x_m]_1) \simeq \mathbb{P}(V^*)$, with V an $(m+1)$-dimensional vector space over K, the generic element belonging to $\sigma_s(X_{m,d})$ is the projective class of a form (a symmetric tensor) of type:

$$F = L_1^d + \cdots + L_r^d, \quad (T = v_1^{\otimes d} + \cdots + v_r^{\otimes d}). \tag{1}$$

The decomposition of a homogeneous polynomial that combines a minimum number of terms and that involves a minimum number of variables is a problem that is having a lot of attentions not only form classical Algebraic Geometry ([1], [7], [5], [6], [9]), but also from applications like Computational Complexity ([8]) and Signal Processing ([10]).

At the Workshop on Tensor Decompositions and Applications (September 13–17, 2010, Monopoli, Bari, Italy), A. Bernardi presented the following result:

([2], Corollary 1) Let $F \in K[x_0, \ldots, x_m]_d$ be such that $\mathrm{sbr}(F) + \mathrm{sr}(F) \leq 2d+1$ and $\mathrm{sbr}(F) < \mathrm{sr}(F)$. Then there are an integer $t \geq 0$, linear forms $L_1, L_2, M_1, \ldots, M_t \in K[x_0, \ldots, x_m]_1$, and a form $Q \in K[L_1, L_2]_d$ such that $F = Q + M_1^d + \cdots + M_t^d$, $t \leq \mathrm{sbr}(F) + \mathrm{sr}(F) - d - 2$, and $\mathrm{sr}(F) = \mathrm{sr}(Q) + t$. Moreover t, $M_1, \ldots, M_t$ and the linear span of L_1, L_2 are uniquely determined by F.

In terms of tensors it can be translates as follows:

1991 *Mathematics Subject Classification.* 15A21, 15A69, 14N15.

Key words and phrases. Waring problem, Polynomial decomposition, Symmetric rank, Symmetric tensors, Veronese varieties, Secant varieties.

The authors were partially supported by CIRM of FBK Trento (Italy), Project Galaad of INRIA Sophia Antipolis Méditerranée (France), Marie Curie: Promoting science (FP7-PEOPLE-2009-IEF), MIUR and GNSAGA of INdAM (Italy).

([2], Corollary 2) Let $T \in S^dV^*$ be such that $\mathrm{sbr}(T) + \mathrm{sr}(T) \leq 2d + 1$ and $\mathrm{sbr}(T) < \mathrm{sr}(T)$. Then there are an integer $t \geq 0$, vectors $v_1, v_2, w_1, \dots, w_t \in S^1V^*$, and a symmetric tensor $v \in S^d(\langle v_1, v_2\rangle)$ such that $T = v + w_1^{\otimes d} + \cdots + w_t^{\otimes d}$, $t \leq \mathrm{sbr}(T) + \mathrm{sr}(T) - d - 2$, and $\mathrm{sr}(T) = \mathrm{sr}(v) + t$. Moreover t, $w_1, \dots, w_t$ and $\langle v_1, v_2\rangle$ are uniquely determined by T.

The natural questions that arose at that workshop from applied people, were about the possible uniqueness of the binary form Q in [2], Corollary 1 (ie. the vector v in [2], Corollary 2) and a bound on the number t of linear forms (ie. rank 1 symmetric tensors). We are finally able to give the most possible complete answer to this question. The main result of this paper is the following.

Theorem 1. *Let $P \in \mathbb{P}^N$ with $N = \binom{m+d}{d} - 1$. Suppose that:*

$$\mathrm{sbr}(P) < \mathrm{sr}(P) \text{ and}$$
$$\mathrm{sbr}(P) + \mathrm{sr}(P) \leq 2d + 1.$$

Let $\mathcal{S} \subset X_{m,d}$ be a 0-dimensional reduced subscheme that realizes the symmetric rank of P, and let $\mathcal{Z} \subset X_{m,d}$ be a 0-dimensional non-reduced subscheme such that $P \in \langle \mathcal{Z}\rangle$ and $\deg \mathcal{Z} \leq \mathrm{sbr}(P)$. There is a unique rational normal curve $C_d \subset X_{m,d}$ such that $C_d \cap (\mathcal{S} \cup \mathcal{Z}) \geq d + 2$. Then, for all points $P \in \mathbb{P}^N$ as above we have that:

$$\mathcal{S} = \mathcal{S}_1 \sqcup \mathcal{S}_2, \qquad \mathcal{Z} = \mathcal{Z}_1 \sqcup \mathcal{S}_2,$$

where $\mathcal{S}_1 = \mathcal{S} \cap C_d$, $\mathcal{Z}_1 = \mathcal{Z} \cap C_d$ and $\mathcal{S}_2 = (\mathcal{S} \cap \mathcal{Z}) \setminus \mathcal{S}_1$.
Moreover C_d, $\mathcal{S}_2$ and $\mathcal{Z}$ are unique, $\deg(\mathcal{Z}) = \mathrm{sbr}(P)$, $\deg(\mathcal{Z}_1) + \deg(\mathcal{S}_1) = d + 2$, $\mathcal{Z}_1 \cap \mathcal{S}_1 = \emptyset$ and $\mathcal{Z}$ is a the unique zero-dimensional subscheme N of $X_{m,d}$ such that $\deg(N) \leq \mathrm{sbr}(P)$ and $P \in \langle N\rangle$.

In the language of polynomials, Theorem 1 can be rephrased as follows.

Corollary 1. *Let $F \in K[x_0, \dots, x_m]_d$ be such that $\mathrm{sbr}(F) + \mathrm{sr}(F) \leq 2d + 1$ and $\mathrm{sbr}(F) < \mathrm{sr}(F)$. Then there are an integer $0 \leq t \leq (d-1)/2$, linear forms $L_1, L_2, M_1, \dots, M_t \in K[x_0, \dots, x_m]_1$, and a form $Q \in K[L_1, L_2]_d$ such that $F = Q + M_1^d + \cdots + M_t^d$, $t \leq \mathrm{sbr}(F) + \mathrm{sr}(F) - d - 2$, and $\mathrm{sr}(F) = \mathrm{sr}(Q) + t$.*
Moreover the line $\langle L_1, L_2\rangle$, the forms $M_1, \dots, M_t$ and Q such that $Q = \sum_{i=1}^q l_i^{d-d_i} m_i$ with l_i's linear forms and m_i's forms of degree d_i such that $\sum(d_i + 1) = s - t$, are uniquely determined by F.

An analogous corollary can be stated for symmetric tensors.

Corollary 2. *Let $T \in S^dV^*$ be such that $\mathrm{sbr}(T) + \mathrm{sr}(T) \leq 2d + 1$ and $\mathrm{sbr}(T) < \mathrm{sr}(T)$. Then there are an integer $0 \leq t \leq (d-1)/2$, vectors $v_1, v_2, w_1, \dots, w_t \in S^1V^*$, and a symmetric tensor $v \in S^d(\langle v_1, v_2\rangle)$ such that $T = v + w_1^{\otimes d} + \cdots + w_t^{\otimes d}$, $t \leq \mathrm{sbr}(T) + \mathrm{sr}(T) - d - 2$, and $\mathrm{sr}(T) = \mathrm{sr}(v) + t$. Moreover the line $\langle v_1, v_2\rangle$, the vectors $v_1, \dots, v_t$ and the tensor v such that $v = \sum_{i=1}^q u_i^{\otimes(d-d_i)} \otimes z_i$ with $u_i \in \langle v_1, v_2\rangle$ and $z_i \in S^{d_i}(\langle v_1, v_2\rangle)$ such that $\sum(d_i + 1) = s - t$, are uniquely determined by T.*

Moreover in Theorem 2 and in Corollary 4, by introducing the notion of linearly general position of a scheme (Definition 1), we can also extend to a geometric description the condition for the uniqueness of the scheme $\mathcal{Z}$ of Theorem 1. We can rephrase their contents in therms of homogeneous polynomials and symmetric tensors in the following Corollary.

Corollary 3. *Fix integers $m \geq 2$ and $d \geq 4$. Fix F an homogeneous polynomial in $m + 1$ variables of degree d ($T \in S^dV$ respectively) such that $\mathrm{sbr}(F) \leq d$ ($\mathrm{sbr}(T) \leq d$). Let $Z \subset \mathbb{P}^m$ be* *any smoothable zero-dimensional scheme such that* $\nu_d(Z)$ *computes* $\mathrm{sbr}(F)$ *(*$\mathrm{sbr}(T)$*). Assume that* Z *is in linearly general position. Then* Z *is the unique scheme computing* $\mathrm{sbr}(P)$ *(*$\mathrm{sbr}(F)$*).*

1. Proofs

The existence of such a scheme $\mathcal{Z}$ was known from [3] and [4] (see Remark 1 of [2]).

Lemma 1. *Fix integers* $m \ge 2$ *and* $d \ge 2$*, a line* $\ell \subset \mathbb{P}^m$ *and any finite set* $E \subset \mathbb{P}^m \setminus \ell$ *such that* $\sharp(E) \le d$*. Then* $\dim(\langle \nu_d(E)\rangle) = \sharp(E) - 1$ *and* $\langle \nu_d(\ell)\rangle \cap \langle \nu_d(E)\rangle = \emptyset$*.*

Proof. Since $h^0(\ell \cup E, \mathcal{O}_{\ell\cup E}(d)) = d + 1 + \sharp(E)$, to get both statements it is sufficient to prove $h^1(\mathcal{I}_{\ell\cup E}(d)) = 0$. Let $H \subset \mathbb{P}^m$ be a general hyperplane containing ℓ. Since E is finite and H is general, we have $H \cap E = \emptyset$. Hence the residual exact sequence of the scheme $\ell \cup E$ with respect to the hyperplane H is the following exact sequence on $\mathbb{P}^m$:

$$0 \to \mathcal{I}_E(d-1) \to \mathcal{I}_{\ell\cup E}(d) \to \mathcal{I}_{\ell,H}(d) \to 0 \qquad (2)$$

Since $h^1(\mathcal{I}_E(d-1)) = h^1(H, \mathcal{I}_{\ell,H}(d)) = 0$, we get the lemma. □

Proof of Theorem 1. All the statements are contained in [2], Theorem 1, except the uniqueness of $\mathcal{Z}$, that $\deg(\mathcal{Z}_1) + \deg(\mathcal{S}_1) = d + 2$ and that $\mathcal{Z}_1 \cap \mathcal{S}_1 = \emptyset$. Let $\ell \subset \mathbb{P}^m$ be the line such that $\nu_d(\ell) = C_d$. Take $Z, S, Z_i, S_i \subset \mathbb{P}^m$, $i = 1, 2$, such that $\nu_d(Z) = \mathcal{Z}$, $\nu_d(S) = \mathcal{S}$, $\nu_d(Z_i) = \mathcal{Z}_i$, and $\nu_d(S_i) = \mathcal{S}_i$. Assume the existence of another subscheme $\mathcal{Z}' \subset X_{m,d}$ such that $P \in \langle \nu_d(\mathcal{Z}')\rangle$ and $\deg(\mathcal{Z}') \le \mathrm{sbr}(P)$. Set $\mathcal{Z}'_1 := \mathcal{Z}' \cap C_d$. The proof of [2], Theorem 1, gives $\mathcal{Z}' = \mathcal{Z}'_1 \sqcup S_2$. Since C_d is a smooth curve, $\mathcal{Z}_1 \cup \mathcal{Z}'_1 \subset C_d$, $S_2 \cap \ell = \emptyset$, and $\mathcal{Z} \cup \mathcal{Z}' = (\mathcal{Z}_1 \cup \mathcal{Z}'_1) \sqcup S_2$, the schemes $\mathcal{Z}$ and $\mathcal{Z}'$ are curvilinear. Hence all subschemes of $\mathcal{Z}$ and $\mathcal{Z}'$ are smoothable. Hence any subscheme of either $\mathcal{Z}$ or $\mathcal{Z}'$ may be used to compute the border rank of some point of $\mathbb{P}^N$. Since $\deg(\ell \cap (Z \cup S)) \ge d + 2$, $\nu_d((Z \cup S) \cap \ell)$ spans $\langle C_d\rangle$. Lemma 1 implies $\langle C_d\rangle \cap \langle \mathcal{S}_2\rangle = \emptyset$. Since $P \in \langle \mathcal{S}_1 \cup \mathcal{S}_2\rangle$ and $\sharp(S) = \mathrm{sr}(P)$, we have $P \notin \langle \mathcal{A}\rangle$ for any $\mathcal{A} \subsetneq \mathcal{S}$. Therefore we get that $\langle \{P\} \cup \mathcal{S}_2\rangle \cap \langle \mathcal{S}_1\rangle$ is a unique point. Call P_1 this point. Similarly, $\langle \mathcal{Z}_1\rangle \cap \langle \mathcal{S}_2\rangle$ is a unique point and we call it P_2. Since $\langle C_d\rangle \cap \langle \mathcal{S}_2\rangle = \emptyset$, the set $\langle C_d\rangle \cap \langle \{P\} \cup \mathcal{S}_2\rangle$ is at most one point. Since $P_i \in \langle C_d\rangle \cap \langle \{P\} \cup \mathcal{S}_2\rangle$, $i = 1, 2$, we have $P_1 = P_2$ and $\{P_1\} = \langle C_d\rangle \cap \langle \{P\} \cup \mathcal{S}_2\rangle$. Since $P_1 = P_2$ and $P_1 \in \langle \mathcal{S}_1\rangle$ and $P_2 \in \langle \mathcal{Z}'_1\rangle$, we have $P_1 \in \langle \mathcal{Z}'_1\rangle \cap \langle \mathcal{S}_1\rangle$. Take any $E \subseteq \mathcal{Z}_1$ such that $P_1 \in \langle E\rangle$. Since $P \in \langle \{P_1\} \cup \mathcal{S}_2\rangle \subseteq \langle E \cup \mathcal{S}_2\rangle$ and $P \notin \langle \mathcal{U}\rangle$ for any $\mathcal{U} \subsetneq \mathcal{Z}$, we get $E \cup \mathcal{S}_2 = \mathcal{Z}$. Hence $E = \mathcal{Z}_1$. Therefore $\mathcal{Z}_1$ computes $\mathrm{sbr}(P_1)$ with respect to C_d. Similarly, $\mathcal{Z}'_1$ computes $\mathrm{sr}(P_2)$ with respect to the same rational normal curve C_d. Since $P_1 = P_2$, we have $\mathcal{Z}'_1 = \mathcal{Z}_1$ (as for all curves we get the uniqueness of $\mathcal{Z}_1$ for $\mathrm{sbr}(P_1) \le \lfloor (d+2)/2\rfloor$). Since $\mathrm{sbr}(P_1) \ne \mathrm{sr}(P_1)$, a theorem of Sylvester gives $\mathrm{sbr}(P_1) + \mathrm{sr}(P_1) = d + 2$, i.e. $\deg(\mathcal{Z}_1) + \deg(\mathcal{S}_1) = d + 2$.

Definition 1. A scheme $Z \subset \mathbb{P}^m$ is said to be in *linearly general position* if for every linear subspace $R \subsetneq \mathbb{P}^m$ we have $\deg(R \cap Z) \le \dim(R) + 1$.

Notice that the next theorem is false if either $d = 2$ or $m = 1$.

Theorem 2. *Fix integers* $m \ge 2$ *and* $d \ge 4$*. Fix* $P \in \mathbb{P}^N$ *Let* $Z \subset \mathbb{P}^m$ *be any smoothable zero-dimensional scheme such that* $P \in \langle \nu_d(Z)\rangle$ *and* $P \notin \langle \nu_d(Z')\rangle$*. Assume* $\deg(Z) \le d$ *and that* Z *is in linearly general position. Then* Z *is the unique scheme* $Z' \subset \mathbb{P}^m$ *such that* $\deg(Z') \le d$ *and* $P \in \langle \nu_d(Z')\rangle$*. Moreover* $\nu_d(Z)$ *computes* $\mathrm{sbr}(P)$*.*

Proof. The existence of a scheme computing $\mathrm{sbr}(P)$ follows from =[2], Remark ++= and the assumption " $\mathrm{sbr}(P) \le d$ ". Fix any scheme $Z' \subset \mathbb{P}^m$ such that $Z' \ne Z$, $\deg(Z') \le d$, $P \in \langle \nu_d(Z')\rangle$, and $P \notin \langle \nu_d(Z'')\rangle$ for any $Z'' \subsetneq Z'$. Assume $Z' \ne Z$. Since $\deg(Z \cup Z') \le 2d + 1$ and $h^1(\mathbb{P}^m, \mathcal{I}_{Z\cup Z'}(d)) > 0$ ([2], Lemma 1), there is a line $D \subset \mathbb{P}^m$ such that $\deg(D \cap (Z \cup Z')) \ge d + 2$. Since Z is in linearly general position and $m \ge 2$, we have $\deg(Z \cap D) \le 2$. Hence $\deg(Z' \cap D) \ge d$.

Hence $\deg(Z') = d$. Since $\deg(Z') = d$, we get $Z' \subset D$. Hence $P \in \langle \nu_d(D) \rangle$. Hence $\mathrm{sr}(P) = d$. As for all curves we get $\mathrm{sbr}(P) \le \lfloor (d+2)/2 \rfloor$. Since $\deg(Z') = d$, we assumed $\deg(Z') \le \mathrm{sbr}(P)$, contradicting the assumption $d \ge 4$. □

Corollary 4. *Fix integers $m \ge 2$ and $d \ge 4$. Fix $P \in \mathbb{P}^N$ such that $\mathrm{sbr}(P) \le d$. Let $Z \subset \mathbb{P}^m$ be any smoothable zero-dimensional scheme such that $\nu_d(Z)$ computes $\mathrm{sbr}(P)$. Assume that Z is in linearly general position. Then Z is the unique scheme computing $\mathrm{sbr}(P)$.*

References

[1] J. Alexander, A. Hirschowitz. Polynomial interpolation in several variables. J. Algebraic Geom. 4 (1995), no. 2, 201–222.
[2] E. Ballico, A. Bernardi, Decomposition of homogeneous polynomials with low rank. Math. Z. DOI 10.1007/s00209-011-0907-6.
[3] A. Bernardi, A. Gimigliano, M. Idà. Computing symmetric rank for symmetric tensors. J. Symbolic. Comput. 46 (2011), 34–55.
[4] J. Buczyński, A. Ginensky, J. M. Landsberg. Determinantal equations for secant varieties and the Eisenbud-Koh-Stillman conjecture. arXiv:1007.0192 [math.AG].
[5] L. Chiantini, C. Ciliberto. Weakly defective varieties. Trans. Amer. Math. Soc. 454 (2002), no. 1, 151–178.
[6] C. Ciliberto, M. Mella, F. Russo. Varieties with one apparent double point. J. Algebraic Geom. 13 (2004), no. 3, 475–512.
[7] A. Iarrobino, V. Kanev. Power sums, Gorenstein algebras, and determinantal loci. Lecture Notes in Mathematics, vol. 1721, Springer-Verlag, Berlin, 1999, Appendix C by Iarrobino and Steven L. Kleiman.
[8] L.-H. Lim, V. De Silva. Tensor rank and the ill-posedness of the best low-rank approximation problem. SIAM J. Matrix Anal. 30 (2008), no. 3, 1084–1127.
[9] J. M. Landsberg, Z. Teitler. On the ranks and border ranks of symmetric tensors. Found. Comput. Math. 10, (2010) no. 3, 339–366.
[10] R. C. Vaughan, T. D. Wooley. Waring's problem: a survey, Number theory for the millennium. III (Urbana, IL, 2000), A K Peters, Natick, MA, (2002), 301–340.

Dept. of Mathematics, University of Trento, 38123 Povo (TN), Italy

GALAAD, INRIA Méditerranée, BP 93, 06902 Sophia Antipolis, France.
E-mail address: ballico@science.unitn.it, alessandra.bernardi@inria.fr