THE CACTUS RANK OF CUBIC FORMS

ALESSANDRA BERNARDI, KRISTIAN RANESTAD

Abstract. We prove that the smallest degree of an apolar 0-dimensional scheme to a general cubic form in $n+1$ variables is at most $2n+2$, when $n \geq 8$, and therefore smaller than the rank of the form. When $n = 8$ we show that the bound is sharp, i.e. the smallest degree of an apolar subscheme is 18.

Introduction

The rank of a homogeneous form F of degree d is the minimal number of linear forms $L_1, \dots, L_r$ needed to write F as a sum of pure d-powers:

$$F = L_1^d + \cdots + L_r^d.$$

Various other notions of rank, such as cactus rank and border rank, appear in the study of higher secant varieties and are closely related to the rank. The cactus rank is the minimal length of an apolar subscheme to F, while the border rank is the minimal r such that F is a limit of forms of rank r. The notion of cactus rank is inspired by the cactus varieties studied in [Buczynska, Buczynski 2010]. For irreducible cubic forms that do not define a cone, the cactus rank is minimal for cubics of Fermat type, e.g. $F = x_0^3 + \cdots + x_n^3$. In this case all three ranks coincide. There are however other cubic forms with minimal cactus rank whose border rank is strictly higher. We show that the cactus rank is smaller than the rank for a general form, as soon as the degree is at least 3 and the number of variables is at least 9, and we show in particular that the cactus rank for the general cubic form in 9 variables is 18, while the rank and the border rank is 19.

The special computation is the first example, to our knowledge, of a computation of cactus rank of a general form, where it is less than the rank. On the other hand the more general result is an important explicit obstruction to the computation of rank.

The rank of forms has seen growing interest in recent years. This work is close in line to [Iarrobino 1994], [Iarrobino, Kanev 1999] and [Elias, Rossi 2011], in their study of apolarity and the local Gorenstein algebra associated to a polynomial. Applications to higher secant varieties can be found in [Chiantini, Ciliberto, 2002], [Landsberg, Ottaviani 2011] and [Buczynska, Buczynski 2010], while the papers [Landsberg, Teitler 2010], [Brachat et al. 2010], [Bernardi et al. 2011] and [Carlini et al. 2011] concentrate on effective methods to compute the rank and to compute an explicit decomposition of a form.

1. Apolar Gorenstein subschemes

We consider homogeneous polynomials $F \in S = \mathbb{C}[x_0, \dots, x_n]$, and consider the dual ring $T = \mathbb{C}[y_0, \dots, y_n]$ acting on S by differention:

$$y_j(x_i) = \frac{d}{dx_j}(x_i) = \delta_{ij}$$

With respect to this action S_1 and T_1 are natural dual spaces and $< x_0, \dots, x_n >$ and $< y_0, \dots, y_n >$ are dual bases. In particular T is naturally the coordinate ring of $\mathbb{P}(S_1)$ the projective space of 1-dimensional subspaces of S_1, and vice versa. The annihilator of F is an ideal $F^\perp \subset T$, and the quotient $T_F = T/F^\perp$ is graded Artinian and Gorenstein.

Definition 1. A subscheme $X \subset \mathbb{P}(S_1)$ is apolar to F if the homogeneous ideal $I_X \subset F^\perp \subset T$.

F admits some natural finite local apolar Gorenstein subschemes. Any local Gorenstein subscheme is defined by an (in)homogeneous polynomial: For any $f \in S_{x_0} = \mathbb{C}[x_1, \dots, x_n]$, the annihilator $f^\perp \subset T_{y_0} = \mathbb{C}[y_1, \dots, y_n]$ defines an Artinian Gorenstein quotient $T_f = T_{y_0}/f^\perp$. Now, $\mathrm{Spec}(T_f)$ is naturally a subscheme in $\mathbb{P}(S_1)$. Taking $f = F(1, x_1, \dots, x_n)$ we show that $\mathrm{Spec}(T_f)$ is apolar to F. In fact, F admits a natural apolar Gorenstein subscheme for any linear form in S.

Any nonzero linear form $l \in S$ belongs to a basis $(l, l_1, \dots, l_n)$ of S_1, with dual basis $(l', l'_1, \dots, l'_n)$ of T_1. In particular the homogeneous ideal in T of the point $[l] \in \mathbb{P}(S_1)$ is generated by $\{l'_1, \dots, l'_n\}$, while $\{l_1, \dots, l_n\}$ generate the ideal of the point $\phi([l]) \in \mathbb{P}(T_1)$, where $\phi : \mathbb{P}(T_1) \to \mathbb{P}(S_1)$, $y_i \mapsto x_i, i = 0, \dots, n$.

The form $F \in S$ defines a hypersurface $\{F = 0\} \subset \mathbb{P}(T_1)$. The Taylor expansion of F with respect to the point $\phi([l])$ may naturally be expressed in the coordinates functions $(l, l_1, \dots, l_n)$. Thus

$$F = a_0 l^d + a_1 l^{d-1} f_1(l_1, \dots, l_n) + \dots + a_d f_d(l_1, \dots, l_n).$$

We denote the corresponding dehomogenization of F with respect to l by F_l, i.e.

$$F_l = a_0 + a_1 f_1(l_1, \dots, l_n) + \dots + a_d f_d(l_1, \dots, l_n).$$

Also, we denote the subring of T generated by $\{l'_1, \dots, l'_n\}$ by $T_{l'}$. It is the natural coordinate ring of the affine subspace $\{l' \neq 0\} \subset \mathbb{P}(S_1)$.

Lemma 1. *The Artinian Gorenstein scheme $\Gamma(F_l)$ defined by $F_l^\perp \subset T_{l'}$ is apolar to F, i.e. the homogenization $(F_l^\perp)^h \subset F^\perp \subset T$.*

Proof. If $g \in F_l^\perp \subset \mathbb{C}[l'_1, \dots, l'_n]$, then $g = g_1 + \dots + g_r$ where g_i is homogeneous in degree i. Similarly $F_l = f = f_0 + \dots + f_d$. The annihilation $g(f) = 0$ means that for each $e \geq 0$, $\sum_j g_j f_{e+j} = 0$. Homogenizing we get

$$g^h = G = (l')^{r-1} g_1 + \dots + g_r, \quad f^h = F = l^d f_0 + \dots + f_d$$

and

$$G(F) = \sum_e \sum_j l^{d-r-e} g_j f_{e+j} = \sum_e l^{d-r-e} \sum_j g_j f_{e+j} = 0.$$

□

Remark 1. (Suggested by Mats Boij) The ideal $(F_l^\perp)^h$ may be obtained without dehomogenizing F. Write $F = l^e F_{d-e}$, such that l does not divide F_{d-e}. Consider the form $F_{2(d-e)} = l^{d-e} F_{d-e}$. Unless $d - e = 0$, i.e. $F = l^d$, the degree $d - e$ part of the annihilator $(F_{2(d-e)})^\perp_{d-e}$ generates an ideal in $(l)^\perp$ and the saturation $\mathrm{sat}(F_{2(d-e)})^\perp_{d-e})$ coincides with $(F_l^\perp)^h$. In fact if $G \in T_{d-e}$ then

$$G(F_{2(d-e)}) = G(l^{d-e} F_{d-e}) = G(l^{d-e}) F_{d-e} + l G(l^{d-e-1} F_{d-e}$$

so $G(F_{2(d-e)}) = 0$ only if $G(l^{d-e}) = 0$.

Apolarity has attracted interest since it characterizes powersum decompositions of F, cf. [Iarrobino, Kanev 1999], [Ranestad, Schreyer 2000]. The annihilator of a power of a linear form $l^d \in S$ is the ideal of the corresponding point $p_l \in \mathbb{P}_T$ in degrees at most d. Therefore $F = \sum_{i=1}^r l_i^d$ only if $I_\Gamma \subset F^\perp$ where $\Gamma = \{p_{l_1}, \dots, p_{l_r}\} \subset \mathbb{P}_T$. On the other hand, if $I_{\Gamma,d} \subset F_d^\perp \subset T_d$, then any differential form that annihilates each l_i^d also annihilates F, so, by duality, $[F]$ must lie in the linear span of the $[l_i^d]$ in $\mathbb{P}(S_d)$. Thus $F = \sum_{i=1}^r l_i^d$ if and only if $I_\Gamma \subset F^\perp$.

Various notions of rank for F are therefore naturally defined by apolarity : The cactus rank $cr(F)$ is defined as

$$cr(F) = \min\{\text{length}\Gamma \mid \Gamma \subset \mathbb{P}(T_1), \dim\Gamma = 0, I_\Gamma \subset F^\perp\},$$

the smoothable rank $sr(F)$ is defined as

$$sr(F) = \min\{\text{length}\Gamma \mid \Gamma \subset \mathbb{P}(T_1) \ \text{ smoothable}, \dim\Gamma = 0, I_\Gamma \subset F^\perp\}$$

and the rank $r(F)$ is defined as

$$r(F) = \min\{\text{length}\Gamma \mid \Gamma \subset \mathbb{P}(T_1) \ \text{ smooth}, \dim\Gamma = 0, I_\Gamma \subset F^\perp\}.$$

Clearly $cr(F) \leq sr(F) \leq r(F)$. A separate notion of border rank, $br(F)$, often considered, is not defined by apolarity. The border rank is rather the minimal r, such that F is the limit of polynomials of rank r. Thus $br(F) \leq sr(F)$. These notions of rank coincide with the notions of length of annihilating schemes in Iarrobino and Kanev book [Iarrobino, Kanev 1999, Definition 5.66]: Thus cactus rank coincides with the scheme length, $cr(F) = l_{\text{sch}}(F)$, and smoothable rank coincides with the smoothable scheme length, $sr(F) = l_{\text{schsm}}(F)$, while border rank coincides with length $br(F) = l(F)$. In addition they consider the differential length $l_{\text{diff}}(F)$, the maximum of the dimensions of the space of k-th order partials of F as k varies between 0 and $\deg F$. This length is the maximal rank of a catalecticant or Hankel matrix at F, and is always a lower bound for the cactus rank: $l_{\text{diff}}(F) \leq cr(F)$.

For a general form F in S of degree d the rank, the smoothable rank and the border rank coincide and equals, by the Alexander Hischowitz theorem,

$$br(F) = sr(F) = r(F) = \left\lceil \frac{1}{n+1}\binom{n+d}{d} \right\rceil,$$

when $d > 2$, $(n,d) \neq (2,4), (3,4), (4,3), (4,4)$. The local Gorenstein subschemes considered above show that the cactus rank for a general polynomial may be smaller. Let

$$N_d = \begin{cases} 2\binom{n+k}{k} & \text{when } d = 2k+1 \\ \binom{n+k}{k} + \binom{n+k+1}{k+1} & \text{when } d = 2k+2 \end{cases} \tag{1}$$

and denote by $\text{Diff}(F)$ the subspace of S generated by the partials of F of all orders, i.e. of order $0, \dots, d = \deg F$.

Theorem 1. *Let $F \in S = \mathbb{C}[x_0, \dots, x_n]$ be a homogeneous form of degree d, and let $l \in S_1 =< x_0, \dots, x_n >$ be any linear form. Let F_l be a dehomogenization of F with respect to l. Then*

$$cr(F) \leq \dim_{\mathbb{C}}\text{Diff}(F_l).$$

In particular

$$cr(F) \leq N_d.$$

Proof. According to Lemma 1 the subscheme $\Gamma(F_l) \subset \mathbb{P}(T_1)$ is apolar to F. The subscheme $\Gamma(F_l)$ is affine and has length equal to

$$\dim_{\mathbb{C}} T_{l'}/F_l^{\perp} = \dim_{\mathbb{C}} \mathrm{Diff(F_l)}.$$

If all the partial derivatives of F_l of order at most $\lfloor \frac{d}{2} \rfloor$ are linearly independent, and the partial derivatives of higher order span the space of polynomials of degree at most $\lfloor \frac{d}{2} \rfloor$, then

$$\dim_{\mathbb{C}} \mathrm{Diff(F_l)} = 1 + n + \binom{n+1}{n-1} + \cdots + \binom{n + \lfloor \frac{d}{2} \rfloor}{n-1} + \cdots + n + 1 = N_d.$$

Clearly this is an upper bound so the theorem follows. □

Question 1. What is the cactus rank $cr(n,d)$ for a general form $F \in \mathbb{C}[x_0,\dots,x_n]_d$?

If Γ computes the rank (resp. smoothable rank or cactus rank) of F, then Γ is locally Gorenstein [Buczynska, Buczynski 2010, proof of Proposition 2.2]. Every local Gorenstein scheme Γ in $\mathbb{P}_T$ is isomorphic to $\mathrm{Spec}(\mathbb{C}[y_1,\dots,y_r]/g^{\perp})$ for some polynomial $g \in \mathbb{C}[x_1,\dots,x_r]$ (cf. [Iarrobino 1994, Lemma 1.2]). If Γ is a local Gorenstein component of an apolar subscheme to F, then the number r of variables in the polynomial g defining Γ is at most n but the degree of g may be larger than the degree of F. In particular, even local apolar subschemes of minimal length may not be of the kind $\Gamma(F_l)$, described above.

2. Cubic forms

If $F \in S$ is a general cubic form, then the cactus rank according to Proposition 1 is at most $2n+2$.

If F is a general reducible cubic form in S and l is a linear factor, then $f = F_l$ is a quadratic polynomial and $\Gamma(f)$ is smoothable of length at most $n+2$: The partials of a nonsingular quadratic polynomial in n variables form a vector space of dimension $n+2$, so this is the length of $\Gamma(f)$. On the other hand let E be an elliptic normal curve of degree $n+2$ in $\mathbb{P}^{n+1}$. Let $T(E)$ be the homogeneous coordinate ring of E. A quotient of $T(E)$ by two general linear forms is artinian Gorenstein with Hilbert function $(1,n,1)$ isomorphic to T_q for a quadric q of rank n. Thus T_f is isomorphic to T_q and $\Gamma(f)$ is smoothable.

Theorem 2. *For a general cubic form $F \in \mathbb{C}[x_0,\dots,x_n]$, the cactus rank is*

$$cr(F) \leq 2n+2.$$

For a general reducible cubic form $F \in \mathbb{C}[x_0,\dots,x_n]$ with $n > 1$, the cactus rank and the smoothable rank are

$$cr(F) = sr(F) = n+2.$$

Proof. It remains to show that for a general reducible cubic form $cr(F) \geq n+2$. If $\Gamma \subset \mathbb{P}_T$ has length less than $n+1$ it is contained in a hyperplane, so $I_\Gamma \subset F^{\perp}$ only if the latter contains a linear form. If $\{F=0\}$ is not a cone, this is not the case. If $\Gamma \subset \mathbb{P}_T$ has length $n+1$, then, for the same reason, this subscheme must span $\mathbb{P}_T$. Its ideal in that case is generated by $\binom{n+1}{2}$ quadratic forms. If F is general, $F_2^{\perp}$ is also generated by $\binom{n+1}{2}$, so they would have to coincide. On the other hand this equality is a closed condition on cubic forms. If $F = x_0(x_0^2 + \cdots + x_n^2)$, then

$$F_2^{\perp} = < y_1y_2, \dots, y_{n-1}y_n, y_0^2 - y_1^2, \dots, y_0^2 - y_n^2 >,$$

In particular $\dim F_2^{\perp} = \binom{n+1}{2}$, But the quadrics $F_2^{\perp}$ do not have any common zeros, so $cr(F) \geq n+2$. The general reducible cubic must therefore also have cactus rank at least $n+2$ and the theorem follows. □

Remark 2. By [Landsberg, Teitler 2010, Theorem 1.3] the lower bound for the rank of a reducible cubic form that depends on $n+1$ variables and not less, is $2n$.

If $F = x_0F_1(x_1,\dots,x_n)$ where F_1 is a quadratic form of rank n, then $cr(F) = sr(F) = n+1$, the same as for a Fermat cubic, while the rank is at least $2n$.

We give another example with $cr(F) = n+1 < sr(F)$. Let $G \in \mathbb{C}[x_1,\dots,x_m]$ be a cubic form such that the scheme $\Gamma(G) = \mathrm{Spec}(\mathbb{C}[y_1,\dots,y_m]/G^\perp)$ has length $2m+2$ and is not smoothable. Denote by $G_1 = y_1(G),\dots,G_m = y_m(G)$ the first order partials of G. Let

$$F = G + x_0x_1x_{m+1} + \cdots + x_0x_mx_{2m} + x_0^2x_{2m+1} \in \mathbb{C}[x_0,\dots,x_{2m+1}].$$

Then

$$F_{x_0} = G + x_1x_{m+1} + \cdots + x_mx_{2m} + x_{2m+1}$$

and

$$\mathrm{Diff}(F_{x_0}) =< F_{x_0}, G_1 + x_{m+1},\dots,G_m + x_{2m}, x_1,\dots,x_m, 1 >$$

so $\dim_{\mathbb{C}}\mathrm{Diff}(F_{x_0}) = 2m+2$. Therefore $\Gamma(F_{x_0})$ is apolar to F and computes the cactus rank of F. Since $\{F = 0\}$ is not a cone, $\Gamma(F_{x_0})$ is nondegenerate, so its homogeneous ideal is generated by the quadrics in the ideal of $F^\perp$. In particular $\Gamma(F_{x_0})$ is the unique apolar subscheme of length $2m+2$. Since this is not smoothable, the smoothable rank is strictly bigger.

By Proposition 2 the cactus rank of a generic cubic form $F \in \mathbb{C}[x_0,\dots,x_n]$ is at most $2n+2$. The first n for which $2n+2$ is smaller than the rank $r(F) = \lceil \frac{1}{n+1}\binom{n+3}{3}\rceil$ of the generic cubic form in $n+1$ variables is $n = 8$, where $r(F) = 19$ and $cr(F) \le 18$.

In the remainder of this paper we show that the bound on the cactus rank is sharp.

Proposition 1. *The cactus rank of a generic homogeneous cubic $F \in \mathbb{C}[x_0,\dots,x_8]$ is 18.*

Proof. The strategy for our proof is to consider the dimension h_d of the Hilbert scheme $Hilb_d(\mathbb{P}^8)$ of length d subschemes of $\mathbb{P}^8$, for $d < 18$. In the third Veronese embedding $\mathbb{P}^8 \to \mathbb{P}^{164}$, the span of each subscheme Γ of length d is a linear space L_Γ of dimension at most $d-1$. The cactus rank of a general cubic form is then the minimal d such that these linear spaces fill $\mathbb{P}^{164}$. So the cactus rank is at least the minimal d such that $h_d + d - 1 \ge 164$.

For this minimum it suffices to consider the subscheme of the Hilbert scheme parameterizing locally Gorenstein schemes, i.e. all of whose componenets are local Gorenstein schemes, cf. [Buczynska, Buczynski 2010]. Furthermore, any local Gorenstein scheme of length at most 10 is smoothable (cf. [Casnati, Notari 2011]), so for our estimate we need only to consider locally Gorenstein schemes with a component Γ_0 of length at least 11. Denote by $Hilb^{loc}_{G,l_0}\mathbb{A}^8_0$ the subscheme of the Hilbert scheme parameterizing local Gorenstein schemes of length l_0 supported at the origin of $\mathbb{A}^8$. Let g_{l_0} be the dimension of $Hilb^{loc}_{G,l_0}\mathbb{A}^8_0$. Comparing with the smoothable subschemes we get that $h_{l_0} = max\{g_{l_0} + 8, 8l_0\}$. In the above inequality $h_{17} + 17 - 1 \ge 164$ we then get (for $11 \le l_0 \le 17$) $g_{l_0} + 8 + 8(17 - l_0) + 17 - 1 = g_{l_0} + 160 - 8l_0 \ge 164$, i.e.

$$g_{l_0} \ge 8l_0 + 4.$$

Now each local Gorenstein scheme in $\mathbb{A}^8$ supported at the origin is a Gorenstein scheme $\Gamma(f) \subset \mathbb{A}^8$ defined by a polynomial $f \in \mathbb{C}[x_1,\dots,x_8]$. In particular the affine coordinate ring of $\Gamma(f)$ is a quotient $\mathbb{C}[y_1,\dots,y_8]/f^\perp$. The length of $\Gamma(f)$ equals the dimension of $\mathrm{Diff}(f)$, the space of partials of f of all orders. For our computations, we therefore consider set of polynomials

$$V_{l_0} = \{f \in \mathbb{C}[x_1,\dots,x_s] \mid \dim_{\mathbb{C}}\mathrm{Diff}(f) = l_0\} \subset \mathbb{C}[x_1,\dots,x_s]$$

modulo the equivalence relation $f \equiv g$ if and only if $\mathrm{Diff}(f) = \mathrm{Diff}(g)$. In particular $g_{l_0} = \mathrm{dim} V_{l_0} - l_0$.

Thus we have shown

Lemma 2. *The cactus rank is* 18*, if*

$$\mathrm{dim} V_{l_0} = \dim\{f \in \mathbb{C}[x_1, \dots, x_8] \mid \dim_{\mathbb{C}} \mathrm{Diff}(f) = l_0\} < 9l_0 + 4$$

for $l_0 = 11, \dots, 17$.

For each polynomial $f \in V_{l_0}$, the grading of the polynomial ring induces a degree-filtration of the vector space of partials. Assume that f has degree δ, then for each $i \leq \delta$ the partials of f of degree at most i form a subspace $\mathrm{Diff}(f)_i$ of $\mathrm{Diff}(f)$ and

$$\mathrm{Diff}(f)_0 \subset \mathrm{Diff}(f)_1 \subset \mathrm{Diff}(f)_2 \subset \ldots \subset \mathrm{Diff}(f)_\delta = \mathrm{Diff}(f)$$

where $\mathrm{Diff}(f)_0 =< 1 >$ and $\mathrm{Diff}(f)_1 \subset< 1, x_1, \dots, x_8 >$.

$\mathrm{Diff}(f)$ is a vector space of polynomials, closed under differentiation. Furthermore there is a natural linear isomorphism

$$\tau : T/f^\perp \to \mathrm{Diff}(f); \quad g \mapsto g(f).$$

Let m be the maximal ideal in the local ring $T/f^\perp$ and let δ be the degree of f. The image of $(0 : m^i)$ under the map τ is precisely $\mathrm{Diff}(f)_{i-1}$, so the Loewy filtration

$$(0 : m^\delta) \subset (0 : m^{\delta-1}) \subset (0 : m^{\delta-2}) \subset \ldots \subset (0 : m) \subset T/f^\perp$$

of $T/f^\perp$ is mapped to the degree-filtration of $\mathrm{Diff}(f)$. Since $(0 : m^i)/(0 : m^{i-1}) \cong (m^{i-1}/m^i)^*$ the integral function

$$H_f(0) = 1, H_f(i) = \dim_{\mathbb{C}} \mathrm{Diff}(f)_i - \dim_{\mathbb{C}} \mathrm{Diff}(f)_{i-1}, i = 1, \dots, \delta$$

coincides with the Hilbert function $h_f(i) = \dim_{\mathbb{C}} m^i/m^{i+1}$ of the associated graded ring $\mathbb{C} \oplus m/m^2 \oplus \cdots \oplus m^\delta$ to the quotient ring $T/f^\perp$.

The order of a partial $g(f)$ of f is defined as the smallest degree term g_i of $g \in T$ such that $g_i(f)$ is nonzero. Thus the image $\tau(m^i) \subset \mathrm{Diff}_{\delta-i}(f)$ is simply the space of partials of order at least i of f.

Let

$$C_{a,i} = \tau((0 : m^{i+1-a}) \cap m^{\delta-i})/\tau((0 : m^{i+1-a}) \cap m^{\delta-i}),$$

i.e. $C_{a,i} \subset \mathrm{Diff}_i(f)$ is the vector space of partial derivatives of f of degree i and order at least $\delta - a - i$. Thus

$$C_{0,i} \subset C_{1,i} \subset \ldots \subset C_{\delta,i} = \mathrm{Diff}_i(f).$$

Let

$$\Delta_{a,f}(i) = \dim_{\mathbb{C}}(C_{a,i}/C_{a-1,i})$$

Iarrobino shows that for each non-negative a, the integral function $\Delta_{a,f}(i)$ is symmetric around $(\delta - a)/2$ [Iarrobino 1994, Theorem 1.5]. Clearly, these functions add up to H_f

$$H_f = \sum_a \Delta_{a,f}$$

so we obtain a symmetric decomposition of the Hilbert function H_f. Furthermore, if

$$f = f_\delta + f_{\delta-1} + \cdots + f_1 + f_0$$

is the decomposition of f into homogeneous summands of degree given by the index, then any partial derivative of f of degree i and order $\delta-a-i$ must be a partial of the summand $f_{\delta-a}+\cdots+f_0$. Therefore

$$H_{f_\delta+\cdots+f_{\delta-a}} = \Delta_{\leq a} = \sum_{b\leq a}\Delta_b.$$

In particular, each partial sum $\Delta_{\leq a}$ is the Hilbert function of a $\mathbb{C}$-algebra quotient of $\mathbb{C}[x_1,\ldots,x_8]$.

To estimate the dimension of V_{l_0} we consider the decomposition of V_{l_0} into subsets according to the Hilbert function and its symmetric decomposition Δ:

$$V_{l_0} = \cup_{H,\Delta}V_{l_0}^{H,\Delta} = \cup_H\{f\in V_{l_0} \mid H_f = H, \Delta_{a,f}=\Delta_a\}$$

where H ranges over all possible Hilbert functions whose total sum of values is l_0, and the union ranges over all possible symmetric decompositions of H. Since the components $V_{l_0}^H$ are disjoint, the dimension of V_{l_0} equals the dimension of the largest component $V_{l_0}^{H,\Delta}$.

We describe the Hilbert function H and its symmetric decomposition by its values:

$$H : (H(0), H(1), \ldots, H(\delta))$$
$$H = \sum_a \Delta_a, \quad \Delta_a(i) = \Delta_a(\delta-a-i).$$

As Hilbert functions H and $\Delta_{\leq a}$ satisfy first of all that $a_i > 0$, for $0\leq i\leq\delta$. Secondly each partial sum $\Delta_{\leq a}$ satisfies the Macaulay growth condition (cf. [Macaulay]): If the i-binomial expansion of $\Delta_{\leq a}(i)$ is

$$\Delta_{\leq a}(i) = \binom{m_i}{i}+\binom{m_{i-1}}{i-1}+\cdots+\binom{m_j}{j}; \quad m_i > m_{i-1} > \cdots > m_j \geq j\geq 1,$$

then

$$\Delta_{\leq a}(i+1) = \binom{m_i+1}{i+1}+\binom{m_{i-1}+1}{i}+\cdots+\binom{m_j+1}{j+1}. \tag{2}$$

We now argue by considering all Hilbert functions H whose sum of values is between 11 and 17, and all possible symmetric decompositions $H=\sum_{i\leq\delta}\Delta_i$, where δ is largest argument with nonzero value for H.

Let us fix the notation,

$$H = (1, H(1), \ldots, H(\delta-1), 1), \tag{3}$$

and let $H_{sum} = 1 + H(1)+\cdots+(\delta-1)+1$ be the sum of the values.

Example: For $H(1)=8, H(2)\geq 5$ and $H_{sum}=17$ the possible Hilbert functions H and their decompositions that satisfy the Macaulay growth conditions are the following: $H=\sum_i\Delta_i$

$$\begin{array}{rcccc} H= & 1 & 8 & 7 & 1\\ \Delta_0= & 1 & 7 & 7 & 1\\ \Delta_1= & 0 & 1 & 0 & 0\end{array}, \quad \begin{array}{rccccc} H= & 1 & 8 & 6 & 1 & 1\\ \Delta_0= & 1 & 1 & 1 & 1 & 1\\ \Delta_1= & 0 & 5 & 5 & 0 & 0\\ \Delta_2= & 0 & 2 & 0 & 0 & 0\end{array}, \quad \begin{array}{rcccccc} H= & 1 & 8 & 5 & 1 & 1 & 1\\ \Delta_0= & 1 & 1 & 1 & 1 & 1 & 1\\ \Delta_1= & 0 & 4 & 4 & 0 & 0 & 0\\ \Delta_2= & 0 & 3 & 0 & 0 & 0 & 0\end{array}$$

$$\begin{array}{rccccc} H= & 1 & 8 & 5 & 2 & 1\\ \Delta_0= & 1 & 2 & 3 & 2 & 1\\ \Delta_1= & 0 & 2 & 2 & 0 & 0\\ \Delta_2= & 0 & 4 & 0 & 0 & 0\end{array}, \quad \begin{array}{rccccc} H= & 1 & 8 & 5 & 2 & 1\\ \Delta_0= & 1 & 2 & 2 & 2 & 1\\ \Delta_1= & 0 & 3 & 3 & 0 & 0\\ \Delta_2= & 0 & 3 & 0 & 0 & 0\end{array},$$

In the appendix we list all decompositions of Hilbert functions with $H_{sum}=17$.

The value $H(1)$ is the embedding dimension of $\Gamma(f)$. We compute an upper bound for the dimension of $V_{l_0}^{H,\Delta}$, the set of polynomials $f = \sum_i f_i$ with Hilbert function H, $H_{sum} = l_0$ and symmetric decomposition $H = \sum_a \Delta_a$, where

$$\Delta_a(i) = \dim C_{a,i}/C_{a-1,i}$$

and $C_{a,i} \subset \mathrm{Diff}_i(f)$ is the vector space of partials of f of degree i and order at least $\delta - a - i$. Inductively, we compute an upper bound d_i for the dimension of the set of summands f_i given $f_\delta + \cdots + f_{i+1}$ for $i = \delta, \delta - 1, \ldots, 1$. Thus $\dim V_{l_0}^{H,\Delta} \leq d_\delta + \cdots + d_1 + 1$.

A polynomial $f \in V_{l_0}^{H,\Delta}$ defines a filtration of S_1 and T_1 defined in terms of the symmetric decomposition Δ. So we fix a filtration first, and then consider the variety of filtrations to find estimate the dimension of $V_{l_0}^{H,\Delta}$.

Given a symmetric decomposition Δ, let

$$b_i = \sum_{j=0}^{i} \Delta_j(1).$$

and consider the filtration

$$< x_1, \ldots, x_{b_0} > \subset < x_1, \ldots, x_{b_1} > \subset \ldots \subset < x_1, \ldots, x_{b_{\delta-2}} > \subset < x_1, \ldots, x_8 > = S_1.$$

Let f be a polynomial with symmetric decomposition Δ of its Hilbert function. Then $x_1, \ldots, x_{b_0}$ span the partials of degree one and order at least $\delta - 1$, and more generally, $x_1, \ldots, x_{b_i}$ span the partials of f of degree one and order at least $\delta - 1 - i$.

Lemma 3. *Let $f = f_\delta + \cdots + f_{\delta-a}$ and assume that $y \in T$ is a linear form such that $y(f_\delta + \cdots + f_{\delta-a+1}) = y(\mathrm{Diff}(f)_1) = 0$, while $y(f) = y(f_{\delta-a}) = g \neq 0$. Then there is a linear form x, with $y(x) = 1, f_{\delta-a} = xg + h$ and $y(g) = y(h) = 0$. Furthermore, for any linear form $y' \in T$, if $y'(\mathrm{Diff}(f)_1) = 0$, then $y'(g) = 0$.*

Proof. Since $y(f_\delta + \cdots + f_{\delta-a+1}) = 0$, we get $y(f) = y(f_{\delta-a})$, and $y(f_{\delta-a}) = g \neq 0$ means that there is a linear form x such that $f_{\delta-a} = xg + h$, with $y(h) = 0$ and $y(x) = 1$. The form g is homogeneous, so if $y(y(f)) = y(y(f_{\delta-a})) = y(g) \neq 0$, then x is a partial of order $\delta - a - 2$ of g, and hence x is a partial of degree one of f contrary to the assumption that $y(\mathrm{Diff}(1)) = 0$. Therefore $y(y(f_{\delta-a})) = y^2(xg + h) = y(g + xy(g)) = 2y(g) + xy^2(g) = 0$, i.e. $2y(g) = -xy(y(g))$. But then $y(g) = 0$, since g is a polynomial. Finally, assume $y' \in T_1$ and $y'(g) = g'$. There is a $u \in T$ such that $u(g) = x'$ and $y'(x') = 1$. Then $uy(f) = x'$, and hence $y'(\mathrm{Diff}(f)_1) \neq 0$. □

We call xg, as in Lemma 3, an exotic summand for f of degree $\delta - a$.

Corollary 1. *$f = f_\alpha + f_\omega$ where $T/f^\perp$ and $T/f_\alpha^\perp$ are isomorphic, and $f_\alpha \in \mathbb{C}[x_1, ..., x_{b_\delta}]$, while f_ω is a finite sum of exotic summands.*

Proof. If xg is an exotic summand, then $y(f) = g = u(f)$ for some $u \in \mathbb{C}[y_1, \ldots, y_{b_\delta}]$. Then $y - u \in f^\perp$, hence f and $f - gx$ have isomorphic apolar rings. □

Corollary 2. *The leading summand of a partial of f of degree $\delta - i$ and order j lies in $\mathbb{C}[x_1, \ldots, x_{b_{i-j}}]$.*

Proof. It suffices to consider partials of order $j = 1$. If g is the leading term of degree $\delta - i$ of a partial of order one, and $g \notin \mathbb{C}[x_1, \ldots, x_{b_{i-1}}]$, then there is a y with $yx_j = 0$ for $j \leq b_{i-1}$ and $y(g) \neq 0$. Let $u \in T$ be a form such that $u(y(g)) = 1$. Then $u(g) = x$ is a linear form with $y(x) = 1$, so x is not in $\mathbb{C}[x_1, \ldots, x_{b_{i-1}}]$. But then x is also a partial of order one of $f_{>\delta-i}$, contrary to the assumption. □

In enumerating polynomials f with given Hilbert function and symmetric decomposition, we write $f = f_\alpha + f_\beta$, where f_α has summands $f_{i,\alpha} \in \mathbb{C}[x_1, \dots, x_{b_i}]$, while f_β is a sum of exotic summands.

In the largest degree δ the summand $f_\delta = f_{\delta,\alpha}$ is a form in $b_0 = \Delta_0(1)$ variables, with a $\Delta_0(2)$-dimensional space of second order partials, so if $\delta \geq 4$ we get

$$d_\delta \leq \binom{b_0 - 1 + \delta}{b_0 - 1} - 1/2(\binom{b_0 + 1}{2} - \Delta_0(2) + 1)(\binom{b_0 + 1}{2} - \Delta_0(2))$$

The second summand is a lower bound for the codimension of the rank $\Delta_0(2)$ locus for a symmetric submatrix of a catalecticant matrix in degrees $(\delta - 2) \times 2$, therefore if $\delta = 3$ we simply take $d_3 \leq \binom{b_0+2}{b_0-1}$.

If $i > 0$, and $\delta - i \geq 4$, then the form $f_{\delta-i}$ has two summands $f_{\delta-i} = f_{\delta-i,i} + f_{\delta-i,\infty}$. The first summand belong to f_α, while the second belongs to f_β.

The first summand $f_{\delta-i,i}$ is a form of degree $\delta - i$ in b_i variables with at most

$$c_i = \Delta_0(\delta - i - 2) + \cdots + \Delta_i(\delta - i - 2)$$

partials of order 2, so it depends on

$$d_{\delta-i,i} = \binom{\delta - i + b_i - 1}{b_i - 1} - \binom{\binom{b_i+1}{2} - c_i + 1}{2}$$

parameters.

The second summand is exotic. It is a sum of products of linear forms in $8 - b_i$ variables and a partial of f of degree $\delta - i - 1$, so it depends on at most

$$d_{d-i,\infty} = (8 - b_i)(\sum_{j=0}^{i} \Delta_j(\delta - i - 1))$$

parameters. So we take $d_{\delta-i} \leq d_{\delta-i,i} + d_{\delta-i,\infty}$.

In degree 3 the estimate is

$$d_3 \leq \binom{b_{\delta-3} + 2}{3} + (8 - b_{\delta-3}) \sum_{j=0}^{\delta-3} \Delta_j(2),$$

while in degree 2 and 1 the estimate is

$$d_2 \leq \binom{b_{\delta-2} + 1}{2} + (8 - b_{\delta-2})(b_{\delta-2}) \quad \text{and} \quad d_1 = 8.$$

We finally turn to the set of filtrations of $< x_1, \dots, x_8 >$ with respect to the symmetric decomposition of H. It is parameterized by the flag variety $Fl(b_0, b_1, \dots, b_\delta, 8)$. The dimension of $Fl(b_0, b_1, \dots, b_\delta, 8)$ is

$$d_\Delta \leq b_0(8 - b_0) + (b_1 - b_0)(8 - b_1) + \cdots + (b_\delta - b_{\delta-1})(8 - b_\delta) = \sum_j \Delta_j(1)(8 - b_j)$$

Together we get:

$$\dim V_{l_0}^{H,\Delta} \leq d_\delta + \sum_{i\geq 0} d_{\delta-i} + d_\Delta = \tag{4}$$
$$= \binom{b_0-1+\delta}{b_0-1} - 1/2(\binom{b_0+1}{2} - \Delta_0(2)+1)(\binom{b_0+1}{2} - \Delta_0(2))$$
$$+ \sum_{i=1}^{\delta-4}(\binom{\delta-i+b_i-1}{b_i-1} - \binom{\binom{b_i+1}{2}-c_i+1}{2})$$
$$+ \binom{b_{\delta-3}+2}{3} + \binom{b_{\delta-2}+1}{2} + (8-b_{\delta-2})(b_{\delta-2}) + 9$$
$$+ \sum_{i=1}^{\delta-3}(8-b_i)(\sum_{k=0}^{i}\Delta_k(\delta-i-1))$$
$$+ \sum_j \Delta_j(1)(8-b_j).$$

Example:

$H=$	1	4	5	4	1	1	1
$\Delta_0=$	1	1	1	1	1	1	1
$\Delta_2=$	0	3	4	3	0	0	0

We get that $\delta=6, l_0=17, \Delta_0(1)=1, \Delta_2(1)=3$ and $(b_0,b_1,b_2,b_3,b_4,b_5,b_6)=(1,1,4,4,4,4,4)$. Therefore $d_1=8, d_2=26, d_3=40, d_4=36, d_5=8, d_6=1, d_\Delta=19$, hence $d=1+d_1+\cdots+d_6+d_\Delta=139$ which is less than $9\cdot 17+4=157$.

The dimension estimate for the other Hilbert functions are computed similarly, with no estimate exceeding the bound $9l_0+4$. We list all of them in the Appendix. The computations are done with Matlab. □

Acknowledgements. The authors would like to thank the Institut Mittag-Leffler (Djursholm, Sweden) for their support and hospitality, and Tony Iarrobino for helpful comments on Gorenstein algebras. The first author was partially supported by Project Galaad of INRIA Sophia Antipolis Méditerranée (France) and by Marie Curie Intra-European Fellowships for Career Development (FP7-PEOPLE-2009-IEF): "DECONSTRUCT".

References

[Bernardi et al. 2011] Bernardi, Alessandra, Gimigliano, Alessandro, Idà, Monica: Computing symmetric rank for symmetric tensors, *J. Symbolic Comput.* **46**, (2011), pp. 34–53.

[Brachat et al. 2010] Brachat, Jerome, Comon, Pierre, Mourrain, Bernard, Tsigaridas , Elias: Symmetric tensor decomposition, *Linear Algebra and Applications* **433**, (2010)1851–1872

[Buczynska, Buczynski 2010] Buczynska, Weronika, Buczynski, Jaroslaw: Secant varieties to high degree Veronese reembeddings, catalecticant matrices and smoothable Gorenstein schemes, arXiv:1012.3563.

[Carlini et al. 2011] Carlini, Enrico, Catalisano, Maria V., Geramita, Anthony V.: The Solution to Waring's Problem for Monomials, arXiv:1110.0745.

[Casnati, Notari 2011] Casnati, Gianfranco, Notari, Roberto: On the irreducibility and the singularities of the Gorenstain locus of the punctual Hilbert scheme of degree 10, *J. Pure Appl. Algebra* **215** (2011), 1243–1254.

[Chiantini, Ciliberto, 2002] Chiantini, Luca, Ciliberto, Ciro: Weakly defective varieties. *Trans. Amer. Math. Soc.* **354** (2002) 151–178.

[Elias, Rossi 2011] Elias, Juan, Rossi, Maria E.: Isomorphism classes of short Gorenstein local rings via Macaulay's inverse system, Preprint, to appear in *Transactions of AMS.*

[Iarrobino 1994] Iarrobino, Anthony: Associated graded algebra of a Gorenstein Artin Algebra, *Mem. Amer. Math. Soc.* **107**, (1994), no 514, Amer. Math. Soc. Providence.

[Iarrobino, Kanev 1999] Iarrobino, Anthony, Kanev, Vassil: Power Sums, Gorentein Algebras and Determinantal Loci, *Lecture Notes in Mathematics* **1721**, Springer-Verlag, Berlin Heidelberg New York (1999).

[Landsberg, Teitler 2010] Landsberg, Joseph M., Teitler, Zach: On the ranks and border ranks of symmetric tensors, Found Comput. Math. **10**, (2010), no 3, 339–366.

[Macaulay] Macaulay, F.S.: Some properties of enumeration in the theory of modular systems, *Proc. London Math. Soc.* **26** (1927),531–555.

[Landsberg, Ottaviani 2011] Landsberg, Joseph M., Ottaviani, Giorgio: Equations for secant varieties to Veronese varieties. arXiv:1006.0180.

[Ranestad, Schreyer 2000] Ranestad, Kristian, Schreyer, Frank-Olaf: Varieties of Sums of Powers. *J. reine angew. Math.* **525** (2000) 147–181

Appendix

In this section we list all the possible Hilbert functions H's for which $11 \leq l_0 \leq 17$ and that satisfy the Macaulay growth condition (2). For all of them we will write the corresponding value of $d = d_\delta + \sum_{i\geq 0} d_{\delta-i} + d_\Delta$ as computed in (4). For the case $l_0 = 17$ we will also present the symmetric decompositions $H = \sum_{i\leq\delta} \Delta_i$. For $11 \leq l_0 \leq 16$ we will not write them because they are already listed in [Iarrobino 1994] §5.F. When the symmetric decompositions of H is not unique we get different values for d's, in that case, if $11 \leq l_0 \leq 16$, we will write only the biggest one.

We explain here the compact notation that we will use in Section 2.7.
Example:

$$\begin{array}{rccccccc} H = & 1 & 4 & 5 & 4 & 1 & 1 & 1 \\ \Delta_0 = & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \Delta_2 = & 0 & 3 & 4 & 3 & 0 & 0 & 0 \end{array}$$

In this case we will write

$$(1,4,5,4,1,1,1) \to (1,1,1,1,1,1,1),(0,3,4,3,0,0,0).$$

The fact that $(1,1,1,1,1,1,1)$ and $(0,3,4,3,0,0,0)$ correspond to Δ_0 and Δ_2 respectively, is clear from the centre of symmetry. In fact for $2 \leq i \leq l$ we have that

$$\Delta_{l-i} = (\Delta_{l-i}(0),\ldots,\Delta_{l-i}(l-i),0,\ldots,0)$$

with $l-i$ zero at the end and $\Delta_{l-i}(j) = \Delta_{l-i}(l-i-j)$ for $0 \leq j \leq l-i$ ([Iarrobino 1994, Theorem 1.5]).

2.1. $l_0 = 11$**.** In this case we have that the bound $9l_0 + 4$ computed in Lemma 2 is 103. The list below shows that $d < 103$ for any possible choice of H with $l_0 = 11$.

$(1,5,4,1)$, $d=78$	$(1,6,3,1)$, $d=73$	$(1,7,2,1)$, $d=65$
$(1,8,1,1)$, $d=53$	$(1,4,3,2,1)$, $d\leq 85$	$(1,4,4,1,1)$, $d=91$
$(1,5,2,2,1)$, $d=80$	$(1,5,3,1,1)$, $d=88$	$(1,6,2,1,1)$, $d=80$
$(1,7,1,1,1)$, $d=66$	$(1,4,2,2,1,1)$, $d=89$	$(1,4,3,1,1,1)$, $d=90$
$(1,5,2,1,1,1)$, $d=86$	$(1,6,1,1,1,1)$, $d=76$	$(1,4,2,1,1,1,1)$, $d=89$
$(1,5,1,1,1,1,1)$, $d=83$	$(1,4,1,1,1,1,1,1)$, $d=87$	

2.2. $l_0 = 12$**.** In this case we have that the bound $9l_0 + 4$ computed in Lemma 2 is 112. The list below shows that $d < 112$ for any possible choice of H with $l_0 = 12$.

$(1,5,5,1)$, $d=89$	$(1,6,4,1)$, $d=82$	$(1,7,3,1)$, $d=73$	$(1,8,2,1)$, $d=61$
$(1,4,3,3,1)$, $d=88$	$(1,4,4,2,1)$, $d\leq 95$	$(1,5,3,2,1)$, $d\leq 91$	$(1,5,4,1,1)$, $d=98$
$(1,6,2,2,1)$, $d=82$	$(1,6,3,1,1)$, $d=91$	$(1,7,2,1,1)$, $d=79$	$(1,8,1,1,1)$, $d=61$
$(1,4,2,2,2,1)$, $d=92$	$(1,4,3,2,1,1)$, $d=99$	$(1,4,4,1,1,1)$, $d=99$	$(1,5,2,2,1,1)$, $d=94$
$(1,5,3,1,1,1)$, $d=96$	$(1,6,2,1,1,1)$, $d=88$	$(1,7,1,1,1,1)$, $d=74$	$(1,4,2,2,1,1,1)$, $d=97$

$(1,4,3,1,1,1,1)$, $d=98$	$(1,5,2,1,1,1,1)$, $d=94$	$(1,6,1,1,1,1,1)$, $d=84$
$(1,4,2,1,1,1,1,1)$, $d=97$	$(1,5,1,1,1,1,1,1)$, $d=91$	$(1,4,1,1,1,1,1,1,1)$, $d=95$

2.3. $l_0=13$. In this case we have that the bound $9l_0+4$ computed in Lemma 2 is 121. The list below shows that $d<121$ for any possible choice of H with $l_0=13$.

$(1,6,5,1)$, $d=94$	$(1,7,4,1)$, $d=83$	$(1,8,3,1)$, $d=70$
$(1,4,3,4,1)$, $d=90$	$(1,4,4,3,1)$, $d\leq 99$	$(1,4,5,2,1)$, $d=100$
$(1,5,3,3,1)$, $d=94$	$(1,5,4,2,1)$, $d\leq 102$	$(1,5,5,1,1)$, $d=109$
$(1,6,3,2,1)$, $d\leq 94$	$(1,6,4,1,1)$, $d=102$	$(1,7,2,2,1)$, $d=81$
$(1,7,3,1,1)$, $d=91$	$(1,8,2,1,1)$, $d=75$	$(1,4,3,2,2,1)$, $d\leq 102$
$(1,4,3,3,1,1)$, $d=106$	$(1,4,4,2,1,1)$, $d=109$	$(1,5,2,2,2,1)$, $d=97$
$(1,5,3,2,1,1)$, $d=105$	$(1,5,4,1,1,1)$, $d=106$	$(1,6,2,2,1,1)$, $d=96$
$(1,6,3,1,1,1)$, $d=99$	$(1,7,2,1,1,1)$, $d=87$	$(1,8,1,1,1,1)$, $d=69$
$(1,4,2,2,2,1,1)$, $d=106$	$(1,4,3,2,1,1,1)$, $d=107$	$(1,4,4,1,1,1,1)$, $d=107$
$(1,5,2,2,1,1,1)$, $d=102$	$(1,5,3,1,1,1,1)$, $d=104$	$(1,6,2,1,1,1,1)$, $d=96$
$(1,7,1,1,1,1,1)$, $d=82$	$(1,4,2,2,1,1,1,1)$, $d=105$	$(1,4,3,1,1,1,1,1)$, $d=106$
$(1,5,2,1,1,1,1,1)$, $d=102$	$(1,6,1,1,1,1,1,1)$, $d=92$	$(1,4,2,1,1,1,1,1,1)$, $d=105$
$(1,5,1,1,1,1,1,1,1)$, $d=99$	$(1,4,1,1,1,1,1,1,1,1)$, $d=103$	

2.4. $l_0=14$. In this case we have that the bound $9l_0+4$ computed in Lemma 2 is 130. The list below shows that $d<130$ for any possible choice of H with $l_0=14$.

$(1,6,6,1)$, $d=110$	$(1,7,5,1)$, $d=96$	$(1,8,4,1)$, $d=81$
$(1,4,4,4,1)$, $d=101$	$(1,4,5,3,1)$, $d\leq 106$	$(1,5,3,4,1)$, $d=97$
$(1,5,4,3,1)$, $d\leq 106$	$(1,5,5,2,1)$, $d\leq 114$	$(1,6,3,3,1)$, $d=97$
$(1,6,4,2,1)$, $d\leq 106$	$(1,6,5,1,1)$, $d=114$	$(1,7,3,2,1)$, $d\leq 94$
$(1,7,4,1,1)$, $d=103$	$(1,8,2,2,1)$, $d=77$	$(1,8,3,1,1)$, $d=88$
$(1,4,3,3,2,1)$, $d\leq 110$	$(1,4,3,4,1,1)$, $d=110$	$(1,4,4,2,2,1)$, $d\leq 112$
$(1,4,4,3,1,1)$, $d\leq 117$	$(1,5,3,2,2,1)$, $d\leq 108$	$(1,5,3,3,1,1)$, $d=112$
$(1,5,4,2,1,1)$, $d=116$	$(1,5,5,1,1,1)$, $d=117$	$(1,6,2,2,2,1)$, $d=99$
$(1,6,3,2,1,1)$, $d=108$	$(1,6,4,1,1,1)$, $d=110$	$(1,7,2,2,1,1)$, $d=95$
$(1,7,3,1,1,1)$, $d=99$	$(1,8,2,1,1,1)$, $d=83$	$(1,4,2,2,2,2,1)$, $d=110$
$(1,4,3,2,2,1,1)$, $d\leq 116$	$(1,4,3,3,1,1,1)$, $d=114$	$(1,4,4,2,1,1,1)$, $d=117$
$(1,5,2,2,2,1,1)$, $d=111$	$(1,5,3,2,1,1,1)$, $d=113$	$(1,5,4,1,1,1,1)$, $d=114$
$(1,6,2,2,1,1,1)$, $d=104$	$(1,6,3,1,1,1,1)$, $d=107$	$(1,7,2,1,1,1,1)$, $d=95$
$(1,8,1,1,1,1))$, $d=77$	$(1,4,2,2,2,1,1,1)$, $d=114$	$(1,4,3,2,1,1,1,1)$, $d=115$
$(1,4,4,1,1,1,1,1)$, $d=115$	$(1,5,2,2,1,1,1,1)$, $d=110$	$(1,5,3,1,1,1,1,1)$, $d=112$
$(1,6,2,1,1,1,1,1)$, $d=104$	$(1,7,1,1,1,1,1,1)$, $d=90$	$(1,4,2,2,1,1,1,1,1)$, $d=113$
$(1,4,3,1,1,1,1,1,1)$, $d=114$	$(1,5,2,1,1,1,1,1,1)$, $d=110$	$(1,6,1,1,1,1,1,1,1)$, $d=100$
$(1,4,2,1,1,1,1,1,1,1)$, $d=113$	$(1,5,1,1,1,1,1,1,1,1)$, $d=107$	$(1,4,1,1,1,1,1,1,1,1,1)$, $d=111$

2.5. $l_0=15$. In this case we have that the bound $9l_0+4$ computed in Lemma 2 is 139. The list below shows that $d<139$ for any possible choice of H with $l_0=15$.

$(1,7,6,1)$, $d=113$	$(1,8,5,1)$, $d=95$	$(1,4,5,4,1)$, $d=111$	$(1,4,6,3,1)$, $d\leq 112$
$(1,5,4,4,1)$, $d\leq 112$	$(1,5,5,3,1)$, $d\leq 119$	$(1,5,6,2,1)$, $d=118$	$(1,6,3,4,1)$, $d=101$
$(1,6,4,3,1)$, $d\leq 110$	$(1,6,5,2,1)$, $d\leq 119$	$(1,6,6,1,1)$, $d=128$	$(1,7,3,3,1)$, $d=97$
$(1,7,4,2,1)$, $d\leq 107$	$(1,7,5,1,1)$, $d=116$	$(1,8,3,2,1)$, $d\leq 91$	$(1,8,4,1,1)$, $d=101$
$(1,4,3,3,3,1)$, $d=118$	$(1,4,3,4,2,1)$, $d\leq 116$	$(1,4,4,3,2,1)$, $d\leq 121$	$(1,4,4,4,1,1)$, $d=121$
$(1,4,5,2,2,1)$, $d=117$	$(1,4,5,3,1,1)$, $d=124$	$(1,5,3,3,2,1)$, $d\leq 116$	$(1,5,3,4,1,1)$, $d=117$

$(1,5,4,2,2,1)$, $d \leq 119$	$(1,5,4,3,1,1)$, $d \leq 124$	$(1,5,5,2,1,1)$, $d = 128$
$(1,6,3,2,2,1)$, $d \leq 111$	$(1,6,3,3,1,1)$, $d = 115$	$(1,6,4,2,1,1)$, $d = 120$
$(1,6,5,1,1,1)$, $d = 122$	$(1,7,2,2,2,1)$, $d = 98$	$(1,7,3,2,1,1)$, $d = 108$
$(1,7,4,1,1,1)$, $d = 111$	$(1,8,2,2,1,1)$, $d = 91$	$(1,8,3,1,1,1)$, $d = 96$
$(1,4,3,2,2,2,1)$, $d \leq 120$	$(1,4,3,3,2,1,1)$, $d = 124$	$(1,4,3,4,1,1,1)$, $d = 118$
$(1,4,4,2,2,1,1)$, $d \leq 126$	$(1,4,4,3,1,1,1)$, $d \leq 125$	$(1,5,2,2,2,2,1)$, $d = 115$
$(1,5,3,2,2,1,1)$, $d \leq 122$	$(1,5,3,3,1,1,1)$, $d = 120$	$(1,5,4,2,1,1,1)$, $d = 124$
$(1,5,5,1,1,1,1)$, $d = 125$	$(1,6,2,2,2,1,1)$, $d = 113$	$(1,6,3,2,1,1,1)$, $d = 116$
$(1,6,4,1,1,1,1)$, $d = 118$	$(1,7,2,2,1,1,1)$, $d = 103$	$(1,7,3,1,1,1,1)$, $d = 107$
$(1,8,2,1,1,1,1)$, $d = 91$	$(1,4,2,2,2,2,1,1)$, $d = 124$	$(1,4,3,2,2,1,1,1)$, $d \leq 124$
$(1,4,3,3,1,1,1,1)$, $d = 122$	$(1,4,4,2,1,1,1,1)$, $d = 125$	$(1,5,2,2,2,1,1,1)$, $d = 119$
$(1,5,3,2,1,1,1,1)$, $d = 121$	$(1,5,4,1,1,1,1,1)$, $d = 122$	$(1,6,2,2,1,1,1,1)$, $d = 112$
$(1,6,3,1,1,1,1,1)$, $d = 115$	$(1,7,2,1,1,1,1,1)$, $d = 103$	$(1,8,1,1,1,1,1,1)$, $d = 85$
$(1,4,2,2,2,1,1,1,1)$, $d = 122$	$(1,4,3,2,1,1,1,1,1)$, $d = 123$	$(1,4,4,1,1,1,1,1,1)$, $d = 123$
$(1,5,2,2,1,1,1,1,1)$, $d = 118$	$(1,5,3,1,1,1,1,1,1)$, $d = 120$	$(1,6,2,1,1,1,1,1,1)$, $d = 112$
$(1,7,1,1,1,1,1,1,1)$, $d = 98$	$(1,4,2,2,1,1,1,1,1,1)$, $d = 121$	$(1,4,3,1,1,1,1,1,1,1)$, $d = 122$
$(1,5,2,1,1,1,1,1,1,1)$, $d = 118$	$(1,6,1,1,1,1,1,1,1,1)$, $d = 108$	$(1,4,2,1,1,1,1,1,1,1,1)$, $d = 121$
$(1,5,1,1,1,1,1,1,1,1,1)$, $d = 115$	$(1,4,1,1,1,1,1,1,1,1,1,1)$, $d = 119$	

2.6. $l_0 = 16$**.** In this case we have that the bound $9l_0 + 4$ computed in Lemma 2 is 148. The list below shows that $d < 148$ for any possible choice of H with $l_0 = 16$.

$(1,7,7,1)$, $d = 135$	$(1,8,6,1)$, $d = 113$	$(1,4,6,4,1)$, $d = 120$
$(1,4,7,3,1)$, $d = 117$	$(1,5,4,5,1)$, $d = 105$	$(1,5,5,4,1)$, $d \leq 122$
$(1,5,6,3,1)$, $d \leq 125$	$(1,6,4,4,1)$, $d \leq 117$	$(1,6,5,3,1)$, $d \leq 124$
$(1,6,6,2,1)$, $d \leq 134$	$(1,7,3,4,1)$, $d = 102$	$(1,7,4,3,1)$, $d \leq 111$
$(1,7,5,2,1)$, $d \leq 121$	$(1,7,6,1,1)$, $d = 131$	$(1,8,3,3,1)$, $d = 94$
$(1,8,4,2,1)$, $d \leq 105$	$(1,8,5,1,1)$, $d = 115$	$(1,4,3,4,3,1)$, $d = 124$
$(1,4,4,3,3,1)$, $d \leq 129$	$(1,4,4,4,2,1)$, $d \leq 127$	$(1,4,5,3,2,1)$, $d = 124$
$(1,4,5,4,1,1)$, $d = 131$	$(1,5,3,3,3,1)$, $d = 124$	$(1,5,3,4,2,1)$, $d = 122$
$(1,5,3,4,2,1)$, $d \leq 122$	$(1,5,4,3,2,1$, $d \leq 128$	$(1,5,4,4,1,1)$, $d \leq 132$
$(1,5,5,2,2,1)$, $d \leq 131$	$(1,5,5,3,1,1)$, $d \leq 137$	$(1,6,3,3,2,1)$, $d \leq 119$
$(1,6,3,4,1,1)$, $d = 121$	$(1,6,4,2,2,1)$, $d \leq 123$	$(1,6,4,3,1,1)$, $d \leq 128$
$(1,6,5,2,1,1)$, $d = 133$	$(1,6,6,1,1,1)$, $d = 136$	$(1,7,3,2,2,1)$, $d \leq 111$
$(1,7,3,3,1,1)$, $d = 115$	$(1,7,4,2,1,1)$, $d = 121$	$(1,7,5,1,1,1)$, $d = 124$
$(1,8,2,2,2,1)$, $d = 94$	$(1,8,3,2,1,1)$, $d = 105$	$(1,8,4,1,1,1)$, $d = 109$
$(1,4,3,3,2,2,1)$, $d \leq 128$	$(1,4,3,3,3,1,1)$, $d = 136$	$(1,4,3,4,2,1,1)$, $d = 129$
$(1,4,4,2,2,2,1)$, $d \leq 130$	$(1,4,4,3,2,1,1)$, $d = 135$	$(1,4,4,4,1,1,1)$, $d = 129$
$(1,4,5,2,2,1,1)$, $d = 131$	$(1,4,5,3,1,1,1)$, $d = 132$	$(1,5,3,2,2,2,1)$, $d \leq 126$
$(1,5,3,3,2,1,1)$, $d = 130$	$(1,5,3,4,1,1,1)$, $d = 125$	$(1,5,4,2,2,1,1)$, $d \leq 133$
$(1,5,4,3,1,1,1)$, $d \leq 132$	$(1,5,5,2,1,1,1)$, $d = 136$	$(1,6,2,2,2,2,1)$, $d = 117$
$(1,6,3,2,2,1,1)$, $d \leq 125$	$(1,6,3,3,1,1,1)$, $d = 123$	$(1,6,4,2,1,1,1)$, $d = 128$
$(1,6,5,1,1,1,1)$, $d = 130$	$(1,7,2,2,2,1,1)$, $d = 112$	$(1,7,3,2,1,1,1)$, $d = 116$
$(1,7,4,1,1,1,1)$, $d = 119$	$(1,8,2,2,1,1,1)$, $d = 99$	$(1,8,3,1,1,1,1)$, $d = 104$
$(1,4,2,2,2,2,2,1)$, $d = 129$	$(1,4,3,2,2,2,1,1)$, $d \leq 134$	$(1,4,3,3,2,1,1,1)$, $d = 132$
$(1,4,3,4,1,1,1,1)$, $d = 126$	$(1,4,4,2,2,1,1,1)$, $d \leq 134$	$(1,4,4,3,1,1,1,1)$, $d \leq 133$
$(1,5,2,2,2,2,1,1)$, $d = 129$	$(1,5,3,2,2,1,1,1)$, $d \leq 130$	$(1,5,3,3,1,1,1,1)$, $d = 128$
$(1,5,4,2,1,1,1,1)$, $d = 132$	$(1,5,5,1,1,1,1,1)$, $d = 133$	$(1,6,2,2,2,1,1,1)$, $d = 121$
$(1,6,3,2,1,1,1,1)$, $d = 124$	$(1,6,4,1,1,1,1,1)$, $d = 126$	$(1,7,2,2,1,1,1,1)$, $d = 111$

$(1,7,3,1,1,1,1,1)$, $d=115$
$(1,8,2,1,1,1,1,1)$, $d=99$
$(1,4,2,2,2,2,1,1,1)$, $d=132$
$(1,4,3,2,2,1,1,1,1)$, $d\le 132$
$(1,4,3,3,1,1,1,1,1)$, $d=130$
$(1,4,4,2,1,1,1,1,1)$, $d=133$
$(1,5,2,2,2,1,1,1,1)$, $d=127$
$(1,5,3,2,1,1,1,1,1)$, $d=129$
$(1,5,4,1,1,1,1,1,1)$, $d=130$
$(1,6,2,2,1,1,1,1,1)$, $d=120$
$(1,6,3,1,1,1,1,1,1)$, $d=123$
$(1,7,2,1,1,1,1,1,1)$, $d=111$
$(1,8,1,1,1,1,1,1,1)$, $d=93$
$(1,4,2,2,2,1,1,1,1,1)$, $d=130$
$(1,4,3,2,1,1,1,1,1,1)$, $d=131$
$(1,4,4,1,1,1,1,1,1,1)$, $d=131$
$(1,5,2,2,1,1,1,1,1,1)$, $d=126$
$(1,5,3,1,1,1,1,1,1,1)$, $d=128$
$(1,6,2,1,1,1,1,1,1,1)$, $d=120$
$(1,7,1,1,1,1,1,1,1,1)$, $d=106$
$(1,4,2,2,1,1,1,1,1,1,1)$, $d=129$
$(1,4,3,1,1,1,1,1,1,1,1)$, $d=130$
$(1,5,2,1,1,1,1,1,1,1,1)$, $d=126$
$(1,6,1,1,1,1,1,1,1,1,1)$, $d=116$
$(1,4,2,1,1,1,1,1,1,1,1,1)$, $d=129$
$(1,5,1,1,1,1,1,1,1,1,1,1)$, $d=123$
$(1,4,1,1,1,1,1,1,1,1,1,1,1)$, $d=127$

2.7. $l_0=17$**.** In this case we have that the bound $9l_0+4$ computed in Lemma 2 is 157. The list below shows that $d<157$ for any possible choice of H with $l_0=17$.

The notation for this section is given at the begin of the Appendix.

$(1,8,7,1)\to(1,7,7,1),(0,1,0,0)$, $d=136$
$(1,4,7,4,1)\to(1,4,7,4,1)$, $d=128$
$(1,5,5,5,1)\to(1,5,5,5,1)$, $d=119$
$(1,5,6,4,1)\to(1,4,6,4,1),(0,1,0,0,0)$, $d=127$
$(1,5,6,4,1)\to(1,4,5,4,1),(0,1,1,0,0)$, $d=131$
$(1,5,7,3,1)\to(1,3,6,3,1),(0,1,1,0,0),(0,1,0,0,0)$, $d=124$
$(1,5,7,3,1)\to(1,3,5,3,1),(0,2,2,0,0)$, $d=130$
$(1,6,4,5,1)\to(1,5,4,5,1),(0,1,0,0,0)$, $d=110$
$(1,6,5,4,1)\to(1,4,5,4,1),(0,2,0,0,0)$, $d=122$
$(1,6,5,4,1)\to(1,4,4,4,1),(0,1,1,0,0),(0,1,0,0,0)$, $d=127$
$(1,6,5,4,1)\to(1,4,3,4,1),(0,2,2,0,0)$, $d=135$
$(1,6,6,3,1)\to(1,3,6,3,1),(0,3,0,0,0)$, $d=118$
$(1,6,6,3,1)\to(1,3,5,3,1),(0,1,1,0,0),(0,2,0,0,0)$, $d=123$
$(1,6,6,3,1)\to(1,3,4,3,1),(0,2,2,0,0),(0,1,0,0,0)$, $d=130$
$(1,6,6,3,1)\to(1,3,3,3,1),(0,3,3,0,0)$, $d=140$
$(1,6,7,2,1)\to(1,2,3,2,1),(0,4,4,0,0)$, $d=137$
$(1,7,4,4,1)\to(1,4,4,4,1),(0,3,0,0,0)$, $d=113$
$(1,7,4,4,1)\to(1,4,3,4,1),(0,1,1,0,0),(0,2,0,0,0)$, $d=119$
$(1,7,5,3,1)\to(1,3,5,3,1),(0,4,0,0,0)$, $d=112$
$(1,7,5,3,1)\to(1,3,4,3,1),(0,1,1,0,0),(0,3,0,0,0)$, $d=118$
$(1,7,5,3,1)\to(1,3,3,3,1),(0,2,2,0,0),(0,2,0,0,0)$, $d=126$
$(1,7,6,2,1)\to(1,2,3,2,1),(0,3,3,0,0),(0,2,0,0,0)$, $d=125$
$(1,7,6,2,1)\to(1,2,2,2,1),(0,4,4,0,0),(0,1,0,0,0)$, $d=137$
$(1,7,7,1,1)\to(1,1,1,1,1),(0,6,6,0,0)$, $d=149$
$(1,8,3,4,1)\to(1,4,3,4,1),(0,4,0,0,0)$, $d=100$
$(1,8,4,3,1)\to(1,3,4,3,1),(0,5,0,0,0)$, $d=102$
$(1,8,4,3,1)\to(1,3,3,3,1),(0,1,1,0,0),(0,4,0,0,0)$, $d=109$
$(1,8,5,2,1)\to(1,2,3,2,1),(0,2,2,0,0),(0,4,0,0,0)$, $d=110$
$(1,8,5,2,1)\to(1,2,2,2,1),(0,3,3,0,0),(0,3,0,0,0)$, $d=120$
$(1,8,6,1,1)\to(1,1,1,1,1),(0,5,5,0,0),(0,2,0,0,0)$, $d=131$

$(1,4,4,4,3,1) \to (1,3,4,4,3,1), (0,1,0,0,0,0),\ d=134$
$(1,4,4,4,3,1) \to (1,3,3,3,3,1), (0,1,1,1,0,0),\ d=135$
$(1,4,4,5,2,1) \to (1,2,3,3,2,1), (0,2,1,2,0,0),\ d=131$
$(1,4,5,3,3,1) \to (1,3,3,3,3,1), (0,0,2,0,0,0), (0,1,0,0,0,0),\ d=133$
$(1,4,5,3,3,1) \to (1,3,3,3,3,1), (0,0,1,0,0,0), (0,1,1,0,0,0),\ d=136$
$(1,4,5,4,2,1) \to (1,2,3,3,2,1), (0,1,1,1,0,0), (0,1,1,0,0,0),\ d=134$
$(1,4,5,4,2,1) \to (1,2,2,2,2,1), (0,2,3,2,0,0),\ d=136$
$(1,4,6,4,1,1) \to (1,1,1,1,1,1), (0,3,5,3,0,0),\ d=140$
$(1,5,3,4,3,1) \to (1,3,3,3,3,1), (0,1,0,1,0,0), (0,1,0,0,0,0),\ d=131$
$(1,5,4,3,3,1) \to (1,3,3,3,3,1), (0,0,1,0,0,0), (0,2,0,0,0,0),\ d=132$
$(1,5,4,3,3,1) \to (1,3,3,3,3,1), (0,1,1,0,0,0), (0,1,0,0,0,0),\ d=136$
$(1,5,4,4,2,1) \to (1,2,3,3,2,1), (0,1,1,1,0,0), (0,2,0,0,0,0),\ d=130$
$(1,5,4,4,2,1) \to (1,2,3,3,2,1), (0,1,0,1,0,0), (0,1,1,0,0,0), (0,1,0,0,0,0),\ d=134$
$(1,5,4,4,2,1) \to (1,2,2,2,2,1), (0,2,2,2,0,0), (0,1,0,0,0,0),\ d=133$
$(1,5,4,4,2,1) \to (1,2,2,2,2,1), (0,2,1,2,0,0), (0,1,1,0,0,0),\ d=137$
$(1,5,4,5,1,1) \to (1,1,1,1,1,1), (0,4,3,4,0,0),\ d=125$
$(1,5,5,3,2,1) \to (1,2,3,3,2,1), (0,2,2,0,0,0), (0,1,0,0,0,0),\ d=131$
$(1,5,5,3,2,1) \to (1,2,2,2,2,1), (0,1,1,1,0,0), (0,2,2,0,0,0),\ d=141$
$(1,5,5,4,1,1) \to (1,1,1,1,1,1), (0,3,4,3,0,0), (0,1,0,0,0,0),\ d=138$
$(1,5,5,4,1,1) \to (1,1,1,1,1,1), (0,3,3,3,0,0), (0,1,1,0,0,0),\ d=142$
$(1,5,6,2,2,1) \to (1,2,2,2,2,1), (0,0,1,0,0,0), (0,3,3,0,0,0),\ d=135$
$(1,5,6,3,1,1) \to (1,1,1,1,1,1), (0,2,3,2,0,0), (0,2,2,0,0,0),\ d=143$
$(1,6,3,3,3,1) \to (1,3,3,3,3,1), (0,3,0,0,0,0),\ d=127$
$(1,6,3,4,2,1) \to (1,2,3,3,2,1), (0,1,0,1,0,0), (0,3,0,0,0,0),\ d=125$
$(1,6,3,4,2,1) \to (1,2,2,2,2,1), (0,2,1,2,0,0), (0,2,0,0,0,0),\ d=126$
$(1,6,4,3,2,1) \to (1,2,3,3,2,1), (0,1,1,0,0,0), (0,3,0,0,0,0),\ d=124$
$(1,6,4,3,2,1) \to (1,2,2,2,2,1), (0,1,1,1,0,0), (0,1,1,0,0,0), (0,2,0,0,0,0),\ d=132$
$(1,6,4,4,1,1) \to (1,1,1,1,1,1), (0,3,3,3,0,0), (0,2,0,0,0,0),\ d=132$
$(1,6,4,4,1,1) \to (1,1,1,1,1,1), (0,3,2,3,0,0), (0,1,1,0,0,0), (0,1,0,0,0,0),\ d=137$
$(1,6,5,2,2,1) \to (1,2,2,2,2,1), (0,0,1,0,0,0), (0,2,2,0,0,0), (0,2,0,0,0,0),\ d=128$
$(1,6,5,2,2,1) \to (1,2,2,2,2,1), (0,3,3,0,0,0), (0,1,0,0,0,0),\ d=136$
$(1,6,5,3,1,1) \to (1,1,1,1,1,1), (0,2,3,2,0,0), (0,1,1,0,0,0), (0,2,0,0,0,0),\ d=135$
$(1,6,5,3,1,1) \to (1,1,1,1,1,1), (0,2,2,2,0,0), (0,2,2,0,0,0), (0,1,0,0,0,0),\ d=142$
$(1,6,6,2,1,1) \to (1,1,1,1,1,1), (0,1,1,1,0,0), (0,4,4,0,0,0),\ d=148$
$(1,7,3,3,2,1) \to (1,2,3,3,2,1), (0,5,0,0,0,0),\ d=112$
$(1,7,3,3,2,1) \to (1,2,2,2,2,1), (0,1,1,1,0,0), (0,4,0,0,0,0),\ d=119$
$(1,7,3,4,1,1) \to (1,1,1,1,1,1), (0,3,2,3,0,0), (0,3,0,0,0,0),\ d=122$
$(1,7,4,2,2,1) \to (1,2,2,2,2,1), (0,0,1,0,0,0), (0,1,1,0,0,0), (0,4,0,0,0,0),\ d=117$
$(1,7,4,2,2,1) \to (1,2,2,2,2,1), (0,2,2,0,0,0), (0,3,0,0,0,0),\ d=124$
$(1,7,4,3,1,1) \to (1,1,1,1,1,1), (0,2,3,2,0,0), (0,4,0,0,0,0),\ d=123$
$(1,7,4,3,1,1) \to (1,1,1,1,1,1), (0,2,2,2,0,0), (0,1,1,0,0,0), (0,3,0,0,0,0),\ d=129$
$(1,7,5,2,1,1) \to (1,1,1,1,1,1), (0,1,1,1,0,0), (0,3,3,0,0,0), (0,2,0,0,0,0),\ d=135$
$(1,7,6,1,1,1) \to (1,1,1,1,1,1), (0,5,5,0,0,0), (0,1,0,0,0,0),\ d=139$
$(1,8,3,2,2,1) \to (1,2,2,2,2,1), (0,0,1,0,0,0), (0,6,0,0,0,0),\ d=101$
$(1,8,3,2,2,1) \to (1,2,2,2,2,1), (0,1,1,0,0,0), (0,5,0,0,0,0),\ d=108$
$(1,8,3,3,1,1) \to (1,1,1,1,1,1), (0,2,2,2,0,0), (0,5,0,0,0,0),\ d=112$
$(1,8,4,2,1,1) \to (1,1,1,1,1,1), (0,1,1,1,0,0), (0,2,2,0,0,0), (0,4,0,0,0,0),\ d=119$
$(1,8,5,1,1,1) \to (1,1,1,1,1,1), (0,4,4,0,0,0), (0,3,0,0,0,0),\ d=123$

$(1,4,3,3,3,2,1) \to (1,2,3,3,3,2,1),(0,2,0,0,0,0,0),\ d=131$
$(1,4,3,3,3,2,1) \to (1,2,2,2,2,2,1),(0,1,1,1,1,0,0),(0,1,0,0,0,0,0),\ d=141$
$(1,4,3,4,2,2,1) \to (1,2,2,2,2,2,1),(0,0,1,1,0,0,0),(0,1,0,1,0,0,0),(0,1,0,0,0,0,0),\ d=134$
$(1,4,3,4,2,2,1) \to (1,2,2,2,2,2,1),(0,2,1,2,0,0,0),\ d=133$
$(1,4,3,4,3,1,1) \to (1,1,1,1,1,1,1),(0,2,2,2,2,0,0),(0,1,0,1,0,0,0),\ d=142$
$(1,4,4,3,2,2,1) \to (1,2,2,2,2,2,1),(0,0,1,1,0,0,0),(0,1,1,0,0,0,0),(0,1,0,0,0,0,0),\ d=133$
$(1,4,4,3,2,2,1) \to (1,2,2,2,2,2,1),(0,1,1,1,0,0,0),(0,1,1,0,0,0,0),\ d=139$
$(1,4,4,3,3,1,1) \to (1,1,1,1,1,1,1),(0,2,2,2,2,0,0),(0,0,1,0,0,0,0),(0,1,0,0,0,0,0),\ d=144$
$(1,4,4,3,3,1,1) \to (1,1,1,1,1,1,1),(0,2,2,2,2,0,0),(0,1,1,0,0,0,0),\ d=147$
$(1,4,4,4,2,1,1) \to (1,1,1,1,1,1,1),(0,1,1,1,1,0,0),(0,2,2,2,0,0,0),\ d=140$
$(1,4,5,2,2,2,1) \to (1,2,2,2,2,2,1),(0,0,1,0,0,0,0),(0,2,2,0,0,0,0),\ d=135$
$(1,4,5,4,1,1,1) \to (1,1,1,1,1,1,1),(0,3,4,3,0,0,0),\ d=139$
$(1,5,3,3,2,2,1) \to (1,2,2,2,2,2,1),(0,0,1,1,0,0,0),(0,3,0,0,0,0,0),\ d=129$
$(1,5,3,3,2,2,1) \to (1,2,2,2,2,2,1),(0,1,1,1,0,0,0),(0,2,0,0,0,0,0),\ d=134$
$(1,5,3,3,3,1,1) \to (1,1,1,1,1,1,1),(0,2,2,2,2,0,0),(0,2,0,0,0,0,0),\ d=142$
$(1,5,3,4,2,1,1) \to (1,1,1,1,1,1,1),(0,1,1,1,1,0,0),(0,2,1,2,0,0,0),(0,1,0,0,0,0,0),\ d=136$
$(1,5,4,2,2,2,1) \to (1,2,2,2,2,2,1),(0,0,1,0,0,0,0),(0,1,1,0,0,0,0),(0,2,0,0,0,0,0),\ d=132$
$(1,5,4,2,2,2,1) \to (1,2,2,2,2,2,1),(0,2,2,0,0,0,0),(0,1,0,0,0,0,0),\ d=137$
$(1,5,4,3,2,1,1) \to (1,1,1,1,1,1,1),(0,1,1,1,1,0,0),(0,1,1,1,0,0,0),(0,1,1,0,0,0,0),(0,1,0,0,0,0,0),\ d=142$
$(1,5,4,4,1,1,1) \to (1,1,1,1,1,1,1),(0,3,3,3,0,0,0),(0,1,0,0,0,0,0),\ d=136$
$(1,5,4,4,1,1,1) \to (1,1,1,1,1,1,1),(0,3,2,3,0,0,0),(0,1,1,0,0,0,0),\ d=140$
$(1,5,5,2,2,1,1) \to (1,1,1,1,1,1,1),(0,1,1,1,1,0,0),(0,0,1,0,0,0,0),(0,2,2,0,0,0,0),(0,1,0,0,0,0,0),\ d=138$
$(1,5,5,2,2,1,1) \to (1,1,1,1,1,1,1),(0,1,1,1,1,0,0),(0,3,3,0,0,0,0),\ d=145$
$(1,5,5,3,1,1,1) \to (1,1,1,1,1,1,1),(0,2,3,2,0,0,0),(0,1,1,0,0,0,0),(0,1,0,0,0,0,0),\ d=139$
$(1,5,5,3,1,1,1) \to (1,1,1,1,1,1,1),(0,2,2,2,0,0,0),(0,2,2,0,0,0,0),\ d=145$
$(1,6,3,2,2,2,1) \to (1,2,2,2,2,2,1),(0,0,1,0,0,0,0),(0,4,0,0,0,0,0),\ d=124$
$(1,6,3,2,2,2,1) \to (1,2,2,2,2,2,1),(0,1,1,0,0,0,0),(0,3,0,0,0,0,0),\ d=129$
$(1,6,3,3,2,1,1) \to (1,1,1,1,1,1,1),(0,1,1,1,1,0,0),(0,1,1,1,0,0,0),(0,3,0,0,0,0,0),\ d=133$
$(1,6,3,4,1,1,1) \to (1,1,1,1,1,1,1),(0,3,2,3,0,0,0),(0,2,0,0,0,0,0),\ d=129$
$(1,6,4,2,2,1,1) \to (1,1,1,1,1,1,1),(0,1,1,1,1,0,0),(0,0,1,0,0,0,0),(0,1,1,0,0,0,0),(0,3,0,0,0,0,0),\ d=131$
$(1,6,4,2,2,1,1) \to (1,1,1,1,1,1,1),(0,1,1,1,1,0,0),(0,2,2,0,0,0,0),(0,2,0,0,0,0,0),\ d=137$
$(1,6,4,3,1,1,1) \to (1,1,1,1,1,1,1),(0,2,3,2,0,0,0),(0,3,0,0,0,0,0),\ d=131$
$(1,6,4,3,1,1,1) \to (1,1,1,1,1,1,1),(0,2,2,2,0,0,0),(0,1,1,0,0,0,0),(0,2,0,0,0,0,0),\ d=136$
$(1,6,5,2,1,1,1) \to (1,1,1,1,1,1,1),(0,1,1,1,0,0,0),(0,3,3,0,0,0,0),(0,1,0,0,0,0,0),\ d=141$
$(1,6,6,1,1,1,1) \to (1,1,1,1,1,1,1),(0,5,5,0,0,0,0),\ d=144$

$(1,7,2,2,2,2,1) \to (1,2,2,2,2,2,1),(0,5,0,0,0,0,0),\ d=116$
$(1,7,3,2,2,1,1) \to (1,1,1,1,1,1,1),(0,1,1,1,1,0,0),(0,0,1,0,0,0,0),(0,5,0,0,0,0,0),$
$d=119$
$(1,7,3,2,2,1,1) \to (1,1,1,1,1,1,1),(0,1,1,1,1,0,0),(0,1,1,0,0,0,0),(0,4,0,0,0,0,0),$
$d=125$
$(1,7,3,3,1,1,1) \to (1,1,1,1,1,1,1),(0,2,2,2,0,0,0),(0,4,0,0,0,0,0),\ d=123$
$(1,7,4,2,1,1,1) \to (1,1,1,1,1,1,1),(0,1,1,1,0,0,0),(0,2,2,0,0,0,0),(0,3,0,0,0,0,0),$
$d=129$
$(1,7,5,1,1,1,1) \to (1,1,1,1,1,1,1),(0,4,4,0,0,0,0),(0,2,0,0,0,0,0),\ d=132$
$(1,8,2,2,2,1,1) \to (1,1,1,1,1,1,1),(0,1,1,1,1,0,0),(0,6,0,0,0,0,0),\ d=108$
$(1,8,3,2,1,1,1) \to (1,1,1,1,1,1,1),(0,1,1,1,0,0,0),(0,1,1,0,0,0,0),(0,5,0,0,0,0,0),$
$d=113$
$(1,8,4,1,1,1,1) \to (1,1,1,1,1,1,1),(0,3,3,0,0,0,0),(0,4,0,0,0,0,0),\ d=117$
$(1,4,3,2,2,2,2,1) \to (1,2,2,2,2,2,2,1),(0,0,1,0,0,0,0,0),(0,2,0,0,0,0,0,0),\ d=136$
$(1,4,3,2,2,2,2,1) \to (1,2,2,2,2,2,2,1),(0,1,1,0,0,0,0,0),(0,1,0,0,0,0,0,0),\ d=139$
$(1,4,3,3,2,2,1,1) \to (1,1,1,1,1,1,1,1),(0,1,1,1,1,1,0,0),(0,0,1,1,0,0,0,0),(0,2,0,0,0,0,0,0),$
$d=138$
$(1,4,3,3,2,2,1,1) \to (1,1,1,1,1,1,1,1),(0,1,1,1,1,1,0,0),(0,1,1,1,0,0,0,0),(0,1,0,0,0,0,0,0),$
$d=142$
$(1,4,3,3,3,1,1,1) \to (1,1,1,1,1,1,1,1),(0,2,2,2,2,0,0,0),(0,1,0,0,0,0,0,0),\ d=144$
$(1,4,3,4,2,1,1,1) \to (1,1,1,1,1,1,1,1),(0,1,1,1,1,0,0,0),(0,2,1,2,0,0,0,0),\ d=137$
$(1,4,4,2,2,2,1,1) \to (1,1,1,1,1,1,1,1),(0,1,1,1,1,1,0,0),(0,0,1,0,0,0,0,0),(0,1,1,0,0,0,0,0),$
$(0,1,0,0,0,0,0,0),\ d=140$
$(1,4,4,2,2,2,1,1) \to (1,1,1,1,1,1,1,1),(0,1,1,1,1,1,0,0),(0,2,2,0,0,0,0,0),\ d=144$
$(1,4,4,3,2,1,1,1) \to (1,1,1,1,1,1,1,1),(0,1,1,1,1,0,0,0),(0,1,1,1,0,0,0,0),(0,1,1,0,0,0,0,0),$
$d=143$
$(1,4,4,4,1,1,1,1) \to (1,1,1,1,1,1,1,1),(0,3,3,3,0,0,0,0),\ d=137$
$(1,4,5,2,2,1,1,1) \to (1,1,1,1,1,1,1,1),(0,1,1,1,1,0,0,0),(0,0,1,0,0,0,0,0),(0,2,2,0,0,0,0,0),$
$d=139$
$(1,4,5,3,1,1,1,1) \to (1,1,1,1,1,1,1,1),(0,2,3,2,0,0,0,0),(0,1,1,0,0,0,0,0),\ d=140$
$(1,5,2,2,2,2,2,1) \to (1,2,2,2,2,2,2,1),(0,3,0,0,0,0,0,0),\ d=134$
$(1,5,3,2,2,2,1,1) \to (1,1,1,1,1,1,1,1),(0,1,1,1,1,1,0,0),(0,0,1,0,0,0,0,0),(0,3,0,0,0,0,0,0),$
$d=136$
$(1,5,3,2,2,2,1,1) \to (1,1,1,1,1,1,1,1),(0,1,1,1,1,1,0,0),(0,1,1,0,0,0,0,0),(0,2,0,0,0,0,0,0),$
$d=140$
$(1,5,3,3,2,1,1,1) \to (1,1,1,1,1,1,1,1),(0,1,1,1,1,0,0,0),(0,1,1,1,0,0,0,0),(0,2,0,0,0,0,0,0),$
$d=138$
$(1,5,3,4,1,1,1,1) \to (1,1,1,1,1,1,1,1),(0,3,2,3,0,0,0,0),(0,1,0,0,0,0,0,0),\ d=133$
$(1,5,4,2,2,1,1,1) \to (1,1,1,1,1,1,1,1),(0,1,1,1,1,0,0,0),(0,0,1,0,0,0,0,0),(0,1,1,0,0,0,0,0),$
$(0,2,0,0,0,0,0,0),\ d=136$
$(1,5,4,2,2,1,1,1) \to (1,1,1,1,1,1,1,1),(0,1,1,1,1,0,0,0),(0,2,2,0,0,0,0,0),(0,1,0,0,0,0,0,0),$
$d=141$
$(1,5,4,3,1,1,1,1) \to (1,1,1,1,1,1,1,1),(0,2,3,2,0,0,0,0),(0,2,0,0,0,0,0,0),\ d=136$
$(1,5,4,3,1,1,1,1) \to (1,1,1,1,1,1,1,1),(0,2,2,2,0,0,0,0),(0,1,1,0,0,0,0,0),(0,1,0,0,0,0,0,0),$
$d=140$
$(1,5,5,2,1,1,1,1) \to (1,1,1,1,1,1,1,1),(0,1,1,1,0,0,0,0),(0,3,3,0,0,0,0,0),\ d=144$
$(1,6,2,2,2,2,1,1) \to (1,1,1,1,1,1,1,1),(0,1,1,1,1,1,0,0),(0,4,0,0,0,0,0,0),\ d=131$
$(1,6,3,2,2,1,1,1) \to (1,1,1,1,1,1,1,1),(0,1,1,1,1,0,0,0),(0,0,1,0,0,0,0,0),(0,4,0,0,0,0,0,0),$

$d = 128$
$(1,6,3,2,2,1,1,1) \to (1,1,1,1,1,1,1,1), (0,1,1,1,1,0,0,0), (0,1,1,0,0,0,0,0), (0,3,0,0,0,0,0,0),$ $d = 133$
$(1,6,3,3,1,1,1,1) \to (1,1,1,1,1,1,1,1), (0,2,2,2,0,0,0,0), (0,3,0,0,0,0,0,0),\ d = 131$
$(1,6,4,2,1,1,1,1) \to (1,1,1,1,1,1,1,1), (0,1,1,1,0,0,0,0), (0,2,2,0,0,0,0,0), (0,2,0,0,0,0,0,0),$ $d = 136$
$(1,6,5,1,1,1,1,1) \to (1,1,1,1,1,1,1,1), (0,4,4,0,0,0,0,0), (0,1,0,0,0,0,0,0),\ d = 138$
$(1,7,2,2,2,1,1,1) \to (1,1,1,1,1,1,1,1), (0,1,1,1,1,0,0,0), (0,5,0,0,0,0,0,0),\ d = 120$
$(1,7,3,2,1,1,1,1) \to (1,1,1,1,1,1,1,1), (0,1,1,1,0,0,0,0), (0,1,1,0,0,0,0,0), (0,4,0,0,0,0,0,0),$ $d = 124$
$(1,7,4,1,1,1,1,1) \to (1,1,1,1,1,1,1,1), (0,3,3,0,0,0,0,0), (0,3,0,0,0,0,0,0),\ d = 127$
$(1,8,2,2,1,1,1,1) \to (1,1,1,1,1,1,1,1), (0,1,1,1,0,0,0,0), (0,6,0,0,0,0,0,0),\ d = 107$
$(1,8,3,1,1,1,1,1) \to (1,1,1,1,1,1,1,1), (0,2,2,0,0,0,0,0), (0,5,0,0,0,0,0,0),\ d = 112$
$(1,4,2,2,2,2,2,1,1) \to (1,1,1,1,1,1,1,1,1), (0,1,1,1,1,1,1,0,0), (0,2,0,0,0,0,0,0,0),$ $d = 143$
$(1,4,3,2,2,2,1,1,1) \to (1,1,1,1,1,1,1,1,1), (0,1,1,1,1,1,0,0,0), (0,0,1,0,0,0,0,0,0),$ $(0,2,0,0,0,0,0,0,0),\ d = 139$
$(1,4,3,2,2,2,1,1,1) \to (1,1,1,1,1,1,1,1,1), (0,1,1,1,1,1,0,0,0), (0,1,1,0,0,0,0,0,0),$ $(0,1,0,0,0,0,0,0,0),\ d = 142$
$(1,4,3,3,2,1,1,1,1) \to (1,1,1,1,1,1,1,1,1), (0,1,1,1,1,0,0,0,0), (0,1,1,1,0,0,0,0,0),$ $(0,1,0,0,0,0,0,0,0),\ d = 140$
$(1,4,3,4,1,1,1,1,1) \to (1,1,1,1,1,1,1,1,1), (0,3,2,3,0,0,0,0,0),\ d = 134$
$(1,4,4,2,2,1,1,1,1) \to (1,1,1,1,1,1,1,1,1), (0,1,1,1,1,0,0,0,0), (0,0,1,0,0,0,0,0,0),$ $(0,1,1,0,0,0,0,0,0), (0,1,0,0,0,0,0,0,0),\ d = 138$
$(1,4,4,2,2,1,1,1,1) \to (1,1,1,1,1,1,1,1,1), (0,1,1,1,1,0,0,0,0), (0,2,2,0,0,0,0,0,0),$ $d = 142$
$(1,4,4,3,1,1,1,1,1) \to (1,1,1,1,1,1,1,1,1), (0,2,3,2,0,0,0,0,0), (0,1,0,0,0,0,0,0,0),$ $d = 138$
$(1,4,4,3,1,1,1,1,1) \to (1,1,1,1,1,1,1,1,1), (0,2,2,2,0,0,0,0,0), (0,1,1,0,0,0,0,0,0),$ $d = 141$
$(1,5,2,2,2,2,1,1,1) \to (1,1,1,1,1,1,1,1,1), (0,1,1,1,1,1,0,0,0), (0,3,0,0,0,0,0,0,0),$ $d = 137$
$(1,5,3,2,2,1,1,1,1) \to (1,1,1,1,1,1,1,1,1), (0,1,1,1,1,0,0,0,0), (0,0,1,0,0,0,0,0,0),$ $(0,3,0,0,0,0,0,0,0),\ d = 134$
$(1,5,3,2,2,1,1,1,1) \to (1,1,1,1,1,1,1,1,1), (0,1,1,1,1,0,0,0,0), (0,1,1,0,0,0,0,0,0),$ $(0,2,0,0,0,0,0,0,0),\ d = 138$
$(1,5,3,3,1,1,1,1,1) \to (1,1,1,1,1,1,1,1,1), (0,2,2,2,0,0,0,0,0), (0,2,0,0,0,0,0,0,0),$ $d = 136$
$(1,5,4,2,1,1,1,1,1) \to (1,1,1,1,1,1,1,1,1), (0,1,1,1,0,0,0,0,0), (0,2,2,0,0,0,0,0,0),$ $(0,1,0,0,0,0,0,0,0),\ d = 140$
$(1,5,5,1,1,1,1,1,1) \to (1,1,1,1,1,1,1,1,1), (0,4,4,0,0,0,0,0,0),\ d = 141$
$(1,6,2,2,2,1,1,1,1) \to (1,1,1,1,1,1,1,1,1), (0,1,1,1,1,0,0,0,0), (0,4,0,0,0,0,0,0,0),$ $d = 129$
$(1,6,3,2,1,1,1,1,1) \to (1,1,1,1,1,1,1,1,1), (0,1,1,1,0,0,0,0,0), (0,1,1,0,0,0,0,0,0),$ $(0,3,0,0,0,0,0,0,0),\ d = 132$
$(1,6,4,1,1,1,1,1,1) \to (1,1,1,1,1,1,1,1,1), (0,3,3,0,0,0,0,0,0), (0,2,0,0,0,0,0,0,0),$ $d = 134$
$(1,7,2,2,1,1,1,1,1) \to (1,1,1,1,1,1,1,1,1), (0,1,1,1,0,0,0,0,0), (0,5,0,0,0,0,0,0,0),$

$d=119$
$(1,7,3,1,1,1,1,1,1)\to(1,1,1,1,1,1,1,1,1),(0,2,2,0,0,0,0,0,0),(0,4,0,0,0,0,0,0,0),$
$d=123$
$(1,8,2,1,1,1,1,1,1)\to(1,1,1,1,1,1,1,1,1),(0,1,1,0,0,0,0,0,0),(0,6,0,0,0,0,0,0,0),$
$d=107$
$(1,4,2,2,2,2,1,1,1,1)\to(1,1,1,1,1,1,1,1,1,1),(0,1,1,1,1,1,0,0,0,0),(0,2,0,0,0,0,0,0,0,0),$
$d=140$
$(1,4,3,2,2,1,1,1,1,1)\to(1,1,1,1,1,1,1,1,1,1),(0,1,1,1,1,0,0,0,0,0),(0,0,1,0,0,0,0,0,0,0),$
$(0,2,0,0,0,0,0,0,0,0)$, $d=137$
$(1,4,3,2,2,1,1,1,1,1)\to(1,1,1,1,1,1,1,1,1,1),(0,1,1,1,1,0,0,0,0,0),(0,1,1,0,0,0,0,0,0,0),$
$(0,1,0,0,0,0,0,0,0,0)$, $d=140$
$(1,4,3,3,1,1,1,1,1,1)\to(1,1,1,1,1,1,1,1,1,1),(0,2,2,2,0,0,0,0,0,0),(0,1,0,0,0,0,0,0,0,0),$
$d=138$
$(1,4,4,2,1,1,1,1,1,1)\to(1,1,1,1,1,1,1,1,1,1),(0,1,1,1,0,0,0,0,0,0),(0,2,2,0,0,0,0,0,0,0),$
$d=141$
$(1,5,2,2,2,1,1,1,1,1)\to(1,1,1,1,1,1,1,1,1,1),(0,1,1,1,1,0,0,0,0,0),(0,3,0,0,0,0,0,0,0,0),$
$d=135$
$(1,5,3,2,1,1,1,1,1,1)\to(1,1,1,1,1,1,1,1,1,1),(0,1,1,1,0,0,0,0,0,0),(0,1,1,0,0,0,0,0,0,0),$
$(0,2,0,0,0,0,0,0,0,0)$, $d=137$
$(1,5,4,1,1,1,1,1,1,1)\to(1,1,1,1,1,1,1,1,1,1),(0,3,3,0,0,0,0,0,0,0),(0,1,0,0,0,0,0,0,0,0),$
$d=138$
$(1,6,2,2,1,1,1,1,1,1)\to(1,1,1,1,1,1,1,1,1,1),(0,1,1,1,0,0,0,0,0,0),(0,4,0,0,0,0,0,0,0,0),$
$d=128$
$(1,6,3,1,1,1,1,1,1,1)\to(1,1,1,1,1,1,1,1,1,1),(0,2,2,0,0,0,0,0,0,0),(0,3,0,0,0,0,0,0,0,0),$
$d=131$
$(1,7,2,1,1,1,1,1,1,1)\to(1,1,1,1,1,1,1,1,1,1),(0,1,1,0,0,0,0,0,0,0),(0,5,0,0,0,0,0,0,0,0),$
$d=119$
$(1,8,1,1,1,1,1,1,1,1)\to(1,1,1,1,1,1,1,1,1,1),(0,7,0,0,0,0,0,0,0,0)$, $d=101$
$(1,4,2,2,2,1,1,1,1,1,1)\to(1,1,1,1,1,1,1,1,1,1,1),(0,1,1,1,1,0,0,0,0,0,0),$
$(0,2,0,0,0,0,0,0,0,0,0)$, $d=138$
$(1,4,3,2,1,1,1,1,1,1,1)\to(1,1,1,1,1,1,1,1,1,1,1),(0,1,1,1,0,0,0,0,0,0,0),(0,1,1,0,0,0,0,0,0,0,0),$
$(0,1,0,0,0,0,0,0,0,0,0)$, $d=139$
$(1,4,4,1,1,1,1,1,1,1,1)\to(1,1,1,1,1,1,1,1,1,1,1),(0,3,3,0,0,0,0,0,0,0,0)$, $d=139$
$(1,5,2,2,1,1,1,1,1,1,1)\to(1,1,1,1,1,1,1,1,1,1,1),(0,1,1,1,0,0,0,0,0,0,0),(0,3,0,0,0,0,0,0,0,0,0),$
$d=134$
$(1,5,3,1,1,1,1,1,1,1,1)\to(1,1,1,1,1,1,1,1,1,1,1),(0,2,2,0,0,0,0,0,0,0,0),(0,2,0,0,0,0,0,0,0,0,0),$
$d=136$
$(1,6,2,1,1,1,1,1,1,1,1)\to(1,1,1,1,1,1,1,1,1,1,1),(0,1,1,0,0,0,0,0,0,0,0),(0,4,0,0,0,0,0,0,0,0,0),$
$d=128$
$(1,7,1,1,1,1,1,1,1,1,1)\to(1,1,1,1,1,1,1,1,1,1,1),(0,6,0,0,0,0,0,0,0,0,0)$, $d=114$
$(1,4,2,2,1,1,1,1,1,1,1,1)\to(1,1,1,1,1,1,1,1,1,1,1,1),(0,1,1,1,0,0,0,0,0,0,0,0),$
$(0,2,0,0,0,0,0,0,0,0,0,0)$, $d=137$
$(1,4,3,1,1,1,1,1,1,1,1,1)\to(1,1,1,1,1,1,1,1,1,1,1,1),(0,2,2,0,0,0,0,0,0,0,0,0),$
$(0,1,0,0,0,0,0,0,0,0,0,0)$, $d=138$
$(1,5,2,1,1,1,1,1,1,1,1,1)\to(1,1,1,1,1,1,1,1,1,1,1,1),(0,1,1,0,0,0,0,0,0,0,0,0),$
$(0,3,0,0,0,0,0,0,0,0,0,0)$, $d=134$
$(1,6,1,1,1,1,1,1,1,1,1,1)\to(1,1,1,1,1,1,1,1,1,1,1,1),(0,5,0,0,0,0,0,0,0,0,0,0),$

$d = 124$

$(1,4,2,1,1,1,1,1,1,1,1,1,1) \to (1,1,1,1,1,1,1,1,1,1,1,1,1), (0,1,1,0,0,0,0,0,0,0,0,0,0),$
$(0,2,0,0,0,0,0,0,0,0,0,0,0)$, $d = 137$

$(1,5,1,1,1,1,1,1,1,1,1,1,1) \to (1,1,1,1,1,1,1,1,1,1,1,1,1), (0,4,0,0,0,0,0,0,0,0,0,0,0),$
$d = 131$

$(1,4,1,1,1,1,1,1,1,1,1,1,1,1) \to (1,1,1,1,1,1,1,1,1,1,1,1,1,1), (0,3,0,0,0,0,0,0,0,0,0,0,0,0),$
$d = 135$

GALAAD, INRIA Méditerranée, 2004 route del Lucioles, BP 93, F-06902 Sophia Antipolis, Cedex France.
E-mail address: alessandra.bernardi@inria.fr

Matematisk institutt, Universitetet i Oslo, PO Box 1053, Blindern, NO-0316 Oslo, Norway
E-mail address: ranestad@math.uio.no