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Abstract

The paper examines alternative strategies for pricing and hedging options on German DAX-index. To this purpose an affine stochastic volatility model is estimated directly on objective probability system through a three step approach. Errors obtained by the implementation of the stochastic volatility model and Black and Scholes with different historical and implied volatility measures are compared and the performance is evaluated in terms of out-of-sample pricing and hedging. The results for DAX-index options market support the estimation on the affine stochastic volatility model in pricing as well as in hedging procedures.

1. INTRODUCTION

Deviations of actual option prices from the benchmark model of Black and Scholes (Black and Scholes 1973) have been widely experienced since volatility term structures and smiles have been observed in the international derivative markets; since then, the effects of post crash fear on investors' risk attitudes and the growing diffusion of optionalities embedded in structured products called for a new approach in option pricing models. Many recent papers, basing on extensions of the classical Black and Scholes model (BS) and allowing for a time varying volatility, investigate different specifications of option pricing models consistent with the empirical evidence of smile and smirk as well as the leverage effect and the volatility persistence. Among these an interesting setup consists in a two diffusion equations system, explaining stock and volatility dynamics, known as stochastic or exogenous volatility (SV) approach, whose implementation involves many econometric issues. The main purpose of this study consists in providing an evaluation of a specified stochastic volatility model on option pricing and hedging. The performance of the proposed strategy is assessed by means of comparison with alternative Black and Scholes implementations. The application is on German option exchange index (DAX) which is the second largest in the world and is scarcely analyzed due to prevalence of American literature. Moreover, an advantage for paper purpose comes from the quotation in this market of an index based on implied volatilities of options on German market index (VDAX), computed as linear interpolation of the implied volatilities of the two sub-indices nearest to 45 days maturity; we will consider it as a benchmark for volatility forecasting.

The paper develops as follows. In section 2 the adopted SV model is presented. Section 3 explains how and why SV parameters are directly estimated on objective probability system via a three step estimation strategy. Section 4 provides estimation results, while section 5 and 6 present result of the application on a large option dataset. Concluding remarks are provided in section 7.

2. THE STOCHASTIC VOLATILITY MODEL

As a quite general setting, the Heston affine SV model is considered (Heston 1993), which assumes the following structural data generating process for the stock price S_t and the volatility v_t:

$$\begin{aligned} d\ln S_t &= \mu\, dt + \sqrt{v_t}\left(\sqrt{1-\rho^2}dz_{1t} + \rho dz_{2t}\right) \\ dv_t &= k\left(\theta - v_t\right)\, dt + \sigma\sqrt{v_t}dz_{2t} \end{aligned} \tag{1}$$

where z_1 and z_2 are two independent Wiener processes and the parameter vector under objective probability is $\phi = (\mu, k, \theta, \sigma, \rho)'$. This model captures several characteristics of the stock return dynamics. The stochastic volatility is modeled via an autonomous process v_t with a drift mean reverting to long run volatility level θ, and a proportional to a constant factor σ^2 variance. The coefficient ρ captures the (negative) association of price and volatility instantaneous variations. In option pricing framework the structural dynamics is transformed in the risk neutral dynamic system under the equivalent martingale measure Q, by means of no arbitrage arguments and assumption of a volatility risk premium proportional to v, as follows:

$$\begin{aligned} d\ln S_t &= r\, dt + \sqrt{v_t}\left(\sqrt{1-\rho^2}dz_{1t}^Q + \rho\, dz_{2t}^Q\right) \\ dv_t &= \left[k\left(\theta - v_t\right) - \eta v_t\right]\, dt + \sigma\sqrt{v_t}dz_{2t}^Q \end{aligned}$$

The new parameter vector $\phi^Q = \left(\phi', \eta\right)'$ contains the further volatility risk premium parameter, η, subtracted to volatility drift. Given the strike K, the time to expiration $T-t$, the risk-free interest and dividend rates, r and d, by means of the transform based approach (Heston 1993), it is possible to obtain a closed price for the European call option,

$$C\left(S_t, K, T-t, r, d, v_t\right) = e^{-d(T-t)} S_t P_1^Q - e^{-r(T-t)} K P_2^Q$$

where explicit formulations for the risk-neutral probability P_1^Q and P_2^Q could be evaluated by the Fourier inversion of two underlying known characteristic functions. Because of the Heston model's weakness to explain fat

tails and cross sectional smirkness, a jump process or a second volatility factor have been recently included extending the (1) (Andersen, Benzoni and Lund 2002, Pan 2002, Chernov, Gallant, Ghysels and Tauchen 2003). Regarding this, it has been showed (Bakshi, Cao and Chen 1997) that stochastic volatility and stochastic volatility with jump models are increasingly better than BS model in the in-sample fit as well as in the out-of-sample. But it has been also observed that adding other feature to a bivariate SV specification (jump or stochastic interest rate) led to second order pricing improvement and the bivariate specification could be considered quite general and robust.

3. ESTIMATION STRATEGY

The implementation of SV formula needs, with respect to BS, parameters' vector estimates and volatility process filtration. Stochastic volatility continuous time model in option pricing framework can be estimated with respect to the objective or directly to the risk neutral diffusion system. There are anyway many problems in the direct estimation of the risk-neutral specification due to highly non linearities in option pricing formula and to the not observability of the risk-neutral dynamics especially when market call/put options on the underlying index have tight liquidity or otherwise when they do not exist at all. The strike prices of the contracts are in fact set by the market makers and changed every day, making difficult to construct a time series of every contract of a statistically sufficient length. Moreover, it has been observed that parameters calibrated directly on options prices rarely have economic meaning, with negative consequence on interpretability, and on-out-of sample performance (Bakshi et al. 1997). On the other side, the alternative strategy to proceed is a three steps approach based on estimation of the structural parameter of the objective system on the underlying data.

The main difficulty in conducting inference for continuous time models from discretely sampled data is that a closed form expression for the discrete transition density generally is not available nor simple to be numerically computed. Simulation based procedures have been suggested in the literature to estimate continuous time process' parameters, such as the Markov chain Monte Carlo approach (Eraker 1998), but when latent variables must be integrated out of the likelihood function, maximum likelihood and Bayesian estimators become computationally too extensive and more flexibility is given by method of moment techniques. In the paper the Efficient Method of Moment (EMM) has been adopted as a reasonable solution of this econometric problem due to its efficiency property (Gallant and Tauchen 1996).

EMM rationale is based upon the main idea to find an analytically tractable approximating function whose likelihood's score is used to deduce moment conditions. EMM presumes that the score generator provides an adequate statistical approximation to the transition density of the data. The standard way to describe the conditional density is to reduce the rate of return process y_t to an innovation process $z_t = \frac{y_t - \mu_t}{\sigma_t}$, adopting a parametric AR structure for conditional mean, μ_t, and an ARCH-type process for the conditional variance, σ_t^2. Therefore, letting $n(\cdot)$ the standard normal distribution, the transition density $p(y_t | x_{t-1}, \phi_0)$ is approximated via a $K-th$ order orthogonal Hermite polynomial representation (H_K), named auxiliary function,

$$f_K\left(y_t \middle| x_{t-1}, \widehat{\xi}\right) = \frac{1}{\sigma_t} \frac{\left[H_K(z_t, x_{t-1})\right]^2 n(z_t)}{\int \left[H_K(z_t, x_{t-1})\right]^2 n(u)\, du}$$

which turns out to be a mixture of normal distributions able to capture excess kurtosis and, to a lesser extent, asymmetry of returns. In short the method proceeds as follows: supposing that y_t is the observed variable and $x_{t-1} = (y_{t-1}, \ldots, y_{t-L})$ the vector of its L past observations, if $f_K(y_t | x_{t-1}, \xi)$ is the approximating auxiliary function for the transition density of observed data $p(y_t | x_{t-1}, \phi_0)$ and $\widehat{\xi}$ is the maximum likelihood estimator of the parameter vector ξ, posing $s_f(\cdot)$ the score of auxiliary function, the moment conditions are:

$$\int s_f(y_t, x_{t-1}, \xi)\, p(y_t | x_{t-1}, \phi_0)\, d(y_t, x_{t-1}) = 0 \qquad (2)$$

Expectations in the (2), even if not analytically, could be computed on the basis of the Central Limit Theorem, by averaging over a N-length Monte Carlo data series $\widetilde{y}_t$ simulated from the structural model $p(y_t | x_{t-1}, \phi)$, so that $\widetilde{m}_N\left(\phi, \widehat{\xi}\right) = \frac{1}{N} \sum_{t=L+1}^{N} s_f\left(\widetilde{y}_t(\phi), \widetilde{x}_{t-1}(\phi), \widehat{\xi}\right)$ and the EMM estimator is

$$\widehat{\phi} = \arg\min \left\{ \widetilde{m}_N\left(\phi, \widehat{\xi}\right)' \widetilde{I}_n^{-1} \widetilde{m}_N\left(\phi, \widehat{\xi}\right) \right\}$$

where the optimal weighting matrix, $\widetilde{I}_n$, corresponds to the estimated information matrix

$$\widetilde{I}_n = \frac{1}{n} \sum_{t=1}^{n} \left[s_f\left(\widetilde{y}_t(\phi), \widetilde{x}_{t-1}(\phi), \widehat{\xi}\right) \right] \left[s_f\left(\widetilde{y}_t(\phi), \widetilde{x}_{t-1}(\phi), \widehat{\xi}\right) \right]'$$

and n is the number of observations. EMM provides a way to test the goodness of fit of structural model; if the selected model is the true data generating process, then $n\left[m_N\left(\widehat{\phi}, \widehat{\xi}\right)' \widetilde{I}_n^{-1} m_N\left(\widehat{\phi}, \widehat{\xi}\right)\right] \longrightarrow \chi^2_{(l_\xi - l_\phi)}$ where l_ξ and l_ϕ are the lengths of parameters vector. Moreover, if the auxiliary model encompasses the structural one, EMM can be as efficient as maximum likelihood.

After estimating the structural diffusion equations parameters, furthers steps of the strategy needs filtering the latent volatility process and calibrating volatility risk premium.

To this purpose, the filtered volatility could be obtained (Gallant and Tauchen 1998) reprojecting a long simulated series conditional to EMM parameters' estimates on to the former semi-nonparametric (SNP) density and looking on the second conditional moment. An alternative procedure consists in finding a conditional model to reproduce all the details of the unobservable data in terms of the observable ones for the two simulated time series. In our opinion, it is quite natural to choose the volatility filter specification into the ARCH-type class. Finally, the volatility risk premium parameter could be directly calibrated on a subset of options data by means

Table 1: DAX Index Return. Summary Statistics

	Mean	Median	Max	Min	St.Dev.	Skew	Kurtosis
y_t	0.044	0.034	7.287	-9.871	1.258	-0.415	7.701
lag	**1**	**2**	**3**	**4**	**5**	**6**	**7**
$\rho(y_t, y_{t-i})$	0.014	-0.029	-0.013	0.009	0.027	-0.056	0.045
$\rho(y_t^2, y_{t-i}^2)$	0.167	0.161	0.158	0.139	0.138	0.105	0.118

of minimization of the quadratic relative loss function

$$\min_{\eta} \sum \left[\frac{C\left(\eta, \widehat{\phi}\right) - C}{C} \right]^2$$

4. EMPIRICAL RESULTS

To perform the empirical test of the proposed strategy, DAX index and call options prices have been used because of their relatively high liquidity with respect to other traded options in European financial markets. The data covers the period from January 1, 1990 to December 31, 2000[1]. Summary statistics in table 1 show that empirical distribution of DAX rate of return is skewed to the left with fat tails suggesting excess kurtosis. There is no autocorrelation in rate of returns, while squared returns are autocorrelated.

Basing upon Akaike and Bayes information criteria a Semi Non Parametric specification (Gallant and Tauchen 1996) corresponding to an AR(0)-ARCH(11)-H(4,0) auxiliary model has been chosen. For EMM estimation, the structural system has been discretised using the Euler scheme, with 24 observations for each day, one for each hour, discarding the firsts 23 and retaining the last one; 50,000 pseudo-observations have been simulated.

Results of EMM procedure (see Table 2) show that the affine stochastic volatility model specification for DAX rate of return is not rejected. Skewness is controlled by the negative correlation parameter and the amount of kurtosis is explained by the volatility diffusion parameter σ. The positive mean reverting coefficient, k, guaranties a steady state distribution.

[1]All data are from Datastream.

Table 2: Emm Estimates

Par	Estim	Std Err	*t* stat.	Par	Estim	Std Err	*t* stat.
μ	0.20101	0.01734	11.592	ρ	-0.53665	0.00553	-96.954
k	4.14211	0.16011	25.871	σ	0.35202	0.00654	53.854
θ	0.04043	0.00040	99.620	η	-5.1	$\chi^2_{(12)}$	10.568

The conditional expected variance has been modelled following an exponential ARCH-X process, as in the following specification

$$E\left[\log\left(v_t\right)\right] = \omega + \beta \log\left(v_{t-1}\right) + \alpha_1 y_t + x_{t-1}\alpha_2$$

The negative risk premium, calibrated on 225 near the money options traded from 1 to 31 January, 2001, acts augmenting the drift of the volatility process and, therefore, options prices.

5. PRICING PERFORMANCE

The proposed strategy has been empirically tested in out-of-sample performance by means of the comparison with the benchmark formula of BS with differently historical volatility measures computed respectively on 1, 2, 4 and 8 weeks windows (h1w, h2w, h4w and h8w) and the VDAX implied volatility index (IV), that, while intrinsically not consistent in varying volatility framework due to BS assumptions dependence, is widely considered to incorporate agents' forward looking expectations. According to existing literature (Bates 2003) and emulating the practitioners, each day call prices have been computed backing out the volatility measures of the day before for each strategy. Option database includes 3009 call prices on DAX index traded from January 1, through June 30, 2001. Performances were controlled, on different maturity and moneyness subclasses, through mean absolute percentage (MAPE) and mean percentage (MPE) pricing errors as precision and bias measures.

The best performances in term of systematic errors correspond to h2w, SV and IV forecasts, but, among these, the historical two-weeks window strategy displays a quite relevant mean absolute percentage error meaning that SV and IV should be preferred (Tab. 3); mean percentage errors are still quite low both for SV and for IV strategies, respectively 2.10% and 5.61%. Moreover SV out-of-sample pricing presents an overall mean absolute relative error of 14.59%, 1.63% less than IV with implied volatility while all the historical measures produce less accurate prices. As expected, there is a greater benefit in using SV instead of IV overall in medium and long maturity options pricing, especially if deep in the money (Tab.4). IV formula with implied volatility is significantly better than SV, excluded that for the obvious short maturity near the money options by which is computed, for 2 to 6 months to maturity near the money options.

6. HEDGING PERFORMANCE

In judging the alternative strategies, hedging errors measures have been empirically computed. The cash position achieving the derivative duplication for the hedge at time t, is

$$X_0(t) = C\left(t, T-t\right) - X_s\left(t\right) S_t \tag{3}$$

Table 3: Pricing and Hedging Performances

	Pricing		Hedging	
	MPE	MAPE	MPE	MAPE
h1w	-9.89	46.43	3.82	45.93
h2w	1.77	40.69	4.42	24.11
h4w	11.45	38.31	5.67	18.22
h8w	28.54	39.78	6.21	13.74
IV	5.61	16.22	2.39	12.72
SV	2.10	14.59	0.09	0.42

Table 4: Pricing Errors by Moneyness and Maturity

	MPE - SV					MAPE SV/IV ratio				
	Maturity									
Moneyness	<2M	2/6M	6M/1Y	>1Y	tot	<2M	2/6M	6M/1Y	>1Y	tot
< 0.94	-	-10.21	-5.25	16.08	1.00	-	0.95	0.78	0.93	0.85
0.94-0.97	-	1.20	-1.11	4.63	1.64	-	1.03	1.04	0.75	1.00
0.97-1.03	1.15	4.69	-1.72	4.83	3.05	1.00	1.03	0.92	0.96	0.99
1.03-1.06	2.72	6.83	-3.12	4.76	4.31	1.04	1.05	0.79	0.94	1.00
>1.06	9.07	3.20	-3.48	5.73	6.02	1.05	1.08	0.94	0.96	1.02
tot	3.95	0.11	-2.48	12.49	2.10	1.02	1.01	0.82	0.92	0.90

where $X_s(t)$ is the number of shares of underlying. This has been implemented for BS strategies posing $X_s(t) = \Delta_s(t, T-t)$. Since in presence of stochastic volatility the market is no more complete and a riskless hedging portfolio can not be assured due to volatility risk source, to preserve comparability of results we choose to implement the single-instrument hedging obtained solving the standard minimum-variance hedging problem. For SV strategy the following expression has been substituted in the formula (3)

$$X_s(t) = \Delta_s(t, T-t) + \rho\sigma\Delta_v(t, T-t)$$

where $\Delta_v(.) = \frac{\partial C(.)}{\partial v}$. Supposing that in practice discrete rebalancing takes place at every time intervals Δt, at time $t + \Delta t$ the hedging error becomes

$$H(t+\Delta t) = X_s(t) S_{t+\Delta t} + X_0(t) e^{r(t)\Delta t} - C(t+\Delta t, T-t-\Delta t)$$

Rebalancing the portfolio once a week, hedging errors measures have been computed for each strategy on the options database. Performances controlled by means of percentage absolute and percentage errors are reported in table 3 while in table 5 a skeptical view of gains by SV for different maturity and moneyness subclasses is presented. It can be viewed that BS formula performs better with implied volatility than with historical ones in hedging strategy as in pricing. The mean absolute percentage error of SV hedging (0.09%) is widely less than every BS strategy; moreover it diminishes, as expected, passing from out of the money to at and in the money calls, and from long to short maturity. In summary, the empirical evidence indicates that the proposed strategy allows relevant improvements respect to BS formula in hedging as in pricing.

7. CONCLUDING REMARKS

In this paper alternative models for the German DAX-index options market are examined. In particular the affine stochastic volatility option

Table 5: Hedging Errors by Moneyness and Maturity

	MPE - SV					MAPE SV/IV ratio				
	Maturity									
Moneyness	<2M	2/6M	6M/1Y	>1Y	tot	<2M	2/6M	6M/1Y	>1Y	tot
< 0.94	-	0.03	0.08	0.17	0.10	-	0.02	0.03	0.04	0.03
0.94-0.97	-	0.05	0.06	0.20	0..07	-	0.02	0.04	0.07	0.03
0.97-1.03	0.04	0.08	0.11	0.16	0.09	0.01	0.02	0.06	0.11	0.04
1.03-1.06	0.02	0.10	0.03	0.19	0.08	0.01	0.04	0.07	0.12	0.05
>1.06	0.01	0.06	0.10	0.08	0.06	0.03	0.03	0.10	0.10	0.07
tot	0.02	0.06	0.08	0.17	0.09	0.01	0.02	0.03	0.05	0.03

pricing model estimated under objective probability system is developed and its performance is compared to the Black and Scholes model with standard volatility measures, such as historical and implied. In terms of out-of-sample pricing the empirical evidence indicates wide improvements respect to BS with historical measures by either implied volatility BS and SV models and SV relatively enhances the prediction of medium and long term options' prices. One week rebalancing hedging errors of SV model are quite smaller than among all the others strategies. In summary, the affine stochastic volatility option pricing model estimated under objective probability system is found to perform quite well either in pricing and hedging German DAX-index options.

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