ISSN 2282-6483

Alma Mater Studiorum - Università di Bologna
DEPARTMENT OF ECONOMICS

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Quaderni - Working Paper DSE N°1043

Parabolic Cylinders and Folk Theorems*

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December 14, 2015

Abstract

We study a class of games featuring payoff functions being parabolic cylinders where best reply functions are orthogonal and therefore the pure-strategy non-cooperative solution is attained as a Nash equilibrium in dominant strategies. We prove that the resulting threshold of the discount factor above which implicit collusion on the Pareto frontier is stable in the infinite supergames is independent of the number of players. This holds irrespective of whether punishment is based on infinite Nash reversion or one-shot stick-and-carrot strategy. We outline two examples stemming from economic theory and one from international relations.

JEL Codes: C73

Keywords: parabolic cylinder; supergame; folk theorem; implicit collusion

*We thank Arsen Palestini for helpful suggestions. The usual disclaimer applies.

1 Introduction

In the theory of non-cooperative games, we are accustomed to think that infinite supergames with discounting may allow players to generate infinitely many equilibria which are Pareto-superior to the Nash equilibrium of the stage game. This is the essential message of generations of folk theorems (see, e.g., Fudenberg and Tirole, 1991, ch. 5). Moreover, the acquired wisdom holds that the stability of implicitly collusive paths is monotonically decreasing in the number of players, the reason for this being that for any given size of the pie to be split among players, the individual slice is itself monotonically decreasing in the number of diners.

We shall show that this property is engendered by the presence of multiplicative effects among players' strategic variables, in the absence of which the stability criterion is indeed a pure number. To do so, we construct a payoff function which takes the shape of a parabolic cylinder producing orthogonal best reply functions, thereby identifying a class of non-cooperative games which are solvable in dominant strategies. Then, we characterize some properties of such payoff function, to be used to investigate the outcome of the infinite supergame based on (i) infinite Nash reversion (Friedman, 1971) or (ii) one-shot stick-and-carrot punishments (Abreu, 1986), alternatively. By doing so, we prove rather surprising results, namely, that

- the critical threshold of the discount factor above which implicit collusion along the Pareto frontier is stable is independent of the number of players in both cases (i-ii); and
- under (i) such threshold is twice as high as under (ii).

The straightforward implication of our results is that relying upon the number of players to assess the stability of the Pareto-efficient outcome may not be a sound proposal. For example, if the game describes oligopolistic interaction, an antitrust agency could be tempted to think that increasing the number of firms in the industry may make collusion less likely. Our result imply that this implication is not systematically reliable.

To corroborate the general analysis (Section 2), which is carried out with no reference to any specific field of social sciences, we outline two examples belonging to the theory of the firm (Section 3) and the intersection between economic geography and the theory of industrial organization (Section 4), and one emerging from the theory of international relation (Section 5). Concluding remarks are in Section 6.

2 The general case

Consider a supergame over discrete time $t = 0, 1, 2, ...\infty$, played by a finite set $\mathcal{N} = (2, 3, ...n)$ of agents. Each player i controls a single variable $x_i \in \mathbb{R}$. Each stage game is played noncooperatively under complete, symmetric and imperfect information. All players discount the future at a common and time-invariant discount factor $\delta \in (0, 1)$. In the whole paper, we confine our attention to pure strategies.

Let the objective function of the i-th player be

$$v_i = \alpha x_i^2 + \beta \sum_{j \neq i} x_j + \gamma x_i + \varepsilon \qquad (1)$$

where $\{\alpha, \beta, \gamma, \varepsilon\}$ is a vector of real parameters. Depending on the sign of α, player i faces either a maximum or a minimum problem. We will come back later to the sign of $\{\beta, \gamma, \varepsilon\}$.

Note that v_i is the equation of a parabolic cylinder in $\mathbb{R}^n$, whose canonical equation is $\overline{v}_i = \alpha x_i^2 + \beta \sum_{j \neq i} x_j$. In our case, the parabolic cylinder is translated by $\gamma x_i + \varepsilon$.

An interesting property of (1) will become relevant in the ensuing analysis. From the *Theorema Egregium* of Gauss (1827), we know that the Gaussian curvature of any given surface does not modify by bending the surface itself without stretching it. That is, the Gaussian curvature is determined by measuring angles, distances and their rates on the surface itself, with no reference whatsoever to the specific way in which the surface is embedded in the 3D Euclidean space. Hence, the Gaussian curvature is an intrinsic invariant property of a surface. In Gauss' own words:

"Si superficies curva in quamcumque aliam superficiem explicatur, mensura curvaturae in singulis punctis invariata manet."

The Gaussian curvature $\mathbf{K}$ of a surface $f(x_i, x_j)$ is defined as the determinant of the Hessian matrix of the function $f(x_i, x_j)$ itself:[1]

$$\mathbf{K} \equiv \det \begin{bmatrix} \frac{\partial^2 f(\cdot)}{\partial x_i^2} & \frac{\partial^2 f(\cdot)}{\partial x_i \partial x_j} \\ \frac{\partial^2 f(\cdot)}{\partial x_j \partial x_i} & \frac{\partial^2 f(\cdot)}{\partial x_j^2} \end{bmatrix} = \frac{\partial^2 f(\cdot)}{\partial x_i^2} \cdot \frac{\partial^2 f(\cdot)}{\partial x_j^2} - \frac{\partial^2 f(\cdot)}{\partial x_i \partial x_j} \cdot \frac{\partial^2 f(\cdot)}{\partial x_j \partial x_i} \quad (2)$$

If $f(\cdot)$ is a cylinder, then $\mathbf{K} = 0$, because it can be flattened onto a plane whose Gaussian curvature is zero. This property obviously extends to the specific case in which $f(\cdot)$ is a parabolic cylinder, as (1).

The Gaussian curvature is no longer nil in presence of a multiplicative effect between x_i and x_j or when $f(\cdot)$ is non-linear in at least one of the $n-1$ x_j's.

In our model, since v_i is additively separable w.r.t. the strategic variables of all players, the extremal of v_i is in correspondence of

$$x_i = x^* = -\frac{\gamma}{2\alpha} \quad (3)$$

for any vector of x_j's. This amounts to saying that reaction functions are orthogonal to each other. At x_i^*, the maximum or minimum of v_i is

$$v_i^* = -\frac{\gamma^2}{4\alpha} + \beta \sum_{j \neq i} x_j + \varepsilon \quad (4)$$

for any vector of x_j's.

[1] For more, see, e.g., do Carmo (1976, chapter 3).

Another relevant property of v_i is the following. The gradient of v_i is

$$\nabla v_i = \left\{ \frac{\partial v_i}{\partial x_1} = \beta, ... \frac{\partial v_i}{\partial x_{i-1}} = \beta, \frac{\partial v_i}{\partial x_i} = 2\alpha x_i + \gamma, \frac{\partial v_i}{\partial x_{i+1}} = \beta, ... \frac{\partial v_i}{\partial x_n} = \beta \right\} \tag{5}$$

which implies the following:

Lemma 1 *If the objective function of player i is a parabolic cylinder, then, for any x_i, the effect of a change in any x_j on v_i, for all $j \neq i$, is (a) constant and (b) independent of n.*

Notice that properties (a-b) do not generally hold true. For instance, if player i's objective function is

$$f_i (x_i, X_{-i}) = \left[\alpha - \gamma x_i - \beta \sum_{j \neq i} x_j - \varepsilon \right] x_i \tag{6}$$

where X_{-i} is the vector of all x_j, $j \neq i$, then $\partial f_i (x_i, X_{-i}) / \partial x_j = -\beta x_i$ and this, combined with the fact that the best reply function of i is $x_i^* = \left(\alpha - \beta \sum_{j \neq i} x_j - \varepsilon \right) / (2\gamma)$, makes $\partial f_i (x_i, X_{-i}) / \partial x_j$ dependent on n. This is because (6) generates a multiplicative effect between x_i and all of the x_j's, which reveals, as noted above, that (6) is not a parabolic cylinder and therefore its Gaussian curvature is not nil. This example illustrates that, when the best replies are not orthogonal, each element of the gradient ∇v_i being a function of the numerosity of players, Lemma 1 stops holding true.

A direct implication of Lemma 1 is:[2]

Proposition 2 *For any vector of pure strategies $(x_i, x_j = x_i + (n-1) k_\ell)$, with $\ell = 1, 2, 3$,*

$$\psi_{ih} = \frac{v_i (x_i, x_i + (n-1) k_3) - v_i (x_i, x_i + (n-1) k_2)}{v_i (x_i, x_i + (n-1) k_h) - v_i (x_i, x_i + (n-1) k_1)} = \frac{k_3 - k_2}{k_h - k_1}, h = 2, 3$$

[2]From Proposition 2 onwards, we follow the common practice of treating n as a continuous variable.

and therefore

$$\frac{\partial \psi_{ih}}{\partial n} = 0.$$

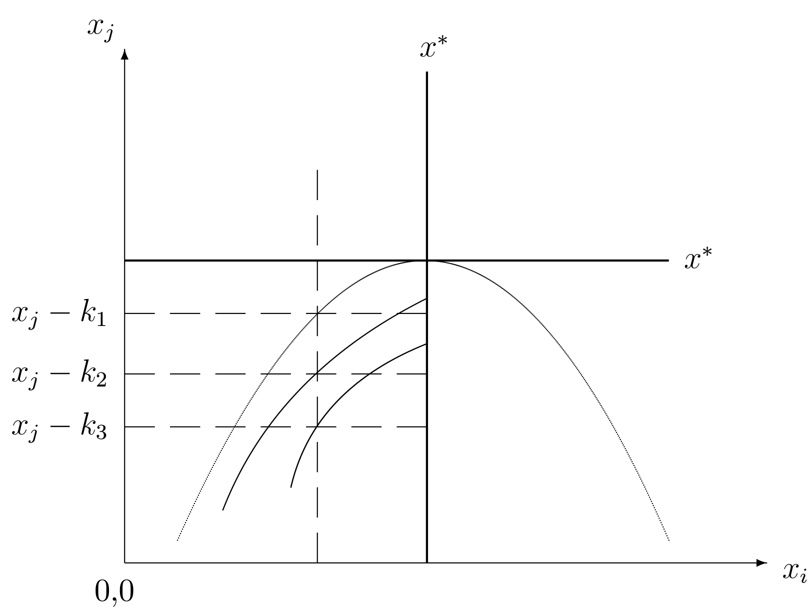
Proof. Suppose player i picks any x_i (which, in general, does not coincide with the best reply x^*, although it may). Then, imagine all players $j \neq i$ choose $x_j = x_i + (n-1)k$, with $k \in \mathbb{R}$. Notice that $x_j = x_i + (n-1)k$ covers the whole strategy space over the real axis, for any player $j \neq i$, by tuning parameter k. Obviously, if $k = 0$, then $x_j = x_i$.

Player i's payoff in correspondence of outcome $(x_i, x_i + (n-1)k)$ is

$$v_i(x_i, x_i + (n-1)k) = (\alpha x_i + \gamma)x_i + \beta(n-1)[x_i + (n-1)k] + \varepsilon \qquad (7)$$

For any triple $k_1 \neq k_2 \neq k_3$, we can define the corresponding payoffs to player i as $v_i(x_i, x_i + (n-1)k_\ell)$ with $\ell = 1, 2, 3$, respectively. This situation is illustrated in Figure 1, where $x_i < x^*$.

Figure 1



Now observe that the ratios

$$\psi_{i3} = \frac{v_i\left(x_i, x_i + (n-1)\,k_3\right) - v_i\left(x_i, x_i + (n-1)\,k_2\right)}{v_i\left(x_i, x_i + (n-1)\,k_3\right) - v_i\left(x_i, x_i + (n-1)\,k_1\right)} = \frac{k_3 - k_2}{k_3 - k_1} \tag{8}$$

$$\psi_{i2} = \frac{v_i\left(x_i, x_i + (n-1)\,k_3\right) - v_i\left(x_i, x_i + (n-1)\,k_2\right)}{v_i\left(x_i, x_i + (n-1)\,k_2\right) - v_i\left(x_i, x_i + (n-1)\,k_1\right)} = \frac{k_3 - k_2}{k_2 - k_1} \tag{9}$$

are both independent of the number of players. ■

One might wonder about the interest of the above analysis. The answer lies in the understanding of ψ_{ih} and its partial derivative w.r.t. the number of players. The interpretation becomes evident as soon as one thinks back to the critical threshold of the discount factor above which implicit collusion becomes sustainable in an infinitely repeated game in which the time-invariant payoff function takes the form specified in (1).

To see this, consider the case in which player i sticks to his best reply, i.e., $x_i = x^* = -\gamma/(2\alpha)$. If all other players choose $k = 0$ and thus play $x_j = x^* = -\gamma/(2\alpha)$, the resulting outcome is the Nash equilibrium of the stage game, yielding the symmetric payoff

$$v^N = \varepsilon - \frac{\gamma^2}{4\alpha} - \frac{\beta\gamma\,(n-1)}{2\alpha} \tag{10}$$

where N stands for *Nash*. If instead all players collude along the frontier of the collective payoff $V = \sum_{i=1}^{n} v_i$, the symmetric strategy optimising V is

$$x^C = -\frac{\beta\,(n-1) + \gamma}{2\alpha} \tag{11}$$

where C mnemonics for *collusion*. The corresponding payoff is

$$v^C = \frac{4\alpha\varepsilon - \beta\,(n-1)\left[\beta\,(n-1) + 2\gamma\right] - \gamma^2}{4\alpha} \tag{12}$$

The unilateral deviation from the collusive outcome takes place along (3) and delivers

$$v^D = \frac{4\alpha\varepsilon - 2\beta\,(n-1)\left[\beta\,(n-1) + \gamma\right] - \gamma^2}{4\alpha} \tag{13}$$

in which D stands for deviation.

As illustrated in Figure 2,

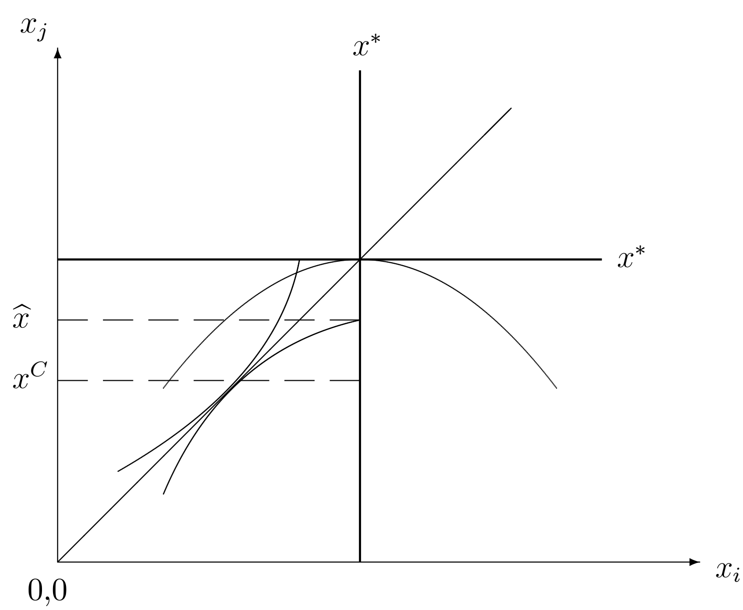
$$x_j = \widehat{x} = -\frac{\beta(n-1)+2\gamma}{4\alpha} \tag{14}$$

univocally solves $v_i(x^*, x_j) = v^C$. That is, $\widehat{x}$ is the strategy that each of the $n-1$ rivals should play in order to grant to player i the same payoff as under collusion when player i himself sticks to his best reply. Using x^*, x^C and $\widehat{x}$, one can easily see that

$$\frac{x^* - \widehat{x}}{x^* - x^C} = \frac{1}{2} \tag{15}$$

which is a special case of the situation illustrated in Proposition 2, where $\widehat{x} = x^* - \beta(n-1)/(4\alpha)$, so that $k = \widehat{k} = -\beta/(4\alpha)$ and $x^C = x^* - \beta(n-1)/(2\alpha)$, so that $k = k^C = -\beta/(2\alpha)$.

Figure 2



Now it is appropriate to recollect the rules of a supergame over an infinite horizon relying on Friedman's (1971) grim trigger strategies, according to which any deviation from the collusive path is followed by the infinite Nash reversion. This implies that collusion is stable iff

$$\frac{v^C}{1-\delta} \geq v^D - \frac{\delta v^N}{1-\delta} \tag{16}$$

that is, for all

$$\delta \geq \frac{v^D - v^C}{v^D - v^N} \equiv \delta_F = \frac{1}{2} \tag{17}$$

where subscript F mnemonics for Friedman. Therefore, if the triple of values chosen for k is $\left(k_1 = 0, k_2 = k^C, k_3 = \widehat{k}\right)$, then the ratio ψ_{i3} is nothing but δ_F, i.e., the critical level of the discount factor under a punishment based on the infinite Nash reversion.

Alternatively, if one-shot stick-and-carrot punishments (also termed as optimal punishments) are used,[3] as in Abreu (1986), the perpetual stability of the collusive path is ensured iff

$$v^D - v^C \leq \delta \left(v^C - v^{OP}\right) \tag{18}$$

while the incentive compatibility constraint that must be satisfied in order for players to simultaneously adopt the optimal punishment is[4]

$$v^D \left(x^{OP}\right) - v^{OP} \leq \delta \left(v^C - v^{OP}\right) \tag{19}$$

where v^{OP} is the payoff resulting from the symmetric adoption of the optimal punishment x^{OP} and $v^D \left(x^{OP}\right)$ is the payoff produced by the optimal unilateral deviation x^* against the punishment. The system of inequalities (18-19) has to be solved w.r.t. the unknowns δ and x^{OP}, yielding

$$\delta \geq \frac{v^D - v^C}{v^C - v^{OP}} \equiv \delta_A = \frac{1}{4} \tag{20}$$

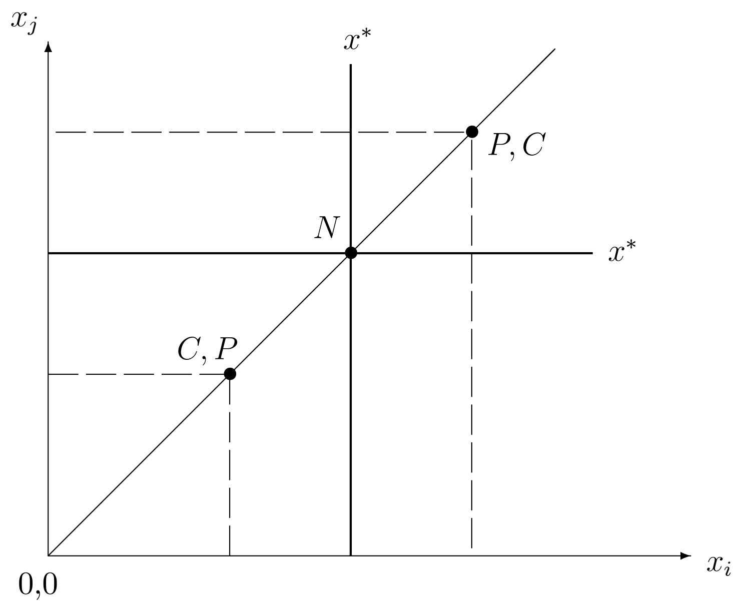
[3] We confine ourselves to single-period optimal punishments. For the analysis of multi-period optimal punishments, see Lambertini and Sasaki (2002).

[4] There exists a third constraint regarding the non-negativity of the continuation payoff. This must be satisfied to ensure that players do not quit the supergame after an initial deviation from collusion. We will explicitly take into account this constraint in the remainder of the paper.

where subscript A stands for Abreu, and $x^{OP} \geq [\beta(n-1)-\gamma]/(2\alpha)$. Note that the lower bound of x^{OP} is $x^* + (n-1)k$ when $k = \beta/(2\alpha) \equiv k^{OP}$. Accordingly, ψ_{i2} is δ_A, i.e., the lowest level of the discount factor stabilising collusion under optimal one-shot punishments if the triple of values of k is $\left(k_1 = \beta/(2\alpha), k_2 = k^C, k_3 = \widehat{k}\right)$.

The interpretation of the property whereby $\partial\psi_{ih}/\partial n = 0$, as from Proposition 2, is therefore that in supergames where the payoff function is a parabolic cylinder, collusive stability does not depend on the number of players, contrary to the acquired wisdom on the basis of which we are accustomed to think that enlarging a cartel size amounts to destabilising it.

Figure 3



Additionally, observe that the value of k^{OP} generating the lower bound of x^{OP} is exactly equal to $-k^C$.[5] This implies that the two segments CN

[5] Hence, the position of C and P w.r.t. N depends on the sign of β/α, that determines the direction along which v_i increases along i's best reply. This fact is reflected in Figure

and NP along the 45° line in Figure 3 have the same length, and therefore $CN/CP = NP/CP = 1/2$, which explains why $\delta_F = 2\delta_A$. Properties (15) and $k^{OP} = -k^C$ follow from any cylinder (including a parabolic one) being broadly defined in differential geometry as any ruled surface spanned by a parametric family of parallel lines.

The foregoing discussion boils down to the following:

Proposition 3 *If the payoff function is a parabolic cylinder, then the stability of implicit collusion in a supergame over an infinite horizon is independent of the number of players, under both Nash and optimal punishments. Moreover, the critical discount factor generated by the infinite Nash reversion is twice as high as that generated by one-shot stick-and-carrot punishments.*

This prompts for an additional question, about the relevance of the class of games featuring this property. We are about to provide three such examples by borrowing well established models from the theory of the firm, from a terrain where industrial organization and economic geography overlap, and international relations. In all of these models, players act non cooperatively over an infinite time horizon.

3 Team production

As for the stage game, here we use a version of the model by Holmström (1982), where the focus is the arising of moral hazard in teams. A set of n agents is employed in a firm whose output q is obtained through the production function $q = z\sum_{i=1}^{n}\sqrt{e_i}$, where $e_i \geq 0$ is agent i's effort and z is a positive parameter. Each agent chooses e_i to maximise utility $u_i = w_i - ce_i$, where $w_i \geq 0$ is wage and c is a positive parameter. The shape of u_i reveals constant work-aversion. The sharing rule is decided by the principal, unable to observe each individual effort. Hence, the principal sets $w_i = q/n$ for all

3, where C and P may exchange positions depending on the sign of β/α.

$q \geq \bar{q} > 0$; and $w_i = 0$ otherwise (as in Groves, 1973). If indeed $w_i = q/n$, the individual utility writes

$$u_i = \frac{z \sum_{i=1}^{n} \sqrt{e_i}}{n} - ce_i \tag{21}$$

which is a a parabolic cylinder corresponding to (1), if one redefines $x_i = \sqrt{e_i}$. However, here the non-negativity of efforts entails that we confine our attention to real and positive portion of the cylinder. Moreover, looking back at (1), $\alpha = -c$, $\beta = z/n$ and $\gamma = \varepsilon = 0$.

The Nash equilibrium of the stage game is delivered by the solution of the following system of first order conditions (FOCs):

$$\frac{\partial u_i}{\partial e_i} = \frac{z}{2n\sqrt{e_i}} - c = 0 \tag{22}$$

whereby, $e^* = z^2/(4c^2n^2)$. The resulting payoff is $u^N = (2n-1)\, z^2/(4cn^2)$. The associated output is $q^N = z^2/(2c)$.

In the context of this model, the collusive outcome is the Pareto-efficient solution, which is attained by imposing the *a priori* symmetry condition upon efforts, $e_i = e$ for all i, and then maximising the generic u_i w.r.t. e. Doing so, one obtains $e^C = z^2/(4c^2)$, yielding $u^C = z^2/(4c)$. The unilateral deviation from e^C takes place along the best reply (22), and is $e^D = e^*$. The resulting payoff is $u^D = [1 + 2(n-1)n]\, z^2/(4cn^2)$.

If deviation is deterred via the infinite Nash reversion as in Friedman (1971), the Pareto-efficient outcome is stable iff

$$\delta \geq \delta_F = \frac{u^D - u^C}{u^D - u^N} = \frac{1}{2} \tag{23}$$

If instead the one-shot stick-and-carrot strategy is adopted, the utility level in the punishment period is $u^{OP} = z\sqrt{e^{OP}} - ce^{OP}$, while the utility generated by a unilateral deviation from the optimal punishment is

$$u^D\left(e^{OP}\right) = \frac{z\left[\sqrt{e^*} + (n-1)\sqrt{e^{OP}}\right]}{n} - ce^*. \tag{24}$$

Solving the inequalities corresponding to (18-19), one obtains $\delta_A = 1/4$ and $e^{OP} \geq z^2 (n-2)^2 / (4c^2 n^2)$.

As for the third constraint, the discounted flow of payoffs accruing to each player from the symmetric punishment phase to doomsday must be non-negative so as to prevent exit. That is,

$$u^{OP} + \frac{\delta u^C}{1-\delta} \geq 0 \tag{25}$$

Using e^C and the lower bound of e^{OP}, we can simplify the l.h.s. of (25) as $z^2 (n^2 - 4 + 4\delta) / [4cn^2 (1-\delta)]$, which is strictly positive for all $\delta \in (0,1)$ and all $n \geq 2$. Hence, in this example, $\delta_F = 2\delta_A = 1/2$, as required by Proposition 3.

Our reformulation of Holmström's (1982) model illustrates that (i) the Pareto-efficient outcome can be sustained forever without implementing the Groves mechanism, provided agents are sufficiently patient; and (ii) the critical threshold of the discount factor is independent of the team size.

4 Agglomeration, externalities and collusion

The acquired wisdom from both the IO literature (Tirole, 1988, pp. 247-48) and policy reports (Ivaldi *et al.*, 2003) is that high market concentration is a facilitating factor for (tacit as well as explicit) collusion. In addition to coordination being likely more difficult in larger groups, the intuition that the incentive to collusion shrinks with too many competitors is fairly simple:

"Since firms must share the collusive profit, as the number of firms increases each firm gets a lower share of the pie. This has two implications. First, the gain from deviating increases for each firm since, by undercutting the collusive price, a firm can steal market shares from all its competitors; that is, having a smaller share each firm would gain more from capturing the entire market. Second, for each firm the long-term benefit of maintaining collusion is reduced, precisely because it gets a smaller share of the collusive profit. Thus the short-run gain from deviation increases, while at the same time the long-run benefit of maintaining collusion is reduced. It is thus more difficult to prevent firms from deviating." (Ivaldi *et al.*, 2003, p. 12)

The source of this property is the externality engendered by a negatively sloped demand function, as it transpires from the above quotation. Hence, *a priori*, one might be induced to conjecture that, if market price is exogenously given for some reasons, this property stops holding and collusion becomes altogether impossible. We are about to show that such conjecture may be falsified if an industry features some other admissible type of externality.

Consider n identical firms selling a homogeneous good whose market price $p > 0$ is exogenous, because of either a perfectly competitive market or regulatory intervention. Firm i's output is q_i, and production involves total costs $C_i = c\left(q_i^2 - b\sum_{j\neq i} q_j\right)$, with $c > 0$ and $b \in (0, 1/(n-1))$. This cost function captures the presence of a linear externality due, for instance, to agglomeration phenomena as in Krugman (1991) and the whole literature on agglomeration and economic geography.

Per-period individual profits writes

$$\pi_i = pq_i - c\left(q_i^2 - b\sum_{j\neq i} q_j\right) \tag{26}$$

Looking back at (1), in (26) we can recognize a parabolic cylinder where $\alpha = -c$, $\beta = b - c$, $\gamma = p$ and $\varepsilon = 0$.

The best reply function, delivering the Nash equilibrium output, is $q^* = p/(2c)$. Nash equilibrium profits are $\pi^N = p\left[p + 2bc(n-1)\right]/(4c)$. If firms collude along the frontier of industry profits, each of them has to solve

$$\max_{q_i} \Pi = \sum_{i=1}^{n} \pi_i \tag{27}$$

whereby the individually optimal collusive output is $q^C = \left[p + bc(n-1)\right]/(2c)$ and the corresponding share of cartel profits is $\pi^C = \left[p + bc(n-1)\right]^2/(4c)$.

The unilateral deviation q^* from collusion delivers

$$\pi^D = \frac{p\left[p + 2bc\left(n-1\right)\right] + 2b^2c^2\left(n-1\right)^2}{4c} \tag{28}$$

If the supergame relies on Friedman's (1971) grim trigger strategies, the critical level of the discount factor is

$$\delta_F = \frac{\pi^D - \pi^C}{\pi^D - \pi^N} = \frac{1}{2} \tag{29}$$

If instead one-shot stick-and-carrot punishment is used, one has to solve the following system of inequalities:

$$\begin{array}{c} \pi^D - \pi^C \leq \delta\left(\pi^C - \pi^{OP}\right) \\ \pi^D\left(q^{OP}\right) - \pi^C \leq \delta\left(\pi^C - \pi^{OP}\right) \end{array} \tag{30}$$

w.r.t. δ and q^{OP}, so as to obtain $\delta_A = 1/4$ and $q^{OP} \geq \left[p - bc\left(n-1\right)\right]/\left(2c\right) \geq 0$ for all $b \in \left(0, 1/\left(n-1\right)\right)$ since the profit margin on the first unit being produced is $p - c$, which must be positive.

The continuation payoff is

$$\pi^{OP} + \frac{\delta\pi^C}{1-\delta} = \frac{p\left[p + 2bc\left(n-1\right)\right] - 3b^2c^2\left(n-1\right)^2 + 4b^2c^2\left(n-1\right)^2\delta}{4c\left(1-\delta\right)} \tag{31}$$

and it must be non-negative to prevent firms to quit the supergame after a deviation from the collusive path. A sufficient condition for the expression on the r.h.s. of (31) to be positive is

$$p\left[p + 2bc\left(n-1\right)\right] - 3b^2c^2\left(n-1\right)^2 \geq 0 \tag{32}$$

which is certainly true because the above inequality is met by all

$$b \in \left(0, \frac{p}{c\left(n-1\right)}\right) \tag{33}$$

where $p/\left[c\left(n-1\right)\right] > 1/\left(n-1\right)$ because $p > c$, as we already know.

As anticipated above, the presence of an externality makes collusive stability independent of cartel size. Therefore, an antitrust authority should keep an eye on these phenomena since agglomeration - if accompanied by cost-reducing externalities traditionally associated with industrial districts - might indeed neutralize the usual destabilising effect of increasing industry fragmentation on implicitly collusive behaviour.

5 Arms race vs disarmament

We consider a supergame among $n \geq 2$ countries, each of them endowed with a given military capacity yielding a utility level $\overline{u} \geq 0$.[6] Country i chooses $a_i \in \mathbb{R}$, to be interpreted as increase in its military capacity if positive, or disarmament if negative. In either direction, a change in the military endowment involves a quadratic cost. The resulting utility function of country i is

$$u_i = \overline{u} + a_i - \sum_{j \neq i} a_j - \frac{b a_i^2}{2} \tag{34}$$

where $\alpha = -b/2$, $\beta = -1$, $\gamma = 1$, $\varepsilon = \overline{u}$. The function (34) is concave and single-peaked in a_i and linearly decreasing in any other country's investment.

The best reply function is $a^* = 1/b$, yielding $u^N = \overline{u} - (2n-3)/(2b)$. The Pareto-efficient strategy vector solving

$$\max_{a_i} U = \sum_{i=1}^{n} u_i \tag{35}$$

is identified by $a_i = a^C = (2-n)/b$ for all i. Note that $a^C \leq 0$ for all $n \geq 2$. That is, the collective agreement along the Pareto frontier entails maintaining the *status quo* unaltered if there are only two powers, or effective disarmament if there are more. The associated utility is $u^C = \overline{u} + (n-2)^2/(2b) > u^N$. Deviating unilaterally from a^C and playing

[6] The literature on arms races is too long to be exhaustively accounted for here. See Shubik (1971); Powell (1990); and Zagare and Kilgour (2000) among many others.

instead a^* delivers the deviation utility

$$u^D = \overline{u} + \frac{2n(n-3)+5}{2b} > u^C \tag{36}$$

Using grim trigger strategies, the resulting critical threshold of the discount factor is again $\delta_F = 1/2$.

To evaluate the outcome of the supergame under the one-shot stick-and-carrot punishment, one has to define the punishment payoff:

$$u^{OP} = \overline{u} - \frac{a^{OP}\left[2(n-1) - ba^{OP}\right]}{2} \tag{37}$$

and the utility generated by a unilateral deviation from the punishment along a^*:

$$u^D\left(a^{OP}\right) = \overline{u} - (n-1)\,a^{OP} + \frac{1}{2b} \tag{38}$$

Then, solving

$$\begin{aligned} u^D - u^C &\leq \delta\left(u^C - u^{OP}\right) \\ u^D\left(a^{OP}\right) - u^C &\leq \delta\left(u^C - u^{OP}\right) \end{aligned} \tag{39}$$

one obtains $a^{OP} \geq n/b$ and $\delta_A = 1/4$. As for the non-negativity constraint on the continuation payoff, it is met by all

$$\overline{u} \geq \frac{2n(n-1)-1}{2b} \tag{40}$$

It is noteworthy that the optimal one-shot punishment describes an intense arms race taking place in a single period. Moreover, confirming the property highlighted in Section 2, $a^* - a^C = (n-1)/b = a^{OP} - a^*$.

6 Concluding remarks

We have shown that when payoff functions are parabolic cylinders, the stability of implicit collusion does not depend on the number of players. Fairly natural applications of our results extend to many issues in industrial economics and have relevant implications for antitrust authorities. Additionally, our frame may accomodate issues pertaining to international relations and possibly also others in different fields of social sciences, with equally relevant implications.

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