UNIVERSITA' DEGLI STUDI DI BOLOGNA

DIPARTIMENTO DI SCIENZE ECONOMICHE

On Lie point symmetries in differential games

Arsen Palestini

Quaderni - Working Papers DSE N° 698

On Lie point symmetries in differential games

Arsen Palestini *
Department of Economics, University of Bologna
Strada Maggiore 45, 40125 Bologna, Italy
palestini@math.unifi.it

Abstract

A technique to determine closed-loop Nash equilibria of n-player differential games is developed when their dynamic state-control system is composed of decoupled ODEs. In particular, the theory of Lie point symmetries is exploited to achieve first integrals of such systems.

JEL Classification: C72, C73

Keywords: differential game, closed-loop Nash equilibrium, Lie point symmetry, infinitesimal generator, first integral

1 Introduction

The present paper introduces a procedure to help the calculation of the optimal trajectories of some differential games in closed form. The analytic tools I will utilize are the Lie point symmetries, which rarely have been used in any game theory framework.

Differential game theory, together with the related applications in management science, economics, engineering and a number of further fields, has been developing very much in the last three decades. A recent survey on the fundamental instruments in use and the foremost models in literature is due to Jørgensen and G. Zaccour (2007). For an exhaustive overview on this

*The author is thankful to Davide Barbieri and Davide Dragone for their very useful comments and suggestions. The usual disclaimer applies.

subject, from both theoretical and applicative standpoints, see also Dockner et al. (2000).

Among the several equilibrium concepts usually investigated, most authors prefer to linger on open-loop and feedback Nash trajectories, whereby the determination of exact closed-loop Nash solutions seldom appears in literature.

Since closed-loop equilibria are feedback equilibria depending on the initial conditions of the associated Cauchy problem, sometimes the determination of the former follows from the computation of the latter (see for instance Fershtman and Kamien, 1987, or Yeung, 1989). In some other setups, particular numerical techniques are developed to show the nonuniqueness for closed-loop Nash solutions (e.g. Mehlmann and Willing, 1983) in particular classes of subgame perfect differential games (Reinganum, 1982).

Alternatively, the investigation of the closed-loop information structure helps the evaluation of the feedback effects along the equilibrium paths of the game (Cellini and Lambertini, 2005).

Several definitions of these special solutions are given (see for example Yeung and Petrosyan, 2006, or Sethi and Thompson, 2000) but a complete treatise on this subject seems to be missing. A partial motivation for this lack of analysis might stem from the fact that feedback equilibria, generally much more complicated to deal with, can be seen as an extension of the closed-loop equilibria, since they do not depend on the initial conditions of the dynamic constraint of the problem at hand.

To the best of my knowledge, no attempt has been made to embed Lie point symmetries and the related infinitesimal generators in a differential game theory framework.

In this paper I will suggest a technique to carry out the closed form integration of the dynamic state-control systems of a class of simultaneous differential games. In the next section I will outline the setup of the games under consideration and state the conditions to achieve a state-control dynamic system of decoupled ODEs. In section 3 I will recall some very preliminary concepts on Lie point symmetries and collect some results. Section 4 features a complete example to which all the above results apply. Section 5 concludes and introduces the possible future developments.

2 The basic setup

Consider an n-player infinite horizon differential game Γ in which the i-th agent, endowed with a profit function $\pi_i(\underline{x}, \underline{u}, t)$ and manoeuvring her unique strategic variable u_i, aims to maximize the following functional objective:

$$J_i \equiv \int_{t_0}^{\infty} e^{-\rho_i t} \pi_i(\underline{x}(t), \underline{u}(t), t) dt \tag{2.1}$$

s.t.:

$$\begin{cases} \dot{x}_s(t) = g_s(x_s(t), u_s(t), t) \\ x_s(t_0) = x_{s0} \end{cases} \tag{2.2}$$

$s = 1, \ldots, n$, where:

- $x_s(t) \in X_s \subseteq \mathbb{R}$, $s = 1, \ldots, n$ are the state variables and all X_s are open subsets of $\mathbb{R}$;
- $u_i(t) \in U_i \subseteq \mathbb{R}$, $i = 1, \ldots, n$ are the control variables of the n players and U_i are all open subsets of $\mathbb{R}$;
- $\underline{u}(t) = (u_1(t), \ldots, u_n(t)) \in U_1 \times \ldots \times U_n$ and $\underline{x}(t) = (x_1(t), \ldots, x_n(t)) \in X_1 \times \ldots \times X_n$ respectively are vectors of control and state variables;
- $\pi_1, \ldots, \pi_n \in C^2(X_1 \times \ldots \times X_n \times U_1 \times \ldots \times U_n \times [t_0, \infty))$;
- $g_s \in C^2(X_s \times U_s \times [t_0, \infty))$, $s = 1, \ldots, n$;
- ρ_i is the intertemporal discount rate for the i-th player;
- the game is played simultaneously.

If we call $\lambda_{is}(t)$ the costate variable associated by the i-th player to the s-th state, the current-value Hamiltonian function of the game Γ will read as follows (from now on, most of the time arguments will be omitted for brevity):

$$H_i(\cdot) = \pi_i(\underline{x}, \underline{u}) + \lambda_{ii} g_i(x_i, u_i) + \sum_{i \neq s} \lambda_{is} g_s(x_s, u_s).$$

Definition 2.1. *A decision rule $u_i^*(\underline{x},\ t) \in U_i$ is a* ***closed-loop strategy*** *if it is continuous in t and uniformly Lipschitz in $\underline{x}$ for each t.*

Definition 2.2. *A vector* $\underline{u}^* = (u_1^*, \ldots, u_n^*) \in U_1 \times \ldots U_n$ *of closed-loop strategies is a* ***closed-loop Nash equilibrium*** *if*

$$J_i(u_1^*, \ldots, u_n^*) \geq J_i(u_1^*, \ldots, u_{i-1}^*, u_i, u_{i+1}^*, \ldots, u_n^*), \quad i = 1, \ldots, n \tag{2.3}$$

holds for all closed-loop strategies u_i.

The determination of an equilibrium information structure generally requires the application of Pontryagin's Maximum Principle. The crucial property of Γ to be exploited is that the i-th dynamic constraint depends only on the related i-th state and on the i-th player's strategic variable, thus implying:

$$\frac{\partial g_s}{\partial u_i} = \frac{\partial g_s}{\partial x_i} = 0 \tag{2.4}$$

for all $s \neq i$, which entails the following first order conditions (FOCs):

$$\frac{\partial H_i}{\partial u_i} = \frac{\partial \pi_i}{\partial u_i} + \lambda_{ii} \frac{\partial g_i}{\partial u_i} = 0, \tag{2.5}$$

for all $i = 1, \ldots, n$.

If we call $\tilde{u}(\underline{x}) = (\tilde{u}_1(\underline{x}), \ldots, \tilde{u}_n(\underline{x}))$ the vector of control variables satisfying (2.5), then the associated adjoint variable dynamic system results in:

$$\dot{\lambda}_{ii} = \left(\rho_i - \frac{\partial g_i}{\partial x_i}\right) \lambda_{ii} - \frac{\partial \pi_i}{\partial x_i}, \tag{2.6}$$

$$\dot{\lambda}_{is} = \rho_i \lambda_{is} - \frac{\partial \pi_i}{\partial x_s} - \sum_{s \neq i} \frac{\partial \pi_i}{\partial u_s} \frac{\partial \tilde{u}_s}{\partial x_s}, \tag{2.7}$$

$i, s = 1, \ldots, n$, $s \neq i$.

The related transversality conditions read as:

$$\lim_{t \to +\infty} e^{-\rho_i t} \lambda_{is}(t) = 0. \tag{2.8}$$

Since no costate variable λ_{is}, where $i \neq s$, appears in (2.5), the $n^2 - n$ adjoint equations (2.7) are not relevant for the determination of the closed-loop Nash equilibrium of Γ.

2.1 The dynamic state-control structure

After deriving (2.5) with respect to time, under suitable regularity hypotheses, and subsequently substituting in (2.6), we obtain a state-control dynamic system of $2n$ equations, possibly but rarely integrable in closed form with standard methods. I will focus on the conditions under which the above transformation of variables is feasible.

Proposition 2.1. *If along every equilibrium trajectory of* Γ *the following conditions hold for every* $i, j = 1, \ldots, n$, $i \neq j$*:*

1. $\displaystyle \frac{\partial^2 \pi_i}{\partial u_i^2} - \frac{\frac{\partial \pi_i}{\partial u_i}}{\frac{\partial g_i}{\partial u_i}} \cdot \frac{\partial^2 g_i}{\partial u_i^2} \neq 0,$

2. $\displaystyle \frac{\partial^2 \pi_i}{\partial u_i \partial u_j} = 0,$

then the dynamic state-control system of Γ *is formed by (2.2) and by the following ODEs:*

$$\begin{pmatrix} \dot{u}_1 \\ \vdots \\ \dot{u}_n \end{pmatrix} = A^{-1} \cdot \begin{pmatrix} h_1(\underline{u}, \underline{x}) \\ \vdots \\ h_n(\underline{u}, \underline{x}) \end{pmatrix}, \qquad (2.9)$$

where $A^{-1} = (b_{ij})$ *is a diagonal* $n \times n$ *matrix with the following entries:*

$$b_{ii} = \left(\frac{\partial^2 \pi_i}{\partial u_i^2} - \frac{\frac{\partial \pi_i}{\partial u_i}}{\frac{\partial g_i}{\partial u_i}} \cdot \frac{\partial^2 g_i}{\partial u_i^2} \right)^{-1},$$

and where

$$\begin{pmatrix} h_1(\cdot) \\ \vdots \\ h_n(\cdot) \end{pmatrix} = \begin{pmatrix} \left(\rho_1 - \frac{\partial g_1}{\partial x_1}\right) \frac{\partial \pi_1}{\partial u_1} + \frac{\partial \pi_1}{\partial x_1} \frac{\partial g_1}{\partial u_1} - \sum_{j=1}^n \frac{\partial^2 \pi_1}{\partial u_1 \partial x_j} \cdot g_j + \frac{\frac{\partial \pi_1}{\partial u_1}}{\frac{\partial g_1}{\partial u_1}} \frac{\partial^2 g_1}{\partial u_1 \partial x_1} \cdot g_1 \\ \vdots \\ \left(\rho_n - \frac{\partial g_n}{\partial x_n}\right) \frac{\partial \pi_n}{\partial u_n} + \frac{\partial \pi_n}{\partial x_n} \frac{\partial g_n}{\partial u_n} - \sum_{j=1}^n \frac{\partial^2 \pi_n}{\partial u_n \partial x_j} \cdot g_j + \frac{\frac{\partial \pi_n}{\partial u_n}}{\frac{\partial g_n}{\partial u_n}} \frac{\partial^2 g_n}{\partial u_n \partial x_n} \cdot g_n \end{pmatrix}. \qquad (2.10)$$

Proof. The derivation of (2.5) yields:

$$\sum_{j=1}^{n}\left[\frac{\partial^2\pi_i}{\partial u_i\partial u_j}\dot{u}_j+\frac{\partial^2\pi_i}{\partial u_i\partial x_j}\dot{x}_j\right]+\dot{\lambda}_{ii}\frac{\partial g_i}{\partial u_i}+\lambda_{ii}\left(\frac{\partial^2 g_i}{\partial u_i^2}\dot{u}_i+\frac{\partial^2 g_i}{\partial u_i\partial x_i}\dot{x}_i\right)=0, \tag{2.11}$$

then, by using (2.6) and (2.5) we obtain:

$$\sum_{j=1}^{n}\left[\frac{\partial^2\pi_i}{\partial u_i\partial u_j}\dot{u}_j+\frac{\partial^2\pi_i}{\partial u_i\partial x_j}\cdot g_j\right]-\left[\left(\rho_i-\frac{\partial g_i}{\partial x_i}\right)\frac{\partial\pi_i}{\partial u_i}+\frac{\partial\pi_i}{\partial x_i}\frac{\partial g_i}{\partial u_i}\right]+ \tag{2.12}$$

$$-\frac{\frac{\partial\pi_i}{\partial u_i}}{\frac{\partial g_i}{\partial u_i}}\left[\frac{\partial^2 g_i}{\partial u_i^2}\dot{u}_i+\frac{\partial^2 g_i}{\partial u_i\partial x_i}\cdot g_i\right]=0, \tag{2.13}$$

relations that we can express in the following matrix form:

$$A\cdot\begin{pmatrix}\dot{u}_1\\ \vdots\\ \dot{u}_n\end{pmatrix}=\begin{pmatrix}\left(\rho_1-\frac{\partial g_1}{\partial x_1}\right)\frac{\partial\pi_1}{\partial u_1}+\frac{\partial\pi_1}{\partial x_1}\frac{\partial g_1}{\partial u_1}-\sum_{j=1}^{n}\frac{\partial^2\pi_1}{\partial u_1\partial x_j}\cdot g_j+\frac{\frac{\partial\pi_1}{\partial u_1}}{\frac{\partial g_1}{\partial u_1}}\frac{\partial^2 g_1}{\partial u_1\partial x_1}\cdot g_1\\ \vdots\\ \left(\rho_n-\frac{\partial g_n}{\partial x_n}\right)\frac{\partial\pi_n}{\partial u_n}+\frac{\partial\pi_n}{\partial x_n}\frac{\partial g_n}{\partial u_n}-\sum_{j=1}^{n}\frac{\partial^2\pi_n}{\partial u_n\partial x_j}\cdot g_j+\frac{\frac{\partial\pi_n}{\partial u_n}}{\frac{\partial g_n}{\partial u_n}}\frac{\partial^2 g_n}{\partial u_n\partial x_n}\cdot g_n\end{pmatrix}, \tag{2.14}$$

where $A=(a_{ij})$ is the $n\times n$ matrix whose coefficients are:

$$a_{ii}=\frac{\partial^2\pi_i}{\partial u_i^2}-\frac{\frac{\partial\pi_i}{\partial u_i}}{\frac{\partial g_i}{\partial u_i}}\cdot\frac{\partial^2 g_i}{\partial u_i^2}, \qquad a_{ij}=\frac{\partial^2\pi_i}{\partial u_i\partial u_j}. \tag{2.15}$$

If the two hypotheses hold, A is nonsingular, hence (2.9) and the kinematic equations (2.2) make up a $2n$-variable state-control dynamic system. □

Corollary 2.1. *If for every $i=1,\dots,n$ we have:*

$$\frac{\partial}{\partial x_s}\left[\left(\frac{\partial^2\pi_i}{\partial u_i^2}-\frac{\frac{\partial\pi_i}{\partial u_i}}{\frac{\partial g_i}{\partial u_i}}\cdot\frac{\partial^2 g_i}{\partial u_i^2}\right)^{-1}\right]=\frac{\partial}{\partial u_s}\left[\left(\frac{\partial^2\pi_i}{\partial u_i^2}-\frac{\frac{\partial\pi_i}{\partial u_i}}{\frac{\partial g_i}{\partial u_i}}\cdot\frac{\partial^2 g_i}{\partial u_i^2}\right)^{-1}\right]=0 \tag{2.16}$$

for all $s\neq i$, then (2.9) and (2.2) form a dynamic system composed of decoupled ODEs.

In the following, I will discuss the cases in which the hypotheses of Corollary 2.1 hold, i.e. on the games whose associated system results in:

$$\begin{cases} \dot{x}_i = g_i(x_i, u_i) \\ \dot{u}_i = h_i(x_i, u_i) \end{cases} \tag{2.17}$$

forming a Cauchy problem with the initial condition of (2.2) and with the appropriate transversality conditions implied by (2.8). A system of $2n$ decoupled ODEs can be thought of as a set of n planar systems. By the time elimination method, every pair of ODEs locally provides an expression of the closed-loop trajectory $u_i^*(x_i^*)$. Lie point symmetries, to which the next Section is devoted, constitute quite a powerful tool for the possible closed form integration of the i-th pair of (2.17).

3 Preliminaries about Lie point symmetries

An exhaustive overview of the current developments and applications of Lie point symmetries can be found in Hydon (2000) and in Starrett (2007). In particular, I will rely on a notation as similar to the one used by Hydon as possible to ease the reading of the upcoming facts.

Definition 3.1. *Given a one-parameter planar Lie symmetry*

$$S_\epsilon(x, u) = (\hat{x}(x, u, \epsilon), \hat{u}(x, u, \epsilon)),$$

the ***tangent vector*** *to the orbit at the point* (x, u) *is the vector*

$$(\xi(x, u),\ \eta(x, u)) = \left(\frac{d\hat{x}}{d\epsilon}_{|\epsilon=0},\ \frac{d\hat{u}}{d\epsilon}_{|\epsilon=0} \right).$$

When an ODE admits a one-parameter Lie group of symmetries, the partial differential operator

$$X = \xi(x, u)\frac{\partial}{\partial x} + \eta(x, u)\frac{\partial}{\partial u}$$

is called the ***infitesimal generator*** *of the Lie group.*

Definition 3.2. *Given the first order ODE*

$$\frac{du}{dx} = \frac{h(x, u)}{g(x, u)} := \omega(x, u), \tag{3.1}$$

we call the ***reduced characteristic*** *the function*

$$\overline{Q}(x, u) := \eta(x, u) - \omega(x, u)\xi(x, u).$$

A solution curve of (3.1) is invariant under a given nontrivial Lie group if and only if $\overline{Q}(x, f(x)) = 0$. In general, one of the standard procedures is based on the determination of an integrand factor to compute a first integral of (3.1). We can summarize the fundamental results in the following:

Proposition 3.1. *If the vector field $(\xi(x,u),\ \eta(x,u))$ satisfies the* ***linearized symmetry condition****:*

$$\frac{\partial \eta}{\partial x} + \left(\frac{\partial \eta}{\partial u} - \frac{\partial \xi}{\partial x}\right)\omega - \frac{\partial \xi}{\partial u}\omega^2 = \xi\frac{\partial \omega}{\partial x} + \eta\frac{\partial \omega}{\partial u} \tag{3.2}$$

then

$$\int \frac{du - \omega(x,u)dx}{\overline{Q}(x,u)} = C \tag{3.3}$$

is the general solution of (3.1).

Proof. See Hydon (pp. 31-37). □

Consider a vector field $(\xi(x), \eta(u))$, where the i-th component only depends on the i-th variable. The condition (3.2) can be simplified as shown by the following results:

Proposition 3.2. *The vector field $(\xi(x), \eta(u))$ verifies the linearized symmetry condition if and only if the following equality holds:*

$$\omega = \frac{\dfrac{\partial(\xi h)}{\partial x} + \eta\dfrac{\partial h}{\partial u}}{\dfrac{\partial(\eta g)}{\partial u} + \xi\dfrac{\partial g}{\partial x}}. \tag{3.4}$$

Proof. If ξ depends only on x and η depends only on u, then (3.2) reduces to:

$$\left(\frac{\partial \eta}{\partial u} - \frac{\partial \xi}{\partial x}\right)\omega = \xi\frac{\partial \omega}{\partial x} + \eta\frac{\partial \omega}{\partial u}, \tag{3.5}$$

and by definition of $\omega(x,u)$, after deriving we obtain:

$$\left(\frac{\partial \eta}{\partial u} - \frac{\partial \xi}{\partial x}\right)hg = \xi\left(g\frac{\partial h}{\partial x} - h\frac{\partial g}{\partial x}\right) + \eta\left(g\frac{\partial h}{\partial u} - h\frac{\partial g}{\partial u}\right),$$

which upon collecting the terms becomes:

$$h\left[\frac{\partial(\eta g)}{\partial u} + \xi\frac{\partial g}{\partial x}\right] = g\left[\frac{\partial(\xi h)}{\partial x} + \eta\frac{\partial h}{\partial u}\right],$$

from which (3.4) follows. □

Proposition 3.3. *If the vector field $(\xi(x), \eta(u))$ is such that $\xi'(x) = \eta'(u)$, then it satisfies (3.2) if and only if*

$$\omega = \frac{\xi \dfrac{\partial h}{\partial x} + \eta \dfrac{\partial h}{\partial u}}{\xi \dfrac{\partial g}{\partial x} + \eta \dfrac{\partial g}{\partial u}}. \tag{3.6}$$

Proof. The left-hand side of (3.5) vanishes so that through some algebra we obtain:

$$\xi g \frac{\partial h}{\partial x} + \eta g \frac{\partial h}{\partial u} = \xi h \frac{\partial g}{\partial x} + \eta h \frac{\partial g}{\partial u},$$

which entails the identity (3.6). □

Note that all vector fields of the kind $(\xi(x), \eta(u)) = (ax+b, au+c)$ verify both previous Propositions, hence they represent a very useful ansatz for the resolution of (2.17), as pointed out by an example in the next Section.

I chose to investigate a differential game which is endowed with a structure slightly different from the usual ones, characterized by a polynomial functional objective with cubic and 4 degree terms.

4 An example

Example 4.1. *Consider a 2-player game, with agents i and j, in the setup that we fixed in Section 2. Call u_i and u_j the control variables, and x_i and x_j the respective states. The i-th agent seeks to maximize the following functional objective w.r.t. her strategic variable:*

$$J_i \equiv \int_0^\infty e^{-\rho_i t}(u_i^2 x_i + x_i^3 - x_j^2 u_j^2)dt$$

subject to:

$$\begin{cases} \dot{x}_i = u_i \\ x_i(0) = x_{i0}, \end{cases}$$

$i = 1, 2$. *For simplicity, the state and control sets are $X_1 = X_2 = U_1 = U_2 = \mathbb{R}$.*

Since the relevant first order derivatives are:

$$\frac{\partial \pi_i}{\partial u_i} = 2u_i x_i, \qquad \frac{\partial g_i}{\partial u_i} = 1, \qquad \frac{\partial \pi_i}{\partial x_i} = u_i^2 + 3x_i^2, \qquad \frac{\partial g_i}{\partial x_i} = 0,$$

then we have:

$$\frac{\partial^2 \pi_i}{\partial u_i^2} - \frac{\dfrac{\partial \pi_i}{\partial u_i}}{\dfrac{\partial g_i}{\partial u_i}} \cdot \frac{\partial^2 g_i}{\partial u_i^2} = 2x_i,$$

vanishing if and only if $\lambda_{ii} \equiv 0$, that would make the problem collapse into a static game and not yield an optimal trajectory.

$$\frac{\partial^2 \pi_i}{\partial u_i \partial u_j} = 0,$$

so all the hypotheses of Proposition 2.1 and of Corollary 2.1 hold. The system (2.9) amounts to:

$$\begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2x_1} & 0 \\ 0 & \frac{1}{2x_2} \end{pmatrix} \begin{pmatrix} 2\rho_1 u_1 x_1 + 3x_1^2 - u_1^2 \\ 2\rho_2 u_2 x_2 + 3x_2^2 - u_2^2 \end{pmatrix}. \tag{4.1}$$

The related state-control dynamic system reads as follows:

$$\begin{cases} \dot{x}_1 = u_1 \\ \dot{u}_1 = \rho_1 u_1 + \dfrac{3x_1}{2} - \dfrac{u_1^2}{2x_1} \\ \dot{x}_2 = u_2 \\ \dot{u}_2 = \rho_2 u_2 + \dfrac{3x_2}{2} - \dfrac{u_2^2}{2x_2} \end{cases}. \tag{4.2}$$

(4.2) is composed of decoupled equations.

Therefore, it is not restrictive to consider any of the two pairs of equations:

$$\frac{du_i}{dx_i} = \rho_i + \frac{3x_i}{2u_i} - \frac{u_i}{2x_i} := \omega_i(x_i, u_i). \tag{4.3}$$

The vector field $(\xi(x_i), \eta(u_i)) = (x_i, u_i)$ verifies the hypothesis of Proposition 3.3; since (3.6) holds, as shown in the following:

$$\frac{x_i \left(\dfrac{3}{2} + \dfrac{u_i^2}{2x_i^2}\right) + u_i \left(\rho_i - \dfrac{u_i}{x_i}\right)}{u_i} = \omega_i(x_i, u_i),$$

then a first integral of (4.3) turns out to be:

$$\int \frac{2x_i u_i du + (u_i^2 - 2\rho_i x_i u_i - 3x_i^2)dx}{3x_i u_i^2 - 2\rho_i x_i^2 u_i - 3x_i^3} = C,$$

i.e., for $i = 1, 2$,

$$\ln \sqrt[3]{3x_i u_i^2 - 2\rho_i x_i u_i - 3x_i^3} + \frac{\rho_i}{3\sqrt{\rho_i^2 + 9}} \times \tag{4.4}$$

$$\times \ln \left[\left(\frac{3x_i + (\rho_i - \sqrt{\rho_i^2 + 9})u_i}{3x_i + (\rho_i + \sqrt{\rho_i^2 + 9})u_i} \right)^2 \left(\frac{3u_i - (\rho_i + \sqrt{\rho_i^2 + 9})x_i}{3u_i - (\rho_i - \sqrt{\rho_i^2 + 9})x_i} \right) \right] = C_i, \tag{4.5}$$

where C_i *is a real constant depending on the* i*-th initial state and control variables.*

5 Concluding remarks

In this paper I proposed some ideas to embed the Lie point symmetry theory in the problem of the determination of exact solutions of the optimal state-control dynamic system generated by a differential game.

I showed that under certain assumptions, it is possible to find an infinitesimal generator of the Lie group leading to an integrand factor for the ODEs of the game.

In my view, this topic deserves further investigation and future research. In particular, the possible next lines of research should concern two main aspects.

First of all, it would be very interesting to check whether such a technique can be applied to some of the economic models I recalled in the Introduction. The example in Section 4 shows the possibility of constructing an infinitesimal generator that does not depend on the intertemporal discount rates. It would be helpful to fix the conditions for an infinitesimal generator to be found for every choice of the parameters usually contained in such models (spillover, reservation price, depreciation rate, and so on).

Moreover, it is undoubtedly worth exploring the connections between the closed-loop and the feedback information structures. There could be some relationships between the Lie symmetry leading to the integration of the state-control dynamic system and the characteristics of the optimal value function solving the Hamilton-Jacobi-Bellman-Isaacs equation. In my opinion, Lie point symmetries might own a hidden potential which has not been exploited all the way thus far.

References

R. Cellini and L. Lambertini (2005). R&D incentives and market structure: dynamical analysis. *Journal of Optimization Theory and Applications*, 126, 1, 85-96.

E. Dockner, S. Jørgensen, N. Van Long, G. Sorger (2000). Differential games in economics and management science. *Cambridge University Press*.

C. Fershtman and M.I. Kamien (1987). Duopolistic competition with sticky prices. *Econometrica*, 55, 5, 1151-1164.

P. E. Hydon (2000). Symmetry methods for differential equations: a beginner's guide. *Cambridge Texts in Applied Mathematics*.

S. Jørgensen and G. Zaccour (2007). Developments in Differential Game Theory and Numerical Methods: Economic and Management Applications. *Computational Management Science*, vol. 4, 2, 159-182.

A. Mehlmann and R. Willing (1983). On nonunique closed-loop Nash equilibria for a class of differential games with a unique and degenerate feedback solution. *Journal of Optimization Theory and Applications*, 41, 463-472.

J. Reinganum (1982). A class of differential games for which the closed-loop and open-loop Nash equilibria coincide. *Journal of Optimization Theory and Applications*, 36, 253-262.

J. Starrett (2007). Solving differential equations by symmetry groups. *American Mathematical Monthly*, vol. 114, 9, 778-792.

S.P. Sethi and G.L. Thompson (2000). Optimal control theory: Applications to management science and economics. *Springer*.

D.W.K. Yeung and L. Petrosyan (2006). Cooperative stochastic differential games, *Springer*.

D.W.K. Yeung (1989). A class of differential games with state-dependent closed-loop feedback solutions. *Journal of Optimization Theory and Applications*, 62, 165-174.