Closed form solutions to generalized logistic-type nonautonomous systems

Giovanni Mingari Scarpello* Arsen Palestini† and Daniele Ritelli‡

Abstract

In this paper the subject is met of providing a two-fold generalization of the logistic population dynamics to a nonautonomous context. First it is assumed the carrying capacity alone pulses the population behavior changing logistically on its own. In such a way we get again the model of Meyer and Ausubel (1999), by them computed numerically, and we solve it completely through the Gauss hypergeometric function. Furthermore, both the carrying capacity and net growth rate are assumed to change simultaneously following two independent logisticals. The population dynamics is then found in closed form through a more difficult integration, involving a (τ_1, τ_2) extension of the Appell generalized hypergeometric function, Al-Shammery and Kalla (2000); about such a extension a new analytic continuation theorem has been proved.

Keywords: Logistic growth generalization, carrying capacity, Appell hypergeometric function.

JEL: C63, B16, J10

1 Introduction

When the logistic equation on population growth was proposed (*Notice sur la loi que la population suit dans son accroissement*, 1838), P. F. Verhulst meant to provide a possible solution to the unrealistic exponential growth forecast by T. Malthus (1798), *An essay on the principle of population.* As a matter of fact, population modelling became of particular interest in the 20^{th} century to biologists urged by limited means of sustenance and increasing human populations. Such a way Verhulst's scheme was rediscovered by A. Lotka and others, as a simple model of a self-regulating population. If $x(t)$ is the population (single species in a closed ecosystem without migrations) at time t, a Verhulst law formulation is:

$$\begin{cases} \dot{x} = rx\left(1 - \dfrac{x}{k}\right) \\ x(0) = x_0. \end{cases}$$

The dot means derivative with respect to time and the *intrinsic growth rate* r is a positive constant measuring the population average net growth rate. In the above equation, "any role of resources is subsumed in the idealized parameters r and k", Grover (1997). In fact, being x^2 representative of the rate of pair interactions, then r/k will provide the rate of them acting as a decrease of population growth. The *carrying capacity* k, due to environmental pressures, stands for the saturation, or maximum sustainable value, of population; so that $r\,(1 - x(t)/k)$ means the *per capita birth rate* at epoch t. The carrying capacity utmost bound of a territory is not fixed: it can spread thanks to new technologies capable of improving the environment productiveness. So that all the growth models for human systems based either on fixed resource limits, or fixed k-values, are unrealistic.

*Dipartimento di Matematica per le Scienze Economiche e Sociali, viale Filopanti, 5, 40126 Bologna, Italy, giovanni.mingari@unibo.it

†Dipartimento di Scienze Economiche, Strada Maggiore, 45 Bologna, Italy, palestini@dm.unibo.it

‡Dipartimento di Matematica per le Scienze Economiche e Sociali, viale Filopanti, 5, 40126 Bologna, Italy, daniele.ritelli@unibo.it

In the above differential equation k has therefore to be replaced by an exogenous function $k(t)$: in such a way realism and complexity of the model are both increased. In fact $k(t)$ can be any function; several variants have been proposed and studied over the years: sinusoidal, exponential, linear, and so on. In the last decades, the continuous acceleration of the technological changes has shown the importance of technology and innovation management for competitive advantage and survival. We define "technology" as a process, technique, or methodology which transforms inputs of labour, capital, information, material, and energy into outputs of greater value. Cohen (1995) presented a model of logistic structure, but whose human carrying capacity $k(t)$ is modelled logistically in order to simulate the invention and diffusion of technologies which lift the k-bounds during time. Therefore one is faced with a population Verhulst-type differential equation pulsed by a logistic carrying capacity, hereinafter Verhulst Logistical Carrying Capacity, VLCC model. It will be recalled that Nkashama, (2000) and (2001), proved that each Verhulst-type equation with positive nonautonomous bounded forcing coefficients has exactly one bounded solution that is positive, and that does not approach the zero solution in the past and in the future. Our interest in the subject is not concerning its purely demographic content, but its economic sense and impliances having a description capability much better than the population dynamics analysis. Let us pass to Watanabe-Kondo-Ouchi-Wei model (2003). It is true that simple logistic growth functions were useful in modelling diffusion process of innovations, but this function is based on imitators behavior than that of innovators. Including innovators behavior too, after innovation with new functionality (namely IT, Information Technology) is diffused, it will be altering the carrying capacity or creating some new one. Meyer and Ausubel (1999) proposed a logistic type differential equation within a dynamic carrying capacity approach to model this diffusion behaviour. In the next section we will integrate a VLCC model in closed form, finding its solution, which of course will be not logistic at all, by means of the machinery of hypergeometric functions. Furthermore we will go on with injecting a better realism into the model. In fact, the population growth studies led to designations of "k-selected and r-selected" populations. The latter produce many offspring, which are comparatively less likely to survive to adulthood. Whereas k-selected species invest more heavily the nurture of fewer offspring, which has a better chance of surviving to adulthood. In unstable or unpredictable environments r-selection predominates, where the ability to reproduce quickly is crucial, and there is little advantage in adaptations leading to successful competition. In stable or predictable environments k-selection predominates, as the ability to compete successfully for limited resources is crucial. In practice, most populations show a mixture of r-selected and k-selected traits: *a population mathematical model is then wanted where both parameters are properly changing during time.* Accordingly, our second model (see section 3), generalizes the VLCC one, assuming that the increase of technology is affecting *both* the control functions $a(t)$ and $b(t)$ of $\dot{x}(t) = a(t)x - b(t)x^2$ where $b(t)$ is a self-regulation reaction due to overcrowding or food shortage. The technological progress improves the environment quality and, as a consequence, the net growth rate $a(t) = r$ will increase during time. The impact of technology on the environment means that the carrying capacity shall increase. Then if all this is modelled through a decreasing logistic law $b(t)$, we will mean the "frictions" tend to decrease due to the environment improved smoothness. Furthermore, $b(t) = r/k$ decline means that the environment capability to sustain people, grows faster than the net rate r, so that $-b(t)$ tends to reduce its effect of demographic deceleration. This further and more difficult model will be solved through a generalization, due to P. Appell, of a hypergeometric function.

2 The Verhulst-type model under Logistical Carrying Capacity

Let us write the general logistic growth differential equation:

$$\begin{cases} \dot{x}(t) = a(t)x(t) - b(t)x^2(t) \\ x(0) = x_0 > 0 \end{cases} \tag{2.1}$$

where $a(t) > 0$ and $b(t) > 0$ are given continuous positive bounded functions of time. Equation (2.1) can be easily drawn back to the quadratures:

$$x(t) = \frac{\exp\left(\int_0^t a(\tau)\mathrm{d}\tau\right)}{\frac{1}{x_0} + \int_0^t b(\tau)\exp\left(\int_0^\tau a(\xi)d\xi\right)\mathrm{d}\tau}. \tag{2.2}$$

Meyer and Ausubel (1999) introduce

$$k(t) = \kappa_1 + \frac{\kappa_2}{1 + \exp\left[-\alpha_m\left(t - t_{m_\kappa}\right)\right]}$$

so that $k(t)$ is solution of the logistic too:

$$\begin{cases} \dot{k}(t) = \alpha_m\left(k(t) - \kappa_1\right)\left(1 - \dfrac{k(t) - \kappa_1}{\kappa_2}\right) \\ k(0) = \kappa_1 + \dfrac{\kappa_2}{1 + \exp\left(\alpha_m t_{m_\kappa}\right)} \equiv \kappa_0 \end{cases}$$

and then consider (2.2) where:

$$a(t) \equiv \alpha, \qquad b(t) = \frac{\alpha}{k(t)}. \tag{2.3}$$

Observe that Watanabe-Kondo-Ouchi-Wei (2003) in their market model plug in a Verhulst-type differential equation a constant net growth rate, whilst the carrying capacity is assumed logistic, but with a freedom degree less than the Meyer-Ausubel one, so that Watanabe et al. model is elementary integrable. On the contrary, we will perform a closed form integration when the carrying capacity $k(t)$ changes along time logistically between two fixed bounds, say a starting value $\kappa_1 > 0$ and a final addition $\kappa_2 > 0$, namely "à la Meyer-Ausubel" which has a further degree of freedom than Watanabe one. We will refer to it as Verhulst Logistical Carrying Capacity model, say VLCC.

In order to integrate (2.2) with (2.3), let us recall something on the Gauss ${}_2\mathrm{F}_1$ hypergeometric function. It was early defined as a x -power series, $|x| < 1$:

$${}_2\mathrm{F}_1\left(\begin{matrix} a,\, b \\ c \end{matrix} \middle| \, x\right) = \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(c)_n}\frac{x^n}{n!}, \tag{2.4}$$

where $(a)_k$ is a Pochhammer symbol: $(a)_k = a(a+1)\cdots(a+k-1)$. The sum of the series (2.4), is the so called "hypergeometric function", but this definition is only suitable when x lies inside the unity circle. It is possible to construct a complex function which is analytic in the complex plane cut along the segment $[1, \infty[$ and which coincides with ${}_2\mathrm{F}_1$ whenever $|x| < 1$. This function is the analytic continuation of ${}_2\mathrm{F}_1$ into the cut plane, and will be denoted by the same symbol. Plugging in (2.4) the expression of Pochhammer symbols through the Gamma function, reversing the order of summation and integration, and minding the binomial expansion, one can arrive, as a first step, at the integral representation theorem:

$${}_2\mathrm{F}_1\left(\begin{matrix} a,\, b \\ c \end{matrix} \middle| \, x\right) = \frac{\Gamma(c)}{\Gamma(c-a)\Gamma(a)}\int_0^1 \frac{t^{a-1}(1-t)^{c-a-1}}{(1-xt)^b}\,\mathrm{d}t,$$

whose validity ranges are: $\mathrm{Re}\, a > \mathrm{Re}\, c > 0$, $|x| < 1$.

The next step is to show that the above integral has meaning and represents an analytic function of x in the plane cut along $[1, \infty[$. On the purpose the reader is referred to pages 238-240 of Lebedev (1972). In the general case where the parameters a, b, c have arbitrary values, the required analytic continuation into the plane cut along $[1, \infty[$ can be obtained as a contour integral by using residue theory to sum the series (2.4). A more elementary method of continuation involves the use of some hypergeometric recurrence relations and can be seen again on the referred Lebedev book.

The differential equation arising from the Meyer Ausubel model can be solved through the ${}_2\mathrm{F}_1$. Minding that κ_1 is the carrying capacity starting value and κ_2 its final addition value, then $\kappa_1 + \kappa_2$ will be the capacity ultimate value, namely its (asymptotic) ceiling. We provide the following:

Theorem 2.1. *The solution to* (2.2) *with* (2.3) *is given by*

$$x(t) = \frac{\kappa_0\, e^{\alpha\, t}}{1 + \kappa_0\, \left(\boldsymbol{I}_1(t) - \boldsymbol{I}_2\right)}.$$

where:

$$\begin{aligned}
\boldsymbol{I}_1(t) &= \frac{e^{t\alpha}}{\kappa_1}\left[\frac{\alpha\, e^{(t-t_{m_\kappa})\alpha_m}}{\alpha+\alpha_m}\, {}_2\mathrm{F}_1\left(\left.\begin{matrix}1,\, 1+\frac{\alpha}{\alpha_m}\\ 2+\frac{\alpha}{\alpha_m}\end{matrix}\right| -\frac{(\kappa_1+\kappa_2)\, e^{(t-t_{m_\kappa})\alpha_m}}{\kappa_1}\right)\right.\\
&+ \left. {}_2\mathrm{F}_1\left(\left.\begin{matrix}1,\, \frac{\alpha}{\alpha_m}\\ 1+\frac{\alpha}{\alpha_m}\end{matrix}\right| -\frac{(\kappa_1+\kappa_2)\, e^{(t-t_{m_\kappa})\alpha_m}}{\kappa_1}\right)\right],\\
\boldsymbol{I}_2 &= \frac{1}{\kappa_1}\left[\frac{\alpha\, e^{-t_{m_\kappa}\alpha_m}}{\alpha+\alpha_m}\, {}_2\mathrm{F}_1\left(\left.\begin{matrix}1,\, 1+\frac{\alpha}{\alpha_m}\\ 2+\frac{\alpha}{\alpha_m}\end{matrix}\right| -\frac{(\kappa_1+\kappa_2)\, e^{-t_{m_\kappa}\alpha_m}}{\kappa_1}\right)\right.\\
&+ \left. {}_2\mathrm{F}_1\left(\left.\begin{matrix}\frac{\alpha}{\alpha_m},\, 1\\ 1+\frac{\alpha}{\alpha_m}\end{matrix}\right| -\frac{(\kappa_1+\kappa_2)\, e^{-t_{m_\kappa}\alpha_m}}{\kappa_1}\right)\right].
\end{aligned}$$

Proof. From the quadrature formula (2.2), minding (2.3) we infer:

$$x(t) = \frac{e^{\alpha\, t}}{\dfrac{1}{\kappa_0} + \displaystyle\int_0^t \frac{\alpha\, e^{\alpha\tau}\left(1 + e^{(\tau - t_{m_\kappa})\alpha_m}\right)}{(\kappa_1+\kappa_2)\, e^{(\tau-t_{m_\kappa})\alpha_m} + \kappa_1}\mathrm{d}\tau}. \tag{2.5}$$

Call $\boldsymbol{I}(t)$ the integral in (2.5). We will express it through the Gauss hypergeometric function ${}_2\mathrm{F}_1$. In fact, passing to a new variable $y = e^{\alpha\tau}$ we get:

$$\boldsymbol{I}(t) = \int_0^{e^t} \frac{\left(y^{\frac{\alpha_m}{\alpha}} + e^{t_{m_\kappa}\alpha_m}\right)}{(\kappa_1+\kappa_2)\, y^{\frac{\alpha_m}{\alpha}} + \kappa_1 e^{t_{m_\kappa}\alpha_m}}\, \mathrm{d}y - \int_0^1 \frac{\left(y^{\frac{\alpha_m}{\alpha}} + e^{t_{m_\kappa}\alpha_m}\right)}{(\kappa_1+\kappa_2)\, y^{\frac{\alpha_m}{\alpha}} + \kappa_1 e^{t_{m_\kappa}\alpha_m}}\, \mathrm{d}y. \tag{2.6}$$

In the first integral of (2.6) we make the normalization $y = e^{\alpha\, t}\, u$, so that:

$$\begin{aligned}
\boldsymbol{I}(t) = &\int_0^1 \frac{e^{t\alpha}\left(e^{t\alpha_m} u^{\frac{\alpha_m}{\alpha}} + e^{t_{m_\kappa}\alpha_m}\right)}{e^{t\alpha_m}\,(\kappa_1+\kappa_2)\, u^{\frac{\alpha_m}{\alpha}} + \kappa_1 e^{t_{m_\kappa}\alpha_m}}\, \mathrm{d}u - \int_0^1 \frac{\left(y^{\frac{\alpha_m}{\alpha}} + e^{t_{m_\kappa}\alpha_m}\right)}{(\kappa_1+\kappa_2)\, y^{\frac{\alpha_m}{\alpha}} + \kappa_1 e^{t_{m_\kappa}\alpha_m}}\, \mathrm{d}y\\
:= &\boldsymbol{I}_1(t) - \boldsymbol{I}_2
\end{aligned} \tag{2.7}$$

what ends the proof.

□

The above formula will provide $x(t)$ trough a power hypergeometric series if and only if the problem data sheet $(\kappa_1, \kappa_2,\, t_{m_\kappa}, \alpha_m)$ meets the inequality:

$$-1 < -\frac{(\kappa_1+\kappa_2)\, e^{-t_{m_\kappa}\alpha_m}}{\kappa_1}$$

and the dynamics can be investigated only for those t-values compliant with the constraint:

$$-1 < -\frac{(\kappa_1+\kappa_2)\, e^{(t-t_{m_\kappa})\alpha_m}}{\kappa_1}.$$

Otherwise the solution of (2.2) with (2.3) will be provided through the Euler analytic continuation or whichever other possible continuation to ${}_2\mathrm{F}_1$, see Becken and Schmelcher (2000). In all cases, the above integral is always the solution to the Meyer Ausubel problem: but outside the series convergence range, integration has to be carried out numerically. Figure 1 shows one of the several tested overlappings as a benchmark of our closed form solution vs. the relevant numerical computation through Mathematica®.

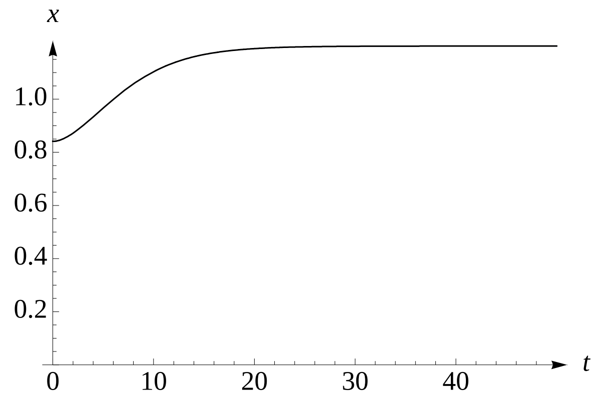

Figure 1: Non-logistical growth vs. time of a population ruled by a Verhulst-type differential equation pulsed by a logistical carrying capacity

3 Verhulst Double Logistical Input Model

Once again, we consider the differential equation (2.1) with its quadrature formula (2.2). Now both $a(t)$ and $b(t)$ will be changing for their part according to single logistic laws:

$$\begin{cases} \dot{a}(t) = \alpha_1 a(t) - \alpha_2 a^2(t), \\ a(0) = a_0 > 0, \end{cases} \qquad \begin{cases} \dot{b}(t) = \beta_1 b(t) - \beta_2 b^2(t), \\ b(0) = b_0 > 0, \end{cases}$$

and then:

$$a(t) = \frac{a_0 \alpha_1 e^{\alpha_1 t}}{\alpha_1 + a_0 \alpha_2 \left(e^{\alpha_1 t} - 1\right)}, \quad b(t) = \frac{b_0 \beta_1 e^{\beta_1 t}}{\beta_1 + b_0 \beta_2 \left(e^{\beta_1 t} - 1\right)}. \tag{3.1}$$

We will refer to such a model as VDLIM. Of course we assume α_i and β_i strictly positive. Such a model describes a population dynamics whose net growth rate and the overcrowding factor are not fixed, but from a starting level evolve with logistic saturation towards an asymptotic level. Notice that if α_1, α_2, β_1, $\beta_2 \to 0$, then the above differential equation will collapse in the classic constant coefficients one. The aim of this section is to solve (2.2) with (3.1), namely to compute the definite integrals (2.2) in closed form, whenever $a(t)$ and $b(t)$ are assigned by (3.1). On the purpose, we need a generalized hypergeometric function. The single variable hypergeometric functions ${}_p\mathrm{F}_q$ generalizes in easy way, i.e. only by rising the number of coefficients, the oldest function ${}_2\mathrm{F}_1$. P. Appell in 1880 had arrived at a further generalization conceiving a double series with four parameters and two variables (w_1, w_2), namely the hypergeometric function F_1 :

$$\mathrm{F}_1 \left(\begin{matrix} a;\ b_1, b_2 \\ c \end{matrix} \middle| w_1, w_2 \right) = \sum_{k,\,\ell=0}^{\infty} \frac{(a)_{k+\ell}\ (b_1)_k (b_2)_\ell}{(c)_{k+\ell}} \frac{w_1^k w_2^\ell}{k!\,\ell!}.$$

The next step, quite recent, is due to Al-Shammery and Kalla (2000) who introduced, see page 193, the function $\mathrm{F}_1^{\tau_1,\,\tau_2}$, where τ_1 and τ_2 are positive constants:

$$\mathrm{F}_1^{\tau_1,\tau_2} \left(\begin{matrix} a;\ b_1, b_2 \\ c \end{matrix} \middle| w_1, w_2 \right) = \frac{\Gamma(c)}{\Gamma(a)} \sum_{k,\,\ell=0}^{\infty} \frac{\Gamma(a + \tau_1 k + \tau_2 \ell)\,(b_1)_k (b_2)_\ell}{\Gamma(c + \tau_1 k + \tau_2 \ell)} \frac{w_1^k w_2^\ell}{k!\ell!}$$

$$= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 \frac{u^{a-1}\,(1-u)^{c-a-1}}{\left(1 - w_1 u^{\tau_1}\right)^{b_1} \left(1 - w_2 u^{\tau_2}\right)^{b_2}}\,\mathrm{d}u. \tag{3.2}$$

In (3.2) Γ is Euler's Gamma function, whereas $(b_j)_k$ denotes again a Pochhammer symbol $(b_j)_k = b_j(b_j+1)\cdots(b_j+k-1)$, and a and c are complex numbers such that $\mathrm{Re}(a) > 0$, $\mathrm{Re}(c-a) > 0$. The above formula's third side provides a remarkable integral representation theorem to $\mathrm{F}_1^{\tau_1,\,\tau_2}$

that holds for any $|w_1|$, $|w_2| < 1$. We will focus on the special case $c - a - 1 = 0$ of (3.2)

$$\int_0^1 \frac{u^{a-1}}{(1 - w_1 u^{\tau_1})^{b_1} (1 - w_2 u^{\tau_2})^{b_2}} \mathrm{d}u = \sum_{k,\ell=0}^{\infty} \frac{(b_1)_k (b_2)_\ell}{a + \tau_1 k + \tau_2 \ell} \frac{w_1^k}{k!} \frac{w_2^\ell}{\ell!}$$

which can be expressed through the Appell generalized function:

$$\int_0^1 \frac{u^{a-1}}{(1 - w_1 u^{\tau_1})^{b_1} (1 - w_2 u^{\tau_2})^{b_2}} \mathrm{d}u = \frac{1}{a} \,\mathrm{F}_1^{\tau_1,\,\tau_2} \left(\begin{matrix} a;\ b_1, b_2 \\ a+1 \end{matrix} \middle| w_1, w_2 \right). \tag{3.3}$$

Having the need of a formula providing a integral stemming from the quadrature relationship (2.2), like we did on the treatment of the *monologistical* model involving ${}_2\mathrm{F}_1$, we have to be concerned on the analytic continuation of $\mathrm{F}_1^{\tau_1,\,\tau_2}$. In order to continue (3.2) analytically in a wider set, no theorem has been found in the literature. As a consequence we developed such a new theorem following a standard technique, e.g. see Gatteschi (1973) pages 48-50, ensuring analiticity in both variables separately. Subsequently we applied the Osgood's Lemma, which implies global analiticity in two variables of the integral function.

Theorem 3.1. *The two variable complex function:*

$$F(w, z) := \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 \frac{u^{a-1} (1-u)^{c-a-1}}{(1 - w u^{\tau})^{b} (1 - z u^{\tau'})^{b'}} \mathrm{d}u \tag{3.4}$$

is analytic with respect to each variable in $\mathbb{C} \setminus \{x \in \mathbb{R} \mid x \geq 1\}$.

Proof. We follow a standard technique, see for instance, Gatteschi (1973), chapter II, pages 49-51. We have to show that the integral (3.4) is an analytic function both in w and in z, uniformly in the plane cut along the real axis from 1 to ∞. First, consider

$$w \in D_{\varepsilon,R,\delta} := \{\xi \in \mathbb{C} \mid \varepsilon \leq |\xi - 1| \leq R,\ |\arg(1-\xi)| \leq \pi - \delta\},$$

where R is arbitrarily large and ε, δ are arbitrarily small positive constants. Fix $\overline{z} \in D_{\varepsilon,R,\delta}$. For $u \in (0,1)$, $u \mapsto u^{a-1}(1-u)^{c-a-1}(1-\overline{z}u^{\tau'})^{-b'}(1-wu^{\tau})^{-b}$ is a continuous function for any w and analytic in w for any u, in particular:

$$|u^{a-1}(1-u)^{c-a-1}(1-\overline{z}u^{\tau'})^{-b'}(1-wu^{\tau})^{-b}| \leq M M' u^{\mathrm{Re}(a)-1}(1-u)^{\mathrm{Re}(c-a)-1},$$

where

$$M = \max_{u \in [0,1]} |(1-wu^{\tau})^{-b}|, \quad M' = \max_{u \in [0,1]} |(1-\overline{z}u^{\tau'})^{-b'}|.$$

Since the integral:

$$\int_0^1 u^{\mathrm{Re}(a)-1}(1-u)^{\mathrm{Re}(c-a)-1} \mathrm{d}u$$

converges for $\mathrm{Re}(a) > 0$, $\mathrm{Re}(c-a) > 0$, (3.4) provides an analytic continuation with respect to the variable w in $D_{\varepsilon,R,\delta}$, which coincides with $\mathbb{C} \setminus \{x \in \mathbb{R} \mid x \geq 1\}$ for ε, $\delta \to 0$, $R \to \infty$. The proof can be completed by swapping the variables and repeating an analogous technique for analiticity with respect to z. ☐

Theorem 3.2. $\mathrm{F}_1^{\tau_1,\tau_2} \left(\begin{smallmatrix} a;\ b_1, b_2 \\ c \end{smallmatrix} \middle| w, z \right)$ *is analytic in* $(\mathbb{C} \setminus \{x \in \mathbb{R} \mid x \geq 1\}) \times (\mathbb{C} \setminus \{x \in \mathbb{R} \mid x \geq 1\})$.

Proof. It is sufficient to apply Osgood's Lemma, see for instance Gunning and Rossi (1965), pages 2-4, which ensures that if a complex-valued function is continuous in an open set $D \subset \mathbb{C}^n$ and is holomorphic in each variable separately, then it is holomorphic in D. ☐

We can then use $\mathrm{F}_1^{\tau_2,\,\tau_2}$ for providing in closed form the solution to (2.1) with (3.1).

Theorem 3.3. *Define*

$$A := a_0 \frac{\alpha_2}{\alpha_1}, \quad B := b_0 \frac{\beta_2}{\beta_1},$$

and assume A, $B < 1$. *Moreover let*

$$w_1 = \frac{A}{A-1} = \frac{a_0\,\alpha_2}{a_0\,\alpha_2 - \ \alpha_1},\ w_2 = \frac{B}{B-1} = \frac{b_0\,\beta_2}{b_0\ \ \beta_2 - \beta_1}, \tag{3.5}$$

so that $A < 1 \implies w_1 < 1$ *and* $B < 1 \implies w_2 < 1$. *Then the solution of differential equation* (2.1) *where the variable coefficents are given by* (3.1) *is*

$$x(t) = \frac{x_0\,\beta_1\,(1-B)\left[1 + A\left(e^{\alpha_1 t - 1}\right)\right]^{1/\alpha_2}}{\beta_1\,(1-B) + b_0\,(1-A)^{1/\alpha_2}\,x_0\,\boldsymbol{H}(t)},$$

where:

$$\boldsymbol{H}(t) = e^{\beta_1 t}\,\mathrm{F}_1^{\beta_1,\,\alpha_1}\left(\begin{matrix}\beta_1;\ -1/\alpha_2, 1\\ 1+\beta_1\end{matrix}\middle|\, w_1 e^{\alpha_1 t}, w_2 e^{\beta_1 t}\right) - \mathrm{F}_1^{\beta_1,\,\alpha_1}\left(\begin{matrix}\beta_1;\ -1/\alpha_2, 1\\ 1+\beta_1\end{matrix}\middle|\, w_1, w_2\right).$$

Remark: Hereinafter we mean to make use of the analytic continuation of $\mathrm{F}_1^{\beta_1,\,\alpha_1}$ provided by the Euler-type integral (3.3), and not the double power series of hypergeometric nature (3.2). Whenever for computational needs one would make use of the power development series, some restrictions on the coefficients have to be imposed. As a matter of fact, the first necessary requirement is:

$$A,\ B < \frac{1}{2} \iff |w_1| < 1,\ |w_2| < 1.$$

But this is not enough for being entitled to use the hypergeometric series, for depending the first term arguments in $\boldsymbol{H}$ on time t; the series development will be possible if:

$$\left|w_1 e^{\alpha_1 t}\right| < 1, \quad \left|w_2 e^{\beta_1 t}\right| < 1.$$

Proof. Using the $a(t)$ expression coming from (3.1) we are lead to an elementary integration for the numerator of (2.2):

$$N(t) = \exp\left(\int_0^t a(s)\mathrm{d}s\right) = \left[1 + a_0 \frac{\alpha_2}{\alpha_1}\left(e^{\alpha_1 t} - 1\right)\right]^{1/\alpha_2}. \tag{3.6}$$

Inserting (3.6) in the definite integral at the denominator of (2.2), we obtain:

$$D(t) = b_0 \int_0^t \frac{e^{\beta_1 s}}{1 + b_0 \frac{\beta_2}{\beta_1}\left(e^{\beta_1 s} - 1\right)}\left[1 + a_0 \frac{\alpha_2}{\alpha_1}\left(e^{\alpha_1 s} - 1\right)\right]^{1/\alpha_2} \mathrm{d}s. \tag{3.7}$$

Now let us change the variable in (3.7) putting $s = \ln y$, and use the shortcut for A and B previously introduced. We find out:

$$D(t) = b_0 \int_1^{e^t} \frac{y^{\beta_1 - 1}\left[1 + A\left(y^{\alpha_1} - 1\right)\right]^{1/\alpha_2}}{1 + B\left(y^{\beta_1} - 1\right)} \mathrm{d}y. \tag{3.8}$$

Moreover, recalling that $A < 1$ we can write (3.8) as:

$$D(t) = b_0 \frac{(1-A)^{1/\alpha_2}}{1-B} \int_1^{e^t} y^{\beta_1 - 1}\left(1 - w_1 y^{\alpha_1}\right)^{1/\alpha_2}\left(1 - w_2 y^{\beta_1}\right)^{-1} \mathrm{d}y. \tag{3.9}$$

To recognize the structure of the Al Shammery-Kalla (2000) representation theorem, let us write the integral in (3.9) as:

$$\int_1^{e^t} f(y)\,dy = \int_0^{e^t} f(y)\,\mathrm{d}y - \int_0^1 f(y)\,\mathrm{d}y,$$

where for short we used $f(y)=y^{\beta_1-1}\left(1-w_1y^{\alpha_1}\right)^{1/\alpha_2}\left(1-w_2y^{\beta_1}\right)^{-1}$. So, from (3.2), we have:

$$\int_0^1 f(y)\,\mathrm{d}y=\frac{1}{\beta_1}\mathrm{F}_1^{\beta_1,\,\alpha_1}\left(\begin{matrix}\beta_1;-1/\alpha_2,1\\1+\beta_1\end{matrix}\middle|\, w_1,w_2\right).$$

Next, from:

$$\int_0^{e^t} f(y)\,\mathrm{d}y=e^t\int_0^1 f(e^ty)\mathrm{d}y,$$

we find:

$$\int_0^{e^t} f(y)\mathrm{d}y=\frac{e^{\beta_1 t}}{\beta_1}\mathrm{F}_1^{\beta_1,\,\alpha_1}\left(\begin{matrix}\beta_1;-1/\alpha_2,1\\1+\beta_1\end{matrix}\middle|\, w_1e^{\alpha_1 t},w_2e^{\beta_1 t}\right)$$

ending our proof. □

Next figure shows a time evolution for a population model whose input data are:

$$x_0=1,\ a_0=1,\ \alpha_1=4,\ \alpha_2=1,\ b_0=1,\ \beta_1=7,\ \beta_2=3$$

and, together with a bilogistic (curve below) solution, the constant coefficients problem:

$$\begin{cases}\dot{x}=4x-\dfrac{7}{3}x^2\\ x(0)=1\end{cases}$$

is solved. The sample case shows that (unlike the constant coefficients case and the logistical carrying capacity case), the bilogistical problem admits a not monotonic solution whose criticality will occur at time $\hat{t}$ so that:

$$\frac{a(\hat{t})}{b(\hat{t})}=\frac{e^{\hat{t}(\alpha_1-\beta_1)}a_0\alpha_1\left(\beta_1+\left(-1+e^{\hat{t}\beta_1}\right)b_0\beta_2\right)}{b_0\left(\alpha_1+\left(-1+e^{\hat{t}\alpha_1}\right)a_0\alpha_2\right)\beta_1}=x(\hat{t}),$$

which of course can by no means be computed analytically. Finally, if only one of two functions $a(t)$ and $b(t)$ is logistic, while the other is a constant, then population dynamics will be described through ${}_2F_1$.

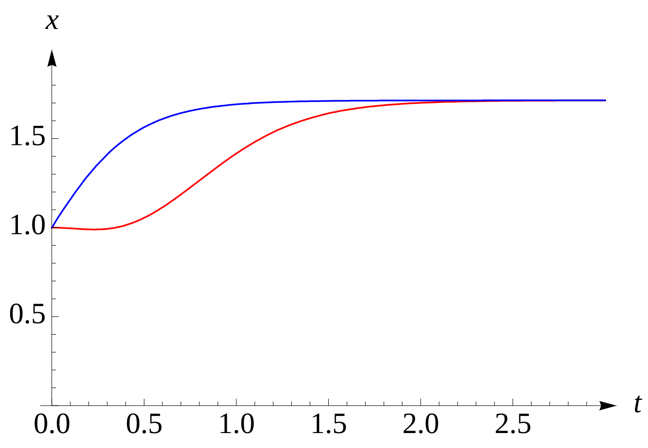

Figure 2: Non-logistical growth (upper blue curve) vs. time of a population ruled by a Verhulst-type ODE pulsed by a VDLIM

4 Concluding remarks

We proved an analytic continuation for the hypergeometric Appell function $F_1^{\tau_1;\tau_2}$, which allows a closed form solution to differential equation, (2.2) with (3.1). However, it should be remarked how the use of hypergeometric functions in finding exact solutions to ordinary differential equations is quite uncommon in economic or population Dynamics. As works where such a class of functions is used, we quote Mingari Scarpello and Ritelli (2003), dealing with growth theory, and besides a very appreciable treatment on the Uzawa-Lucas two-sector model of endogenous growth, due to Boucekkine and Ruiz-Tamarit (2008). On such special functions they think:

> Researchers in economic dynamics should use at last these powerful tools, which can be decisive if one aims at getting beyond the computational and/or local approaches tipically adopted in economics

what is fully shared by us.

References

Al-Shammery, A.H. and Kalla, S.L. (2000). An extension of some hypergeometric functions of two variables. *Revista de la Academia Canaria de Ciencias*, 12, 189-196.

Becken, W. and Schmelcher, P. (2000). The analytic continuation of the Gaussian hypergeometric function ${}_2F_1$ for arbitrary parameters. *Journal of Applied and Computational Mathematics.* 126, 449-478.

Boucekkine, R. and Ruiz-Tamarit, J.R. (2008). Special functions for the study of economic dynamics: the case of the Lucas-Uzawa model. *Journal of Mathematical Economics*, 44, 33-54.

Cohen, J.E. (1995). Population growth and the Earth's human carrying capacity. *Science*, 269, 341-346.

Gatteschi, L. (1973) *Funzioni speciali*, UTET, Torino.

Grover, J.P. (1997). *Resource Competition*, Chapman & Hall, Boca Raton.

Gunning, R.C. and Rossi, H. (1965). *Analytic functions of several complex variables*, Prentice-Hall, Upper Saddle River.

Lebedev, N.N. (1972). *Special functions and their applications*, Dover, New York.

Meyer, P.S. and Ausubel, J.H. (1999). Carrying capacity: a model with logistically varying limits. *Technological Forecasting and Social Change*, 61, 209-214.

Mingari Scarpello, G. and Ritelli, D. (2003). The Solow model improved through the logistic manpower growth law. *Annali Università di Ferrara sezione VII Scienze Matematiche*, 49, 73-83.

Nkashama, M.N. (2000). Dynamics of logistic equations with non-autonomous bounded coefficients. *Electronic Journal of Differential Equations*, 2000, 1-8.

Nkashama, M.N. (2001). Bounded and almost-periodic solutions of nonlocal perturbation of nonautonomous logistic-type equations. *Acta Applicandae Mathematicae*, 65, 283-293.

Watanabe, C., Kondo, R., Ouchi, N and Wei, H. (2003). Formation of IT features through interaction with institutional systems: empirical evidence of unique epidemic behaviour. *Technovation*, 23, 205-219.