A fuzzy model for sensitivity analysis in real options

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Abstract

This paper adopts a promising concept of uncertainty, incorporating both stochastic processes and fuzzy theory to capture the somewhat vague and imprecise ideas the manager has about the future expected cash flows, the profitability of the project, the costs of the project and many other variables involved in an investment decision.

Thus, uncertainty in real option valuation can be faced introducing fuzziness in the fundamental items of the classical approach.

In particular, three examples of real options are examined and the computational experiments are performed. It is shown that fuzziness can play the role of a sensitivity analysis of the real option value with respect to the key decisional variables.

Keywords: Fuzzy Numbers, Parametric Representation, Real Options.

JEL Classification: D81, G31.

1 Real options in a fuzzy environment

Real options theory (ROT) is by now recognized as a most appropriate valuation technique for corporate investment decisions because of its distinctive ability to take into account management's flexibility to adapt ongoing projects in response to uncertain technological and market conditions. Since Myers' ([23]) pioneering idea of viewing firm's discretionary future investment opportunities as real options – that is, the right but not the obligation to undertake some business decision (e.g. deferring, abandoning, expanding, contracting operations, etc.) at a cost during a certain period of time -, a vast literature has developed, which elaborate both theoretical and empirical methods for quantifying the values of various real (call or put) options embedded in investment opportunities. It has also raised a harsh criticism of traditional discounted cash flow techniques for missing the added value of the project-associated options. Dixit and Pindyck ([13]) develop a systematic treatment of ROT, providing the fundamentals of this method, using particularly dynamic programming and its connections with contingent claims analysis, and also emphasize the market implications of such valuation of investment decisions under uncertainty. Trigeorgis ([34], [35] and [36]) provides a taxonomy of real options that maps different categories of investments into the space of different types of financial options. Amran and Kulatilaka [3] and Copeland and Antikarov [9] offer an extensive exposition of this approach and show that ROT has reached advanced textbook status.

Building on the financial option-like features of many corporate investments we can recognize the basic variables on which the value of real options depend, that is, (i) the underlying asset, which is the current value of (gross) expected future operating cash flows, (ii) the exercise price, which is the cost of the project; (iii) the time to expiration of the option, that is the time up to which the project can be undertaken (either finite or infinite); (iv) the standard deviation of the value of the underlying risky asset; (v) the risk-free rate of interest over the life of the option. All the above-mentioned variables are uncertain and therefore various stochastic models have been introduced in ROT to deal with the uncertainty surrounding most corporate decisions. For example, expected future operating cash flows are assumed to evolve according to a geometric Brownian motion, or to a combined Brownian motion and Poisson jump process, if we want to allow for the possibility that at some random time profits abruptly change, maybe due to market competition or other events.

Yet, it is well recognized that reality is more complex and there are some different degrees of perceived uncertainties, which make a precise calculation of how these variables evolve rather difficult. The imprecision associated with the subjective judgement and estimation of future cash flows, which is typical of management's project decisions, needs to be incorporated in the treatment of uncertainty. As will be explained in detail later, our paper introduces a more appropriate and promising concept of uncertainty, incorporating both standard stochastic processes and the theory of fuzzy sets. In this way, we are able to capture the somewhat vague and imprecise ideas the manager possesses about the future expected cash flows, the profitability of the project, the costs of the project, etc.

In what follows we are going to present a few cases of real options that will be evaluated within a fuzzy setting; more specifically, the present values of expected cash flows and expected costs are estimated by fuzzy numbers. To the best of our knowledge, such an approach has never been discussed in the literature, with the exception of Carlsson and Fuller [8]. However, Carlsson and Fuller interpret the possibility of making an investment decision in terms of a European option, while the appropriate analogy is with an American option. They state that the use of probability theory to account for uncertainty can be true in the case of efficient markets of financial options but can produce misleading meanings in the case of real options where uncertainty has a different nature and can be better formulated through a fuzzy approach. Their fundamental result is a formula for fuzzy real option values that involves the possibilistic mean value (introduced in [7]) and variance of fuzzy numbers.

Although in our contribution the stochastic and the fuzzy approach coexist, our point of view differs from Carlsson and Fuller because we deal with American options and elaborate a computing methodology which is more general and can represent the shape of the value functions.

The estimated present value of future net cash flows of the project follows a stochastic process and we model the uncertainty of its parameters across intervals of values. The intervals are built with differentiated levels of uncertainty; given a crisp value, the levels produce a shape that can be characterized by asymmetries or nonlinearities depending on subjective beliefs and available information of the decision maker. It follows that fuzzy parameters play the lead role in a sensitivity analysis that starts gradually from a null variation to the greatest variation of the uncertainty consistent with data.

The paper is organized as follows. In section 2 we present some fundamental elements of fuzzy theory which will be used in the numerical implementation of real option models. In section 3 we describe the introduction of fuzziness in three examples of real options, that is, the option to defer investment, the option to abandon and a case of sequential options. Section 4 collects some of the computational experiments that have been performed in order to capture how and how much fuzziness affects the decision to invest/disinvest. Finally, section 5 concludes.

2 Fundamentals of fuzzy numbers and fuzzy arithmetic

We recall now some fundamental aspects of fuzzy numbers and arithmetic because of its crucial role in the sensitivity analysis modelling.

Fuzzy numbers are a very powerful and flexible way to describe uncertainty or possibilistic values for given variables for which a precise quantification is not possible or one is interested in evaluating the effects of variations around a specified value. In fact a fuzzy number models quite well the different specifications of intervals around a given precise value. A fuzzy number is defined, informally, as a "cascade" of intervals, which start with a given number and grow increasing to a final interval which gives the most uncertain set of possible values. The levels of the cascade are usually parametrized by a parameter $\alpha \in [0,1]$ which represents the so called membership value (or possibilistic degree) of a given interval, with the convention that $\alpha = 1$ corresponds to the exact certain value (the core of the fuzzy number) while $\alpha = 0$ corresponds to the highest uncertainty (the support of the fuzzy number). With the same convention on α we can say that $1 - \alpha$ is the level of uncertainty of the corresponding interval. As we will see, the use of fuzzy arithmetic with fuzzy numbers allows a model to analyze the effects of increasing uncertainty in the key variables of the application.

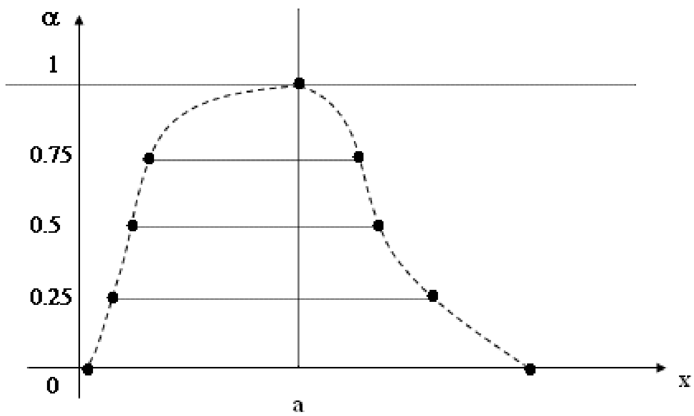


Figure 1: a fuzzy number as a "cascade" of intervals representing increasing uncertainty around the given value a.

In the numerical implementation of section 4 we take advantage of the LU parametric representation introduced in [17], specified in [29] and extensively detailed in [31].

Definition 1 *In the unidimensional case, a fuzzy quantity u is called a fuzzy number if $\exists \widehat{u} \in \mathbb{R}$ such that $core(u) = \{\widehat{u}\}$, and is called a fuzzy interval if $\exists \widehat{u}^-, \widehat{u}^+ \in \mathbb{R}$, $\widehat{u}^- < \widehat{u}^+$ such that $core(u) = [\widehat{u}^-, \widehat{u}^+]$. In particular, the $\alpha -$cuts of a fuzzy number or interval are nonempty, compact intervals of the form*

$$[u]_\alpha = [u_\alpha^-, u_\alpha^+] \subset \mathbb{R}. \tag{1}$$

The usual notation for an LR-fuzzy quantity is $u = \langle a, b, c, d \rangle_{L,R}$ for an interval, and $u = \langle a, b, c \rangle_{L,R}$ for a number. We refer to functions $L(.)$ and $R(.)$ as the left and right branches (shape functions) of u, respectively. On the other hand, the *level-cuts* of a fuzzy number are "nested" closed intervals and this property is the basis for the LU representation (L for lower, U for upper).

Definition 2 *An LU-fuzzy quantity (number or interval) u is completely determined by any pair $u = (u^-, u^+)$ of functions $u^-, u^+ : [0,1] \longrightarrow \mathbb{R}$, defining the end-points of the $\alpha -$cuts, satisfying the three conditions:(i) $u^- : \alpha \longrightarrow u_\alpha^- \in \mathbb{R}$ is a bounded monotonic nondecreasing left-continuous function $\forall\alpha \in]0,1]$ and right-continuous for $\alpha = 0$;(ii) $u^+ : \alpha \longrightarrow u_\alpha^+ \in \mathbb{R}$ is a bounded monotonic nonincreasing left-continuous function $\forall\alpha \in]0,1]$ and right-continuous for $\alpha = 0$;(iii) $u_\alpha^- \leq u_\alpha^+ \ \forall\alpha \in [0,1]$.*

The support of u is the interval $[u_0^-, u_0^+]$ and the core is $[u_1^-, u_1^+]$. If $u_1^- < u_1^+$ we have a fuzzy interval and if $u_1^- = u_1^+$ we have a fuzzy number. We refer to the functions $u_{(.)}^-$ and $u_{(.)}^+$ as the lower and upper branches on u, respectively.

The obvious relation between u^-, u^+ and the membership function μ_u is

$$\mu_u(x) = \sup\{\alpha | x \in [u_\alpha^-, u_\alpha^+]\}. \tag{2}$$

In particular, if the two branches $u_{(.)}^-$ and $u_{(.)}^+$ are continuous invertible functions then $\mu_u(.)$ is formed by two continuous branches, the left being the increasing inverse of $u_{(.)}^-$ on $[u_0^-, u_1^-]$ and the right the decreasing inverse of $u_{(.)}^+$ on $[u_1^+, u_0^+]$.

To model the monotonic branches u_α^- and u_α^- we start with an increasing shape function p such that $p(0) = 0$ and $p(1) = 1$ and a decreasing function q such that $q(0) = 1$ and $q(1) = 0$, with the four numbers $u_0^- \leq u_1^- \leq u_1^+ \leq u_0^+$ defining the support $\left[u_0^-, u_0^+\right]$ and the core $\left[u_1^-, u_1^+\right]$ and we define

$$\begin{aligned} u_\alpha^- &= u_1^- - (u_1^- - u_0^-)p(\alpha) \text{ and} \\ u_\alpha^+ &= u_1^+ - (u_1^+ - u_0^+)q(\alpha) \text{ for all } \alpha \in [0,1]. \end{aligned} \tag{3}$$

The two shape functions p and q, as suggested in [31], are selected in a family of parametrized monotonic functions where the parameters are related to the first derivatives of p and q in 0 and 1; there are many ways to define p and q as illustrated in [29].

The use of the mentioned parametrization allows easy arithmetic operations.

In cases where u_α^- and u_α^+ are required to be more flexible than a single shape function, we can always proceed piecewise over a decomposition of the interval $[0,1]$ into N sub-intervals $[\alpha_{i-1}, \alpha_i]$ for $i = 1, 2, ..., N$. For each decomposition we require (in the differentiable case) $4(N+1)$ parameters to satisfy the following conditions:

$$\begin{aligned} u &= (\alpha_i; u_i^-, \delta u_i^-, u_i^+, \delta u_i^+)_{i=0,1,...,N} \text{ with} \\ u_0^- &\leq u_1^- \leq ... \leq u_N^- \leq u_N^+ \leq u_{N-1}^+ \leq ... \leq u_0^+ \text{ (data)} \\ \delta u_i^- &\geq 0, \delta u_i^+ \leq 0 \text{ (slopes).} \end{aligned} \tag{4}$$

and on each sub-interval $[\alpha_{i-1}, \alpha_i]$ we use the data $u_{i-1}^- \leq u_i^- \leq u_i^+ \leq u_{i-1}^+$ and the parameters $\delta u_{i-1}^-, \delta u_i^- \geq 0$ and $\delta u_{i-1}^+, \delta u_i^+ \leq 0$. In this way we can obtain a wide set of fuzzy numbers.

The simplest representation is obtained on the trivial decomposition of the interval $[0,1]$, with $N = 1$ (without internal points) and $\alpha_0 = 0, \alpha_1 = 1$. In this simple case, u can be represented by a vector of 8 components

$$u = (u_0^-, \delta u_0^-, u_0^+, \delta u_0^+; u_1^-, \delta u_1^-, u_1^+, \delta u_1^+) \tag{5}$$

where $u_0^-, \delta u_0^-, u_1^-, \delta u_1^-$ are used for the lower branch u_α^-, and $u_0^+, \delta u_0^+, u_1^+, \delta u_1^+$ for the upper branch u_α^+.

In the research of the value of a real option, the fundamental step is the computation of fuzzy-valued functions. Given a standard function $y = f(x_1, x_2, ..., x_n)$ of n real (crisp) variables $x_1, x_2, ..., x_n$, its fuzzy extension is obtained to evaluate the effect of uncertainty on the x_j modelled by the corresponding fuzzy number u_j i.e. for each level α by the interval $\left[u_{j,\alpha}^-, u_{j,\alpha}^+\right]$ giving the possible values of x_j for that level.

Let $v = f(u_1, u_2, ..., u_n)$ denote the fuzzy extension of a continuous function f in n variables; for each level α the resulting interval $[v_\alpha^-, v_\alpha^+]$ represents the propagation of uncertainty from the variables x_j to the variable y. In particular, if the uncertainty on the original variables is modelled by fuzzy numbers, the obtained v is yet a fuzzy number starting from a single value (at level $\alpha = 1$) to the most uncertain interval (at level $\alpha = 0$). In this way the fuzzy calculus represents a tool to evaluate the sensitivity of a given dependent variable to the uncertainty in the independent variables.

It is well known that the fuzzy extension of f to normal upper semicontinuous fuzzy intervals (with compact support) has the level-cutting commutative property, i.e. the $\alpha - cuts$ $[v_\alpha^-, v_\alpha^+]$ of v are the images of the $\alpha - cuts$ of $(u_1, u_2, ..., u_n)$ and are obtained by solving the box-constrained optimization problems

$$(EP)_\alpha : \begin{cases} v_\alpha^- = \min\left\{f(x_1, x_2, ..., x_n) | x_k \in [u_{k,\alpha}^-, u_{k,\alpha}^+],\ k = 1, 2, ..., n\right\} \\ v_\alpha^+ = \max\left\{f(x_1, x_2, ..., x_n) | x_k \in [u_{k,\alpha}^-, u_{k,\alpha}^+],\ k = 1, 2, ..., n\right\}. \end{cases} \tag{6}$$

With the exception of simple elementary cases for which the optimization problems above can be solved analytically, the direct application of *(EP)* is difficult and computationally expensive.

We now consider the extension of a multivariate differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ to a vector of n fuzzy numbers $u = (u_1, u_2, ..., u_n)$ with $k{-}th$ component

$$u_k = (u_{k,i}^-, \delta u_{k,i}^-, u_{k,i}^+, \delta u_{k,i}^+)_{i=0,1,...,N} \text{ for } k = 1, 2, ..., n$$

by the LU representation. In all the computations below we will adopt the EP method, but also if other approaches are adopted, the representation still remains valid.

Let $v = f(u_1, u_2, ..., u_n)$ and $v = (v_i^-, \delta v_i^-, v_i^+, \delta v_i^+)_{i=0,1,...,N}$ be its LU representation; the $\alpha - cuts$ of v are obtained by solving the box-constrained optimization problems (6).

For each $\alpha = \alpha_i$, $i = 0, 1, ..., N$ the min and the max (6) can occur either at a point whose components $x_{k,i}$ are internal to the corresponding intervals $[u_{k,i}^-, u_{k,i}^+]$ or are coincident with one of the extremal values; denote by $\widehat{x}_i^- =$

$(\widehat{x}_{1,i}^{-}, ..., \widehat{x}_{n,i}^{-})$ and $\widehat{x}_i^{+} = (\widehat{x}_{1,i}^{+}, ..., \widehat{x}_{n,i}^{+})$ the points where the min and the max take place; then

$$v_i^{-} = f(\widehat{x}_{1,i}^{-}, \widehat{x}_{2,i}^{-}, ..., \widehat{x}_{n,i}^{-}) \text{ and } v_i^{+} = f(\widehat{x}_{1,i}^{+}, \widehat{x}_{2,i}^{+}, ..., \widehat{x}_{n,i}^{+})$$

and the slopes δv_i^{-}, δv_i^{+} are computed (as f is differentiable) by

$$\begin{aligned}
\delta v_i^{-} &= \sum_{\substack{k=1 \\ \widehat{x}_{k,i}^{-} = u_{k,i}^{-}}}^{n} \frac{\partial f(\widehat{x}_{1,i}^{-}, ..., \widehat{x}_{n,i}^{-})}{\partial x_k} \delta u_{k,i}^{-} + \sum_{\substack{k=1 \\ \widehat{x}_{k,i}^{-} = u_{k,i}^{+}}}^{n} \frac{\partial f(\widehat{x}_{1,i}^{-}, ..., \widehat{x}_{n,i}^{-})}{\partial x_k} \delta u_{k,i}^{+} \quad (7) \\
\delta v_i^{+} &= \sum_{\substack{k=1 \\ \widehat{x}_{k,i}^{+} = u_{k,i}^{-}}}^{n} \frac{\partial f(\widehat{x}_{1,i}^{+}, ..., \widehat{x}_{n,i}^{+})}{\partial x_k} \delta u_{k,i}^{-} + \sum_{\substack{k=1 \\ \widehat{x}_{k,i}^{+} = u_{k,i}^{+}}}^{n} \frac{\partial f(\widehat{x}_{1,i}^{+}, ..., \widehat{x}_{n,i}^{+})}{\partial x_k} \delta u_{k,i}^{+}.
\end{aligned}$$

The main and possibly critical steps in the algorithm above is the solution of the optimization problems (6), depending on the dimension n of the solution space and on the possibility of many local optimal points. A detailed analysis of this aspect is in [30].

3 Fuzziness in real options

Real options are basically classified by the type of project they describe, here we detail three examples: the option to defer investment, the option to abandon and a case of sequential options. In particular, the computational experiments will be devoted to the application of the examples to real world data.

3.1 Option to defer investment

The *option to defer investment* is an American call option on the present value of the completed expected cash flows with the exercise price being equal to the required outlay. A project that can be postponed allows learning more about potential project outcomes before making a commitment. A seminal contribution on the option to defer is McDonald and Siegel [22] where the optimal time to invest and an explicit formula for the value of the option to invest are derived for an irreversible project whose net profits follow a geometric Brownian motion. Similarly, Paddock, Siegel and Smith in [25] examine the option to defer in valuing off-shore petroleum leases, Tourinho [32] in valuing reserves of natural resources, Ingersoll and Ross study in [18] the decision to wait in view of the possible beneficial impact on project value of a potential future interest rate decline.

In the *option to defer investment* a firm is supposed to consider the following investment opportunity: at any time t the firm can pay some estimated cost K to install an investment project whose expected future net cash flows conditional on undertaking the project have an estimated present value Π. The installation of such project is irreversible. Let Π follow a geometric Brownian motion of the form:

$$d\Pi = \Pi(\mu dt + \sigma dW_t) \tag{8}$$

where $\mu < r$ is the appreciation rate, r is the risk-free interest rate and σ is the volatility ($\mu \in R, \sigma > 0$) and W is a standard Wiener process. For simplicity, let us assume that the time to expiration of this investment opportunity is infinite, which facilitates the derivation of a closed-form solution.

If $V = V(\Pi)$ is the option value then it holds:

$$\frac{1}{2}\sigma^2\Pi^2 V''(\Pi) + \mu\Pi V'(\Pi) - rV = 0$$

for $\Pi < \Pi^*$ with the initial condition $V(0) = 0$ and smooth-pasting $V(\Pi^*) = \Pi^* - K, V'(\Pi^*) = 1$. The solution is:

$$V(\Pi) = (\Pi^* - K)\left(\frac{\Pi}{\Pi^*}\right)^{\phi} \tag{9}$$

with $\Pi^* = K\frac{\phi}{\phi-1}$ and $\phi = \frac{1}{2} - \frac{\mu}{\sigma^2} + \left(\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}\right)^{\frac{1}{2}} > 1$.

3.2 Option to abandon

The *option to abandon* for a salvage value (or best alternative use) is formally equivalent to an American put on current project value with an exercise price equal to the resale value of its capital equipment and other assets (see Myers and Majd [24]). So is the *option to contract* a project scale by selling a fraction of it at a given price. The abandonment flexibility is important when choosing among alternative production technologies with different purchase-cost to resell-cost ratios. Instead of abandoning a project permanently managers might evaluate the option to temporarily shut down, for example if the output price does not cover the variable costs of production due to unfavorable market conditions. Seminal papers dealing with this case are McDonald and Siegel [22] and Brennan and Schwartz [5]. In Trigeorgis [34] an option to expand is viewed as a call to acquire an additional part ($x\%$) of the project value V by incurring a cost E as exercise price. The investment opportunity with the option to expand can be computed as $V + max(xV - E, 0)$. Similarly, the managers may reduce the scale of operations (by $y\%$) reducing the investment outlay of F, yielding $max(F - yV, 0)$.

In the *option to abandon*, a firm has a net operating profit Π per period that follows equation (8) with $r > \mu$, since we study a closure problem. As Π falls, the firm will at some point close down. Let K denote the estimated liquidation value of the firm's stock of capital.

If $V = V(\Pi)$ is the option value then it holds:

$$\frac{1}{2}\sigma^2\Pi^2 V''(\Pi) + \mu\Pi V'(\Pi) - rV + \Pi = 0$$

for $\Pi > \Pi^*$ with the final condition $\lim_{\Pi\to\infty}\left(V(\Pi) - \frac{\Pi}{r-\mu}\right) = 0$ and with $V(\Pi^*) = K, V'(\Pi^*) = 0$. The solution is:

$$V(\Pi) = \frac{\Pi}{r-\mu} + \left(K - \frac{\Pi^*}{r-\mu}\right)\left(\frac{\Pi}{\Pi^*}\right)^{\psi} \tag{10}$$

with

$$\Pi^* = K\frac{(r-\mu)\psi}{\psi - 1} \tag{11}$$

and

$$\psi = \frac{1}{2} - \frac{\mu}{\sigma^2} - \left(\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}\right)^{\frac{1}{2}} < 0. \tag{12}$$

3.3 Sequential Option

All real options may be part of phased investments. Projects that can be developed in phases typically fit into the category of options on options, or *compound options*. At the end of each phase there is the option to stop or to defer the project, to expand or to contract the project scale, but each phase is an option that is contingent on the earlier exercise of other options. The general valuation formula for compound options has been obtained by Geske [16] for European options. Sequential options are often used to represent corporate growth options (see Brealey and Myers [4], Kester [20], Pindyck [26] and Chung and Charoenwong [11]) and options to expand (Agliardi [2]).

Switching options are portfolios of American call and put options that permit to switch between two modes of operations; an example is the option to exit and re-enter a market, or to close and then restart operations (see Abel, Dixit, Eberly and Pindyck [1]). Brennan and Schwartz [5] determine the combined value of the options to shut-down and restart a mine and to abandon it for salvage, recognizing that partial irreversibility resulting from the costs of switching the mine operating state can create inertia effects, so that it may be optimal to remain in the same operating state even if short-term cash flows seem to favour early switching.

More generally, investment projects involve a collection of multiple real options whose value may interact. Trigeorgis in [34] has emphasized that real options may also interact with financial flexibility (or the option to default on debt payments deriving from limited liability).

The third example of real option that we approach is the option to abandon when disinvestment happens in two stages rather than one (*sequential option*). Consider a project that generates a total operating profit flow equal to Π. The firm's stock of capital is estimated to be K_2 in liquidation. However, when the state variable falls the manager has the option of scaling down its activities by releasing the amount of capital $K_2 - K_1$. If the capital stock is only K_1, then the operating profit is $(1-y)\Pi$, with $0 < y < 1$. Finally, when profits fall even further then the firm is closed down and the remaining stock of capital K_1 is released. It is assumed that the model parameters are such that the first-best policy of the firm is to shrink first before closing down, that is, $\frac{K_2-K_1}{K_1} > \frac{y}{1-y}$.

If $V = V(\Pi)$ is the option value then it holds:

$$V_2(\Pi) = \begin{cases} \frac{\Pi}{r-\mu} + \left(K_2 - K_1 - \frac{y\Pi^{**}}{r-\mu}\right)\left(\frac{\Pi}{\Pi^{**}}\right)^{\psi} + \left(K_1 - \frac{(1-y)\Pi^*}{r-\mu}\right)\left(\frac{\Pi}{\Pi^*}\right)^{\psi} \\ \text{for } \Pi > \Pi^{**} \\ \\ K_2 - K_1 + V_1(\Pi) \\ \text{for } \Pi^* < \Pi < \Pi^{**} \\ \\ K_2 - K_1 \\ \text{for } \Pi < \Pi^* \end{cases} \tag{13}$$

where $V_1(\Pi)$ is the value $V(\Pi)$ in (10), that is $V_1(\Pi) = \frac{\Pi(1-y)}{r-\mu} + \left(K_1 - \frac{(1-y)\Pi^*}{r-\mu}\right)\left(\frac{\Pi}{\Pi^*}\right)^{\psi}$, and ψ , Π^* and Π^{**} are again the same as in the option to abandon, that is, $\Pi^* = \frac{\psi(r-\mu)K_1}{(\psi-1)(1-y)}$ and $\Pi^{**} = \frac{\psi(r-\mu)(K_2-K_1)}{(\psi-1)y}$.

3.4 Fuzzification Model

Our formalization of the valuation of real options schedules the presence of fuzziness in three fundamental steps. First of all, in the stochastic differential equation (8) driving the dynamics of Π, we assume μ, σ and the initial value of Π to be fuzzy and W_t remains a standard Brownian motion (a lot of work on this framework has been done by Yoshida in [37] and Feng in [15]). The second item where fuzziness comes out is in the valuation function of the option (obtained with the extension principle) that depends not only on Π, σ and μ but also on r and K, which we assume to be fuzzy too.

Clearly, fuzziness affects the crucial threshold value Π^*: as soon as Π reaches the threshold value Π^*, the firm finds it optimal to invest (case of the option to defer investment) or disinvest and liquidate (case of the option to abandon). Thus, the decision is based on the threshold value, which depends on all the parameters of the model.

In the valuation method based on fuzzy variables, $\{\Pi_t\}, t \geq 0$ is assumed to be a fuzzy stochastic process, which is specified by the following membership function:

$$\mu_{\Pi_t(\omega)}(x) = max\{1- \mid (x - \widehat{\Pi}_t(\omega))/\beta_t(\omega) \mid, 0\},$$

that is, the fuzzy random variable Π_t is of the triangular type, with centre $\widehat{\Pi}_t(\omega)$, and left-width and right-width $\beta(\omega)$. The assumption of fuzziness is related to the manager's subjective belief about the future profitability of the project. The choice of a triangle-type shape is not restrictive at all and is introduced for simplicity only. Observe that the fuzziness in the process increases as $\beta(\omega)$ becomes bigger. The α-cuts of $\Pi_t(\omega)(x)$ are $\Pi^{\pm}_{t,\alpha}(\omega) = \left[\Pi^{-}_{t,\alpha}(\omega), \Pi^{+}_{t,\alpha}(\omega)\right] =$

$\left[\widehat{\Pi}_t(\omega) - (1-\alpha)\,\beta(\omega), \widehat{\Pi}_t(\omega) + (1-\alpha)\,\beta(\omega)\right]$. It is also reasonable to assume that K is a fuzzy number. In the case of an option to defer, K is the estimated liquidation value of the firm's stock of capital and is affected by depreciation, fluctuating market evaluation and taxation regimes. In the case of an option to abandon, K denotes the investment cost and has many components which can change during the waiting period, due to various unpredictable circumstances.

The extension principle is then applied when in the formula (7) the vector $\widehat{x}_i$ is equal to $(\widehat{\mu}_i, \widehat{\sigma}_i, \widehat{r}_i, \widehat{K}_i)$ and some of the partial derivatives that define the slopes of the representation are nothing else than the first order Greeks, in particular:

$$\frac{\partial f(\widehat{\mu}_i, \widehat{\sigma}_i, \widehat{r}_i, \widehat{K}_i)}{\partial \sigma} \text{ is the Vega}$$

$$\frac{\partial f(\widehat{\mu}_i, \widehat{\sigma}_i, \widehat{r}_i, \widehat{K}_i)}{\partial r} \text{ is the Rho.}$$

The degree of the uncertainty and the way in which it is spread from the model, play a central role in the analysis of the real option. The nonlinearities entering in the definition of $V(\Pi)$ in (9), (10) and in (13) are the main cause of such effects and they can propagate or contract uncertainty. It is very important to perceive the magnitude and the type of these effects. In particular we are interested in the analysis of how the various kinds of uncertainties inserted into the parameters will produce the corresponding uncertainties in Π^*, $V^* = V(\Pi^*)$, Π^{**} and $V^{**} = V(\Pi^{**})$.

As soon as information (on μ, σ, r, K) is modelled by fuzzy numbers, Π^* and V^* also become fuzzy and the degree of fuzziness is related to the $\alpha - cuts$ $\left[\Pi_\alpha^{*^-}, \Pi_\alpha^{*^+}\right]$ and $\left[V_\alpha^{*^-}, V_\alpha^{*^+}\right]$ for a given degree of possibility α.

The maximal uncertainty corresponds to the support at $\alpha = 0$, given by:

$$\left[\Pi_0^{*^-}, \Pi_0^{*^+}\right] \text{ and } \left[V_0^{*^-}, V_0^{*^+}\right].$$

Due to the nonlinearity of Π^* and V^*, the $\alpha - cuts$ are not necessarily symmetric and, for a given uncertainty on the input values μ, σ, r and K, they have different left and right variations.

It is immediate to argue that V^* is symmetric if and only if

$$\Delta V_\alpha^{*^+} = \Delta V_\alpha^{*^-}, \forall \alpha \in [0, 1[$$

where

$$\Delta V_\alpha^{*^+} = V_\alpha^{*^+} - \widehat{V}^* \quad \Delta V_\alpha^{*^-} = \widehat{V}^* - V_\alpha^{*^+};$$

The quantity $\Delta V_\alpha^{*^+}$ represents the possible increase in $\widehat{V}^*$ due to uncertainty and analogously, $\Delta V_\alpha^{*^-}$ measures the possible (absolute) decrease in $\widehat{V}^*$. The same argument can be applied to Π_α^* and $\widehat{\Pi}^*$, defining the quantities

$$\Delta \Pi_\alpha^{*^+} = \Pi_\alpha^{*^+} - \widehat{\Pi}^* \quad \Delta \Pi_\alpha^{*^-} = \widehat{\Pi}^* - \Pi_\alpha^{*^+}.$$

An index that measures the propagation of uncertainty on the right and left sides is the following asymmetry ratio S (such that $0 < S < 2$); for a given value of α we can compute:

$$S_\alpha = \frac{\Delta {\Pi_\alpha^*}^+}{\Delta {\Pi_\alpha^*}^-}.$$

If $\alpha = 1$ we set $S_1 = 1$; if α decreases to zero, both numerator and denominator will increase with different magnitudes reported by their ratio: when $S_\alpha > 1$, it means that, for the given level α of uncertainty, the right semi-interval is larger than the left one, in other words it is more possible to obtain bigger values than the crisp one instead of smaller; when $S_\alpha < 1$, the reverse holds.

For a given $\alpha \in [0, 1[$, we consider the following quantities ε_α^-, ε_α^+ satisfying the relations:

$$\frac{\Delta {V_\alpha^*}^-}{\widehat{V}^*} = \varepsilon_\alpha^- \frac{\Delta {\Pi_\alpha^*}^-}{\widehat{\Pi}^*} \qquad\qquad \frac{\Delta {V_\alpha^*}^+}{\widehat{V}^*} = \varepsilon_\alpha^+ \frac{\Delta {\Pi_\alpha^*}^+}{\widehat{\Pi}^*} \tag{14}$$

We conventionally define $\varepsilon_1^- = \varepsilon_1^+ = 1$ for level $\alpha = 1$. The two functions ε^-, $\varepsilon^+ : [0, 1] \to \mathbb{R}^+$ can be interpreted as a sort of "elasticity to uncertainty" because ε_α^- and ε_α^+ measure the relative lower and upper (absolute) variations of V^*, relative to Π^*, due to the $\alpha-$degree uncertainty in μ, σ, r and K. If $\varepsilon_\alpha^- < 1$ then the possible decrease in V^* is less than the possible decrease in Π^*; if $\varepsilon_\alpha^- > 1$ then the possible (relative) decrease in V^* due to uncertainty is greater then the corresponding decrease in Π^*. Analogously, if $\varepsilon_\alpha^+ < 1$ then a possible relative increase in V^* is less then an increase in Π^* and if $\varepsilon_\alpha^+ > 1$ a possible increase in V^* is greater than the relative increase in Π^*.

It can be interesting to describe the possible patterns of the pair $(\varepsilon_\alpha^-, \varepsilon_\alpha^+)$ for a fixed $\alpha \in [0, 1[$; for example, when $\varepsilon_\alpha^- < 1$ and $\varepsilon_\alpha^+ < 1$ we have a situation where uncertainty in the input parameters produces more (relative) uncertainty on Π^* than on V^* in both lower and upper directions, i.e. Π^* is fuzzier (more uncertain) than V^*.

On the contrary, when $\varepsilon_\alpha^- < 1$ and $\varepsilon_\alpha^+ > 1$ the lower uncertainty of V^* is smaller than the lower uncertainty of Π^*, but the reverse is true on the upper side, i.e. greater values of V^* are possible corresponding to greater values of Π^*.

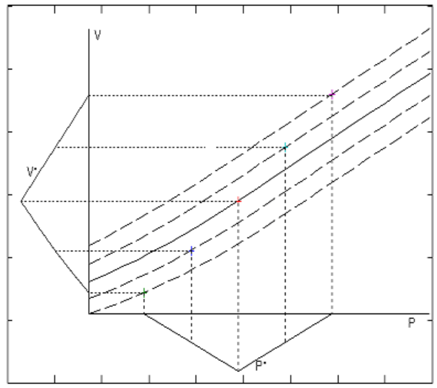


Figure 2: shows the variation of Π^* with respect to V^* when four parameters are modelled as fuzzy numbers

4 Computational experiments

We test the fuzziness effect in the three examples of real options by running several computational experiments; we show only those results that are useful to justify our methodology.

4.1 The option to defer investment

The robustness of the fuzzy model for the option to defer investment is tested with four set of real data that we call, for short, Test1, Test2, and Test3, referring to three different industrial sectors. Test1 refers to an investment decision in the human genome sciences project (HGSI) whose data are taken from the Human Genome project database (details in http://www.ornl.gov/sci/techresources/

Human_Genome/home.shtml). Test2 refers to an investment decision in a big infrastructure, that is the Eurotunnel project (details in [12]); Test3 deals with the case of an investment in new capacity in the public-utility sector, i.e. the electricity market (details in [27]). Values are the following:

	Test1	Test2	Test3
μ	0.01	0.025	0.03
σ	0.048	0.183	0.173
r	0.044	0.06	0.08
Π	2518.9	2312	160
K	704.9	8865	600

We show the shape of V^* in the four cases of real data and the preliminary consideration attains the fact that it exists a uniformity in the results about the

ROT behavior even if the cases under consideration belong to deeply different industrial areas.

In figures concerning the behavior of Π^* we report the three different cases that we will denote as: Allfuzzy (dashed line) when the parameters μ, σ, r and K are fuzzy, Kcrisp (dotted line) when μ, σ, r are fuzzy and K is crisp and finally Kfuzzy (straight line) when μ, σ, r are crisp and K is the unique source of uncertainty.

4.1.1 Results for Test1

Figure 3 shows that the greatest uncertainty occurs when only the parameter K is fuzzy.

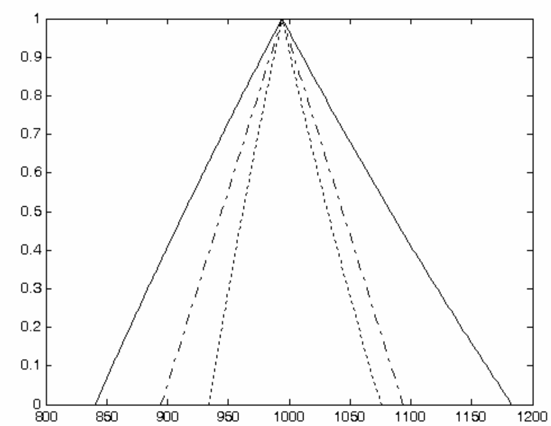


Figure 3: Π^* for Test1

Table 1, Table 2 and Table 3 report values of the $\alpha - cut$ for Test1 in the Allfuzzy, Kcrisp and Kfuzzy case respectively.

Table 1

Level	Π^-	$d\Pi^-$	Π^+	$d\Pi^+$	S
1.0	994.28	168.62	994.28	-168.62	1
0.75	953.16	160.49	1037.56	-177.75	1.052
0.5	913.97	153.21	1083.26	-188.04	1.108
0.25	876.5	146.66	1131.69	-199.72	1.167
0	840.58	140.75	1183.25	-213.04	1.229

Table 2

Level	Π^-	$d\Pi^-$	Π^+	$d\Pi^+$	S
1.0	994.28	69.19	994.28	-69.19	1
0.75	977.6	64.34	1012.25	-74.66	1.0773
0.5	962.07	60	1031.67	-80.83	1.1608
0.25	947.56	56.11	1052.74	-87.86	1.2512
0	933.98	52.61	1075.68	-95.89	1.35

Table 3

Level	Π^-	$d\Pi^-$	Π^+	$d\Pi^+$	S
1.0	994.28	99.43	994.28	-99.43	1
0.75	969.43	99.43	1019.14	-99.43	1
0.5	944.57	99.43	1043.99	-99.43	1
0.25	919.71	99.43	1068.86	-99.43	1
0	894.86	99.43	1093.71	-99.43	1

Observe that in the Kfuzzy case (Table 3) the threshold value Π^* displays a symmetric shape in all analyzed projects because Π depends linearly on K ($S = 1$). In the Allfuzzy and Kcrisp cases, instead, we can observe an asymmetric pattern.

At level 0.5 the average values are 998.615 in Allfuzzy and 996.87 in Kcrisp, which are larger than the crisp value 994.28. Since on average the fuzzy threshold value is larger then without fuzziness, just considering the crisp value the decision to invest would be too early. Figure 4 shows the graphical behavior of fuzzy V^* in the Allfuzzy case; the little crosses point the optimal values Π^*.

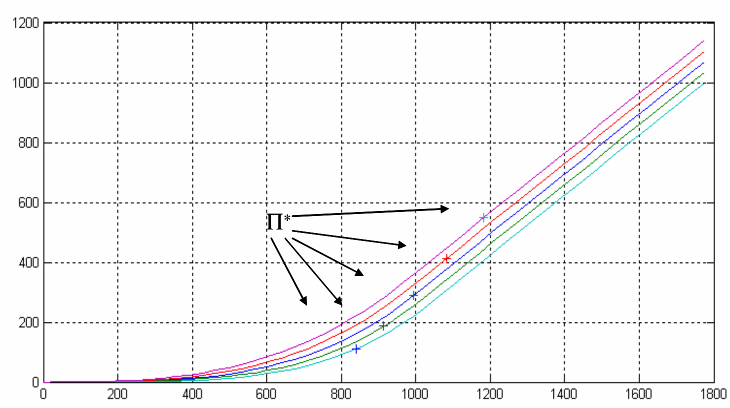


Figure 4: Test1 when parameters μ, σ, r and K are fuzzy.

It is evident that fuzziness implies a certain degree of freedom in the choice of Π^*. Figure 5 illustrates V^* as a sequence of fuzzy numbers.

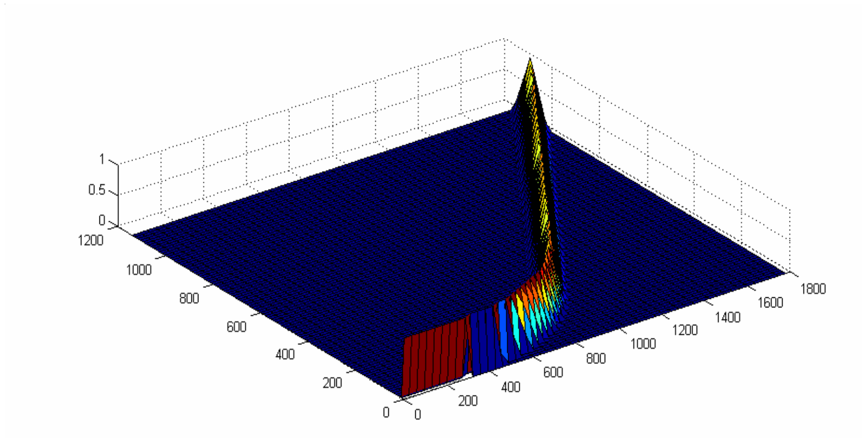
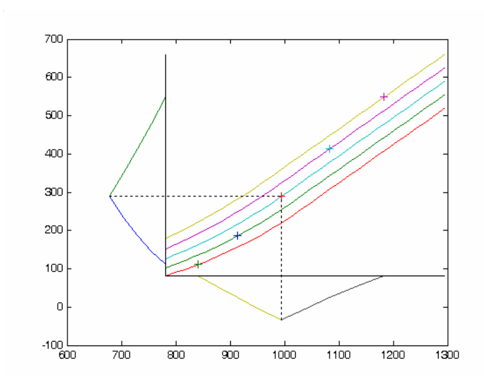
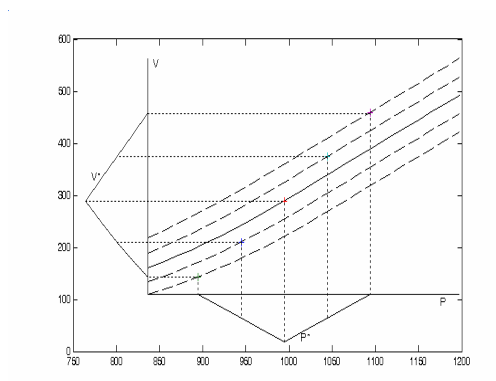


Figure 5: V^* in Allfuzzy case

We are now interested in the portion of the (Π, V) −plane corresponding to the values of Π^* and in the values of ε_α^-, ε_α^+ defined in (14); we report data for Test1 comparing two cases.





Allfuzzy

Level	ε^-	ε^+
1	1	1
0.75	5.52	4.16
0.5	6.34	3.65
0.25	7.39	3.26
0	8.75	2.96

Kfuzzy

Level	ε^-	ε^+
1	1	1
0.75	6.46	5.24
0.5	7.17	4.76
0.25	8.05	4.38
0	9.14	4.06

In the Allfuzzy case we obtain values of ε_α^-, ε_α^+ that are always greater than one but smaller than in Kfuzzy case and this suggest the fact that if V^* decreases, it decreases faster than Π^* when the uncertainty is only in K. At the same time when V^* increases, its increase is greater than for Π^* especially in Kfuzzy case.

4.1.2 Results for Test2

Figure 6 shows the behavior of Π^* in the three different cases: again the biggest uncertainty occurs in the Kfuzzy case (straight line) when μ, σ, r are crisp and K is the unique source of uncertainty.

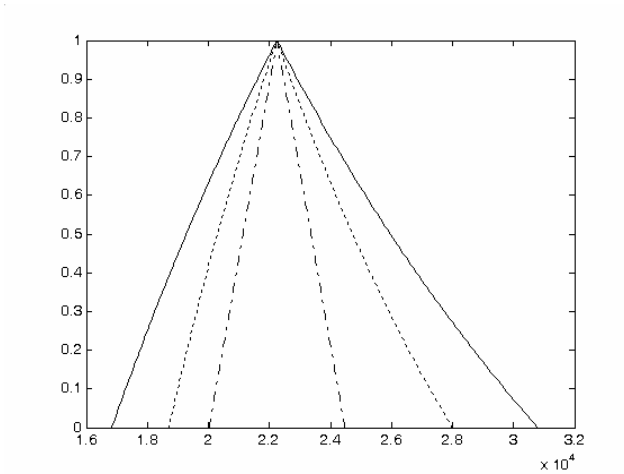


Figure 6: Π^* for Test 2

Table 4 and Table 5 report the values of the $\alpha - cut$ in the Allfuzzy and Kcrisp case for Test2.

Table 4

Level	Π^-	$d\Pi^-$	Π^+	$d\Pi^+$	S
1.0	22249.62	6600.61	22249.62	-6600.61	1
0.75	20686.18	5926.56	23997.55	-7407.79	1.118
0.5	19277.6	5357.68	25967.74	-8386.07	1.251
0.25	18000.33	4872.98	28209.16	-9587.99	1.402
0	16835.43	4456.47	30786.48	-11087.96	1.577

Table 5

Level	Π^-	$d\Pi^-$	Π^+	$d\Pi^+$	S
1.0	22249.62	4375.64	22249.62	-4375.64	1
0.75	21216.59	3902.46	23412.24	-4942.99	1.125
0.5	20292.21	3503.64	24731.18	-5631.38	1.268
0.25	19459.82	3164.32	26241.08	-6478.03	1.431
0	18706.04	2873.18	27987.71	-7535.63	1.619

If we compute again the average values at level 0.5, they are 22622.67 in Allfuzzy and 22511.695 in Kcrisp, which are larger than the crisp value 22249.62. It follows that in the Test2 project it is confirmed the suggestion to wait for the decision to invest.

The parameter S is again always bigger than 1, indicating that bigger values are more possible than smaller values. This aspect is recurring in all simulations and it probably derives from the shape of the function V that assumes bigger values always on the right part of its graph.

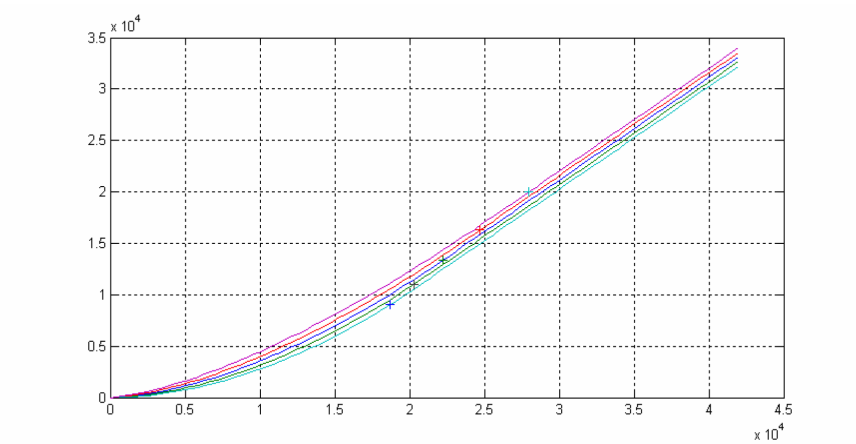


Figure 7: Eurotunnel project (Test2) in the Kcrisp case

4.1.3 Results for Test3

The last project we consider for an option to defer investment is Test3; the relative values of Π^* are reported in Figure 8:

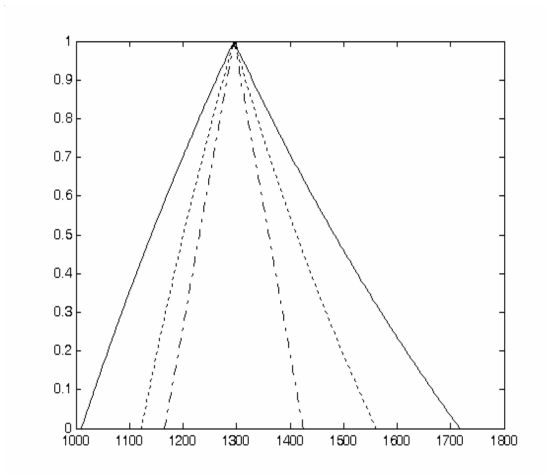


Figure 8: Π^* for Test4

Table 6, Table 7 report values of the $\alpha - cut$ for Test3 in the Allfuzzy and Kcrisp case respectively.

Table 6

Level	Π^-	$d\Pi^-$	Π^+	$d\Pi^+$	S
1.0	1295.31	338.81	1295.31	−338.81	1
0.75	1214.47	308.69	1384.31	−374.24	1.101
0.5	1140.61	282.85	1482.98	−416.32	1.213
0.25	1072.75	260.52	1593.18	−466.86	1.338
0	1010.11	241.06	1717.32	−528.32	1.48

Table 7

Level	Π^-	$d\Pi^-$	Π^+	$d\Pi^+$	S
1.0	1295.31	209.28	1295.31	−209.28	1
0.75	1245.61	188.85	1350.55	−233.35	1.112
0.5	1200.64	171.36	1412.36	−261.99	1.236
0.25	1159.73	156.26	1482.03	−296.43	1.377
0	1122.34	143.14	1561.2	−338.36	1.537

and the graphical representation of V^* in Allfuzzy case is in Figure 9.

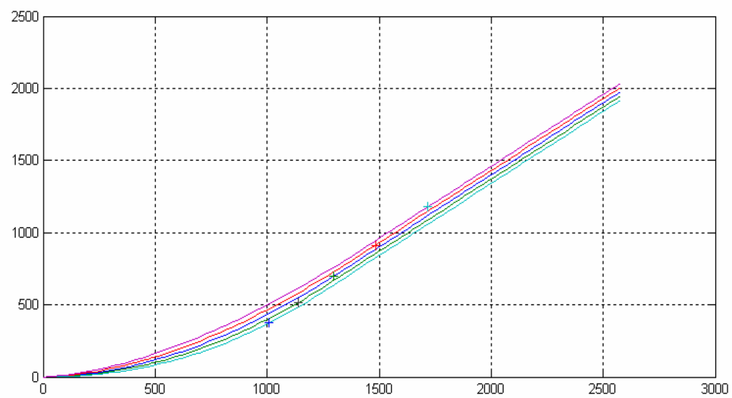


Figure 9: Test3 in the Allfuzzy case

Some further considerations concerning the $\alpha - cut$ values in all the data set enable us to state that our model allows us to describe how the investment decision is actually affected by a perceived increase in "fuzziness". For a pessimistic (optimistic) firm an increase in fuzziness decreases (increases) the perceived value of the project in comparison with the crisp value. On average - for most decision makers- an increase in fuzziness has a positive impact on the investment opportunity, i.e. it increases the perceived value of the project. As a consequence, the decision to invest is delayed in comparison with the absence of fuzziness. However, for pessimistic decision-makers imprecise information about the project value becomes available over time, which makes waiting with investment less valuable. Thus, for pessimistic firms higher fuzziness erodes the subjective value of the investment opportunity. Notice that this result is in keeping with the literature on real options and ambiguity aversion (see, for example, Trojanowska and Kort in [33]). It contrasts with the impact of volatility in the standard real option theory.

4.2 The option to abandon

The robustness of the fuzzy model for the option to abandon is tested with two sets of real data:

	Test4	Test5
μ	0.02	0.01
σ	0.30	0.30
r	0.05	0.05
Π	1200	713
K	325	325

They refer to information systems (IS) projects and data are elaborated from structured interviews with IS project managers (details in [19]).

4.2.1 Results for Test4

The interpretation of Figures 10 and 13 is the same as in the option to defer investment: we represent with a straight line the shape of Π^* in Kfuzzy case, with a dotted line the Kcrisp case and with a dashed line the Allfuzzy case.

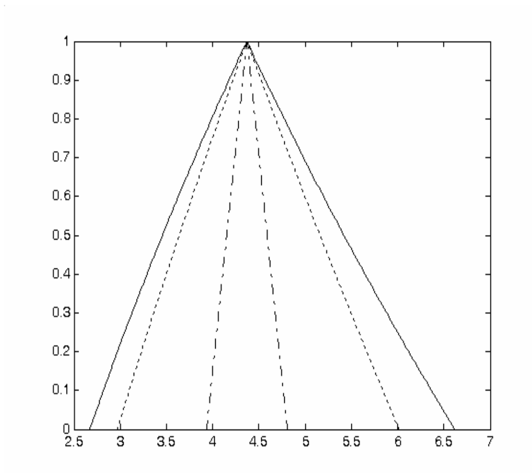


Figure 10: Π^* in Test4

Table 8, Table 9 and Table 10 report values of the $\alpha - cut$ for Test4 in the Allfuzzy, Kcrisp and Kfuzzy case respectively.

Table 8

Level	Π^-	$d\Pi^-$	Π^+	$d\Pi^+$	S
1.0	4.37	1.96	4.37	-1.96	1
0.75	3.9	1.82	4.88	-2.1	1.072
0.5	3.46	1.69	5.42	-2.24	1.150
0.25	3.05	1.57	6.0	-2.39	1.234
0	2.67	1.45	6.62	-2.55	1.323

Table 9

Level	Π^-	$d\Pi^-$	Π^+	$d\Pi^+$	S
1.0	4.37	1.52	4.37	-1.52	1
0.75	3.99	1.46	4.76	-1.58	1.041
0.5	3.64	1.4	5.16	-1.64	1.083
0.25	3.3	1.34	5.58	-1.71	1.128
0	2.97	1.28	6.02	-1.77	1.174

Table 10

Level	Π^-	$d\Pi^-$	Π^+	$d\Pi^+$	S
1.0	4.37	0.437	4.37	-0.437	1
0.75	4.26	0.437	4.48	-0.437	1
0.5	4.15	0.437	4.59	-0.437	1
0.25	4.04	0.437	4.7	-0.437	1
0	3.93	0.437	4.81	-0.437	1

we find that the average values are 4.438 in Allfuzzy and 4.4 Kcrisp, which are larger than the crisp value 4.37. Since on average the fuzzy threshold value is larger than without fuzziness, just considering the crisp value the decision to disinvest would be too late. Firms are more "uncertain" regarding the possibility of when the economy will recover, hence the exit trigger is higher than without fuzziness, implying that firms disinvest earlier.

The fuzzy solution profiles for V^* in the Kcrisp are in Figure 11, whereas Figure 13 shows graphically the fuzzy nature of V^* in the Kfuzzy case.

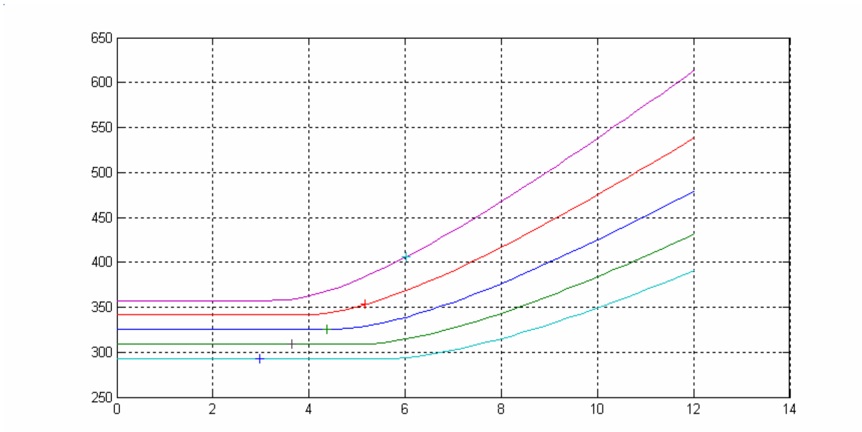


Figure 11: Test4 in the Kcrisp case

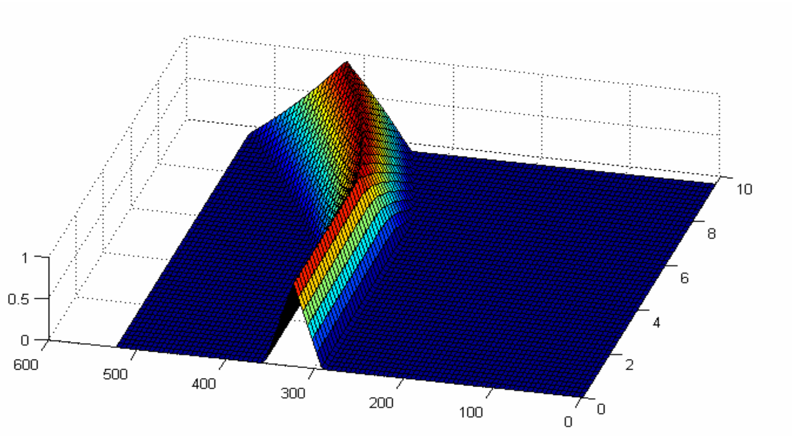


Figure 12: Test4 in the Kfuzzy case

4.2.2 Results for Test5

Taking under consideration the second data set called Test5, the behavior of Π^* in the three different cases is in Figure 13.

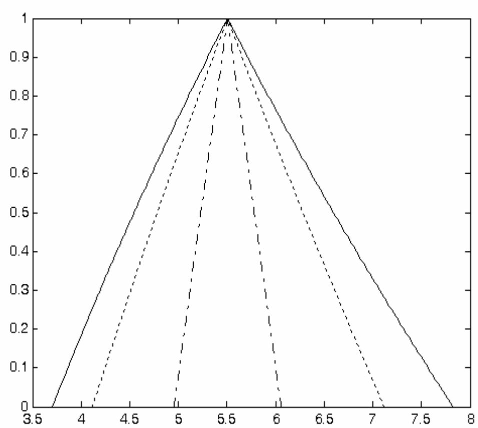


Figure 13: Π^* with Test5 data

Table 11 and Table 12 report values of the $\alpha - cut$ for Test5 in the Allfuzzy and Kcrisp case respectively.

Table 11

Level	Π^-	$d\Pi^-$	Π^+	$d\Pi^+$	S
1.0	5.51	2.05	5.51	-2.05	1
0.75	5.01	1.92	6.03	-2.18	1.066
0.5	4.54	1.8	6.6	-2.32	1.136
0.25	4.11	1.68	7.19	-2.46	1.21
0	3.70	1.56	7.83	-2.61	1.289

Table 12

Level	Π^-	$d\Pi^-$	Π^+	$d\Pi^+$	S
1.0	5.51	1.5	5.51	-1.5	1
0.75	5.14	1.44	5.89	-1.55	1.038
0.5	4.78	1.39	6.28	-1.61	1.077
0.25	4.44	1.33	6.69	-1.67	1.117
0	4.12	1.28	7.12	-1.72	1.159

and the graphical representation of V^* is reported in Figure 14.

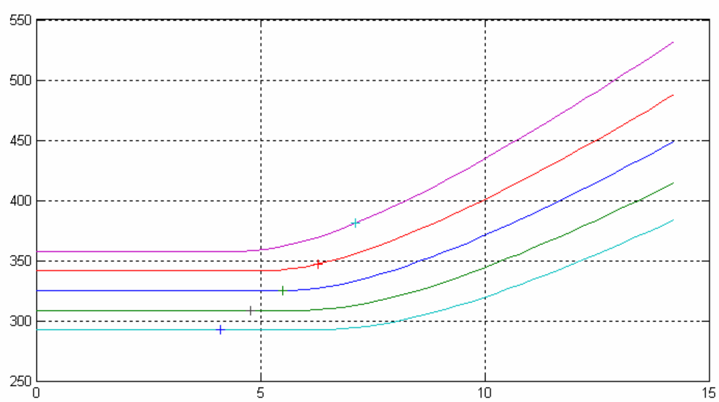


Figure 14: Test5 in the Kcrisp case

4.3 The case of sequential options

Let us test the robustness of the fuzzy model for sequential options using the same data of section 4.2:

	Test6
μ	0.02
σ	0.30
r	0.05
Π	1200
K_1	155
K_2	325

where we have two values of K because disinvestment happens in two stages rather than one; Figure 15 displays the case Allfuzzy (dashed line) is still the one in which μ, σ, r, K_1, K_2 and y are fuzzy, Kcrisp is the one in which μ, σ, r, y are fuzzy and K_1, K_2 are crisp (dotted line), finally Kfuzzy is the case when μ, σ, r, y are crisp and K_1, K_2 are fuzzy (straight line).

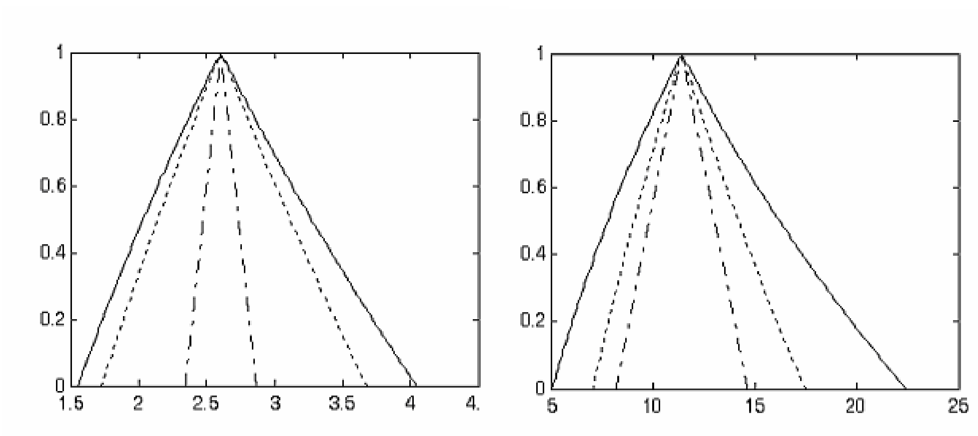


Figure 15: Π^* and Π^{**} in the Allfuzzy, Kcrisp and Kfuzzy

Table 13 and Table 14 report $\alpha - cut$ values of Π^* and Π^{**} respectively, for Test6 in the Allfuzzy case

Table 13

Level	Π^-	$d\Pi^-$	Π^+	$d\Pi^+$	S
1.0	2.61	1.23	2.61	-1.23	1
0.75	2.31	1.13	2.93	-1.33	1.082
0.5	2.04	1.05	3.27	-1.43	1.171
0.25	1.78	0.96	3.64	-1.54	1.267
0	1.55	0.88	4.05	-1.66	1.37

Table 14

Level	Π^-	$d\Pi^-$	Π^+	$d\Pi^+$	S
1.0	11.43	8.35	11.43	-8.35	1
0.75	9.48	7.27	13.66	-9.55	1.146
0.5	7.78	6.3	16.22	-10.89	1.314
0.25	6.32	5.42	19.12	-12.37	1.506
0	5.06	4.64	22.42	-14.02	1.727

Table 15 and Table 16 report $\alpha - cut$ values of Π^* and Π^{**} respectively, for Test6 in the Kcrisp (K_1 and K_2) case.

Table 15

Level	Π^-	$d\Pi^-$	Π^+	$d\Pi^+$	S
1.0	2.61	0.97	2.61	-0.97	1
0.75	2.37	0.92	2.85	-1.02	1.051
0.5	2.14	0.88	3.12	-1.07	1.105
0.25	1.93	0.83	3.39	-1.12	1.162
0	1.73	0.79	3.68	-1.18	1.221

Table 16

Level	Π^-	$d\Pi^-$	Π^+	$d\Pi^+$	S
1.0	11.43	5.12	11.43	-5.12	1
0.75	10.2	4.72	12.76	-5.56	1.085
0.5	9.06	4.35	14.21	-6.03	1.176
0.25	8.02	4.01	15.78	-6.53	1.276
0	7.06	3.69	17.48	-7.08	1.384

A deeper investigation on the profiles of the solution in all the cases can be done by observing Figure 16 and 17.

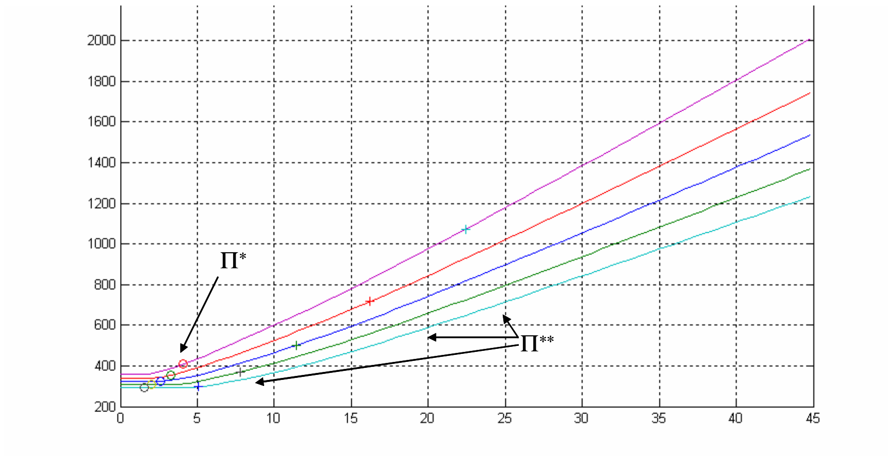


Figure 16: Fuzzy Solution Profiles in the Allfuzzy case,
little circles indicate Π^* and crosses indicate Π^{**}

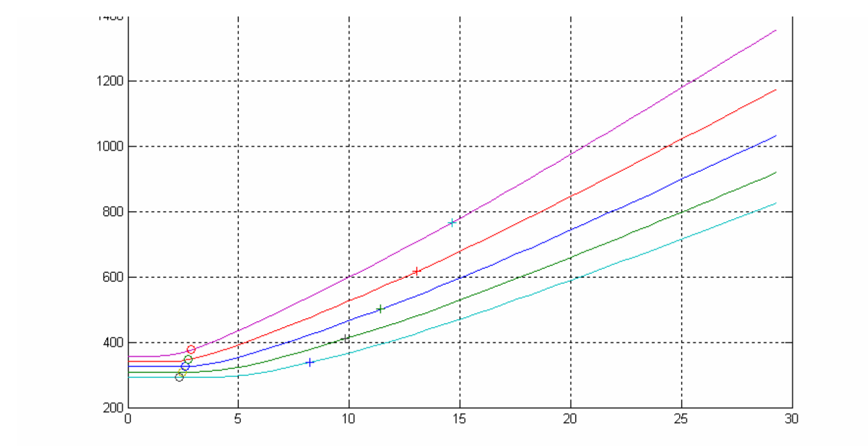


Figure 17: Fuzzy Solution Profiles in the Kfuzzy case (triangular fuzzy curves)

The two threshold values Π^* and Π^{**} show the greatest asymmetry in the Allfuzzy case: in both cases of Π^* and Π^{**}, the size of the asymmetry is bigger in the right part and bigger for Π^{**} than for Π^*. The uncertainty outcomes seems to be more influent in the decision regarding Π^{**} than in the decision about the liquidation Π^*.

5 Concluding remarks

Fuzziness in real world exists in many fields and, especially in human sciences like economics, fuzzy mathematics can provide rigorous models (a detailed motivation of its use is in Zadeh [39]). We model the uncertainty involved in the decision rule when valuing technique for corporate investment decisions; in particular we refer to real options theory because it is overall recognized as the most appropriate. Uncertainty is modelled through fuzzy numbers represented in the LU model; when including fuzziness, the decision rule moves away from the original one and the choice to delay or not the investment becomes a key feature of the fuzzy model.

References

[1] A.B. Abel, A. Dixit, J.C. Eberly, R. Pindyck, Options, the Value of Capital, and Investment, The Quarterly Journal of Economics, (1996), 753-77.

[2] R. Agliardi, Options to expand: some remarks, Finance Research Letters, (2006), 3, 65-72.

[3] M. Amran, N. Kulatilaka, Real Options: Managing Strategic Investment in an Uncertain World, Harvard Business School Press, Boston MA,(1999).

[4] Brealey, Richard and Stewart Myers, Principles of Corporate Finance: 6th Edition (Boston, MA: Irwin McGraw-Hill), (2000).

[5] M. Brennan, E. Schwartz, Evaluating Natural Resources Investment, Journal of Business, (1985), 135-57.

[6] J.J. Buckley, The Fuzzy Mathematics of Finance, Fuzzy Sets and Systems, 21, (1987), 57-73.

[7] C. Carlsson, R. Fullér, On Possibilistic Mean Value and Variance of Fuzzy Numbers, Fuzzy Sets and Systems, 122, (2001), 315-326.

[8] C. Carlsson, R. Fullér, A Fuzzy Approach to Real Option Valuation, Fuzzy Sets and Systems, 139, (2003), 297-312.

[9] T. Copeland, V. Antikaron, Real Options. A Practitioner's Guide, Texere, New York, (2001).

[10] G. Cortazar, E. S. Schwartz, A Compound Option Model of Production and Intermediate Inventories, Journal of Business, (1993), 517-40.

[11] K.L. Chung, C. Charoenwong, Investment Options, Assets in Place and the Risk of Stocks, Financial Management, (1991), 21-33.

[12] A. Damodaran, The Promise and Peril of Real Options, Working Paper, Stern School of Business, 1999.

[13] A. Dixit, R. Pindyck , Investment under Uncertainty, Princeton University Press, (1994), Princeton.

[14] D. Dubois, H. Prade (ed), Fundamentals of Fuzzy Sets, Kluwer, Boston, The Handbooks of Fuzzy Sets Series, 2000.

[15] Y. Feng, Fuzzy stochastic differential systems. Fuzzy Sets and Systems, 115, (2000), 351–363.

[16] R. Geske, The Valuation of Compound Options, Journal of Financial Economics, (1979), 63-81.

[17] M.L. Guerra, L. Stefanini, Approximate Fuzzy Arithmetic Operations Using Monotonic Interpolations, Fuzzy Sets and Systems,150/1, (2005), 5-33.

[18] J. Ingersoll, S. Ross, Waiting to Invest: Investment under Uncertainty, Journal of Business, (1992), 1-29.

[19] J. Kenneally, Y. Lichtenstein, The optional value of IS projects: A study of an IS portfolio at a multinational manufacturer, Tenth European Conference on Information Systems (ECIS'2002), Gdańsk, Poland, 2002.

[20] W. C. Kester, Today's Options for Tomorrow's Growth, Harvard Business Review, (1984), 153-60.

[21] S. Majd, R. Pindyck , Time to Build, Option Value and Investment Decisions, Journal of Financial Economics, (1987), 7-27.

[22] R. McDonald, D. Siegel, The Value of Waiting to Invest, The Quarterly Journal of Economics, 101, 4 (1986), 707-728.

[23] S. C. Myers, Determinants of Corporate Borrowing, Journal of Financial Economics, (1977), 147-76.

[24] S. C. Myers, S. Majd, Abandonment value and Project Life, Advances in Futures and Options Research, (1990), 1-21.

[25] L. Paddock, D. Siegel, J. Smith, Option Valuation of Claims on Physical assets: the Case of Offshore Petroleum Leases, The Quarterly Journal of Economics, (1988), 479-508.

[26] R. Pindyck, Irreversibility, Uncertainty and Investment, Journal of Economics Literature, (1991), 1110-48.

[27] F.A. Roques, N.S. Savva, Price Cap Regulation and Investment Incentives under Demand Uncertainty, Cambridge Working Papers in Economics 0636, 2006.

[28] E. Schwartz, M. Moon, Evaluating Research and Development Investment in Project Flexibility, Agency and Competition, Oxford University Press, Oxford, (2000).

[29] L. Stefanini, L.Sorini, M.L.Guerra, Parametric Representations of Fuzzy Numbers and Applications to Fuzzy Calculus, Fuzzy Sets and Systems, 157/18, (2006), 2423-2455.

[30] L.Stefanini, Differential Evolution methods for LU-fuzzy Arithmetic, Proceedings EUSFLAT 2007, Ostrawa, September 2007.

[31] L. Stefanini, L. Sorini, M.L. Guerra, "Fuzzy Numbers and Fuzzy Arithmetics", in W. Pedrycz, A. Skowron, and V. Kreinovich (Eds.): HANDBOOK OF GRANULAR COMPUTING, John Wiley & Sons, 2008, 249-283.

[32] O. Tourinho, The Option Value of Reserves of Natural Resources, WP 94, University of California, Berkeley, (1979).

[33] M. Trojanowska, P. Kort, Optimal Investment under Ambiguity: the worst case for real options, 10th International Conference on Real Options, New York, (2006).

[34] L. Trigeorgis, Real options and Interactions with Financial Flexibility, Financial Management, (1993), 202-24.

[35] L. Trigeorgis, The Nature of Option Interactions and Valuation of Investments with Multiple Real Options, Journal of Financial and Quantitative Analysis, (1993), 309-26.

[36] L. Trigeorgis, Real Options. Managerial Flexibility and Strategy in Resource Allocation, The MIT Press, (1996).

[37] Y.Yoshida, M. Yasuda, J.Nakagami and M.Kurano, A new evaluation of mean value for fuzzy numbers and its application to American put option under uncertainty, Fuzzy Sets and Systems, 157, (2006), 2614-2626.

[38] L.A. Zadeh, Fuzzy Sets, Information and Control, 8, (1965), 338-353.

[39] L.A. Zadeh, Is there a need for fuzzy logic?, Information Sciences, 178, (2008), 2751-2779.