Advertising with Spillover Effects in a Differential Oligopoly Game with Differentiated Goods

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Abstract

We consider a differentiated oligopoly where firms compete *à la* Cournot in the market phase, and each firms may invest in advertising activity, to enlarge its market size. Each firm's advertising effort has positive external effects on the market size of all rivals. We derive the open-loop (and the coincident closed-loop) Nash equilibrium, and the optimal behaviour of a cartel involving all firms setting both quantities and advertising efforts so as to maximize joint profits. The comparative assessment of these equilibria shows that a cartel may produce a steady state where social welfare is higher than the social welfare level associated with the non-cooperative setting. This is due to the positive externalities from advertising activity.

Keywords: advertising, differential games, capital accumulation, closed-loop equilibria, externality.

JEL Classification: D43, D92, L13, M37.

1 Introduction

We present a dynamic oligopoly model with differentiated products, where firms compete *à la* Cournot in the market phase, and each firm invests in advertising activities aimed at enlarging its market size. The advertising investment carried out by any firm spills over to the rivals. This hypothesis seems to be quite realistic, but has not yet fully investigated by the available literature.[1] Our idea is that advertising activity for, say, a mobile telephone benefits all producers of telephones, even if products are differentiated. Under this respect, advertising is, at least to some extent, a public good. Consequently, the market regulation prescriptions, concerning the competition in the market phase as well as the competition in advertising efforts, are not trivial.

We consider a dynamic framework, where the advertising efforts have long-lasting effects. As to the market decision, we consider an oligopoly, with a positive degree of substitutability between goods, under linear market demand function; it is well known that this competition model includes monopolistic competition and homogenous oligopoly as particular cases (see Spence, 1976, Vives 1985, Cellini and Lambertini, 1998). We show that this dynamic system may converge to a steady state. The determinants of the steady state market sizes and advertising efforts are first analyzed in the non-cooperative game, using the open-loop and the closed-loop Nash equilibria as solution concepts. As a matter of fact, these equilibria coincide in our framework, since the problem at hand is state-linear (see Dockner *et al.*, 2000). Thus, the solution of the differential oligopoly game benefits both from the easy analytical procedure of the open-loop concept and from the time consistency property of the closed-loop concept.

Then, we analyze the optimal decisions by a (symmetric) cartel made up by all firms in the market, aiming at the maximum joint profits with respect to both output and advertising effort. Market sizes, investments

[1] The literature on dynamic models of advertising can be broadly partitioned into two classes: the first establishes a direct relationship between the rate of change in sales (or market shares) and the advertisin efforts of firms (see, e.g., Leitmann and Schmitendorf, 1978, Feichtinger, 1983, and Erickson,1985); the second considers advertisemet as an instrument to increase the stock of goodwill or reputation and establishes a link between the advertising efforts of a firm and her market demand (Sethi, 1977, Fershtman, 1984, and Jørgensen and Zaccour, 1999, Piga, 2000). See Jorgensen (1982), Erickson (1991), Feichtinger, Hartl and Sethi (1994), Dockner *et al.* (2000, ch. 11) for exhaustive surveys.

and production decisions are compared across the steady states of all the aforementioned regimes.

A relevant result is that social welfare in steady state may well be larger in the cartel equilibrium, than in the non-cooperative Cournot equilibrium. This is due to the externality effect affecting firms' advertising activities, and the associated inefficiency affecting the Cournot setting.

The structure of the paper is as follows. Section 2 illustrates the basic setup. Section 3 first shows that the open-loop information structure and the closed-loop information structure are immaterial to the result of this differential game, since the game is a linear state one. Then, it develops the solution of the Nash equilibrium and studies the dynamic properties of the steady state. Section 4 deals with the optimal cartel behaviour. The comparative assessment of all equilibria is in Section 5. Section 6 briefly concludes.

2 The basic setup

We consider an oligopoly game played over $t \in [0, \infty)$. Define the set of firms (players) as $\mathbb{P} \equiv \{1, 2, 3, ...N\}$. Each firm i produces one good, that is differentiated with respect to the good produced by any other firm . Let p_i denote the price of good i, and x_i the quantity of good i. Firm i faces the following demand function, borrowed from Spence (1976) and employed by Singh and Vives (1984), Vives (1985), Lambertini (1997), Cellini and Lambertini (1998, 2002), *inter alia*:

$$p_i(t) = A_i(t) - Bx_i(t) - D \sum_{j \neq i} x_j(t) \tag{1}$$

Variable A_i describes the market size or the reservation price for good i. B and D are parameters, with $0 \leq D \leq B$. Notice that parameter D captures the degree of substitutability between any pair of different goods. In the limit case $D = 0$, goods are independent and each firm becomes a monopolist. In the opposite limit case $D = B$, the goods produced by different firms are perfect substitutes and the model collapses into the homogeneous oligopoly model.

We assume that the market size may be increased by firms, through advertising activities. More precisely, we assume that the efforts made by any firm affects its own market size, as well as the market size of the rivals. In particular, the dynamics of the market size of firm i is described by the following equation:

$$\frac{dA_i(t)}{dt} \equiv \dot{A}_i\ (t) = k_i(t) + \gamma \sum_{j \neq i} k_j(t) - \delta A_i(t) \tag{2}$$

where k_h is the effort in advertising made by firm h, $0 \leq \gamma \leq 1$ is a parameter capturing the external effect of the advertising of a firm on the market size of different firms, $\delta \geq 0$ is a “depreciation” parameter indicating that market size shrinks as time goes by.

Advertising activities are assumed to have decreasing marginal productivity, i.e., they entail a quadratic cost $c_i(k_i(t)) = \frac{\alpha}{2}(k_i(t))^2$ with $\alpha > 0$. Moreover, production entails a constant marginal production costs: $c_i(x_i(t)) = cx_i(t)$ with $c > 0$.

Quantities $\{x_i\}_{i=1}^N$ and advertising efforts $\{k_i\}_{i=1}^N$ are the control variables, while market sizes $\{A_i\}_{i=1}^N$ are state variables. Each player chooses the path of her control variables over time, in order to maximize the present value of her profit flow, subject to (i) the motion laws regarding the state variables, and (ii) the initial conditions. Formally, the problem of player i may be written as follows. The objective function is:

$$\max_{x_i(t), k_i(t)} J_i \equiv \int_0^\infty \pi_i(t) e^{-\rho t} dt \tag{3}$$

where the factor $e^{-\rho t}$ discounts future gains, and the discount rate ρ is assumed to be constant and common to all players. Instantaneous profits are:

$$\pi_i(t) = \left(A_i(t) - Bx_i(t) - D \sum_{j \neq i} x_j(t) \right) x_i(t) - \frac{\alpha}{2}(k_i(t))^2 - cx_i(t) \tag{4}$$

Function (3) is subject to the set of N dynamic constraints of type (2), and to the set of initial conditions $\{A_i(0) = A_{i0}\}_{i=1}^N$. To solve the problem, we have to introduce specific hypotheses about the information structure of players.

3 The non-cooperative oligopoly

The available literature on differential games applied to firms' behaviour mainly concentrates on two kinds of solution concepts: open-loop and closed-loop equilibria. In the former case, firms precommit their decisions on the control variables to a path over time, i.e., they design the optimal plan at the initial date and then stick to it forever. The resulting open-loop Nash equilibrium is only weakly time consistent and therefore, in general, it is not subgame perfect. In the latter, firms do not precommit to any path and their strategies at any instant depend on the preceding history. In this situation, the information set used by firms in setting their strategies at any given time is often simplified to be only the current value of the stocks of state variables at that time. The relevant equilibrium concept, in this case, is the closed-loop memoryless Nash equilibrium, which is strongly time consistent and therefore subgame perfect. A refinement of the closed-loop Nash equilibrium, which is known as the feedback Nash equilibrium, can also be adopted as the solution concept: while in the closed-loop memoryless case the initial and current levels of all state variables are taken into account, in the feedback case only the current stocks of states are considered.[2]

The literature on differential games devotes a considerable amount of attention to identifying classes of games where the closed-loop equilibria degenerate into open-loop equilibria. This interest is motivated by the following reason. Whenever an open-loop equilibrium is a degenerate closed-loop equilibrium, then the former is also subgame perfect; therefore one can rely on the open-loop equilibrium which, in general, is much easier to derive than the closed-loop one.[3] We show later that, in the present model, the coincidence between closed-loop and open-loop Nash equilibrium arises.

We start by considering the solution of the oligopoly game under the closed-loop information structure. The dynamic optimization problem requires to consider the following Hamiltonian function for each player:

[2] For oligopoly models where firms follow feedback rules, see Simaan and Takayama (1978), Fershtman and Kamien (1987, 1990), Dockner and Haug (1990), *inter alia*. For a clear exposition of the difference among these equilibrium solutions see Başar and Olsder (1982, pp. 318-327, and chapter 6, in particular Proposition 6.1) or Dockner *et al.* (2000).

[3] Classes of games where this coincidence arises are illustrated in Clemhout and Wan (1974); Reinganum (1982); Mehlmann and Willing (1983); Dockner, Feichtinger and Jørgensen (1985); Fershtman (1987); Fershtman, Kamien and Muller (1992). For an overview, see Mehlmann (1988), Fershtman, Kamien and Muller (1992), and Dockner *et al.* (2000).

$$\begin{aligned}
\mathcal{H}_i(t) \equiv\; & e^{-\rho t}\left[\left(A_i(t) - Bx_i(t) - D\sum_{j\neq i} x_j(t)\right)x_i(t) - \frac{\alpha}{2}(k_i(t))^2 - cx_i(t)\right. \\
& +\lambda_{ii}(t)\cdot\left(k_i(t) + \gamma\sum_{j\neq i} k_j(t) - \delta A_i(t)\right) + \\
& \left. + \sum_{j\neq i}\lambda_{ij}(t)\cdot\left(k_j(t) + \gamma\sum_{h\neq j} k_h(t) - \delta A_j(t)\right)\right]
\end{aligned} \tag{5}$$

where $\lambda_{ij}(t) = \mu_{ij}(t)e^{\rho t}$ is the costate variable (evaluated at time t) associated by player i with state variable A_j. The first order conditions and the adjoint equations (for each player i) are:

$$\frac{\partial \mathcal{H}_i(.,t)}{\partial x_i(t)} = 0\ ; \tag{6}$$

$$\frac{\partial \mathcal{H}_i(.,t)}{\partial k_i(t)} = 0\ ; \tag{7}$$

$$-\frac{\partial \mathcal{H}_i\,(.)}{\partial A_j(t)} - \sum_{h\neq j}\frac{\partial \mathcal{H}_i(.)}{\partial x_h(t)}\frac{\partial x_h^*(t)}{\partial A_j(t)} - \sum_{h\neq j}\frac{\partial \mathcal{H}_i(.)}{\partial k_h(t)}\frac{\partial k_h^*(t)}{\partial A_j(t)} = \frac{\partial \lambda_{ij}(t)}{\partial t} - \rho\lambda_{ij}\ \forall\, j = 1, 2...N. \tag{8}$$

These conditions have to be considered along with the initial conditions $\{A_i(0) = A_{i0}\}_{i=1}^{N}$ and the transversality conditions, which set the final value of the state and/or co-state variables. In problems defined over an infinite time horizon, it is usual to set

$$\lim_{t\to\infty}\ \lambda_{ij}(t)\cdot A_j(t) = 0 \text{ for all } j\ . \tag{9}$$

>From (6) and (7) we obtain respectively:

$$A_i(t) - c - 2Bx_i(t) - D\sum_{j\neq i} x_j(t) = 0 \tag{10}$$

$$-\alpha k_i(t) + \lambda_{ii}(t) + \gamma\sum_{j\neq i}\lambda_{ij}(t) = 0 \tag{11}$$

and then, under the symmetry hypothesis $\sum_{j \neq i} x_j(t) = (N-1)x$, the optimal rules:

$$x_i^*(t) = \frac{A_i(t) - c}{2B + D(N-1)} \tag{12}$$

$$k_i^*(t) = \frac{\lambda_{ii}(t) + \gamma(N-1)\lambda_{ij}(t)}{\alpha} \tag{13}$$

In the adjoint equations (8), the terms

$$\frac{\partial \mathcal{H}_i(.)}{\partial x_h(t)} \frac{\partial x_h^*(t)}{\partial A_j(t)}, \; \frac{\partial \mathcal{H}_i(.)}{\partial k_h(t)} \frac{\partial k_h^*(t)}{\partial A_j(t)} \tag{14}$$

capture the strategic interaction among firms through the feedback from states to controls, which is by definition absent under the open-loop solution concept. Whenever the expressions in (14) are zero for all j, then the closed-loop memoryless equilibrium collapses into the open-loop Nash equilibrium. This happens in the present model because the optimal values of the control variables of each player are not affected by state variables different from its own market size. With reference to the labels used by Dockner *et al.* (2000, ch. 7), the problem at hand is a "linear state" differential game, and the closed-loop solution collapses into an open-loop solution, which is strongly time consistent, or Markov perfect (see also, e.g., Driskill and McCafferty, 1989, pp. 327-28).

Accordingly, conditions (8) rewrite as follows:

$$-\frac{\partial \lambda_{ii}(t)}{\partial t} = x_i(t) - \lambda_{ii}(t)\delta - \lambda_{ii}(t)\rho \tag{15}$$

$$-\frac{\partial \lambda_{ij}(t)}{\partial t} = -\lambda_{ij}(t)\delta - \lambda_{ij}(t)\rho \tag{16}$$

Differentiating (13) with respect to time, plugging (15) and rearranging, we obtain:

$$\frac{\partial k_i^*(t)}{\partial t} = \frac{1}{\alpha}\left[(\rho + \delta)\alpha k_i(t) - \frac{A_i(t) - c}{2B + D(N-1)}\right] \tag{17}$$

The dynamic equation (17), alongside with the dynamic constraint under symmetry condition, $dA_i(t)/dt = (1 + (N-1)\gamma)k_i(t) - \delta A_i(t)$, may be rewritten, in matrix term, as follows (subscript i is omitted, as it is redundant under symmetry):

$$\begin{bmatrix} \dot{k} \\ \\ \dot{A} \end{bmatrix} = \begin{bmatrix} (\delta+\rho) & -\dfrac{1}{\alpha[2B+D(N-1)]} \\ 1+(N-1)\gamma & -\delta \end{bmatrix} \begin{bmatrix} k \\ \\ A \end{bmatrix} + \begin{bmatrix} \dfrac{c}{2B+D(N-1)} \\ \\ 0 \end{bmatrix} \tag{18}$$

In order to look for a steady state, we solve the system of equations $\dot{k}= 0$, $\dot{A}= 0$. The graphical counterpart of each of these equations is a straight line with a positive slope in the space (A, k). These lines do not necessarily intersect in the positive ortant, i.e., the steady state may be meaningless, from an economic point of view. The necessary and sufficient condition for obtaining an economically meaningful intersection point between the two loci, i.e., an intersection in the positive ortant, is

$$\alpha\delta(\rho+\delta)(2B+D(N-1)) < 1+(N-1)\gamma. \tag{19}$$

We assume that this condition is fulfilled (figure 1); otherwise, no meaningful steady state exists (figure 2).[4] The steady state solutions are:

$$\begin{aligned} A^{SS}_{OLIG} &= \frac{c\,(1+(N-1)\gamma)}{1+(N-1)\gamma-\alpha\delta(\rho+\delta)(2B+D(N-1))}\,; \\ k^{SS}_{OLIG} &= \frac{\delta c}{1+(N-1)\gamma-\alpha\delta(\rho+\delta)(2B+D(N-1))}\,. \end{aligned} \tag{20}$$

As to the dynamic properties of the steady state, notice now that, in (18), the trace of the Jacobian matrix is $\rho > 0$, while the determinant is $[1+(1-N)\gamma]\,/\,[\alpha(2B+D(N-1))] - \delta(\rho+\delta)$, the sign of which is unambiguously negative, as long as condition (19) holds. Consequently, the steady state is a saddle. The dashed line in figure 1 represents the stable arm of the dynamic system to the steady state.

[4] It is interesting to note that, when $\gamma = 0$, no steady state exists. This result replicates the findings in Cellini and Lambertini (2001): in that model, in the absebce of externality, the linearity of market demand is inconsistent with the existence of steady states.

Figure 1 : Dynamics in the space (A, k) with convergence to a saddle equilibrium

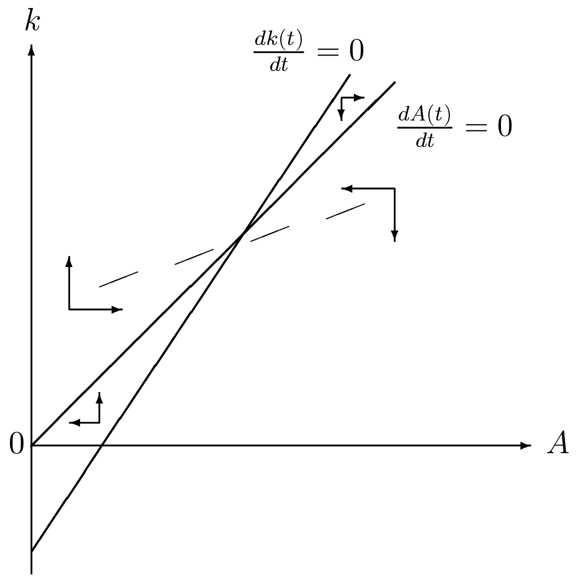
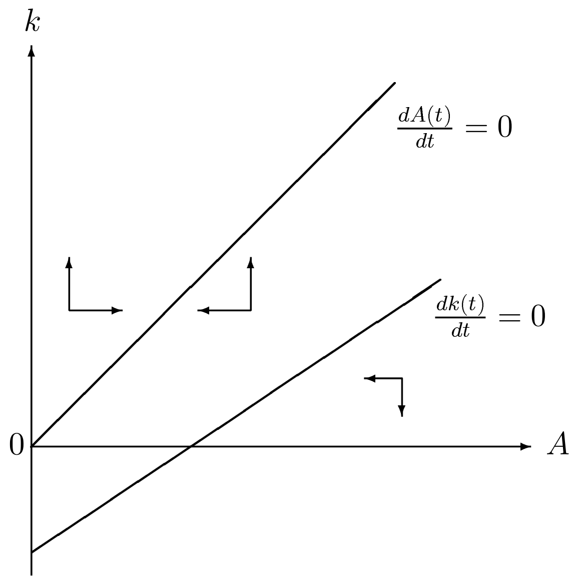


Figure 2 : Dynamics in the space (A, k) without convergence



It is straightforward to interpret the economic effects of parameters on the steady state values of investment and advertising. In particular, the larger is the spillover effect γ, the smaller is the individually optimal advertising effort, and the smaller is the market size in steady state. Moreover, the degree of substitutability between goods, D, has a positive effect both on the steady state market size, and on the steady state advertising investment. This is motivated by the fact that, *ceteris paribus*, a higher product substitutability implies a smaller market power, and this leads firms to increase market size. The effect of B is similar, since the larger is B, the higher is the price elasticity of demand, and the smaller is the market power of any firm.

The effect of the number of of firms, N, upon the optimal steady state values, is not clear-cut, because two opposite effects are in operation: on the one side, the larger is N, the harsher is market competition, that makes it more convenient to have larger market size; on the other hand, the larger is N, the larger is the external effect on other firms, that makes it less convenient to invest in advertising.

4 The cartel

In this section, we study the model under the hypothesis that all the firms in the market build a cartel where all decision variables are set cooperatively, in order to maximize the present value of the joint profits. We label this case as full cartelisation. The dynamic problem can be summarized as follows. We have to find the optimal plans of each of the n (symmetric) firms concerning individual production x and investment k, in order to achieve the maximum present value of the flows of future joint profits. Formally, the problem is

$$\max_{x(t),k(t)} J \equiv \int_0^\infty n\pi(t)e^{-\rho t}dt \tag{21}$$

where the instantaneous profit is:

$$\pi(t) = [A(t) - c]x(t) - B(x(t))^2 - D(n-1)(x(t))^2 - \frac{\alpha}{2}(k(t))^2 \tag{22}$$

Function (21) is subject to the dynamic constraint $\dot{A}(t) = (1+(N-1)\gamma)k(t) - \delta A(t)$, and to the set of initial conditions $\{A_i(0) = A_{i0} = A_0\}_{i=1}^N$ .Notice that we have also posed that the market sizes of all firms are equal, even at the initial time. The choice of the information structure is pointless in this case, since full cooperation among firms is assumed.The Hamiltonian function is:

$$\begin{aligned}\mathcal{H} &= e^{-\rho t} N \left\{ [A(t) - c]x(t) - (B + D(n-1))(x(t))^2 - \frac{\alpha}{2}(k(t))^2 \right. \\ &\quad \left. + \lambda(t)[(1 + (N-1)\gamma)k(t) - \delta A(t)] \right\} . \qquad (23)\end{aligned}$$

where $\lambda(t) = \mu(t)e^{\rho t}$ is the costate variable (evaluated at time t) associated with the state variable A. The first order conditions and the adjoint equation for any firm i (subscript omitted) are:

$$\frac{\partial \mathcal{H}(.)}{\partial x(t)} = 0 \Rightarrow (A(t) - c) - 2(B + D(N-1))x = 0 \qquad (24)$$

$$\frac{\partial \mathcal{H}(.)}{\partial k(t)} = 0 \Rightarrow -\alpha k + \lambda(1 + (N-1)\gamma) = 0; \qquad (25)$$

$$-\frac{\partial \mathcal{H}(.)}{\partial A(t)} = \frac{\partial \lambda(t)}{\partial t} - \rho\lambda \Rightarrow \frac{\partial \lambda(t)}{\partial t} = (\rho + N\delta)\lambda - nx \qquad (26)$$

Moreover, $A(0) = A_0 > 0, \lim_{t\to\infty} \lambda(t)A(t) = 0$. >From (24) and (25) we derive:

$$x^*_{FC}(t) = \frac{A(t) - c}{2B + 2D(N-1)} \qquad (27)$$

$$k^*_{FC}(t) = \frac{\lambda(t)(1 + \gamma(N-1))}{\alpha} \qquad (28)$$

where the subscript FC stays for *full cartelization*. By differentiating (28) w.r.t. time, and by plugging (26), we derive:

$$\frac{\partial k^*_{FC}(t)}{\partial t} = (\rho + \delta)k(t) - \frac{N[1 + (N-1)\gamma][A(t) - c]}{2\alpha[B + D(N-1)]} \qquad (29)$$

The dynamic equation (29), along with the dynamic constraint, may be rewritten, in matrix term, as follows :

$$\begin{bmatrix} \dot{k} \\ \dot{A} \end{bmatrix} = \begin{bmatrix} (\rho + N\delta) & -\frac{N[1+(N-1)\gamma]}{2\alpha[B+D(N-1)]} \\ 1 + (N-1)\gamma & -\delta \end{bmatrix} \begin{bmatrix} k \\ A \end{bmatrix} + \begin{bmatrix} \frac{N[1+(N-1)\gamma]c}{2\alpha[B+D(N-1)]} \\ 0 \end{bmatrix} \qquad (30)$$

The qualitative solution of this dynamic system is similar to the solution of the oligopoly case. An economically meaningful steady state $\dot{k}=0, \dot{A}=0$, $A>0, k>0$, may exist. The necessary and sufficient condition is

$$2\alpha\delta(\rho+N\delta)(B+D(N-1)) < N[1+(n-1)\gamma]^2. \tag{31}$$

Under this condition, the analytical steady state solution values are:

$$\begin{aligned} A_{FC}^{SS} &= \frac{Nc\left(1+\gamma(N-1)\right)^2}{N\left(1+(N-1)\gamma\right)^2 - 2\alpha\delta(\rho+N\delta)(B+D(N-1))}; \\ k_{FC}^{SS} &= \frac{\delta Nc\left(1+\gamma(N-1)\right)}{N\left(1+\gamma(N-1)\right)^2 - 2\alpha\delta(\rho+N\delta)(B+D(N-1))}. \end{aligned} \tag{32}$$

In system (18), the trace of the Jacobian matrix is $\rho+(N-1)\delta>0$,while the determinant is unambiguously negative, as long as condition (31) holds. Consequently, the steady state is a saddle also in this case.

The qualitative effects of parameters on the steady state are similar to the case of oligopoly. The larger are B and D, the larger is the steady state market size, and the advertising efforts; the larger are the market sizes. The effect of the number of firms is ambiguous

5 Comparative assessment

Let us consider the steady state values of market sizes and advertising efforts, under oligopolistic competition, and under cartelization. By simple substitution it is immediate to compute the steady state levels of production, under the two regimes.

$$\begin{aligned} x_{OLIG}^{SS} &= \frac{c(N-1)\gamma}{[2B+D(N-1)][1+(N-1)\gamma-\alpha\delta(\rho+\delta)(2B+D(N-1)]}; \\ x_{FC}^{SS} &= \frac{c[N\left(1+\gamma(N-1)\right)^2-1]}{2\left[B+D(N-1)\right]\left[N\left(1+(N-1)\gamma\right)^2-2\alpha\delta(\rho+N\delta)(B+D(N-1))\right]}. \end{aligned} \tag{33}$$

It is immediate to note that the comparison of the magnitudes of all variables at hand is not simple: it is not clear whether the steady state market size is larger under Cournot competition or full cartelization. This is due to the opponent forces that play a role in this problem. From the one side, cartelization should lead to a smaller market size, taking advantage from collusion in the market size; on the opposite side, the internalization of the external benefits of advertising should lead to a larger efforts in advertising and to a larger market size for any firm. A consequent ambiguity emerges as concerns the steady state levels of production. Not surprisingly, the social welfare in steady state is not necessarily smaller under the cartel than under competition. Indeed, let us define social welfare as the sum of consumers' surplus and profit, in any markets; formally:

$$SW = N\left[(A-c)x - (\frac{B}{2} + D(N-1))x^2 - \frac{\alpha}{2}k^2\right] \tag{34}$$

Substituting the appropriate values, the social welfare under oligopolistic competition and under cartelization turns out to be:

$$\begin{aligned} SW_{OLIG} &= N \cdot \left[\frac{3B(A^{SS}_{OLIG}-c)^2}{2[2B+D(N-1)]^2} - \frac{\alpha}{2}\frac{\delta^2(A^{SS}_{OLIG})^2}{[1+(N-1)\gamma]^2}\right]; \qquad (35) \\ SW_{FC} &= N \cdot \left[\frac{(A^{SS}_{FC}-c)^2}{4[B+D(N-1)]^2}\left(\frac{3}{2}B + D(N-1)\right) - \frac{\alpha}{2}\frac{\delta^2(A^{SS}_{FC})^2}{[1+(N-1)\gamma]^2}\right]. \end{aligned}$$

It is clear that these values depend on seven parameters: $B, D, c, \gamma, \alpha, \rho, N$, and the social welfare under the full cartelization may be larger than under the competitive oligopoly. To give just one numerical example, set $B = 1$, $D = 0.5$, $c = 1$, $\gamma = 0.3$, $\alpha = 1$, $\rho = 0.2$, and consider two alternative values for N. When $N = 3$,it turns out to be $SW_{OLIG} > SW_{FC}$; when $N = 5$, the opposite inequality holds. In this case, for given values of other parameters, the larger is the number of firms, the larger the gain from cartelization, since firms (and society) may benefit to a larger extent from the internalization of the external effects of advertisement.

The policy implication is straightforward: forbidding collusive agreements among firms when the advertising activity has external effects could result to be socially inefficient. This result resembles the analogous findings concerning different investment activities, e.g. investment in R&D or investment in product differentiation, to be operated by firms in a pre-market phase. Also in that cases, the full cartelisation (and *a fortiori* a Research Joint Venture) may well produce allocations, whose social surplus is larger than the social surplus associated with the allocations deriving from oligopolistic competition in all the phases of the economic game among firms (see d'Aspremont and Jacquemin, 1988; Kamien, Muller and Zang, 1992; Suzumura, 1992, *inter alia*, or - in the same vein - Norman and Thisse, 1996).

6 Conclusions

In the foregoing analysis, we have shown that the consideration of the positive external effects from advertising of a firm upon other firms' market size has not a trivial impact on the conclusion about the social desirability of cartel behaviour of firms. In particular, we have taken a differential game approach to study the dynamic optimal plans of firms that produce differentiated goods and make advertising efforts. The advertising efforts of any firm have long-lasting effects on her own market sizes, as well as on the market size af all the rival firms. Under this respect, advertising is, at least to some extent, a public good. We have focused on the steady state solutions of the dynamic system under (usual) symmetry conditions, and we have shown that, when a meaningfull steady state exists, it is a saddle point. Thanks to the positive external effects of advertisement, it is no longer true that the cartel behaviour, i.e. the behaviour of firms aiming at the maximum joint profit, is necessarily detrimental to the steady state social welfare. This result is common to the conclusion of other investigations, where the relevance of pre-market behaviour, like investment in product or process innovation, can lead to the conclusion that the cartelization is socially desirable.

A different interest in the present model could be motivated, in our opinion, by the fact that its formalization is of "linear state" type, and hence the open-loop Nash equilibrium is time-consistent, since it is a degenerate closed-loop equilibrium. Consider, under this respect, that, the available literature on differential games devotes consideable efforts to propose models that are both time-consistent (as under the closed-loop information structure) and easily solvable (like under the open-loop equilibrium information structure).

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