A two-sector model of the business cycle: a preliminary analysis[¤]

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Abstract

In this paper a two-sector dynamic model of business ‡uctuations is presented. It is a disequilibrium dynamic model with two laws of evolution (dynamic laws) built into it: prices of commodities change according to the market disequilibrium of supply and demand, while quantities change according to the stock disequilibrium and the shifting of the degree of utilization of productive capacity away from its target value. Investment by …rms is modelled by a nonlinear accelerator.

Non linearity in the investment function makes the equilibria of the model unstable and causes growing disproportionalities between the two sectors; business ‡uctuations are the outcome of the switching

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of the system to a di¤erent regime that allows to reduce the existing disproportionality.

The di¤erent regimes into which the economy may be found are a situation of overheating and one of depression. A fundamental role in the switching of the economy is played by two crucial features of the capital good sector: its limited productive capacity and the time-lag required to increase it.

1 Introduction

In the following pages we present a two-sectors dynamic model with one sector producing the …xed capital good and the other one producing the consumption good. The model is framed in continuous time; we shall consider the case when adjustment of productive capacity takes time and is not instantaneous.

The aim of our model is to provide a mathematical formulation for the multisector theories of business ‡uctuations according to which the causes of periodical crises are to be searched in periods of overproduction, in the consumption goods sector especially; these are, in turn, due to the di¤erent lengths of the production processes in the di¤erent sectors of the economic system and to the disproportion among di¤erent sectors caused by these di¤erent lengths[1].

1.1 Di¤erent markets for di¤erent goods

Since the working of each single market depends on the characteristics of the traded good[2], the two sectors turn out to be asymmetric. Nowadays the typical end-products of the manufacturing industry are rather specialized and manufacturing and selling come in substance under the same control[3]; as a consequence "the selling department is able to set a selling price and make it e¤ective by holding stocks"[4]. Firms in the consumption goods sector, therefore, throw in inventories to …ll gaps between demand and supply and take their short run decisions to produce in order to approach stock equilibrium[5]. Following Hicks[6], we assume that …rms try to keep inventories near to a target value.

[1] Earlier contributions include Aftalion (1913),(1927), Fanno (1931), Tugan Baranowskj (1913) and Hicks (1974) among the others. To our knowledge the only attempt to formalize such class of theories has been the simple, but interesting model of Delli Gatti Gallegati (1991).

[2] For a thorough discussion of this issue we refer to Hicks (1989), chapters 1 to 4.

[3] See Hicks (1989), chapter 3; for a discussion of the strategic rôle of pricing in highly concentrated industries we refer also to Judd and Petersen (1986).

[4] Hicks (1989), p. 24.

[5] For the concept of stock equilibrium we refer the reader to Hicks (1965), ch. VIII, and Hicks (1974), ch. 1.

[6] See (1974), ch. 1.

On the other hand we assume that …rms in the capital goods sector produce only to order and hence take their short run decision to produce knowing demand. They do not hold inventories, therefore, and change prices to o¤set possible discrepancies between demand and their (maximum) productive capacity[7].

1.2 Investment and speci…city of …xed capital

As far as investment is concerned, we assume that pro…ts are normally reinvested in the sector where they arose, except when pro…ts in one sector are so low that the remuneration of capital falls below its normal level[8]. Moreover, when a disequilibrium in the consumption goods market is signalled by an anomalous capacity utilization rate the investment is corrected by an accelerator[9], while the capital goods sector may ration its supply when its productive capacity does not match the demand. In this sense the consumption goods sector may be formally treated, at least to a large extent, as a sector of a model of complete disequilibrium[10].

It is clear that the capital goods sector plays a key rôle when either rationing occurs or the rate of remuneration of capital is di¤erent for the two sectors, whereas its development follows the development of the consumption goods sector along the equilibrium growth path[11].

Even if only one capital goods sector is considered in the model, we assume that allocations are irreversible; once the allocation has been made, the capital goods become speci…c and can not be traded. This means, in particular, that capital goods can not be moved from one sector to the other[12].

2 Assumptions of the model

In our model …rms within each sector are alike and this will allow us to make use of the representative …rm's concept.

Since one of our aims is to analyse the emergence of bottlenecks due to the length of the capital accumulation process, we are led to assume a …xed

[7] These assumptions presuppose that the rate of utilization of productive capacity is normally below its maximum value in both sectors. We refer to Duménil and Lévy (1991) for a discussion on this point.

[8] The concept of normal remuneration is made precise in formulae (9) and (14).

[9] The accelerator in a multisector model of business ‡uctuation has been introduced by Aftalion (1913), Fanno (1931) and Hicks (1974).

[10] See, for instance, Duménil and Lévy (1991).

[11] The evolution of the system is thus fully endogenous. Our model allows for the description of exogenous evolution too, in particular of a technical progress which changes the coe¢cients of the input matrix, we hope to develop this issue in a future paper.

[12] The assumption of irreversible allocations has a long tradition in macroeconomic modelling; we refer for instance to Solow et al (1966) and Bliss (1968).

coe¢cient rule: any concrete unit of capital has a given output capacity and requires a given complement of labour, i.e. there is no substitutability between labour and capital.

On the other hand labour is perfectly elastic and the real wage is constant througout the analysis.

Thus we assume that the total output Y_i of the i-th sector is given by a Leontief type production function with complementary inputs:

$$Y_i = \min \left\{ b_i K_i; \frac{b_i}{l_i} L_i \right\}; \tag{1}$$

with L_i denoting the input of labour in the sector, b_i the productivity of capital and b_i/l_i the productivity of labour, $i = 1; 2$.

The stock of …xed capital of the i-th sector, K_i, depreciates at a constant rate δ (the same in both sectors[13]); if I_i denotes the gross investment in the i-th sector then the evolution of the stock of …xed capital engaged in production, $\dot{K}_i$ is given by:

$$\dot{K}_i = I_i - \delta K_i: \tag{2}$$

Let w be the real wage and p_1 the price of the consumption good; the income of the workers employed in the i-th sector, W_i, is therefore:

$$W_i = w p_1 L_i: \tag{3}$$

We assume no saving out of wages; therefore the whole income (3) will be spent on the consumption goods market.

2.1 The consumption good sector

In the consumption good sector production can take place only if the condition

$$Y_1 p_1 - W_1 > 0 \tag{4}$$

holds; this may be rewritten in the following form:

$$b_1 - w l_1 > 0: \tag{5}$$

Condition (5) means that the real wage w is compatible with the productive technology employed in the …rst sector[14]. The evolution of inventories S_1 in the consumption good sector depends on the gap between total output Y_1 and demand for consumption goods:

$$\dot{S}_1 = Y_1 - D_1: \tag{6}$$

[13] The case of a sector dependent depreciation rate can be treated with minor changes.

[14] Condition (4) has to be regarded as a surplus condition and not as a pro…t condition.

Net pro…ts in the consumption good sector, Π_1, are de…nded as the di¤erence between the value of total output, Y_1p_1, and the sum of the wages paid to the labour-force and the depreciation of …xed capital, $W_1 + \delta K_1 p_2$:

$$\Pi_1 = Y_1p_1 - (W_1 + \delta K_1 p_2); \tag{7}$$

where p_2 is the (unit) price of the …xed capital good. Pro…ts are divided into two parts, one for consumption, Π_1^c, and the remaining one for accumulation, Π_1^a; therefore

$$\Pi_1 = \Pi_1^c + \Pi_1^a. \tag{8}$$

Concerning the pro…ts allocated to consumption we assume that this part is proportional to the value of the stock of productive capital of the sector; therefore:

$$\Pi_1^c = dK_1p_2; \tag{9}$$

where $d > 0$ is the dividend.

Remark 1 We stress the point that accumulated pro…ts, Π_1^a, need not be positive. In this case …rms in the sector are not only unable to make (positive) net investment but also to replace depreciated …xed capital. Moreover, the surplus condition (5) does not guarantees enough output to pay the scheduled remuneration (9) of the …xed capital.

When this happens this remuneration in paid resorting to the inventories, whenever they are su¢cient[15].

Let W_2 be the total wages and Π_2^c the distributed pro…ts of the capital good sector; the demand for consumption goods, D_1, is given by:

$$D_1 = \frac{\Pi_1^c + \Pi_2^c + W_1 + W_2}{p_1}. \tag{10}$$

Total capital engaged in the production of consumption goods, C_1, is the sum of the value of …xed capital and the value of inventories:

$$C_1 = K_1p_2 + S_1p_1. \tag{11}$$

Furthermore let $u = Y_1/(b_1K_1)$ be the capacity utilization rate, $s = S_1/(b_1K_1)$ the inventories ratio and $\kappa = C_1/K_1$ the unit capital engaged in the production of the consumption goods.

[15] The assumption that …xed productive capital is speci…c to the sector prevents capitalists to move it from a sector to the other even when the former looses its pro…tability. In this event the surplus condition (5) guarantees that going on producing diminishes losses.

The model covers the case of capitalists that are entrepreneurs in one of the two sectors or, more generally, capitalists and entrepreneurs in each sector belong to the same class; in this case the gross pro…t is distributed as remuneration and accumulated pro…t.

2.1.1 Output and price decisions in the consumption goods sector

Production and price decisions of firms in the consumption goods sector are determined on the one side by the need to avoid that sudden jumps of the demand may cause unacceptable changes of prices of the goods[16] and on the other hand to avoid exhaustion as well as excessive accumulation of stocks, which are used as an alternative to price changes in order to offset disequilibria in the market for consumption goods. We assume therefore that, under normal conditions, firms have a margin of unused productive capacity , i.e. $0 < \bar{u} < 1$, with $\bar{u}$ representing the normal degree of capacity utilisation[17] and u the actual one.

In the same manner we assume that firms have a target value for the inventory level, $\bar{s}$. When the actual level of inventories, s, is below its normal value, $\bar{s}$, firms realize that there is disequilibrium on the market for consumption goods and react either by increasing output or by increasing prices or both; a perfectly symmetric situation arises when the actual level of inventories s is above its normal value $\bar{s}$. Firms choose, therefore, their behaviour according to the disequilibrium signals that they observe, i.e. market disequilibrium ($\bar{s} \neq s$) and disequilibrium in production ($\bar{u} \neq u$).

As far as the decision to produce of firms in the consumption goods sector is concerned we assume that:

$$\dot{u} = F(u; s); \qquad (12)$$

where $F \in C^1$ is a function[18] such that $F(\bar{u}; \bar{s}) = 0$, $\partial F/\partial u < 0$, $\partial F/\partial s < 0$ and with $\lim_{u \to 0^+} F(u; s) = +\infty$ and $\lim_{u \to 1^-} F(u; s) = -\infty$ for any $s > 0$. Equation (12) says that firms in the consumption goods sector increase the utilization of their productive capacity if inventories and capacity are below their target (or normal) values and decrease it in the opposite case.

Concerning prices we suppose that p_1 is adjusted by the firms according to the level of inventories, i.e. to the disequilibrium observed on the market;

$$\dot{p}_1 = p_1 g_1(s); \qquad (13)$$

where $g_1 \in C^1$ is a function such that $g_1(\bar{s}) = 0$, $dg_1/ds = g_1' < 0$ and with $\lim_{s \to +\infty} g_1(s) = -\infty$ and $\lim_{s \to 0^+} s g_1(s) = +\infty$[19].

In such a way we are able to study the problem of the dynamic behaviour of the economy far from the equilibrium position or, alternatively, when the normal or long-term equilibrium of the economy is (locally) unstable.

[16] Since the price is a strategic characteristic of the good (see subsection 1.1 for further details) too sudden and wide changes may result unacceptable.

[17] Obviuosly $\bar{u} \gg 1$, a possible lower bound may be for instance 0.8.

[18] This hypotheses on the behaviour of firms in the consumption good sector exclude that the maximum productive capacity $b_1 K_1$ can be ever reached; in this sector, therefore, we always have $Y_1 = (b_1/l_1) L_1$ (see (1)).

[19] This condition means that firms decide strong price increase to prevent exhaustion of resources.

2.2 The capital goods sector

The relations connecting total output, Y_2, employment, L_2, total wages, W_2, and profits, Π_2, of the capital goods sector are analogous to the corresponding ones in the consumption goods sector:

$$\Pi_2 = Y_2 p_2 - (W_2 + \delta K_2 p_2) \tag{14}$$

$$\Pi_2^c = d K_2 p_2 \tag{15}$$

with Π_2^c denoting distributed profits in the capital goods sector and $\Pi_2^a = \Pi_2 - \Pi_2^c$ profits saved and accumulated.

Since it is impossible to use inventories in order to pay dividends, distributed profits in the capital goods sector can not exceed the net profits of the sector:

$$\Pi_2^c = \min\{d K_2 p_2, Y_2 p_2 - W_2\}. \tag{16}$$

We remark explicitly that the assumption of an equal dividend per unit of fixed capital d in both sectors should not lead to the erroneous conclusion that the rate of profit is the same in both sectors as well[20].

In this case too we assume that the production of capital goods take place only if the surplus condition

$$Y_2 p_2 - W_2 > 0 \tag{17}$$

holds (see(4));such a condition entails an upper bound for the relative price $p = p_2/p_1$, i.e.

$$p \leq \frac{b_2}{w l_2}. \tag{18}$$

We can interpret this condition as an upper bound for the nominal wage rate wp_1 that guarantees its compatibility with the technical production conditions and the price of the capital good.

2.2.1 Production and price decisions in the capital goods sector

Firms in the capital goods sector have no reason to hold inventories, as we are assuming that they produce to order; therefore $S_2 = 0$ and the supply of capital goods always coincides with total output Y_2. Production decisions are formalized in this way:

$$Y_2 = \begin{cases} D_2 & \text{if} \quad D_2 \leq b_2 k_2 \\ b_2 K_2 & \text{if} \quad D_2 > b_2 K_2 \end{cases}; \tag{19}$$

where:

$$D_2 = I_1^a + I_2 \tag{20}$$

[20] The case of two different dividends in the two sectors can be treated along the same lines, with only minor changes.

is the demand for the capital good and with I_2 that denotes gross investment in the capital goods sector. Production of the capital goods is equal to demand when demand (20) does not exceed the productive capacity of the sector. When demand of the capital goods exceed productive capacity we are in a situation of rationing; in this case we assume that it is the demand of capital goods coming from the consumption sector that remain partly unfulfilled and therefore ex-post investment in the consumption goods sector is given by $I_1 = b_2 K_2 - I_2$.

Thus the different nature of the produced goods and the resulting different structure of their market[21] determine quite different criteria for the decision maker. This is true not only for output decisions, but also for price decisions. In particular there is no point for a firm in the capital goods sector to have a margin of unused productive capacity, because it knows demand and has no problem with stock equilibrium.

In view of this we assume that firms in the capital goods sector set the price p_2 according to a Walrasian adjustment mechanism:

$$\dot{p}_2 = p_2 g_2 \left(\frac{D_2 - Y_2}{Y_2} \right); \tag{21}$$

where $g_2 \in C^1$ is a function that satisfies $g_2(0) = 0$, $g_2' > 0$ and $\lim_{s \to +\infty} g_2(s) = +\infty$.

Remark 2 We explicitly notice that the normalization in equation (21) is the same used in the definition of u and s, since $Y_2 = b_2 K_2$ whenever $D_2 > Y_2$.

2.3 Investment decisions

Investment decisions of the firms play a fundamental rôle in the dynamic evolution of the economy. Two aspects must be taken into account:

1. the financing of investment, i.e. the allocation of capital (a sum of purchasing power) to the firms by the capitalists; this finance is the sum of the capital set free by the production process, δK_i $(i = 1; 2)$, and of the profits accumulated;

2. the actual use by the firms of the sum so determined.

Point 1 requires the specification of a mechanism according to which capitalists allocate their purchasing power to firms. In our model we assume that capitalists do not know the rate of profit of the different sectors; they only know the rate at which investments are remunerated[22]. We further

[21] We refer the reader to Hicks (1989).

[22] This assumption must be modified if we are assuming that some entrepreneurs are also capitalists (see remark (1) and the relative note).

assume that the rate at which investment is remunerated is the same for both sectors, at least as long as resources of the sector are enough. Therefore profits not spent on consumption are generally reinvested in the sector where they originated[23]. However, when the profitability of a sector declines, profits may become insufficient to ensure the target remuneration; in this case investment moves towards the other sector. The level of investment that firms may realize by using the purchasing power obtained from the capitalists is, therefore:

$$I_i = \max\left\{\delta K_i + \frac{\Pi_i^a}{\pi_i}; 0\right\}; \qquad (22)$$

with $\pi_1 = \pi = p_2 + p_1 b_1 s$ representing the (money) capital required to increase productive capacity by one unit in the consumption goods sector and $\pi_2 = p_2$ representing the money capital required by an additional unit of productive capacity in the capital goods sector[24].

It must be pointed out that the structure of the economy in terms of the ratio between productive capacity of the consumption goods sector and capital goods sector is bound to change when the price system does not ensure a uniform rate of profit across sectors, even if profits are normally reinvested within the sector[25].

As far as investment in the capital goods sector is concerned, we assume that the net investment equals accumulated profit:

$$I_2 = \max\left\{\delta K_2 + \frac{\Pi_2^a}{\pi_2}; 0\right\}: \qquad (23)$$

The level of investment in the consumption goods sector, as determined by (22), is corrected by the firms by taking into account market's disequilibrium. The desired gross investment in the first sector, I_1^a, equals, therefore, accumulated profits under normal conditions, is accelerated[26] when utilization of capacity exceeds its normal value, $u > \bar{u}$, and is decelerated in the

In this case they obviously know the rate of profit of their sector, but they can observe only the rate of remuneration in the other sector. If the latter does not exceed the former these capitalists will reinvest their profit in their sector.

[23] It is well known that internal finance has been the dominant source of finance historically as well as during the post-World war II era. Cfr. Judd and Petersen (1986).

[24] The reader should remember that the capital goods sector doesn't accumulate inventories; cfr. equation (19).

[25] We may term this case the self financing case.

[26] On can ask why we have not introduced an accelerator like (24) in the capital goods sector too. The reason is that, owing to the assumption of a two-sectors economy the accelerator in the first sector necessarily acts as a "decelerator" in the second one; besides the increase of the formal difficulty, the introduction of a second accelerator would not change the instability result of the normal equilibrium points, but it would rather increase such instability.

opposite situation, $u < \bar{u}$. In formal terms:

$$I_1^a = \max\left\{\left(\delta K_1 + \frac{\Pi_1^a}{\kappa}\right) h(u)\,;0\right\}; \tag{24}$$

where $h \in C^1$ is a function such that:

$$h(u) \begin{cases} > 1 & \text{if} \quad u > \bar{u} \\ < 1 & \text{if} \quad u < \bar{u} \end{cases} :$$

Equation (24) defines ex-ante gross investment in the consumption goods sector since the supply of capital goods may be insufficient to cope with demand[27].

2.4 Relative variables and compatibility conditions

The general price level and the fixed capital stocks in both sectors depend only on the initial conditions; it is therefore possible to express all relations introduced in sections 2.1.1 and 2.2.1 by means of the rate of capacity utilization u, the ratio of inventories s, the relative price p and the ratio of fixed capital stocks[28] $z = K_2/K_1$. We can normalize prices in such a way that we have $p = p_1$ and $p_2 = 1$. The model we consider in this paper is therefore well suited to deal with the aspect of the stability analysis related to proportions, following Duménil and Lévy's definition[29].

We want at this point give formal conditions which determine the transition from one regime to another: for instance when rationing of capital goods occurs or when the profitability of one of the two sectors is so low that investment in that sector is no longer convenient. Let us consider the consumption goods sector first; we set:

$$\beta = \frac{1}{\kappa}\left(1 - \frac{wl_1}{b_1}\right) p \tag{25}$$

$$\alpha = \delta - \frac{d + \delta}{\kappa} \tag{26}$$

[27] $\delta K_1 + \Pi_1^a/\kappa$ is the normal investment, i.e. investment which takes place under normal conditions. For a discussion of this concept we refer to Hicks (1974).

[28] z represents the distribution of productive capacity among the two sectors.

[29] The problem of stability of the capitalist economy must be dealt with, according to Duménil and Lévy (see, for example, Duménil and Lévy (1993)) , from two points ov view, stability in proportions and stability in dimension. The problem of stability in proportions refers to the relative variables that characterizes the equilibrium of the economy, i.e. relative prices of commodities and relative quantities (or the structure of sectoral productive capacity); this type of analysis generalizes the classical (and marxian) analysis of the formation of production prices (and the corresponding production levels) as a result of the working of competition within the economy. The problem of stability in dimensions deals with business fluctuations and crises, i.e. with the behaviour of the economy when far from equilibrium. A necessary (albeit not sufficient) condition for analyzing the aspect of stability in dimensions is the explicit consideration of monetary relations, above all the problems related to the creation of money.

and by using (3), (7), (25), (26) in equation (24) and taking into account total profits, $\Pi = \Pi_1 + \Pi_2$, and (9) we can write:

$$I_1^* = \max\{(\alpha + b_1\beta u)\, h(u)\, K_1; 0\}. \tag{27}$$

Finally, if we denote the growth rate of the fixed capital stock of the consumption goods sector by ρ then from (2) and (27) we can write:

$$\dot{K}_1 = \rho K_1,$$

with $\rho = (\alpha + b_1\beta u)\, h(u) - \delta$.

As far as the capital goods sector is concerned we may write, when productive capacity is large enough to avoid rationing and dividends can be regularly distributed:

$$I_2 = \max\left\{\left(1 - \frac{wl_2}{b_2}p\right) Y_2 - dK_2; 0\right\}. \tag{28}$$

On the other hand, from the condition of nonnegativity of output, $Y_2 \geq 0$, and from (19),(28) we can write:

$$Y_2 = \max\left\{\frac{b_2}{wl_2 p}(I_1 - dK_2); 0\right\}. \tag{29}$$

Since we have assumed that rationing does not occur, ex-ante and ex-post investment are the same, $I_1 = I_1^*$; therefore by substituting (27) in (29) we have:

$$Y_2 = \frac{b_2}{wl_2 p}[(\alpha + b_1\beta u)\, h(u) - dz]\, K_1 \tag{30}$$

and, recalling (3):

$$W_2 = [(\alpha + b_1\beta u)\, h(u) - dz]\, K_1. \tag{31}$$

We have already remarked that (30) and (31) hold when $I_1 > dK_2$; this condition is satisfied if and only if:

$$z < \frac{\alpha + b_1\beta u}{d} h(u). \tag{32}$$

The preceding inequality can be rewritten using the definition of relative price. In order to have $Y_2 > 0$ it is obviously required that $I_1 > 0$ (see equation (29)); and this is true if and only if:

$$p > \frac{d}{(b_1 - wl_1)\, u + \delta b_1 s}. \tag{33}$$

Firms in the capital goods sector are able to distribute profits as planned (9) if and only if $I_2 > 0$; this is true if the condition:

$$z < \left(1 - \frac{wl_2}{b_2}p\right)\frac{\alpha + b_1\beta u}{d} h(u) \tag{34}$$

holds. Obviously (34) $\Rightarrow$ (32) $\Rightarrow$ (33).

We conclude this section by observing that rationing on the capital goods market does not occur if and only if:

$$z \geq \frac{\alpha + b_1 \beta u}{d + w l_2 p} h(u). \tag{35}$$

Hence, in order for both conditions (34) and (35) to be met together, we have to require that:

$$p \leq \frac{b_2 - d}{w l_2}. \tag{36}$$

The preceding inequality can be interpreted as an upper bound on the nominal wage pw, just like condition (18); it should be obvious that (36) implies (18).

3 Analysis of the dynamical system

3.1 The dynamical system

In section 2.1.1 and 2.2.1 we have introduced equations for the evolution of the capacity utilization rate u and the evolution of the prices of commodities, p_1 and p_2. In this section we derive an equation for the evolution of the ratio of inventories s in terms of u, p_1 and p_2, when there is no rationing in the capital goods market and both sectors are profitable enough. By combining (6) and (2) and using (10),(9),(16),(3),(9),(31) and (27) we can write:

$$\dot{s} = \frac{\kappa}{b_1 p} (\alpha + b_1 \beta u)(1 - h(u)). \tag{37}$$

Furthermore from (2) it is easy to obtain an equation for the evolution of the relative stock of fixed capital z:

$$\dot{z} = -\left((\alpha + b_1 \beta u) h(u) + \frac{d b_2}{w l_2 p} \right) z + \frac{b_2 - w l_2 p}{w l_2 p} (\alpha + b_1 \beta u) h(u).$$

Finally from (13) and (21) we obtain an equation for the evolution of the relative price. We have therefore obtained a fourth dimensional nonlinear dynamical system in continuous time in the state variables u, s, p and z:

$$\begin{cases} \dot{u} = F(u; s) \\ \dot{s} = G(u; s; p) \\ \dot{p} = g_1(s) p \\ \dot{z} = H(u; s; p; z) \end{cases} \tag{38}$$

where, in order to simplify notation, we have defined[30] the following functions:

$$G(u; s; p) = \frac{\kappa}{b_1 p} (\alpha + b_1 \beta u)(1 - h(u))$$

[30] Function G depends on s also via α, β and κ.

and:

$$H(u;s;p;z) = \left(-(\alpha + b_1\beta u)\,h(u) + \frac{db_2}{wl_2 p}\right) z + \frac{b_2 - wl_2 p}{wl_2 p}(\alpha + b_1\beta u)\,h(u)\,.$$

3.2 Equilibria

We can state the following:

Theorem 1 The dynamical system (38) has an infinity of equilibrium points; all of them are of the form $(\bar u;\bar s;\bar p;\bar z)$ where $\bar u$ and $\bar s$ are the normal values of the capacity utilization rate and of the ratio of inventories respectively and the equilibrium values for the relative price $\bar p$ and for the relative stock of fixed capital $\bar z$ satisfy (34), (35) and (2.34) and:

$$\bar z = (b_2 - wl_2\bar p)\,\frac{\left(\alpha + b_1\bar\beta\right)\bar u}{\left[\left(\alpha + b_1\bar\beta\right)\bar u - wl_2\bar p + db_2\right]}; \qquad (39)$$

with α and $\bar\beta$ evaluated at $\bar p$ and $\bar\mu = 1 + b_1\bar s\bar p$. All these equilibria are unstable[31].

We give a sketch of the proof in the mathematical appendix.

3.3 Local dynamics

In this section we analyze the evolution of the system near an equilibrium point (39) by using the adiabatic principle[32], since the existence of a zero eigenvalue[33] makes the use of the linear approximation to perform the analysis not possible. For this purpose it is necessary to diagonalize the jacobian matrix of the dynamical system, J; we introduce, therefore, the matrix $B = (v_1; v_2; v_3 v_4)$, with v_i $(i = 1;\dots;4)$ representing the eigenvectors of the jacobian matrix. Therefore $B^{-1}JB = \mathrm{diag}(\lambda_i)$ $\quad (i = 1;\dots;4)$. Let us introduce the new variables[34] x_i $\quad (i = 1;\dots;4)$.

In the new coordinate system the axes x_1 and x_4 are the two eigenspaces corresponding to the negative eigenvalues; hence by the adiabatic principle we may approximate the solution of the dynamical system by setting $\underline{x}_1 = \underline{x}_4 = 0$. Obvious simplifications allow us to write:

$$0 = F(u;s) + \frac{\partial F(\bar u;\bar s)}{\partial s}\frac{G(u;s;p)}{\lambda_1} \qquad (40)$$

[31] It is clear that these equilibria are ray equilibria, i.e. equilibria in proportions, in the sense discussed by Boggio (1993).

[32] We refer the reader to Haken (1977) and Zhang (1991) for an exposition of this method of analysis.

[33] A zero eigenvalue makes the equilibria not hyperbolic; it is not possible therefore to use the Hartman-Grobman theorem.

[34] These variables are analogous to the principal coordinates of Goodwin (1982); we refer the reader to the mathematical appendix for the explicit expression of the transformation.

$$\dot{z} = -\frac{\partial H(\mu)}{\partial p}\frac{1}{H_1(\mu)}\frac{g(s)p}{H_1(\eta)} + \frac{H_2(\eta)}{H_1(\eta)}; \qquad (41)$$

where the following notation has been used:

$$H_1(\eta) = (\alpha + b_1\beta u)\,h(u) + \frac{db_2}{wl_2p}$$

$$H_2(\eta) = \frac{b_2 - wl_2p}{wl_2p}(\alpha + b_1\beta u)\,h(u);$$

with $H(\eta) = H_1(\eta)\,z + H_2(\eta)$ and where $\eta = (u; s; p; z)$.

Clearly equation (40) describes the evolution of the relative price p when the evolution of the rate of capacity utilization u and of the ratio of inventories s are known; equation (41) yields the evolution of the relative fixed capital stock.

It is worth noticing that, for any given initial condition for the relative price p the projection on the $(u; s)$ plane of the characteristic curves of the dynamical system (38) is qualitatively independent of the initial condition for the relative fixed capital stock.

3.4 Depression and overheating

The preceding analysis suggests that the system can evolve according to two different regimes. The first regime is characterized by a falling ratio of inventories and an increasing rate of capacity utilization; if the bounds (34), (35) and (36) did not hold there would be a $t = \bar{t}$ such that:

$$\begin{cases} \lim_{t\to\bar{t}} p(t) = +\infty \\ \lim_{t\to\bar{t}} s(t) = 0 \\ \lim_{t\to\bar{t}} u(t) = \bar{u} \end{cases}; \qquad (42)$$

where $\underline{u} < \bar{u} < 1$. The assumption that $\lim_{s\to 0^+} sg_1(s) = +\infty$ together with:

$$\lim_{t\to\bar{t}} G(u; s; p) = \left(1 - \frac{wl_1}{b_1}\right)(1 - h(\bar{u})) \in R_+$$

yields:

$$\lim_{t\to\bar{t}} s(t)\,p(t) = +\infty. \qquad (43)$$

We call this type of evolution overheating regime.

The second type of asymptotic behaviour of the solutions of the dynamical system (38) takes place when the ratio of inventories is ever rising while the capacity utilization rate is falling; in this case it is easy to show that there exists a time $\hat{t}$ such that:

$$\begin{cases} \lim_{t \to \hat{t}^-} p(t) = 0 \\ \lim_{t \to \hat{t}^-} s(t) = +\infty \\ \lim_{t \to \hat{t}^-} u(t) = \hat{u} \end{cases}; \tag{44}$$

where $0 < \hat{u} < \bar{u}$. In this case too the condition $\lim_{s \to +\infty} sg_1(s) = -1$ yields:

$$\lim_{t \to \hat{t}^-} s(t)\,p(t) = 0. \tag{45}$$

We call this alternative behaviour the depression regime.

4 The depression regime

4.1 Effects of the depression regime

We start by analysing the depression regime. In this section we prove that, as long as the depression lasts and worsens, an imbalance between the two sectors arises and the production of capital goods becomes less and less profitable. The proof relies on showing that condition (34) no longer holds and this does mean that the contraction of investment in the consumption goods sector causes an overcapitalisation of the capital goods sector and its increasing loss of profitability.

Theorem 2 If the initial conditions lead to a depression regime (44),(45) then there exists $t = \hat{t}_2$ such that[35]:

$$I_2(t) > 0 \qquad t < \hat{t}_2 \tag{46}$$

and

$$I_2\left(\hat{t}_2\right) = 0. \tag{47}$$

Proof. Conditions (44) and (45) yield:

$$\begin{cases} \lim_{t \to \hat{t}^-} \kappa(t) = 1 \\ \lim_{t \to \hat{t}^-} \alpha(t) = -d \\ \lim_{t \to \hat{t}^-} \beta(t) = 0 \end{cases}$$

Moreover if $I_2(t) > 0$ did hold for every $t > 0$ then condition (34) would hold and hence both $I_1(t) > 0$ and $z(t) > 0$ would hold for every $t > 0$,

[35] This result is in accordance with the widely recognized evidence that slumps hit the capital goods sector more rapidly and deeply than the consumption goods sector.

since (34) $\Rightarrow$ (33). In this case the left hand side of (34) would be positive for every $t > 0$.

On the other hand $\lim_{t \to \hat{t}^-} \alpha(t) + b_1\beta(t) = -d < 0$ and thus the right hand side of (34) would become zero at least in a point t_a; this would mean that $I_2(t_a) = 0$, again by (34). Whence the contradiction. ■

Remark It is evident that $\hat{t}_2$ is the time when investment in the capital goods sector is no more profitable; the fall of the rate of capacity utilization has caused a contraction of investment in the consumption goods sector that results in an increasing excess of the productive capacity of the other sector, that eventually can not remunerate the fixed capital any longer.

4.2 The system during the depression

To understand the evolution of the system in the depression regime, we analyse the changes that the dynamical system (38) undergoes when $I_2(t) = 0$. Since (33) holds also in a right neighbourhood $\hat{t}_2$, from (19), (29), and (27) we can determine the total output of capital goods:

$$Y_2 = (\alpha + b_1\beta u)\, h(u)\, K_1.$$

>From (16) we obtain that the distributed profits of the capital goods sector in a time interval following the instant $\hat{t}_2$ are reduced to:

$$\begin{aligned} \Pi_2^c &= Y_2 - wpL_2 \\ &= \left(1 - \frac{wl_2}{b_2}p\right)(\alpha + b_1\beta u)\, h(u)\, K_1. \end{aligned} \quad (48)$$

Since the conditions under which the consumption goods sector operates, remain unchanged throughout the instant $\hat{t}_2$, we can easily determine the demand of consumption goods from (3), (7), (10), and (48). By the same reasoning of section 2.4 it is easy to realize that the evolution equation for the ratio of inventories holds also after $\hat{t}_2$.

On the other hand the evolution equation for the relative fixed capital stock is different, since from (48), (14), (15) it is easy to see that, after instant $\hat{t}_2$, we have $\Pi_2^a = -\delta K_2$; the fall in the profitability of the capital goods sector prevent firms to replace even the depreciated fixed capital. We have, therefore, the following differential system:

$$\left\{ \begin{aligned} \dot u &= F(u; s) \\ \dot s &= \frac{\cdot}{b_1 p}(\alpha + b_1\beta u)(1 - h(u)) \\ \dot p &= g_1(s)\, p \\ \dot z &= -(\alpha + b_1\beta u)\, h(u)\, z \end{aligned} \right. . \quad (49)$$

The considerations that led to (44), (45) and the proof of theorem 2 hold also with (49). The diminishing profitability in the capital goods sector do not change the evolution of the depression; the fall in the relative price p and the increase of the inventory ratio s continue also after $\hat{t}_2$ as long as the profitability of the consumption goods sector peters out. Thus eventually $I_1^a = 0$; this certainly happens for $t > \hat{t}_1$ (with $\hat{t}_1 > \hat{t}_2$) due to the proof of theorem 2 and (27).

After $\hat{t}_1$ investment in the consumption goods sector and production in the capital goods sector comes to an end together with the distribution of profits and the payment of wages in the second sector. On the contrary we have supposed that firms in the consumption goods sector go on distributing the planned dividends (9) by resorting to inventories.

To write down the new dynamical system we begin by noticing that if $I_1 = I_2 = 0$ then $\dot{z} = 0$.

Since total output of capital goods is zero, total wages and distributed profits of this sector are obviously zero, $W_2 = \Pi_2^c = 0$, while the analogous quantities in the consumption goods sector are still given by (7), (3), and (9) respectively; the differential systems is therefore the following:

$$\begin{cases} \dot{u} = \tilde{F}(u; s) \\ \dot{s} = \left(1 - \frac{wl_1}{b_1}\right) u - \frac{d}{p} + \delta s \\ \dot{p} = g_1(s)\, p \\ \dot{z} = 0 \end{cases} . \tag{50}$$

4.3 Existence of a turning point

>From the third equation of (50) it is easy to see that the decline of the relative price continues also in a time interval following $\hat{t}_1$, when the production of capital goods does not take place, since $s(t) > \bar{s}$ holds in a right neighbourhood of $\hat{t}_1$; however the second equation of (50) together with (45) proves that the ratio of inventories increases up to $\hat{t}_0$ (with $\hat{t}_0 > \hat{t}_1$), while after that time it begins to decline. The higher the level has gone up, the quicker its fall will be; it continues at least until $s(t) = \bar{s}$; at this moment the fall of the relative price p stops and a new increase starts.

The capacity utilization rate also stops falling a shorter time after $\hat{t}_0$ and starts again increasing while the inventories level declines. All these facts allow the system to approach one of the equilibrium points (39).

We feel it is worth noting that all these changes are completely endogenous to the system and we do not need to suppose either a change in agents expectations or in their behaviour in order for them to take place.

5 The overheating regime

5.1 Effects of the overheating

We move now to the analysis and description of the overheating regime. When the relative price is boosted by a demand which exceeds supply of consumption goods and reduces inventories, the consumption goods sector increases its profits and its productive capacity as well, since expectations are in favour not only of a further increase of demand but even of an increase exceeding the one in the capital goods sector. All this can be easily seen by considering that (42) and (43) yield:

$$\begin{cases} \lim_{t \to \bar{t}} \kappa(t) & = & +\infty \\ \lim_{t \to \bar{t}} \alpha(t) & = & \delta \\ \lim_{t \to \bar{t}} \beta(t) & = & +\infty \end{cases} . \tag{51}$$

Therefore $\dot{z}$ soon becomes negative.

In full analogy with the considerations leading to the proof of theorem 2 we can show that condition (36) can not hold for every $t > 0$ when (51) holds. There exists, therefore, an instant t_0 such that either rationing of capital goods occurs or the capital goods sector is no longer able to distribute the planned profits.

The first alternative occurs when the size of the consumption goods sector exceeds that of the capital goods sector so much that the latter has no longer enough production capacity in order to fully satisfy the demand for its products. In this case the capital goods sector turns out to be undercapitalised with respect to the consumption goods sector.

The second alternative occurs when the price of consumption goods reaches such a high level that the capital goods sector is no longer able to accumulate profits, moreover it is compelled to reduce the distributed part under the planned level. In this case the capital goods sector turns out to be overcapitalised with respect to the consumption goods sector.

5.2 The fall of profits in the capital goods sector

We start by examining the second alternative. The differential system is again (49); therefore the capacity utilization rate u, the ratio of inventories s and the relative price p do not change their evolutions after t_0, while the relative fixed capital stock z continues to fall. There exists, therefore, $t_1 > t_0$ such that (35) doe not hold for $t > t_1$ and rationing of capital goods starts. After t_1 ex-post investment I_1 in the consumption goods sector equals the maximum productive capacity of the capital goods sector, since in this sector no investment is taking place after t_0. Therefore $I_1 = b_2 K_2$ and $\dot{K}_1 = b_2 K_2 - \delta K_1$ while $I_2 = 0$ and $\dot{K}_2 = -\delta K_2$.

As far as total output, employment, total wages and distributed profits in the consumption goods sector are concerned, (3), (9) still hold, while in the capital goods sector we have now the following relations:

$$Y_2 = b_2 K_2 \quad (52)$$

$$L_2 = l_2 K_2 \quad (53)$$

$$W_2 = w l_2 p K_2 \quad (54)$$

$$\Pi_2^c = \left(1 - \frac{w l_2}{b_2}\right) b_2 K_2 . \quad (55)$$

The meaning of (52) and (53) is obvious; according to (52) the capital goods sector has reached its mamimum capacity utilization rate while (53) yields the employment required to maintain such a level of production. Total wages (54) are determined using (53) and (3) while distributed profits Π_2^c are given by (16).

In order to determine the change of inventories in the present case we begin by determining the demand for consumption goods using (9), (21), (10), (54), and (55). Then having in mind the evolution of the capital stock K_1 and the second equation in (49), we obtain the evolution equation for the inventories level.

The evolution equation for the relative price p is:

$$\frac{\dot p}{p} = \frac{\dot p_1}{p_1} - \frac{\dot p_2}{p_2} = g_1(s) - g_2\left(\frac{D_2 - Y_2}{Y_2}\right);$$

where obviously $g_2 > 0$, while the evolution of the relative fixed capital stock z can be easily deduced from the growth rate of the system.

After t_1 the dynamical system is therefore the following:

$$\begin{cases} \dot u = F(u; s) \\ \dot s = \left[\left(1 - \frac{wl_1}{b_1}\right) u - \frac{d}{b_1 p} - \frac{b_2}{b_1 p} z - s(b_2 z - \delta)\right] \\ \dot p = \left[g_1(s) - g_2\left(\frac{(\alpha + b_1 \beta u)}{b_2} h(u) \frac{1}{z} - 1\right)\right] p \\ \dot z = -b_2 z^2 \end{cases} . \quad (56)$$

The relative fixed capital stock z and the ratio of inventories s are declining, at least for a while, also after t_1, whereas the relative price p and the capacity utilization rate u are increasing; however, within a certain time, either the right hand side of the equation for the evolution of inventories changes its sign or the right hand side of the equation for the evolution of the relative price changes its sign before that of the preceding one. In the first case the ratio of inventories s enters a growth phase while p begins to fall, since the term $g_1(s)$ is decreasing whereas the term $g_2\left(\frac{(\alpha+b_1\beta u)}{b_2} h(u) z^{-1} - 1\right)$ is still increasing. In the second case the relative price p starts falling. In any case we denote by t_2 the instant when either alternative occurs.

The first alternative occurs when the reaction (21) of the price of the capital goods to the disequilibrium between supply and demand is not too strong; in this case, after instant t_2 the system is driven towards one of the equilibrium points (39). As in the case of depression, the regime described by the differential system (56) has two different possible ends. Either the activity level of the consumption goods sector diminishes until the investment I_1^a planned by firms falls under the maximum productive capacity of the second sector again, so that we come back to the differential system (49); or the relative price falls so quickly that the capital goods sector recovers its profitability and can start investment again. The evolution of the system in the last case is described in the following section.

The recovery of profitability and the beginning of a new phase of investment in the capital goods sector after t_2 is the most likely outcome also when the second alternative prevails and thus we are led to the differential system (61) below. The less likely outcome is that the evolution of inventories undergoes a change in direction so that the system is brought back to the situation described by (49).

5.3 Insufficient output of capital goods

When the capital goods sector still makes sufficient profit in order both to pay the planned dividends and to make (positive) gross investment, but its productive capacity does not succed in satisfying the demand, we must compute the shares in total output of capital goods of both sectors, having in mind that the demand of the second sector is satisfied first. From the rationing condition and (52) we have:

$$I_1 + I_2 = b_2 K_2 : \tag{57}$$

The accumulated profits of the capital goods sector can be determined using the expression giving the accumulated profits in the second sector:

$$\Pi_2^c = \left(1 - \frac{w l_2}{b_2} p\right) b_2 K_2 - (d + \delta) K_2 ; \tag{58}$$

while gross investment I_2 carried out in the capital goods sector is given by:

$$I_2 = \left(1 - \frac{w l_2}{b_2} p - \frac{d}{b_2}\right) b_2 K_2 : \tag{59}$$

Comparing (57) and (59) we can determine also the gross investment I_1 in the consumption goods sector:

$$I_1 = (w l_2 p + d) K_2 : \tag{60}$$

The dynamical system for $t > t_0$ is therefore the following[36]:

[36] We recall that in section 5.1 we used t_0 to denote the instant from which the production of capital goods is not enough to satisfy demand; we have also seen in section 5.2 that the dynamics described by (61) may be reached also in an instant following t_0.

$$
\begin{cases}
\dot{u} = F(u; s) \\
\dot{s} = G(u; s; p; z) \\
\dot{p} = H(u; p; z) \\
\dot{z} = K(u; s; p; z)
\end{cases}; \tag{61}
$$

where:

$$
\begin{aligned}
G(u; s; p; z) &= \left(1 - \frac{wl_1}{b_1}\right) u + \delta s - \frac{d}{b_1 p} - \frac{wl_2 p + d}{b_1 p} z \,(1 + b_1 ps) \\
H(u; p; z) &= \left[g_1(s) - g_2 \left(\frac{(\alpha + b_1 \beta u)\, h(u)}{b_2 z} - \frac{wl_2 p + d}{b_2} \right) \right] p \\
K(u; s; p; z) &= (b_2 - wl_2 p - d)\, z - (wl_2 p + d)\, z^2 :
\end{aligned}
$$

The differential system (61) describes the evolution of the system as long as the capital goods sector makes profits:

$$
p \leq \frac{b_2 - d}{wl_2} \tag{62}
$$

and the disequilibrium between demand and supply persists in the market for capital goods;

$$
z \leq \frac{(\alpha + b_1 \beta u)\, h(u)}{wl_2 + d}: \tag{63}
$$

The evolution described by the differential system (61) is more complex than those we have analysed in the former cases. Our first result is about the existence of a turning point in the rising trend of the relative price p.

Theorem 3 There exists $t_3 > t_0$ such that $\dot{p} > 0$ for $t_3 < t$ and $\dot{p} < 0$ for $t_3 > t$.

Proof. Assume, on the contrary, that $\dot{p} > 0$ for every $t > t_0$; then there exists an instant $t_a \in \left[t_0; \bar{t}\right]$ such that $\dot{z} < 0$ for $t > t_a$. Whence $\lim_{t \to \bar{t}} z(t) = 0$. Therefore we have that:

$$
\lim_{t \to \bar{t}} \frac{(\alpha + b_1 \beta u)\, h(u)}{b_2 z} - \frac{wl_2 p + d}{b_2} = +\infty : \tag{64}
$$

But conditions (13) and (21) on the dynamics of prices p_1 and p_2 and in particular the assumption that $\lim_{s \to 0^+} g_1(s) = +\infty$ together with $\lim_{s \to +\infty} g_2(s) = -\infty$ lead to a contradiction. ■

5.4 Stationary disequilibrium

The evolution given by (61) has not only the outcome described above[37]. In fact we are going to show that the differential system (61) may have an

[37] I.e. the return toward the equilibria (39).

equilibrium point $\mu^* = (\hat{u}^*; \hat{s}^*; \hat{p}^*; \hat{z}^*)$. Because of the condition $H(\hat{p}^*; \hat{z}^*) = 0$ there exists a trade-off between the relative price $\hat{p}^*$ and the relative fixed capital stock $\hat{z}^*$ at such equilibrium point:

$$\hat{z}^* = \frac{b_2 - wl_2\hat{p}^* - d}{wl_2\hat{p}^* + d}. \tag{65}$$

The same holds true for the condition $G(\hat{u}^*; \hat{s}^*; \hat{p}^*; \hat{z}^*) = 0$. By substituting this condition into equation $K(\hat{u}^*; \hat{s}^*; \hat{p}^*; \hat{z}^*) = 0$ we have:

$$g_1(\hat{s}^*) = g_2\left(\frac{wl_2\hat{p}^* + d}{b_2}\left(h(\hat{u}^*) - 1\right)\right); \tag{66}$$

using again the above conditions together with $F(\hat{u}^*; \hat{s}^*) = 0$ we obtain the equilibrium values for u, s, and p; we get, in particular:

$$\hat{s}^* = \frac{(b_1 - wl_1)\hat{p}^*\hat{u}^* - (b_2 - wl_2\hat{p}^*)}{(b_2 - wl_2\hat{p}^* - d - \delta)}. \tag{67}$$

It is easy to prove that a solution of the system:

$$\begin{cases} \hat{s}^* = \dfrac{(b_1 - wl_1)\hat{p}^*\hat{u}^* - (b_2 - wl_2\hat{p}^*)}{(b_2 - wl_2\hat{p}^* - d - \delta)} \\ F(\hat{u}^*; \hat{s}^*) = 0 \end{cases} \tag{68}$$

exists for every $\hat{p}^*$ such that:

$$\hat{p}^* \geq \frac{b_2}{(b_1 - wl_1)\bar{u} + wl_2}; \tag{69}$$

if:

$$\bar{u} > \frac{(d + \delta)wl_2}{(b_1 - wl_1)(b_2 - d - \delta)}. \tag{70}$$

The preceding condition is satisfied if the constant $\bar{u}$ that appears in (42) is sufficiently close to 1; we anyway assume that this is the case. This solution is also unique if the equilibrium relative price $\hat{p}^*$ is not too large; to be precisely:

$$\hat{p}^* < \frac{b_2 - (d + \delta)}{wl_2}. \tag{71}$$

In any case we denote with $(u(\hat{p}^*); s(\hat{p}^*))$ either the unique solution of the system (68) or the solution such that the capacity utilization rate $\hat{u}^*$ is larger and the ratio of inventories $\hat{s}^*$ is smaller. It is clear that solutions with $\hat{u}^*$ small and $\hat{s}^*$ large are not interesting for the case of the overheating regime. Finally we substitute $(u(\hat{p}^*); s(\hat{p}^*))$ into equation (66) and determine the equilibrium relative price $\hat{p}^*$.

The existence of at least one solution $\hat{p}^*$ satisfying (62) and such that the corresponding vector $(\hat{u}^*; \hat{s}^*; \hat{p}^*; \hat{z}^*)$ satisfies (63) obviously depends on the particolar form of the function g_1 and g_2. We think that there is no point in specifying these conditions, as they have no significant economic interpretation. We prefer, instead, to analyse the stability of such a point, when it exists.

To do this we compute the jacobian matrix J^* of the system in the equilibrium point $(\bar u^*; \bar s^*; \bar p^*; \bar z^*)$, having in mind relations (65), (66), (67) and the second of (68) linking the coordinates of this point.

It is easy to see[38] that the roots of the corresponding characteristic polynomial $P^*(\lambda)$ obviously depend on the explicit expression of the derivatives appearing in the third row of matrix J^*; if we assume, for instance, that at the equilibrium point the functions g_1 and g_2 are both stationary then the third row is null and among the roots of the characteristic polynomial one is obviously zero, another is negative and two of them have negative real part. Hence the stability of the equilibrium point depends crucially on the second derivative.

It is therefore clear that the equilibrium point, whenever it exists, may be locally attractive[39]. In this case the system may remain in an overheating regime and both prices p_1 and p_2 rise at the same rate of growth, while the capacity utilization rate remains above the desired target and inventories remain below their normal level.

Last but not least the disequilibrium in the capital goods market never peters out. We call this situation overheating stationary disequilibrium.

6 Conclusions

The main purpose of the paper has been to analyse the stability of equilibrium of a capitalist economy by considering the different regimes that characterize its working: normal equilibrium, overheating and depression.

The problem has been dealt with, at the analytical level, in terms of a four dimensional deterministic dynamical system framed in continuous time with the degree of capacity utilization of the consumption goods sector, the stock of inventories of the consumption goods sector, the relative price of commodities and the structure of the fixed capital stocks as state variables.

The dynamical system has a continuum of equilibrium points $(\bar u; \bar s; \bar p; \bar z)$ where $\bar u; \bar s$ characterize the normal equilibrium in real terms and $\bar p = \Psi(\bar z)$. >From the analytical point of view the existence of a continuum of equilibria depends on the fact that quantities do not react directly on prices.

>From the economic point of view the normal equilibrium of the model exhibits the usual properties of such a configuration of the economy: productive capacity of the sectors is utilized at the normal level, stocks of in-

[38] We sketch a proof in the mathematical appendix.

[39] In this case too we are dealing with the local stability analysis of an equilibrium in proportions in the sense discussed by Boggio (1993).

ventories are at their normal level and prices of commodities are constant, i.e. there is no inflation.

The existence of a one dimensional manifold of equilibria is perhaps unusual but far from been surprising as the effects of disproportion among sectors can be balanced by a suitable price structure, at least within given bounds. These bounds can not be overcome without undermining profit and growth rates of some sectors, together with the equilibrium of the system.

The stability analysis shows that the normal equilibrium is (locally) asymptotically unstable; the economy can evolve, therefore, according to two different regimes. The first regime is what we call overheating; its main features are a stock of inventories that diminish continuously in time, an increasing degree of capacity utilization and a relative price continuously increasing. The second regime is what we call depression; its characteristics are a rising level of inventories, falling prices of consumption goods and a decreasing degree of capacity utilization.

Let us consider the case of depression. The model shows that as the economy stays in such a regime for a sufficiently long time then an increasing disproportion between sectors arises that makes investment in the capital goods sector less and less profitable. In other words we have that the reduction of investments in the consumption goods sector determines an overdimensioned capital goods sector; the excess of productive capacity that, in fact, characterises the capital goods sector will eventually bring to a halt the accumulation process.

Things are the other way around in the case of the overheating regime. The increasing relative price causes a redistribution of profits in favour of the consumption goods sector;this eventually causes problems to the capital goods sector either on the production side or on the profit side. The consequence of this is either the rationing of the capital goods sector or the arrest of accumulation due to a lack of profits.

7 Mathematical appendix

7.1 Proof of theorem 1

Simple calculations yield the following jacobian matrix J of the linearized system, evaluated at one of the equilibrium points (39):

$$J = \begin{pmatrix} \frac{\partial F(\mu)}{\partial u} & \frac{\partial F(\mu)}{\partial s} & 0 & 0 \\ -\frac{\mu}{\beta b_1}\left(\alpha + b_1^{-1}\mu u\, h'\right) & 0 & 0 & 0 \\ 0 & \beta\mu\mu' g_1 & 0 & 0 \\ \frac{\partial H(\mu)}{\partial u} & \frac{\partial H(\mu)}{\partial s} & \frac{\partial H(\mu)}{\partial p} & \frac{\partial H(\mu)}{\partial z} \end{pmatrix}. \qquad (72)$$

To determine the eigenvalues of the jacobian matrix J it is sufficient to write the explicit expression of the last partial derivative of the function H in the equilibrium point:

$$\frac{\partial H(\mu)}{\partial z} = -\left(\alpha + b_1^{-1}\bar{u} + \frac{db_2}{wl_2\bar{p}}\right);$$

hence the characteristic polynomial of the linearized system is:

$$\begin{aligned} P(\lambda) &= \lambda\left(\alpha + b_1^{-1}\bar{u} + \frac{db_2}{wl_2\bar{p}} + \lambda\right) \\ &\quad \left(\lambda^2 - \frac{\partial F(\mu)}{\partial u}\lambda + \frac{1}{b_1\bar{p}}\frac{\partial F(\mu)}{\partial s}\left(\alpha + b_1^{-1}\bar{u}\right)h'\right): \end{aligned}$$

The eigenvalues of the linearized system are:

$$\begin{aligned} \lambda_1 &= \frac{1}{2}\left(\frac{\partial F(\mu)}{\partial u} - \sqrt{\left(\frac{\partial F(\mu)}{\partial u}\right)^2 - 4\frac{1}{b_1\bar{p}}\frac{\partial F(\mu)}{\partial s}\left(\alpha + b_1^{-1}\bar{u}\right)h'}\right) \\ \lambda_2 &= \frac{1}{2}\left(\frac{\partial F(\mu)}{\partial u} + \sqrt{\left(\frac{\partial F(\mu)}{\partial u}\right)^2 - 4\frac{1}{b_1\bar{p}}\frac{\partial F(\mu)}{\partial s}\left(\alpha + b_1^{-1}\bar{u}\right)h'}\right) \\ \lambda_3 &= 0 \\ \lambda_4 &= -\left(\alpha + b_1^{-1}\bar{u} + \frac{db_2}{wl_2\bar{p}}\right): \end{aligned}$$

Hence having in mind the hypotheses (32) on the sign of the partial derivatives of the function F we have $\lambda_1 < 0$, $\lambda_4 < 0$, $\lambda_3 = 0$ and $\lambda_2 > 0$. All equilibria are therefore unstable.

7.2 Proof of (40) and (41)

To apply the adiabatic principle it is necessary to diagonalise the Jacobian matrix J; we thus introduce the matrix B defined by:

$$B = (v_1; v_2; v_3; v_4);$$

where v_i $(i = 1; \ldots; 4)$ are the eigenvectors of the matrix of the linearized system:

$$v_1 = \begin{pmatrix} \frac{\partial F(\mu)}{\partial s} \\ \lambda_1 - \frac{\partial F(\mu)}{\partial u} \\ \frac{\bar{p}g_1'}{\lambda_1}\left(\lambda_1 - \frac{\partial F(\mu)}{\partial u}\right) \\ \frac{\frac{\partial H(\mu)}{\partial p}\frac{\bar{p}g_1'}{\lambda_1}\left(\lambda_1 - \frac{\partial F(\mu)}{\partial u}\right)}{\alpha + b_1^{-1}\bar{u} + db_2/wl_2\bar{p} + \lambda_1} \end{pmatrix} \quad v_2 = \begin{pmatrix} \frac{\partial F(\mu)}{\partial s} \\ \lambda_2 - \frac{\partial F(\mu)}{\partial u} \\ \frac{\bar{p}g_1'}{\lambda_2}\left(\lambda_2 - \frac{\partial F(\mu)}{\partial u}\right) \\ \frac{\frac{\partial H(\mu)}{\partial p}\frac{\bar{p}g_1'}{\lambda_2}\left(\lambda_2 - \frac{\partial F(\mu)}{\partial u}\right)}{\alpha + b_1^{-1}\bar{u} + db_2/wl_2\bar{p} + \lambda_2} \end{pmatrix}$$

$$v_3 = \begin{pmatrix} 0 \\ 0 \\ \alpha + b_1^{-1}\bar{u} + \frac{db_2}{wl_2\bar{p}} \\ \frac{\partial H(\mu)}{\partial p} \end{pmatrix} \quad v_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}:$$

Thus we have $B^{-1}JB = \mathrm{diag}(\lambda_i)$ $(i = 1, \dots, 4)$. We introduce the new variables[40]:

$$x = B^{-1}(\eta - \mu),$$

where $x = (x_1, x_2, x_3, x_4)'$ and $(\eta - \mu) = (u - \bar{u}, s - \bar{s}, p - \bar{p}, z - \bar{z})$, i.e.

$$
\begin{aligned}
u - \bar{u} &= \frac{\partial F(\mu)}{\partial s} x_1 + \frac{\partial F(\mu)}{\partial s} x_2 \\
s - \bar{s} &= -\lambda_2 x_1 - \lambda_1 x_2 \\
p - \bar{p} &= -\frac{\lambda_2}{\lambda_1} \bar{p}\bar{g}_1' x_1 - \frac{\lambda_1}{\lambda_2} \bar{p}\bar{g}_1' x_2 + \left(\alpha + b_1^{-1}\bar{u} + \frac{db_2}{wl_2\bar{p}}\right) x_3 \\
z - \bar{z} &= -\frac{\partial H(\mu)}{\partial p} \frac{\bar{p}\bar{g}_1'}{\alpha + b_1^{-1}\bar{u} + \frac{db_2}{wl_2\bar{p}}} \left(\frac{\lambda_2}{\lambda_1} x_1 + \frac{\lambda_1}{\lambda_2} x_2\right) + \frac{\partial H(\mu)}{\partial p} x_3 + x_4
\end{aligned}
$$

In the new coordinate system the axes x_1 and x_4 are the two eigenspaces corresponding to the negative eigenvalues; hence, by the adiabatic principle, we may approximate the solution of the dynamical system (38) by setting:

$$\underline{x}_1 = \underline{x}_4 = 0.$$

Therefore:

$$
\begin{aligned}
\underline{u} &= \frac{\partial F(\mu)}{\partial s} \underline{x}_2 \\
\underline{s} &= -\lambda_1 \underline{x}_2 \\
\underline{p} &= -\frac{\lambda_1}{\lambda_2} \bar{p}\bar{g}_1' \underline{x}_2 + \left(\alpha + b_1^{-1}\bar{u} + \frac{db_2}{wl_2\bar{p}}\right) \underline{x}_3 \\
\underline{z} &= -\frac{\partial H(\mu)}{\partial p} \frac{\bar{p}\bar{g}_1'}{\alpha + b_1^{-1}\bar{u} + \frac{db_2}{wl_2\bar{p}}} \frac{\lambda_1}{\lambda_2} \underline{x}_2 + \frac{\partial H(\mu)}{\partial p} \underline{x}_3.
\end{aligned}
$$

After obvious simplifications we obtain the expression of the text.

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