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Rational Economic Agents as Formal Logical Systems

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ABSTRACT: We describe economic agents as formal logical systems of the first order, and are then able to show that for any ordinary geometric description of an agent as a preordering over a given choice set within the set of reals, there exists a corresponding logical description in the space of first order-formal systems, and *viceversa*. This approach allows to distinguish formally among propositions known to economic agents and propositions concerning them. Also, we show that for any finite commodity space one can build consistent agents as first-order formal systems, and hence an economy defined as a collection of such systems, both of which can be mapped into the standard topological framework.

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1. *Introduction*

In economic theory, an agent is usually identified by his preferences and his consumption possibility set, both of which are typically assumed to admit of a numerical representation. These, combined with some environmental variables (like endowments and prices), deliver optimal programmes as functions of the environment. Equilibrium is then defined as a state of the environment such that these programmes are consistent with each other. This paper is concerned with the way we describe rational agents as arriving at optimal programmes.

The economist's standard description of rational choice involves the maximization of some objective function subject to constraints – this being allegedly the way rationality is embodied in the economic model. Thus if ϕ_i is the demand set of consumer i, it is the outcome of a rational choice insofar as it arises from some maximization problem: the usual topological description of agent i is meant to describe his rationality. More precisely, the latter is embodied in the properties of Euclidean spaces and the implied definition of a distance function: rational behaviour is assessed as a set of properties mapped into a metric space. An alternative way to describe rationality is to adopt a more pointedly logical approach, and saying that an agent picking up an optimal programme within some prespecified choice set is effectively being endowed with the ability of drawing inferences from given premises: the outcome is rational insofar as it allows no contradiction. In this sense, ϕ_i is the outcome of an inference which has prices, preferences, endowments, etc., as its premises. The main problem we address in this paper is establishing a formal connection between these two approaches.

To be sure, that connection is somehow implied by the logical foundations of set theory, which in turn is used by economists as a matter of course (Stigum, 1990). Moreover, mathematical logic is increasingly being used to tackle foundational problems in game theory – where indeed agents are sometimes identified as formal systems, i.e., statement-generating formal mechanisms (e.g., Binmore, 1992). However, how a geometric representation of rationality relates to a logical definition of it, is something worth spelling out at length in its own right. Given the definition of an agent as a complete preordering on some subset of $\mathbb{R}^L$, we shall prove that it is possible to throw a bridge between that definition, and one which sees agents as formal systems which are extensions of Primitive Recursive Arithmetics with some desirable properties (like consistency).

To the economist's ordinary description, we can associate a formal logical system, and viceversa. The construction of such a mapping should be seen as a first methodological step towards a more general framework. First, looking at an economic agent as a formal system yields a clear distinction between *provability*, *consistency* and *truth* of propositions concerning him. If a given statement can be proved by a formal system T, it is something agent T can draw from his axioms, and may then be seen (metatheoretically) as known to him. A more powerful formal system S will be able to prove all that T can prove, as well as some other statement δ having T as its object: S may then prove that T and δ are consistent: metatheoretically, an external observer (S) knows that δ and T are compatible with no implication that T knows δ. Finally, a formal notion of truth is avaliable, to the effect that a statement (proved) by T will be true, depending on the (formalized) interpretation one bestows upon it. Secondly, one major implication is that an (exchange) economy can be described as a collection of formal systems, each of which will be inconsistent with any other. This may tell us something on economic equilibria. Indeed, using this modeling strategy we were able to show elsewhere (Benassi and Gentilini, 1997) that a competitive equilibrium implies some specific degree of consistency among the formalized expression of the equilibrium statements of the various agents.

As a final remark, we stress that we rely mainly on the power of the syntactical instruments of classical mathematical logic, which appears to us to fit well with Walrasian competitive modelling. Notice in particular that we treat preferences as given (i.e., as described by a set of ordering axioms), and accordingly refrain from going into the questions raised by the so-called preference logics (Moutafakis, 1987). Thus we do not take up the semantic (i.e., substantive) problem of what rational preferences (should) be, but try to work out the syntactical consequences of classical rationality *vis à vis* economic choice. Also, we are mainly concerned with a single agent, and do not model strategic interaction – although this should be in principle amenable to our approach. Indeed, it is game theory that at present raises questions, the answer to which are framed in the language of mathematical logic: computability is just the first item coming to mind, as in Binmore (1987) or Anderlini (1989). In a sense, all controversial points which are likely to emerge under both headings are assumed away within the axiomatic structure of the model.

The plan of the paper is as follows. An informal presentation of our set up and main results is given in Section 2. Section 3 presents the core of the paper, by establishing a formal connection between the typical topological description of a rational economic agent and his representation as a formal system. Section 4 takes up some possible extensions, addressing in particular the problem of consistency among agents seen as formal systems, as well as the constructive definition of an economy which may be derived therefrom. Concluding remarks are gathered in section 5. Finally, an appendix is also presented, where some intermediate results and definitions are collected which are used throughout the text.

2. *An informal presentation*

In this section we introduce the main ideas, and develop informally the argument presented in later sections. An economy is usually described as a collection of sets of numbers, obeying some desirable properties. In a L-commodity world, a standard description (e.g., Hildebrand and Kirman, 1988) would include a finite set of agents $\mathbb{I} \subset \mathbb{N}$, such that each $i \in \mathbb{I}$ is identified by a consumption set $X_i \subseteq \mathbb{R}^L_+$, a binary (preference) relation $\succeq_i \subset X_i \times X_i$ having some properties such that $\succeq_i \in \mathcal{P}$ (a class of preordering relations on X_i), and some initial resources $w_i \in \mathbb{R}^L_+$ – all of which make up the characteristics of agent i. An (exchange) economy is then a mapping $\mathcal{E} : I \to \mathcal{P} \times \mathbb{R}^L_+$ which takes characteristics into agents, such that $\sum_i w_i > 0$. If we take as a reference a *competitive* exchange economy, agent i faces a budget set $B_i(p, w_i) \subset \mathbb{R}^{\ell}_+$ which summarizes his perception of the environment. The latter includes variables beyond his control: his given resources w_i and a price vector $p \in \mathbb{R}^L_{++}$ which embodies the equilibrium signaling mechanism typical of the competitive model.

Consider now the choice of trader i. We usually say that this is given by the best element (according to $\succeq_i$) in B_i: hence, it will be some set of nonnegative real numbers ϕ, such that

$$\phi(\succeq_i, p, w_i) = \{x \in B_i(p, w_i) \cap X_i \mid (x, z) \in \succeq_i \text{ for every } z \in B_i(p, w_i) \cap X_i\} \tag{1}$$

Let now $\widehat{w}_i$ and $\widehat{p}$ be particular descriptions of the environment. Then we could say that agent i, given his characteristics $C^i = (X_i, \succeq_i, \widehat{w}_i)$ and a price $\widehat{p}$, *infers* his optimal programme $\widehat{\phi}$. If we let $\Delta = (C^i, \widehat{p})$ and the symbol $\vdash_i$ mean "agent i infers", this idea may be conveyed by writing

$$\Delta \vdash_i \widehat{\phi} \tag{2}$$

In fact, we are endowing our agent with an inference rule implicit in "utility maximization": this rule ensures that the (structure of his) inference is correct. Indeed, should one state that at $(\widehat{p}, \widehat{w}_i)$ agent i chooses some $x \notin \widehat{\phi}$, we would say that this is not a 'correct' inference (he is not being 'rational').

2.1. *On formal systems*[1]

From a formal point of view, we describe an agent's choices as the result of operations he performs over numbers: in our example, Δ and $\widehat{\phi}$ are in fact (sets of) numbers, and our agent is able to derive the latter from the former. Indeed, the way this derivation is performed embodies our idea of rationality.

In order to make this idea more precise, we first need a language: for example, the formula in (2) may well have the same meaning as equation (1), but this is clearly conveyed by a different language. In general, we define a formal language $\mathcal{L}$ as a set of symbols which enables us to make precise such notions as "alphabet" or "proposition". Formally, a language $\mathcal{L}$ is a triplet $(A\ell, W, F)$: $A\ell$ is the *alphabet* of $\mathcal{L}$, i.e. a given set of symbols; W is the set of *terms* which can be legitimately formed by symbols of $\mathcal{L}$ in $A\ell$; F is the set of *formulae* which can be legitimately formed by combining terms. This is the canonical usage we adopt for the purpose of this paper. Given some language $\mathcal{L}$, we need some inference rules to use it. Thus we may use our language to state the propositions "agent i observes $\widehat{p}$ and $\widehat{w}$" and "agent i chooses a vector in $\widehat{\phi}$"; we may also state that the former implies the latter – so long as $\mathfrak{L}$ includes such possibility, this is indeed meaningful. But we need an inference rules to be sure that this is a *valid* implication: more precisely, we need an *inferential apparatus*. An inferential apparatus $\mathfrak{D}$ is a pair ($\mathcal{A}$, $\mathfrak{R}$): $\mathcal{A}$ is the set of *axioms* – in fact a suitably chosen subset of the set F of formulae; $\mathfrak{R}$ is the set of *inference rules*, i.e. of legitimate ways of connecting any two formulae – drawing the latter (conclusion) from the former (premise).

A pair $(\mathfrak{L},\mathfrak{D}) = S$ is called a *formal system* – that is, a statement-generating formal mechanism. Let Γ be a set of formulae in S (i.e., $\Gamma \subset F$), and γ a formula in S (i.e., $\gamma \in F$): we say that γ can be derived from Γ if and only if there is a finite sequence $\{\gamma_k\}_{k=1}^{n}$ of formulae of S such that any γ_k in the sequence is either an axiom of S, or a element of Γ, or else can be derived by applying any of the rules in $\mathfrak{R}$ to any formulae γ_j preceding γ_k in the sequence; and, of course, $\gamma_n = \gamma$. If this is indeed the case, we write formally $\Gamma \underset{S}{\vdash} \gamma$. The formulae in Γ are the *premises*, and the sequence $\{\gamma_k\}_{k=1}^{n}$ the *derivation*, of γ. Note that it may well be that $\Gamma = \emptyset$: that is, γ can be derived in S with no reference to a specific subset of F. This we write as $\underset{S}{\vdash} \gamma$: in some sense, γ is in this case stronger, which we signal by calling it a *theorem* of S, and the sequence $\{\gamma_k\}_{k=1}^{n}$ its *demonstration* or *proof* within S.

In principle, one may think of different formal systems. For reasons which will be apparent, we shall work within the so-called standard formal logic *of the first order*, although the setting we shall develop can be extended to various non-standard logical systems in a natural way. This choice should be understood as dictated by the use we shall make of formal systems as describing economic agents. To see the rationale behind this, let us go back to the language $\mathfrak{L}$ of S. Its alphabet $A\ell$ will typically include the standard logical boolean connectives ($\vee$ and $\wedge$ for "or" and "and", $\sim$ for "not", and $\leftrightarrow$ for "is equivalent to"), as well as the standard quantifiers $\exists$ and $\forall$, which together make up the operators of the so-called predicate calculus (Shoenfield, 1967, p.14). The other main symbols within $A\ell$ are typically: (a) constants a_i ($i = 1,2,...$), to be seen as *specific* individual objects – e.g., the number 3; (b) variables x_i ($i = 1,2,...$), to be seen as *generic* individual objects – e.g., an even number; (c) function letters f_i^m ($i = 1,...$; $m = 1,...$), the use of which will be clear presently; (d) predicative letters P_i^n ($i = 1,...$; $m = 1,...$), which describe properties of specific or generic individual objects, where n is the number of "slots" the property requires: e.g., P_i^2 might be the property "being greater than" (which calls for two slots, as it applies to pairs of numbers) – in which case $P_i^2 a_1 a_5$ means that the number identified by a_1 is greater than the one identified by a_5. This being so, a term $w \in W$ is defined to be either a constant, or a variable, or else a functional letter – the latter is used to define a new term starting from other terms. A formula is generated by combining either terms or formulae; thus the simplest formulae (so-called atomic) are those describing properties of individual objects (like $P^3_j w_1 w_2 w_3$), while a new formula may result from applying logical connectives to other formulae (if α and β are formulae, so is $\alpha \vee \beta$). A formula may also contain quantifiers which are applied to individual variables – in which case it is sometimes called a sentence: e.g., $\exists x_i P^1_j(x_i)$ is a sentence ("there is at least one object x_i with the property P^1_j"). *A formal systems where the quantifiers $\exists$ and $\forall$ are applied only to individual variables (which are terms), and not to formulae, is said to be of the first order* (Shoenfield, ch.2): this is the kind of logical systems we shall be concerned with. We do so, as this allows us to rely on canonical "completeness" theorems, the meaning of which can be introduced as follows.

Any formal system S is to be clear about some relevant features. The most significant of these can be listed as follows: (a) being consistent; (b) being complete; (c) being endowed with a formal model; (d) being recursively axiomatized; (e) being decidable. These are the *metatheoretical properties* of S, which are presently taken up informally (Shoenfield, 1967, ch.s 3-6; Barwise,1977,ch.A1).

Let a formula of the form "γ and not γ" be defined as a contradiction ($\gamma \wedge \sim \gamma$, where "and" ($\wedge$) and "not" ($\sim$) are in the alphabet of S). A formal system S is ***consistent*** if no contradiction can be derived as a theorem of S. If S is inconsistent, all formulae of S (the whole set F) are theorems of S. A formal system S is ***syntactically complete*** (*sy*-complete) if for each γ, either γ or $\sim \gamma$ can be derived in S. Consistency and *sy*-completeness are *syntactical* properties of a formal system: the latter may or may not be consistent or *sy*-complete, depending *only* on the way it is built – i.e., on what its inferential apparatus regards as legitimate axioms and inference rules, and on the legitimate ways to form terms and formulae starting from its alphabet. However, languages are used to describe realities – they should have a meaning, i.e. a semantics.

Formally, an *interpretation* of the language $\mathcal{L}$ of S is a pair $M = (D, g)$ (Barwise, 1977, ch.A1). Loosely speaking, D is a nonempty set which defines the domain of the interpretation – the universe about which $\mathcal{L}$ is assumed to speak; g a function which assigns a meaning in D to terms and formulae in S. For example, a formula γ of S might have the form Pa, where P is some property and a is a constant; let (D_1, g_1) a particular interpretation of $\mathcal{L}$ in S, according to which $g_1(P)$ is the set of primes and $g_1(a) = 3$. Thus the proposition "3 is a prime" may be rendered as $g_1(a) \in g_1(P)$: since 3 in indeed a prime, we say that Pa is true under the interpretation (D_1, g_1) – of course, Pa may be false under a different interpretation, e.g. one (D_1, g_2) such that $g_2(a) = 4$. This procedure can be generalized to more complex situations, and gives us a formalized notion of *truth*. In general, if γ is *true* under the interpretation M, we say that M is a ***model*** of γ; by extension, if Γ is a set of formulae in S, an interpretation M is a model of Γ only iff it is a model for any $\gamma \in \Gamma$. One key point we shall exploit in what follows is the relationship between the syntactical and the semantical levels: inconsistent systems cannot have a model. Moreover, the following is a canonical result (Shoenfield, 1967, p.43):

THEOREM I (existence of a model): *If S is consistent, it has at least one model M.*

This establishes a connection between a syntactical property (consistency) and a semantic one (the existence of a model). Moreover, consistent formal systems exhibit in general the following

PROPERTY (correctness): *All theorems of S are true in all models of S.*

This paves the way to an outstanding feature of first order systems. If we confine ourselves to correct systems of the first order (which we shall do in the sequel), we can avail ourselves of a stronger property known as semantic completeness (*sem*-completeness). Indeed, the following can be proved (Shoenfield, 1967, p.43):

THEOREM II (*sem*-completeness): *If S is a first order consistent system and proposition γ is true in all models of S, then γ is a theorem of S.*

This explains why we shall work with logical systems of the first order: if correct, these systems exhibit a strong connection between their formal properties and the interpretation that can be bestowed upon them. Differently, higher order systems do not admit *sem*-completeness with respect to the class of standard models (Takeuti, 1987, p.162). Also, the difference between *sy*-completeness and *sem*-completeness should be emphasized: the former depends only on the way a system S is built, while the latter connects such formal structure with external worlds which these formal structure can describe.

The last properties of formal systems we are interested in, are whether it is ***recursively axiomatizable*** and ***decidable*** (Shoenfield, 1967, pp.123 ff). The latter refers to the procedure used to establish whether a formula is a theorem: S is (recursively) decidable if there exist recursive methods (which we need not specify here) which, in a finite number of steps, allows us to decide whether any γ is or not a theorem of S. Now, S is recursively axiomatizable if there exists another formal system S' with the same theorems as S, such that its set of axioms $\mathcal{A}$' is itself decidable – i.e., given any γ' we can decide in a finite number of steps whether γ' is, or is not, in $\mathcal{A}$'. These two notions are connected by the following (Shoenfield, 1967, p.132)

THEOREM III (decidable systems): *If S is sy-complete and recursively axiomatizable, then it is itself decidable.*

However, it should be noticed that most relevant logical systems are undecidable, which may be seen as a consequence of the celebrated Gödel's incompleteness theorems. To introduce the latter, we remark that in economics we use numerical representations to describe the agents' behaviour: it is then sensible to concentrate on a particular class of formal systems which are canonical in mathematical logic, and are used to study numerical representations of reality. This is the *Primitive Recursive Arithmetics* (*PRA*) (Shoenfield, 1967, ch.6; Smorynski, 1985, p.16; Barwise, 1977, p.840). Informally, *PRA* is a formal system with canonical language and inferential apparatus, which includes the predicate calculus together with an extended language and a set of own axioms: these allow *PRA* to express natural numbers and to prove their properties. Its inferential apparatus includes the *induction rule* on quantifier-free formulae (Shoenfield, 1967, p.204; Takeuti, 1987, p.76), which adds greatly to *PRA*'s proof power. It is worth stressing that *PRA* includes the definition of all recursive functions (Smorinski, 1985, ch.0; Shoenfield, 1967, ch.6): as is well known, the latter are the mathematical tools describing each effective operation on numbers, or objects which may be codified by numbers. Accordingly, *PRA* can describe (and prove) actions performed by rational agents, which we can think of as *computable* – indeed, the celebrated Church's thesis states that *each computable process is recursive* (Shoenfield, 1967, p.119). Moreover, as Gödel's theorems show, *PRA* also defines many predicates and functions which are *not* recursive: we shall use them to describe non-computable choices of rational agents.

Saying that S belongs to the class *PRA* means that S will have the same language, inference rules and axioms as *PRA*, *together with a set of own predicative and function letters and a set of proper axioms*, which are meant to make S fit for economic analysis. Like *PRA*, S will be recursively axiomatizable. Suppose now we have a procedure which associates to any symbol or finite sequence of symbols of the language $\mathfrak{L}$ of S, say e, a positive natural number, $\#(e)$, called the Gödel number (G-number) of e (Smorynski, 1977, p.837). Thus one such number may be associated to formula γ of S, and another number to the proof of γ in S – which, as we saw above, is a finite sequence of formulae. In these circumstances, it can be proved that a recursive binary predicate may be defined in *PRA*, denoted $\mathrm{Prov}_S(m,n)$, whose meaning is as follows: "m is the G-number of a proof in S of the formula γ of S, whose G-number is n". Accordingly, the formula $\mathrm{Prov}_S(\#(\delta),\#(\gamma))$ says: "δ is a proof of γ in S". To this we may append a quantifier, to get the proposition $\exists x \mathrm{Prov}_S(x,\#(\gamma))$: "there exists a proof of γ in S", which is rendered for short as $\mathrm{Pr}_S(\gamma)$. That is, we can define a *provability predicate* $\mathrm{Pr}_S(\cdot)$, stating the provability in the given system S of a formula (Smorynski, 1977, p.837; 1985, p.40). In general, this will not be recursive: if it is, then S is decidable in the sense defined above. However, no S of the *PRA* class is decidable. One important consequence of all this is that statements about provability (and hence consistency) of S can be formed within S. Indeed, the latter shares the same language and inference rules of PRA, within which, as we have seen, $\Pr_S(\cdot)$ can be defined. Thus saying that S is consistent amounts simply to writing $\sim \Pr_S(\gamma \wedge \sim \gamma)$ ("γ and $\sim \gamma$ cannot be proved in S"); let $Coer(S)$ be any such statement – clearly, $\Pr_S(\gamma \wedge \sim \gamma)$ will mean that S is inconsistent, that is $\sim Coer(S)$.[2] The celebrated Gödel's incompleteness theorems establish that for all of formal systems which belong to PRA, being consistent implies being *sy*-incomplete, which can be summarized (quite informally) as (Smorinsky, 1977, p.825):

THEOREM IV (Gödel): *If S is consistent and belongs to the class PRA, than it can prove neither Coer(S), nor $\sim$ Coer(S).*

One major consequence of the theorem is, it is possible to prove that any such system admits of a (countably infinite) set of non-isomorphic models. Indeed, if S is consistent, by theorem I it has at least one model within which $Coer(S)$ is true, but there have to be models where $Coer(S)$ is false: as a matter of fact, if $Coer(S)$ was true in all models of S, it would be a theorem of S by theorem II (*sem*-completeness), which in turn would contradict theorem IV. The property of having many different models is a result we shall exploit heavily in what follows.

2.2. *Formal systems and economic theory*

We are going to identify an agent $i \in I$ with a formal system, T_i – including of course the language of T_i, his axioms and his inference rules. Note that we are basically saying two different things here. First, once we have a language and an inferential apparatus for T_i, we can say that T_i is able to prove some propositions within that language: going beck to our example, $\Delta \vdash_i \widehat{\phi}$ means that $\widehat{\phi}$ can be proved within T_i starting from Δ. Second, when we interpret it as "agent i, identified by C^i and observing $\widehat{p}$, chooses an optimal action in $\widehat{\phi}$", we (*as outside observers*) are describing i's behaviour by assigning to him both the language $\mathfrak{L}^i$ and the inferential apparatus $\mathfrak{D}^i$: in doing so we are using some higher-level formal system which allows us to prove statements about T_i, as well as about any other T_j, $j \in \mathbb{I}$.

One advantage of looking at the inner logical structure of rational economic agents is having a clear distinction between *provability*, *consistency* and *truth* of propositions concerning him. Suppose we identify an agent by a formal system $T_i = (\mathfrak{L}^i, \mathfrak{D}^i)$. We may assume that all agents share the same language ($\mathfrak{L}^i = \mathfrak{L}$), though not necessarily the same inference rules – the former is somehow a minimal part of the idea of "rational". However, an agent's axioms will describe the premises from which he is assumed to draw his conclusions: only some of these will be equal across agents – e.g., in a competitive market the price is the same to all; but there will have to be agent-specific axioms, regarding his preferences and wealth: indeed, this is what makes agents different from one another.

Now consider some statement γ_i of T_i. If this can be proved in T_i or is a theorem of T_i, it is something the agent can draw from his axioms: e.g., from a price $\widehat{p}$ and his characteristics C^i agent i can draw the conclusion that his optimal choice lies in $\widehat{\phi}$. Thus we suggest that *a statement provable by* T_i *is (metatheoretically) known to agent i.* The idea of some γ being provable within T_i is different from γ being consistent with T_i. Consider some more powerful formal system S having the same language and inference rules as T_i, but a wider set of axioms: it will then be able to prove all the theorems of T_i, as well as other theorems having T_i as their object. In particular, given some formula δ of S, S may be able to prove that T_i and δ are consistent: *metatheoretically, an external observer (system S) knows that* δ *and* T_i *are compatible with no implication that i knows* δ. Now notice that we shall work within a framework where all T_i's will belong to the *PRA* class: they will be extensions of the standard *PRA* system, built in such a way that their inference rules and axioms ensure consistency. Thus all T_i's will be unable to prove contradictions, which implies (by theorem I) that each of them will admit of at least one interpretation (model) M_i where any theorem γ_i will be true. However, as belonging to the *PRA* class, each of them will be *sy*-incomplete: hence, it will allow for a whole set of non-isomorphic models. All this will also be true of the more poweful system S.

Using the modeling strategy outlined above, we shall prove that it is possible to describe agents as formal systems which are extensions of *PRA* and (a) exhibit some desirable properties, notably consistency, and (b) can be naturally mapped into a standard topological framework to deliver the economist's ordinary description – accordingly, agent i's typical characteristics within our exchange economy, $C^i = (X_i, \succeq_i, \widehat{w}_i)$, can be read as the topological description of the formal system T_i. The construction of such a mapping should be seen as a first methodological step towards a more general framework. Indeed, one major implication is that an economy E can be described as a collection of formal systems $\{T_i\}_{i=1}^{I}$, each of which is *sy*-incomplete, admits of a set of infinite non-isomorphic models, and – crucially – is inconsistent with any other. One important implication, pursued elsewhere (Benassi and Gentilini, 1997), concerns competitive equilibria. Using the tools of mathematical logic, we can define a *model equilibrium* as a set of models $\{M_i^*\}_{i=1}^{I}$ (one for each formal system T_i), such that each M_i is a model of the specific T_i plus the equilibrium choice statements of all the T_j's ($j \neq i$): that is, M_i ensures the that T_i be consistent with the equilibrium choices of all agents. We are then able to show that all competitive equilibria for the economy $\{C^i\}_{i=1}^{I}$ are model equilibria for E.

3. *A rational agent as a formal system*

The typical framework to model rational economic choice involves (a) defining a set X of alternatives from which the agent is assumed to choose, and (b) defining a ranking among such alternatives – a preordering which allows pairwise comparisons between elements x of X. In this section we provide a logical formulation of both (a) and (b), within a definition of an agent as a formal system $T_i = (\mathcal{L}, \mathfrak{D}^i)$. The index $i \in \mathbb{I} = \{1, ..., I\}$ identifies the specific agent one is concerned with.[3]

To start with, it is useful to be clear about the geometric representation we have in mind:

DEFINITION D.1 (economic agent): *An economic agents is a pair* $(\succeq_i, X_i)$, *where* $X_i \subset \mathbb{R}^L$ *and* $\succeq_i \subset X_i \times X_i$ *is a binary preference relation.*

We let $\succeq_i$ and X_i satisfy Debreu's (1959) standard assumptions. D.1 is meant to describe an agent choosing a bundle of L commodities indexed by $\ell \in \mathbb{L} = \{1, ..., L\}$ and identified by real numbers: building a mapping which allows one to pass from D.1 to the following definition D.2 and *viceversa*, is one major object of this paper.

DEFINITION D.2 (economic agent): *An economic agent is a first order formal system* T_i *which includes the Primitive Recursive Arithmetic extended to integers PRA(Z), characterized by:*

(a) a language $\mathcal{L}$;

(b) a set of axioms $\mathcal{A}(T_i)$;

(c) a set of inference rules $\mathfrak{R}(T_i)$.

The system $PRA(Z)$ is simply an extension of PRA which allows considering negative integers (Forcheri, Gentilini, Molfino, 1996, sec.2). The language $\mathcal{L}$ of T_i is an extension of the language of $PRA(Z)$ and, as already remarked, it is common to all agents, and accordingly unindexed. Together with the basic language of $PRA(Z)$, $\mathcal{L}$ includes all predicative and function letters defined in the sequel, which are specific to the problem at hand. We shall denote as $PRA(E)$ the conservative extension of $PRA(Z)$ with the symbols of the language $\mathcal{L}$, and call it *the basic logical system* for the economy $E = \{T_i\}_{i=1}^{I}$.

Starting from $\mathcal{A}$, agent T_i makes use of $\mathfrak{R}$ to prove his set of theorems $\mathrm{Th}(T_i)$, which are meant to describe the whole knowledge of T_i. Surely this set includes all theorems of $PRA(E)$, as well as all tautologies in $\mathcal{L}$. Of course, this definition is far too generic to be useful: we have to specify axioms in $\mathcal{A}$ which make T_i fit to represent economic agents in the spirit of D.1.

3.1. *Some auxiliary definitions*

We first need to expand the standard $PRA(Z)$ language with two function letters (defining new terms starting from other terms) which play a key role in what follows:

AUXILIARY DEFINITION AD.1: *Within PRA(E) a binary functional letter* quot(· , ·) *is defined, such that* quot(*a*,*b*) *is a recursive term codifying the fraction s/t of* $\mathbb{Q}$ *(the set of rationals), where s/t is the reduced form of a/b, as the G-number of a finite sequence of symbols in the following manner:* #(quot,*s*,*t*), *while assuming that* quot *is a function letter different from each letter of PRA(Z).*

Thus, given a pair of natural numbers like (2,3), #quot(2,3) corresponds to the fraction 2/3. We refer to the Appendix for technicalities (Def B.1): our definition allows terms of the quot form to simulate some useful properties of rationals in $\mathbb{Q}$.

AUXILIARY DEFINITION AD.2: *Within PRA(E) a unary functional letter L*-vect(·) *is defined, such that if* ω *is a sequence of PRA(E) terms of the form* $\{\text{quot}(a_\ell, b_\ell)\}_{\ell=1}^{L}$, *then L*-vect($\omega$)=#($\omega$), *i.e., the G-number of a sequence of L terms of the* quot *form.*

Again, we refer to the Appendix (Def B.5) for a complete definition of *L*-vect(·).

These auxiliary definitions will be used later on to derive our results. We can already notice, however, that mappings can be built which take quot-terms into rational numbers, and ℓ-vect(·)-terms into $\mathbb{Q}^L$-vectors of the form $(a_\ell/b_\ell)_{\ell=1,\ldots,L}$. Such mappings will be *external* to each T_i and to $PRA(E)$, and they will be used below (propositions P.1 and P.2) to argue that an external observer can establish a meaningful connection between the topological and the logical descriptions of rational economic agents. Finally, note that, given any operation (like sum $+$, or product $\cdot$) and relation (like greater than $>$) between ratios in $\mathbb{Q}$, we shall use the corresponding starred symbols for operations ($+^*$, $\cdot^*$) and relations ($>^*$) between terms of the type quot (App.: Defs B.2, B3, B.4). Similarly, we define operations and relations between term of the form *L*-vect, the symbols of which will be double-starred to mimic operations (like sum, scalar product, etc.) or relations (like greater than, or equal to) among vectors in $\mathbb{Q}^L$ (App.: Defs B.6, B.7, B.8, B.9, B.10).

3.2. *Choice sets*

We begin by endowing each agent with a suitable mechanism to store information. This is done by

DEFINITION D.3 (commodity choice function): *Within* T_i, *a suitable set of axioms* $\Omega_i \subset \mathcal{A}(T_i)$ *defines a function* $f\colon \mathbb{I}\times\mathbb{N}\times\mathbb{L}\mapsto\mathbb{N}$. *The function f does not belong to the* *language of PRA(Z), and we call it the commodity choice function.*

The intended meaning of D.3 is as follows: given $f(i;n,\ell)$, $i \in \mathbb{I}$ identifies the agent at hand, while $\ell \in \mathbb{L}$ is a commodity index. The second argument of f is a natural number $n \in \mathbb{N}$ which identifies a *memory cell* of agent i: our agent is supposed to be able to consider several statements together, and store them using natural numbers as indices. The value q of f is a natural number, that is, a $PRA(E)$-term of the quot form as defined in AD.1. Thus $f(i;n,\ell) = q$ means: agent i considers a quantity of commodity ℓ equal to the rational number identified by the term q. The latter is just a way to assign a label to i's choices: though immaterial to our logical framework, it is the key to our connection between D.1 and D.2. In general, f is defined by a set of suitable axioms common to all i's, plus a set of specific axioms defining the properties of f for each $i \in \mathbb{I}$. Thus $f(i;n,\ell)$ will differ from $f(j;n,\ell)$ – this is what makes the former the choice function of T_i, as opposed to that of T_j. Note also that f need not be recursive – should it be so, as we shall see, it could not express the features of agent T_i as opposed to those of other agents. Euristically, f does not describe actual (or computable) choices: it is a storing mechanism we use to define the logical equivalent to the standard choice set. This is done in the following

DEFINITION D.4 (choice-set theorems): *For any given memory cell $n \in \mathbb{N}$, a choice-sentence of T_i (cs_i) is a sentence of the form*

$$f(i;n,1) = r_1 \ \wedge ... \wedge f(i;n,L) = r_L$$

where r_ℓ is a term of the form $\mathrm{quot}(a_\ell, b_\ell)$. *A choice-set theorem of T_i is a theorem of T_i which is a cs_i.*

The set of choice-set theorems which can be proved by T_i, $\mathbb{C}_i$ say, is clearly a subset of $\mathrm{Th}(T_i)$. It includes a denumerable infinity of such theorems. Indeed, by running from $n = 1$ to $n = \infty$, we map the whole set of possible choices known to T_i thanks to his storing mechanism. The following observations are worth stressing:

REMARK R.1: If T_i is consistent, for any given memory cell n there is at most one choice-set theorem: indeed, from $f(i;n,\ell) = r$ and $f(i;n,\ell) = s$ (with r and s demonstrably different from one another), we can easily derive a contradiction. Notice also that:

(a) By suitable existential quantifiers, T_i can prove choice-set theorems of the form

$$\exists x_1 ... \exists x_L (q_1 \leq^* x_1 \leq^* r_1 \wedge ... \wedge q_L \leq^* x_L \leq^* r_L \wedge f(i;n,1) = x_1 \wedge ... \wedge f(i;n,L) = x_L),$$

where we recall that starred relations are for quot terms (App., Def B.4). The above are statements identifying intervals within which the quantity r_ℓ of commodity ℓ can vary.

(b) We can define a recursive predicate

$$ch_i(k) \leftrightarrow \text{"}k \text{ is the G-number of a } cs_i\text{"},$$

which expresses choice sentences. Whenever no confusion is arising, we shall use indifferently a sentence A and its code $\#(A)$: thus $ch_i(A)$ is read as "A is a cs_i". Clearly, we can represent the L-uple $(r_1, ..., r_L)$ of quot terms within a cs_i with a recursive term of the form L-vect, as defined in AD.2 (App.: Def B.12).

Before establishing an explicit connection between the set of choice-set theorems $\mathfrak{C}_i$ and the standard choice set $X_i \subseteq \mathbb{R}^L$, we still need a further technical step. This is provided by the following

AUXILIARY DEFINITION AD.3: *Let the following be defined:*

(*i*) *a function* E^q*: Im* quot $\mapsto \mathbf{Q}$, *which assigns to each* quot *term of* $PRA(E)$ *the correspondiong reduced fraction in the set of rationals;*

(*ii*) *a function* E^L*: Im* ℓ-vect $\mapsto \mathbf{Q}^L$, *which assigns to each* L-vect *term of* $PRA(E)$ *the corresponding vector in* $\mathbf{Q}^L$*;*

(*iii*) *the geometric part of any given* cs_i, *A say,* geo($\#A$), *as the recursive term* ℓ-vect($\#(r_1, ..., r_L)$)*;*

(*iv*) *the geometric part of the set* $\mathfrak{C}_i$ *of choice-set theorems of* T_i, Geo($\mathfrak{C}_i$), *as the set of* $\mathbf{Q}^L$*-vectors that an external operator who includes* $PRA(E)$ *can associate to the* geo($\#A$) *of all* cs_i*'s A in* $\mathfrak{C}_i$*: this the operator can do, by using the* E^q *and* E^L *mappings defined above* (App.: Prop B.2 and Def B.11).

(*v*) *Then we say that* $\{A_k\}_{k \in N}$, $A_k \in \mathfrak{C}_i$, *is a Cauchy sequence of choice-set theorems of* T_i *iff* $\{E^L(\text{geo}(\#A_k))\}_{k \in N}$ *is a Cauchy sequence of points in* $\mathbf{Q}^L$.

The prefix *Im* recalls that the concerned mapping is the image of the logical term in the space of rationals. We are now in the position to make precise how one can pass from the logical framework $\mathfrak{C}_i$ to the topological description X_i, and viceversa. This is done in the following two propositions. First we show that given a formal system T_i, we can associate to it a standard consumption set:

PROPOSITION P.1 (Choice set): *Let* T_i *be consistent. Then we can associate to* T_i *a subset* X_i *of* $\mathbb{R}^L$, *such that:*

(*i*) X_i *is closed on* $\mathbb{R}^L$*;*

(*ii*) *given any point* $x \in X_i$, *either x is the image through* E^L *of the geometric part of a choice-set theorem* $t \in \mathfrak{C}_i$, *or there exists a Cauchy sequence of choice-set theorems of* T_i *such that* $\{E^L(\text{geo}(\#A_k))\}_{k \in N}$ *tends to x in the Euclidean metric of* $\mathbb{R}^L$.

PROOF: By definition Geo($\mathcal{C}_i$) is a subset of $\mathbb{Q}^L$; we can then apply (outside PRA) the canonical procedure of completing a metric space, by adding the limits of Cauchy-sequences of points in Geo($\mathcal{C}_i$). As is well known from standard topology, we obtain a closed set X_i in $\mathbb{R}^L$, so the thesis is proved by construction.□

We shall dwell on the possibility of linking to T_i some specific features of X_i (like convexity) in Sections 3.4 and 4, after studying the logical features of T_i. The following proposition connects the standard topological decription of an economic agent to his formalization as a logical system.

PROPOSITION P.2 (Choice set): *Let the geometric description of an economic agent be identified by a pair* $(\succeq_i, X_i)$, *with* $X_i \subset \mathbb{R}^L$, *closed and including a dense subset with coordinates in* $\mathbb{Q}$, *and* $\succeq_i \subset X_i \times X_i$ *a binary preference relation, as from* D.1. *Then we can define (outside* PRA*) a* T_i *satisfying definitions* D.2, D.3 *and* D.4, *which is consistent and such that:*

(i) X_i *is the Euclidean metric completion of* Geo($\mathcal{C}_i$), *and for every* $x \in X_i$ *there is a Cauchy sequence* $\{A_k\}_{k \in N}$ *of choice-set theorems of* T_i *such that* $\{E^L(\text{geo}(\#A_k))\}_{k \in N}$ *tends to* x *in the Euclidean metric of* $\mathbb{R}^L$;

(ii) Geo($\mathcal{C}_i$) *is ordered by a preordering which is isomorphic to* $\succeq_i$.

PROOF: Let H_i be a subset of X_i with coordinates in $\mathbb{Q}$, such that its metric completion is X_i. For any $x \in H_i$ consider the ℓ-vect term (AD.2), corresponding to x in $PRA(E)$: let this be L-vect($\#(\text{quot}(c_\ell, d_\ell)_{\ell = 1, ..., L})$). As H_i is denumerable, we can index these terms in a sequence $\{t_k\}_{k \in N}$ of terms in $PRA(E)$. We now add to the axioms of $PRA(Z)$ the denumerable set of axioms $\Omega_i = \left\{ \wedge_{r = 1, ..., L}(f(i; k, r) = \pi_r^{L**}(t_k)) : k \in \mathbb{N} \right\}$, which define the commodity choice function f for an agent T_i: f is a function letter external to the language of $PRA(Z)$, and π_r^{L**} is a recursive operation which projects the r-th component of L-vect terms (App.:, Def B.9). Clearly, f is a function, and $PRA(Z)$ cannot prove the negation of the axioms in Ω_i. Hence, the system $T_i = PRA(Z) + \Omega_i$ is consistent by the Reduction Theorem (Shoenfield, 1967, p.42).[4] By construction, it satisfies conditions (i) and (ii): indeed, there results Geo($\mathcal{C}_i$) $= H_i$, and $\succeq_i$ orders also the subset H_i of X_i.□

Propositions P.1 and P.2 throw a bridge between the standard description of economic agents and our logical treatment of individual rationality. However, it should be noticed that the features of T_i pinned down by proposition P.2 become relevant from a logical point of view only when endowed with other properties (discussed later), with respect to which the axiomatization given by P.2 is too coarse: e.g., T_i as it results from P.2 is *not* in general recursively axiomatizable, which however turns out to be an essential property for an effective logical treatment of economic agents. Notice finally that endowing T_i with these additional properties will not jeopardize the results established in propositions P.1 and P.2.

We devote the rest of this section to endow the commodity choice function f (defined in D.3) with some logical properties which make it fit for our analysis: these will be embodied in the axioms set $\mathcal{A}(T_i)$. In particular, we shall assume that agent i's choice-set theorems *cannot* be theorems of the basic logical system $PRA(E)$, which is tantamount to make the following basic

ASSUMPTION A.1 (Properties of f): *The cs function f is in general not recursive, and hence agent T_i cannot be represented as a Turing machine* (Barwise, 1977, ch.C1, p.530). *The graph $\{f(i;k,r)=b\}$ of $f(i;\cdot,\cdot)$, resulting from T_i's choice-set theorems, is not $PRA(E)$-decidable, nor is the set $\mathfrak{C}_i$ of choice-set theorems of T_i. This implies that the system $PRA(E)$ is not able to establish whether any given cs_i of T_i is a theorem of T_i.*

We recall that a set Γ of sentences is $PRA(E)$-decidable iff$PRA(E)$ proves the graph of its chracteristic function $\chi_\Gamma(\cdot)$, where χ_Γ is so defined: $\chi_\Gamma(m)=1$ iff m is the G-number of a sentence which is in Γ, $\chi_\Gamma(m)=0$ otherwise.

Now notice that the very simple structure of a cs_i does not imply the information it can convey is as simple. Indeed, the opposite is true, since we are working with logical systems embodying the arithmetization of their own metatheories. Thus the atomic formula $f=k$ may include information which is very complex: as an example, consider the function defined very simply as

$f(n)=1$ if n is the G-number of a theorem of the Set Theory ZF;

$f(n)=0$ if n is not one such G-number;

This is a function of the language of Arithmetic which would encompass all the information on ZF (Devlin, 1979; Barwise, 1977, ch.B1), albeit our choice-set theorems will not reach this limit); also, notice that a number n can be the Gödel code of a formula complex *ad libitum*, which can be unambiguously retrieved starting from n. Thus the structure of choice-set theorems is apparently simple, but the syntax we use is so designed as to include the metatheory of formal systems which are recursively axiomatizable. A formula like $f=a$ will typically imply within T_i some complex statement about goods or agents: indeed, it will be natural to stipulate that (the description of) all peculiarities of the rational agent T_i be concentrated in his choice-set theorems – which will accordingly include what the basic logical system $PRA(E)$ does not say. In essence, as we shall see, f will be the main channel through which we formalize the *logical inconsistency* of different agents, which we see as the basis of economic competitition (Section 4). As a final remark, notice that assumption A.1 rules out that individual choice be predictable, in the sense that there is no method of Turing-computing them (Barwise, 1977, ch.C1): rational choice does not mean computable choice.

3.3. *Preferences*[5]

We now turn to the preferences of our agent, starting from the following

DEFINITION D.5 (preferences): *Within* T_i, *a suitable set of axioms* $\Phi_i \subset \mathcal{A}(T_i)$, *called preference axioms of* T_i, *defines a binary predicative letter* $P_i(\cdot, \cdot)$. *For each* i, Φ_i *includes the following axioms:*

(*i*) $\forall x(L\text{-vect-f}(x) \rightarrow P_i(x,x))$;

(*ii*) $\forall x\forall y\forall z(L\text{-vect-f}(x) \wedge L\text{-vect-f}(y) \wedge L\text{-vect-f}(z) \wedge P_i(x,y) \wedge P_i(y,z) \rightarrow P_i(x,z))$

(*iii*) $\forall x\forall y(\sim L\text{-vect-f}(x) \vee \sim L\text{-vect-f}(y) \rightarrow (P_i(x,y) \leftrightarrow \bot))$

Moreover, Φ_i *can include axioms which specify the particular properties of* P_i *in* T_i.

REMARK R.2: The predicate L-vect-f$(\cdot)$, defined in the Appendix (Def B.13), means: ℓ-vect-f$(t) \leftrightarrow$ "t is a term of form L-vect". The intended meaning of $P_i(\cdot, \cdot)$ is a property of pairs of terms corresponding to vectors in $\mathbb{Q}^L$ which we use to denote quantity of goods. Thus the meaning of (*ii*) is the standard axiom of transitive preferences. The properties of P_i mentioned in D.5 select the area in $\mathbb{Q}^L$ over which T_i 's preferences are established. Notice that we did *not* rule out recursiveness of P_i, on which later.

We can now connect definitions D.4 (choice-set theorems) and D.5 (preferences) to define a theorem which assigns a predicate letter P_i to pairs of T_i choice-sentences: thus we formalize the idea that preference relations are applied to alternatives in the set of choice-set theorems. Accordingly, we have

DEFINITION D.6 (preference theorems): *Let* A *and* B *be choice-sentences for* T_i. *Then*

(*i*) *a preference theorem is a* T_i*-theorem of the form:*

$$\mathrm{Pr}_{T_i}(A) \wedge \mathrm{Pr}_{T_i}(B) \rightarrow P_i(\mathrm{geo}(A), \mathrm{geo}(B));$$

(*ii*) *an* ***actual*** *preference theorem is a* T_i*-theorem the form:*

$$P_i(\mathrm{geo}(A), \mathrm{geo}(B)).$$

REMARK R.3: We recall that geo(A) is the recursive term ℓ-vect$(\#(r_1, ..., r_L))$, which represents the geometric part of the sentence A: $f(i;n,1) = r_1 \wedge ... \wedge f(i;n,L) = r_L$, with

$r_\ell = \mathrm{quot}(a_\ell, b_\ell)$ (App.: Def B.12). We also recall that $\mathrm{Pr}_{T_i}(\gamma)$ means $\exists x \mathrm{Prov}_{T_i}(x, \#(\gamma))$: there exists a proof of γ in T_i.

Through its actual preference theorems, T_i imposes a preference preordering over vectors in $\mathbb{Q}^L$, representing quantity of goods: in general, all this is independent of the axioms in Ω_i which in D.3 define commodity choice functions, and hence of the set of choice-set theorems, $\mathfrak{C}_i$. Also, we do not require of P_i any of the peculiar properties which were typical of the commodity choice function f: indeed, the predicate P_i may be recursive. Finally, we notice that T_i may impose the preordering over sectors in $\mathbb{Q}^L$ *external* to Geo($\mathfrak{C}_i$); conversely, nothing so far implies that Geo($\mathfrak{C}_i$) be ordered by P_i. We shall introduce in section 3.4 a completeness axiom for P_i over Geo($\mathfrak{C}_i$) (Axiom 7), and we shall show (Section 4) that it is consistent with what we impose on T_i.

3.4. *Choice sets and preferences: summary*

We now want to take stock of the formalization of economic agents built so far. To do so, we first give a summary of the axioms and rules of inference which are to be seen as standard in our framework (Definition D.7). We then gather some examples of statements that an economic agent can make and, finally, we add to the list of standard axioms a further set of axioms (Definition D.8), which may be used to describe specific geometric properties – whose compatibility with the system T_i we shall prove in general. Our standard rules and axioms are defined in

DEFINITION D.7 (Standard axioms and rules): *For every agent T_i, we define the following T_i-axioms and rules as standard:*

AX.1: *All axioms and inference rules required to prove all theorems of $PRA(E)$, all theorems of the standard Provability Logic* (Smorynsk, 1985, p.9; Barwise, 1977, p,827) *for the predicate $\mathrm{Pr}_{T_i}(\cdot)$, and,* a fortiori, *all theorems of the first order predicate calculus in the language $\mathfrak{L}$* (Shoenfield, 1967, ch.2). *There follows that every T_i can avail itself of an axiom or rule of induction over atomic formulae* (Smorynski, 1985, p.21; Takeuti, 1987, p.76).

AX.2: *The commodity choice axioms in Ω_i (definition* D.4*), where the term $f(i;x,y)$ is defined; as already established (A.1), the graph of f is not PRA-provable. The individual features of T_i are embodied in f, so that on these axioms will rest the inconsistency among agents.*

AX.3: *The preference axioms in Φ_i. These include the preordering axioms (definition* D.5*), which establish the properties of the preference predicate $P_i(\cdot,\cdot)$. Such axioms may be $PRA(E)$-provable whenever preferences are recursive.*

AX.4: *The finite set of axioms* Com_i, *which describe commodities and can be common to all* T_i*'s. For every commodity, they define a predicate* ℓ-com(t), *to be read as "t is a recursive term representing a quantity of commodity* ℓ*". There is no particular constraint on this set of axioms.*

Finally, we can sum up the significance of our sets of axioms in the following

ASSUMPTION A.2: *The axioms from* AX.1 *to* AX.4 *must be such as to make* T_i:[6]

(*i*) *recursively axiomatizable;*

(*ii*) *undecidable and sy-incomplete;*

(*iii*) *PRA*(*E*)*-undecidable*

From the set of standard axioms and rules, T_i can derive his theorems. However, these will not be confined to choice-set theorems; indeed the standard axioms, the language of *PRA*(*E*) and T_i's inferential apparatus make it possible that T_i say something *about* his choices. In particular, recall from Section 2.1 that T_i includes *PRA*(*E*): hence, it can represent a metatheory, and in T_i a provability predicate can be defined like $\mathrm{Pr}_{T_k}(b) \leftrightarrow$"$b$ is the G-number of a theorem B of", for every agent T_k of E (Smorynski, 1985 and 1977; Gentilini, 1992 and 1997). Moreover, if T_i proves the theorems of the standard provability logic for the predicate $\mathrm{Pr}_{T_i}(\cdot)$, then $\mathrm{Pr}_{T_k}(\cdot)$ becomes a mathematically sound epistemic operator: the formula $\mathrm{Pr}_{T_k}(B)$ may be interpreted as "T_k believes B". Hence, T_i may "think" about his own choices and utter statements where someone else's choice occur. However, T_i being syntactically able to state something about the knowledge of other agents T_k *does not imply* (as will be clear in section 4) *that he posesses a semantically correct knowledge of the other agents, neither does it imply strategic interaction.*[7]

We collect here some simple examples of statements an agent can make:

EXAMPLE E.1: $\forall x \exists y (f(x;y,1) = b_1 \wedge ... \wedge f(x;y,L) = b_L)$, which reads as: For every agents there is a memory cell where the choices of the L goods take the values $(b_1, ..., b_L)$.
EXAMPLE E.2: $\forall x \exists y (f(3;u,x) = y \rightarrow f(7;u,x = -2y))$, which reads as: Given the memory cell u, the quantity chosen by agent $i = 3$ will be twice that of agent $i = 7$, with opposite sign.
EXAMPLE E.3: $\mathrm{Pr}_{T_8}($5-com$(t) \wedge t = 6 \rightarrow \forall x(f(3;x,7) = 10) \rightarrow (\exists y(f(9;y,1) = 66 \vee \mathrm{Pr}_{T_7}($5-com$(t) \wedge t = 44)$, which may be put as: Suppose agent $i = 8$ thinks that the quantity of commodity 5 being 6 implies that agent 3's choice on commodity 7 will always be 10:

then either there is a memory cell such that agent 9's choice on commodity 1 is 66, or agent 7 thinks that commodity 5 is 44"

We now show that we can add to the standard axioms, by making axioms assigning geometric properties to choice-set theorems and preferences of T_i. One of our key points is consistency of these axioms with each other and with the standard axioms.[8] Also, it is important to show how non-trivial geometric conditions can be translated within $PRA(E)$, to allow a canonical projection of the logical structure of our agent into a geometric structure (including assumptions like convexity, etc.).

DEFINITION D.8 (Geometrical T_i-axioms): *The axioms of T_i which define the geometrical properties of the set of choice-set theorems and of preferences:*

AX.5 (convexity of choice-set theorems): $\forall x \forall y \forall w \forall z\{[\mathrm{Pr}_{T_i}(x) \wedge \mathrm{Pr}_{T_i}(y) \wedge ch_i(y) \wedge ch_i(x) \wedge \mathrm{geo}(x) \neq {}^{**}\mathrm{geo}(y) \wedge \text{quot-f}(w) \wedge L\text{-vect-f}(z) \wedge \mathrm{quot}(0,1) < {}^{*}w < {}^{*}\mathrm{quot}(1,1) \wedge z = {}^{**}(w\S^{**}\ \mathrm{geo}(x) + {}^{**}(\mathrm{quot}(1,1) - {}^{*}w)\S^{**}\mathrm{geo}(y))] \rightarrow \exists v[ch_i(v) \wedge \mathrm{Pr}_{T_i}(v) \wedge (\mathrm{geo}(v) = {}^{**}z)]\}$, which reads as: "for any pair of distinct choice-set theorems of T_i, x and y, and for any vector $z \in \mathbb{Q}^L$ lying on the segment between geo(x) and geo (y), there is a choice-set theorem of T_i, v, such that its geometric part geo(v) equals z". This axiom yields a set Geo($\mathbb{C}_i$) which is convex, and hence a convex choice-set X_i.

AX.6 (convexity of preferences): $\forall x \forall y \forall w \forall z\{[L\text{-vect-f}(x) \wedge L\text{-vect-f}(y) \wedge P_i(x,y) \wedge \text{quot-f}(w) \wedge L\text{-vect-f}(z) \wedge \mathrm{quot}(0,1) < {}^{*}w < {}^{*}\mathrm{quot}(1,1) \wedge z = {}^{**}(w\S^{**}x + {}^{**}(\mathrm{quot}(1,1) - {}^{*}w)\S^{**}y)] \rightarrow P_i(z,y)\}$, which reads as: "If T_i prefers the vector y to x in $\mathbb{Q}^L$, then any $\mathbb{Q}^L$-vector on the segment connecting x to y is preferred to x".

AX.7 (completeness of preferences on Geo($\mathbb{C}_i$)): $\forall x \forall y[\mathrm{Pr}_{T_i}(x) \wedge \mathrm{Pr}_{T_i}(y) \wedge ch_i(x) \wedge ch_i(y) \rightarrow (P_i(\mathrm{geo}(x), \mathrm{geo}(y)) \vee P_i(\mathrm{geo}(y), \mathrm{geo}(x)))]$, which reads as: "If x and y are choice-set theorems of T_i, they are ordered by P_i". This axioms establishes that the preordering P_i be complete on the region of $\mathbb{Q}^L$ selected by the choice-set theorems of T_i.

AX.8 (non-satiation on Geo($\mathbb{C}_i$)): $\forall x\{\mathrm{Pr}_{T_i}(x) \wedge ch_i(x) \rightarrow \exists y[ch_i(y) \wedge \mathrm{Pr}_{T_i}(y) \wedge \mathrm{geo}(x) \neq {}^{**}\mathrm{geo}(y) \wedge P_i(\mathrm{geo}(x), \mathrm{geo}(y))]\}$, which read as: "For any choice-set theorem x of T_i there exists another choice-set theorem y which is preferred to x".

AX.9 (boundedness of choice set, Ax-lim$_i$): $\exists y\{\text{quot-f}(y) \wedge \forall x[\mathrm{Pr}_{T_i}(x) \wedge ch_i(x) \rightarrow (\mathrm{geo}(x) \cdot {}^{**}\mathrm{geo}(x) < {}^{*}(y \cdot {}^{*}y))]\}$, which reads as: "There exists an element y of $\mathbb{Q}$ which bounds the norm of any choice-theorem x of T_i". This axiom establishes that the choices of T_i be within a disk in $\mathbb{Q}^L$, which yields a bounded Geo($\mathbb{C}_i$) and hence a compact X_i, given that the latter is closed by construction (proposition P.1).

AX.10 (non-boundedness of choice set): $\sim$ Ax-lim$_i$, which corresponds to the negation of the former axiom and yields an unbounded X_i in $\mathbb{Q}^L$.

AX.11 (existence of a P_i-maximum under constraint): $\forall x \forall y \forall u[\mathrm{Pr}_{T_i}(u) \wedge ch_i(u) \wedge L$-vect-f$(x) \wedge$ quot-f$(y) \wedge A(x,y) \wedge \mathrm{geo}(u) \cdot {}^{**}x \leq {}^{*}y \wedge (\exists a \mathrm{Pr}_{T_i}(a) \wedge\ ch_i(a) \wedge \mathrm{geo}(a) \cdot {}^{**}x = {}^{*}y)$ $\rightarrow \exists z\ (\mathrm{Pr}_{T_i}(z) \wedge ch_i(z)\ \ \wedge \mathrm{geo}(z) \cdot {}^{**}x \leq {}^{*}y \wedge P_i(\mathrm{geo}(u), \mathrm{geo}(z)))]$, which reads as: "Given the $\mathbb{Q}^L$-vector x and the $\mathbb{Q}$-scalar number y, if the hyperplane $\nu \cdot {}^{**}x = {}^{*}y$ (ν is a vector of variables in $\mathbb{Q}^L$) intersects the set Geo($\mathfrak{C}_i$) of choice-set theorems of T_i at least in the choice-set theorem a, then there exists a choice-set theorem z of T_i which lying under the given hyperplane, where $A(x,y)$ is a formula imposing some constraints on the area of $\mathbb{Q}^L$ within which the constraint hyperplane is allowed to vary". Notice that for any axiomatization of T_i, $A(x,y)$ is meant to be fixed. This axiom establishes that, given a constraint on T_i's choices, the choice-set theorems satisfying that constraint have a P_i-maximum. This axiom plays a key role when one discusses optimal consumption plans under wealth constraints (Benassi and Gentilini, 1997).

AX.12 (existence of a viable choice-set theorem): $\forall x \forall y[\ell$-vect-f$(x) \wedge$ quot-f$(y) \wedge A(x,y)$ $\rightarrow \exists z(\mathrm{Pr}_{T_i}(z) \wedge ch_i(z) \wedge \mathrm{geo}(z) \cdot {}^{**}x \leq {}^{*}y)]$, which reads as: "Given any $\mathbb{Q}^L$-vector x and any $\mathbb{Q}$-scalar number y, there exists a choice-set theorem z of T_i which lies under the hyperplane $\nu \cdot {}^{**}x = {}^{*}y$ (where ν is a vector of variables in $\mathbb{Q}^L$), where $A(x,y)$ is a formula imposing some constraints on the area of $\mathbb{Q}^L$ within which the constraint hyperplane is allowed to vary". Again, for any axiomatization of T_i $A(x,y)$ is meant to be fixed. This axiom guarantees that, given a constraint on the set of choice-set theorems, there is at least one of the latter which satisfies it.

REMARK R.4: The predicate quot-f($\cdot$), similarly to L-vect-f($\cdot$) mentioned in remark R.2, is formally defined in the Appendix (Def B.13): it is a recursive predicate such that quot-f(t) means "t is the code of a term of the quot form".

REMARK R.5: Axioms AX.11 and AX.12 are meaningful only insofar as the restriction $A(x,y)$ is given on the constraint.

4. *On consistency and further extensions: the set of different agents and the logical existence of an exchange economy*

In this section we want to define some basic logical features of the set of formal systems $E = \{T_1, ..., T_I\}$, thought of as representation of an economy. The first requirement is that any system (agent) T_i be *consistent*: this will be proved in the following theorem TH.1, and it is the foremost requiremenent for E to exist. From the theorem of existence of a model (Theorem I, Section 2.1), there follows that *each system (agent)* T_i *admits of at least one model in the canonical semantics for first order theories*; that is, for any T_i there is at least one world within which all theorems of T_i are true.

As to the mutual consistency among agents, we require that *systems (agents) be in general inconsistent with each other: for aver pair i,k,* $T_i \cup T_k$ *is inconsistent.* This accords to intuition: there are at least two agents with at least a pair of inconsistent ideas. Indeed, the Craig-Robinson theorem (Shoenfield, 1967, p.79) ensures that there are incompatible opinions between inconsistent agents:

THEOREM V (Craig-Robinson): *If* T_i *and* T_k *are first-order theories and* $T_i \cup T_k$ *is inconsistent, then there exists a sentence A such that* $\vdash_{T_i} A$ *and* $\vdash_{T_k} \sim A$.

Actually, in general we would think of a competitive economy as a social system populated by agents with different and inconsistent beliefs. Our formal language does include this possibility. For example,

EXAMPLE E.4 (inconsistent beliefs about commodities): (2-com(t) $\wedge\, t = 6$)→(3-com(y) $\wedge\, y = 8$), that is, "if commodity 2 is 6, then commodity 3 is 8"; (2-com(t) $\wedge\, t = 6) \wedge \sim$ (3-com(y) $\wedge\, y = 8$), that is "commodity 2 is 6 and commodity 3 is not 8".
EXAMPLE E.5 (inconsistent beliefs about behaviour): (2-com(t) $\wedge\, t = 5$) →$\forall x(f(70,x,3) = 20)$, that is, "if commodity 2 is 5, the choice of agent 70 on commodity 3 is 20"; (2-com(t) $\wedge\, t = 5) \wedge \forall x(f(70,x,3) = 56)$, that is, "commodity 2 is 5 and the choice about it of agent 70 is 56".

The agents' mutual inconsistency will be expressed formally by the commodity choice function f, as will be clear from the proofs of theorems TH.1 and TH.2, below. As a consequence, the agents' inconsistency with each other (and its holding with respect to the whole Th(T_i), i.e., with respect to the whole of each agent) translates into our logical framework the idea that society admits competition among them. Of course, competition does not rule out for significant portions of Th(T_i) to be consistent across i's: indeed, an equilibrium for our economy may be definied as a conjunction of all agents' optimal choices that are consistent (Benassi and Gentilini, 1997). One consequence of the agents' inconsistency is that *the system formed by the union of the* T_i*'s has no model*, that is, there exists no world where *all* theorems Th(T_i), $i = 1,..., I$, hold true simultaneously. And actually, therein lies the significance of any equilibrium existence proof, following the above definition.[9]

Consistency of each agent and inconsistency across agents do not exhaust the preliminary requirement we ask of a definition of the economy E. We still have to be precise about each agent's proving capabilities with respect to other agents. We already know *via* Gödel's theorems (Smorynski, 1977) that no T_i can prove his own consistency $Coer(T_i)$; but clearly no equilibrium is meaningful in any sense if any agent has so strong a proving ability, as to be able to prove the others' consistency. Thus we make

ASSUMPTION A.3 (metatheoretical parity hypothesis): *No agent T_i can prove metatheoretical properties of T_k, which T_k is unable to prove himself – even though can state them in his language. In particular:*

(*i*) *no agent can prove another agent's consistency or inconsistency;*

(*ii*) *no agent can prove $\exists z Coer(T_z)$, i.e. the existence of at least a consistent agent.*

Without the metatheoretical parity hypothesis (MPH), the formal systems (agents) we are concerned with would lose their logical homogeneity.[10] Finally, we repeat the conditions already mentioned in Section 3: the language of the T_i's is common to all – all can avail themselves of the language $\mathfrak{L}$. Indeed, it is having the same syntax that makes it meaningful to speak of different beliefs.[11]

We can now sum up all this in the following

DEFINITION D.9 (economy): *An economy E is a I-uple of formal systems $\{T_1, ..., T_I\}$, such that:*

(*i*) *each T_i includes rules and axioms of the form from* AX.1 *to* Ax.12 *established in* D.7 *and* D.8, *with assumption* A.2 *and is consistent;*

(*ii*) *for any pair i, k, $T_i \cup T_k$ is inconsistent;*

(*iii*) *all $T_1, ..., T_I$ satisfy* MPH (Assumpion A.3);

(*iv*) *the language of every T_i $(i = 1, ..., I)$ is $\mathfrak{L}$* (Definition D.2).

We say that these are necessary condition for the existence of the economy E. Accordingly, the following are logical existence theorems for E.

THEOREM TH.1 (existence of agents): *Consider a Euclidean space of commodities of dimension $L \subset \mathbb{N}$, with $L > 1$, finite and arbitrarily fixed. Then there exists a denumerable infinity of systems T_i, different and with the same language $\mathfrak{L}$, such that any T_i contains axioms and rules of the form from* AX.1 *to* AX.12 *with the exclusion of* AX.9, *or else of the form from* AX.1 *to* AX.12 *with the exclusion of* AX.10 (*definitions* D.7 *and* D.8), *and*

T_i *is consistent.*

PROOF: The proof is by construction: we construct a T_i satisfying our requirements and prove its consistency; then we show that one can build a denumerable infinity of such T_i.

A. We first build a T_i by considering the alternative excluding AX.9. Let $V_1, \ldots, V_L$ be L consistent and $PRA(E)$-undecidable theories: suppose that V_ℓ be Arithmetics with the induction rule restricted to at most $L+1$ quantifiers (see Takeuti, 1987, p.116); then each V_ℓ is consistent. Indeed, given the quasi-constructive proof by transfinite induction on numerable ordinals provided by Gentzen for the consistency of PA Arithmetic (Takeuti, 1987, p.101), and given that PA is an oversystem for all V_ℓ's, we can consider the consistency of V_ℓ syntactically proved. Moreover, since $PRA(E)$ is a conservative extension (Shoenfield, 1967, p.41) of $PRA(Z)$ *via* only the language $\mathcal{L}$, and $PRA(Z)$ is a subsystem of all the V_ℓ's, by definition $PRA(Z)$ can prove neither the graph of the characteristic function of V_ℓ's theorems, nor V_ℓ's consistency.

Given $i \in \{1, \ldots, I\} = \mathbb{I}$ (where the index i identifies the ith agent), we can define L new functional letters ψ_ℓ: $\mathbb{N} - L\text{-VECT} \mapsto \mathbb{N}$, where $L\text{-VECT} = \{k \in \mathbb{N}: L\text{-vect-f}(k) \text{ holds}\}$ is the set of terms which codify vectors and $\ell \in \{1, \ldots, L\} = \mathbb{L}$, given by

$$\psi_\ell(z) = i \qquad \text{iff } z \text{ is a code of a formula and } \mathrm{Pr}_{V_\ell}(z);$$

$$\psi_\ell(z) = I + 1 \qquad \text{iff } z \text{ is a code of a formula and } \sim \mathrm{Pr}_{V_\ell}(z);$$

$$\psi_\ell(z) = 0 \qquad \text{otherwise;}$$

ψ_ℓ is a version of the characteristic function of the theorems of V_ℓ, and we include the definition of ψ_ℓ in AX.2 of T_i.

We now define a new functional letter g: $\mathbb{I} \times \mathbb{N} \times \mathbb{L} \mapsto \mathbb{N}$, having fixed i in the first factor of the domain:

$$g(i,k,\ell) = \mathrm{quot}(2^{2k}5^{\ell}13^{i},\ 1) \qquad \text{iff } \psi_\ell(k) = i \quad \text{and } \sim L\text{-vect-f}(k);$$

$$g(i,k,\ell) = \mathrm{quot}(3^{2k}7^{\ell}11^{i}, 1) \qquad \text{iff } \psi_\ell(k) = I + 1 \quad \text{and } \sim L\text{-vect-f}(k);$$

$$g(i,k,\ell) = 0 \qquad \text{iff } L\text{-vect-f}(k).$$

We include the definition of g in AX.2 of T_i. Notice that, by the properties of Z as a factorial ring, the numbers $2^{2k}5^{\ell}13^{i}$ and $3^{2k}7^{\ell}11^{i}$ belong to disjoint sets, which are not empty for any k, ℓ, i. Hence, also $g(i,k,\ell)$ can be seen as a version of the characteristic function of the theorems of V_ℓ. It is apparent that both definitions of ψ_ℓ and g can be immediately translated into a finite number of $PRA(E)$-formulae.

We now give the axioms defining the commodity choice function f, which identify T_i as an individual. Given i in the first factor of the domain, the functional letter $f: \mathbb{I} \times \mathbb{N} \times \mathbb{L} \mapsto \mathbb{N}$ is given by the following procedure. We shall divide the second factor in the domain of f in a finite number of disjoint parts $\{M_s\}$, and define f on $\mathbb{I} \times M_s \times \mathbb{L}$, supposing we have defined f on $\mathbb{I} \times M_{s-1} \times \mathbb{L}$; moreover, each set M_s can be described as a formula of $PRA(E)$:

STEP 1: let M_1 be the part of $\mathbb{N}$ given by "$(u \in M_1)$ iff $\sim \ell$-vect-f(u)"; we then define:

$$f(i,u,\ell) = g(i,u,\ell) \quad \text{iff} \quad u \in M_1$$

which we can translate as a $PRA(E)$-formula:

$$f(i,u,\ell) = g(i,u,\ell) \leftrightarrow \sim L\text{-vect-f}(u)$$

(we shall denote the formula $\sim L$-vect-f(u) with $A1(u)$). Notice that the values of f on $\mathbb{I} \times M_1 \times \mathbb{L}$ are terms of the form quot, corresponding (*via* the function Ext defined in the Appendix) to fractions of the form $n/1$, i.e. to integers. There follows that, if the memory cell $u \in M_1$, then agent T_i's choice-set theorems

$$f(i,u,1) = q_1 \wedge ... \wedge f(i,u,L) = q_L$$

(notice that in such a choice-sentence the cell u must be the same for all joined terms) select terms $q_1, ..., q_L$ of the quot form which, *via* the function Ext (App.) yield points of $\mathbb{Q}^L$ with integer co-ordinates, corresponding to values of the function g. Clearly, these are points in the positive orthant of $\mathbb{Q}^L$, not lying within a bounded region. Hence, they are a part of Geo($\mathbb{C}_i$) with integer co-ordinates, yielding a discrete subset of $\mathbb{Q}^L$.

STEP 2: Let M_2 be the part of $\mathbb{N}$ given by "$(u \in M_2)$ iff $A2(u)$", where $A2(u)$ is the following formula of $PRA(E)$:

$$L\text{-vect-f}(u) \wedge \{\exists z \exists y \exists w[\sim L\text{-vect-f}(z) \wedge \sim L\text{-vect-f}(y) \wedge x \neq {}^{**}y \wedge \text{quot-f}(w) \wedge$$
$$\text{quot}(0,1) < {}^{*}w < {}^{*}\text{quot}(1,1) \wedge u = {}^{**}(w\S^{**}L\text{-vect}(\#((g(i,z,1),...,g(i,z,L))) +^{**}$$
$$(\text{quot}(1,1) - {}^{*}w)\S^{**}L\text{-vect}(\#((g(i,y,1),...,g(i,y,L))))] \vee B2(u)\},$$

and $B2(u)$ is the formula

$$\exists y \exists w[\sim \text{quot-f}(y) \wedge \text{quot-f}(w) \wedge \text{quot}(0,1) < {}^{*}w < {}^{*}\text{quot}(1,1) \wedge$$
$$u = ((\text{quot}(1,1) - {}^{*}w)\S^{**}L\text{-vect}(\#((g(i,y,1),...,g(i,y,L))))].$$

We can now define

$$f(i,u,\ell) = \pi_\ell^{L**}(u) \quad \text{iff} \quad u \in M_2,$$

which we can translate as a $PRA(E)$-formula

$$f(i,u,\ell) = \pi_\ell^{L**}(u) \leftrightarrow A2(u).$$

(we recall that π_ℓ^{L**} is the operation of projection; see App.). Thus, $A2(u)$ selects the terms u of ℓ-vect form, which identify a point of $\mathbb{Q}^L$ lying on the line joining two points of $\mathbb{Q}^L$, which in turn are the geometric part of the choice-set theorems given by the values of f defined in STEP 1; $B2(u)$ adds, for technical reasons, the points on the lines joining the values of f defined in STEP 1 with the origin $(0,...,0)$. There follows that, if the memory cell $u \in M_2$, then T_i's choice-set theorems

$$f(i,u,1) = q_1 \wedge ... \wedge f(i,u,L) = q_L$$

select the terms $q_1,...,q_L$ of the form quot which, *via* the function Ext (Appendix), yield a point of $\mathbb{Q}^L$ lying on the line joining two points of Geo($\mathbb{C}_i$) defined in STEP 1.

STEP 3: Let M_3 be the part of $\mathbb{N}$ given by "$(u \in M_3)$ iff $A3(u)$", where $A3(u)$ is the following formula of $PRA(E)$:

$$L\text{-vect-f}(u) \sim A1(u) \wedge \sim A2(u) \wedge \{\exists z \exists y \exists w[A2(z) \wedge A2(y) \wedge z \neq {}^{**}y \wedge \text{quot-f}(w)$$
$$\wedge \quad \text{quot}(0,1) < {}^*w < {}^*\text{quot}(1,1) \wedge u = {}^{**}(w\S^{**}L\text{-vect}(\#((f(i,z,1),...,f(i,z,L))) +^{**}$$
$$(\text{quot}(1,1) - {}^*w)\S^{**}L\text{-vect}(\#((f(i,y,1),...,g(i,y,L))))]\},$$

(notice that in STEPs 1 and 2, f has been defined over $\mathbb{I} \times (M_1 \cup M_2) \times \mathbb{L}$, with formulae of $PRA(E)$ and hence we can use it to define f over $\mathbb{I} \times M_3 \times \mathbb{L}$ in this part of the domain: the situation is the same as that of the definition of a function F with the classical recursion clause, where the definition of F over further parts of the domain is based on the definition of F over the former parts). We can now define

$$f(i,u,\ell) = \pi_\ell^{L**}(u) \quad \text{iff} \quad u \in M_3,$$

which we can translate as a $PRA(E)$-formula

$$f(i,u,\ell) = \pi_\ell^{L**}(u) \leftrightarrow A3(u).$$

That is, $A3(u)$ selects the terms u of form L-vect, which identify a point of $\mathbb{Q}^L$ lying on the line joining two points of $\mathbb{Q}^L$, which in turn are the geometric part of the choice-set theorems given by the values of f defined in STEP 2.There follows that, if the memory cell $u \in M_3$, then T_i's choice-set theorems

$$f(i,u,1) = q_1 \wedge ... \wedge f(i,u,L) = q_L$$

select the terms $q_1,...,q_L$ of the form quot which, *via* the function Ext (Appendix), yield a point of $\mathbb{Q}^L$ lying on the line joining two points of Geo($\mathbb{C}_i$) defined in STEP 2.

STEP 4: Finally, let M_4 be the part of $\mathbb{N}$ given by "$(u \in M_4)$ iff $A4(u)$", where

$A4(u)$ is the following formula of $PRA(E)$:

$$\sim (A1(u) \vee A2(u) \vee A3(u)).$$

We can now define

$$f(i,u,\ell) = \text{quot}(0,1) \quad \text{iff} \quad u \in M_4,$$

which we can translate as a $PRA(E)$-formula

$$f(i,u,\ell) = \text{quot}(0,1) \leftrightarrow A4(u).$$

There follows that, if the memory cell $u \in M_4$, then T_i's choice-set theorems

$$f(i,u,1) = q_1 \wedge \ldots \wedge f(i,u,L) = q_L$$

select $q_\ell = \text{quot}(0,1)$ for all ℓ, and hence its geometric part corresponds to the origin $(0,\ldots,0)$ of $\mathbf{Q}^L$.

This concludes the definition of f, that is the construction of AX.2. *By construction, f is defined by a recursive set of formulae in $\mathfrak{L}$; moreover, Geo($\mathfrak{C}_i$) is by construction a convex and unbounded subset of $\mathbf{Q}^L$. Finally, the graph of f is $PRA(E)$-unprovable and hence, a fortiori, f is not recursive*: indeed, if $PRA(E)$ was able to prove the graph of f, it should also prove it for its restriction g; however, this is impossible: as we have seen, for any given ℓ, g is a version of the characteristic function of the theorems of a theory V_ℓ which is $PRA(E)$-undecidable.

We now define the preference axioms AX.3: they define a predicative letter P_i (different from all those already in $PRA(Z)$), include the pre-ordering axioms given in D.5, and also include the following, which specifies preferences for T_i:

$$\text{ch}_i(x) \wedge \text{ch}_i(y) \wedge P_i(\text{geo}(x), \text{geo}(y)) \leftrightarrow (\text{geo}(x) \cdot {}^{**}\text{geo}(x)) \leq {}^{*}(\text{geo}(y) \cdot {}^{**}\text{geo}(y))$$

This states that any two vectors of $\mathbf{Q}^L$ are preferred by T_i according to the increasing order of their euclidean norm. The corresponding indifference surfaces in $\mathbf{Q}^L$ are ball-shaped and centered in the origin. P_i is recursive, but this is allowed by the definition of AX.3. Moreover, we notice that P_i *induces a convex preordering on* Geo($\mathfrak{C}_i$), *as such is the preordering of euclidean norms of vectors in* $\mathbf{Q}^L$. Also, P_i *induces a complete preordering on* Geo($\mathfrak{C}_i$), *as such is the preordering of euclidean norms of vectors in* $\mathbf{Q}^L$. *Finally, given unboundedness of* Geo($\mathfrak{C}_i$), *such preordering is unsatiable.*

We now give the commodity axioms of AX.4. They are actually arbitrary, and it is enough to give an example which is consistent:

$$\forall w[\exists y(f(k,y,w+1) = \text{quot}(2,1) \cdot {}^{*}b) \leftrightarrow (w\text{-com}(t) \wedge t = b)],$$

where b is a fixed closed term of the form quot, which states the following property for all commodities: "for any w, the w-indexed commodity equals b iff there is a memory cell y such that in y for the commodit index $w+1$ agent k chooses the quantity $2b$".

We can now take stock of our results so far, and state that T_i *with its axioms from* AX.1 *to* AX.4 *is consistent and satisfies assumption* A.1, *that is, it is recursively axiomatizable, undecidable, sy-incomplete,* $PRA(E)$*-undecidable; moreover, its consistency is based upon a syntactical proof by induction on denumerable ordinals.* Indeed, we can assume $PRA(E)$ is proved to be consistent *via* Gentzen's syntactic proof by induction on denumerable ordinals up to ϵ_0 (Takeuti, 1987, p.10). Moreover, the defining axioms of f are consistent with each other, since f is really a function and a point in the domain is not given different values. $PRA(E)$ cannot prove the negation of a T_i-theorem of the form $f = t$ which is not a $PRA(E)$-theorem, since it has no specific information on the functional letter f; *a fortiori*, if $f = t$ was a $PRA(E)$-theorem, it would not prove its negation due to its consistency. Hence, by the reduction theorem (Shoenfield, 1967, p.42), $PRA(E) +$ AX.2 is consistent. Likewise, $PRA(E) +$ AX.2 + AX.3 and $PRA(E) +$ AX.2 + AX.3 + AX.4 are both consistent: in both cases the extended system lacks the necessary information to prove the negation of the axioms extending it. Hence, T_i's consistency depends crucially on the consistency of $PRA(E)$, which is proved. Also, T_i is recursively axiomatizable, since $PRA(E)$ is recursively axiomatized and the function f is defined from $PRA(E)$ starting from a finite set of axioms. Finally, T_i is undecidable and *sy*-incomplete as an extension of $PRA(E)$; by construction of AX.2, is also $PRA(E)$-undecidable.

We now take up consistency when including the geometric axioms AX.5 to AX.9. Notice that by construction Geo($\mathfrak{C}_i$) obeys all such axioms; such construction can certainly be formalized within the ZF set theory(Devlin, 1979), so that ZF is able to yield a model of

$$T_i = PRA(E) + \sum_{k=1}^{8} \text{AX}.k + \text{AX}.10,$$

which accordingly is consistent.

We now take up consistency when also including AX.11 and AX.12. This is meaningful only if we impose restrictions on the constraint, fixing the formula $A(x, y)$ within each axiom. To fix ideas, *we suppose the constraint hyperplanes in* $\mathbb{Q}^L$ *cut the positive orthant in such a way that the origin lies below.* Then, since Geo($\mathfrak{C}_i$) is within the positive orthant and it includes the origin, for given constraint hyperplanes AX.11 is satisfied. As to AX.12, *we shall further suppose that constraint hyperplanes in* $\mathbb{Q}^L$ *cut all the edges of the positive orthant, having positive coordinate for each point of edge-intersection.* Then the convexity of Geo($\mathcal{C}_i$), together with its unboundedness, imply at least one maximum-norm point among all those which lie below the constraint. Further, given an arbitrary constraint hyperplane *via* a suitable form for $A(x,y)$ in AX.11 and AX.12, it would be possible to assign values of f, *with no substantial change in the structure of the definition of f given here*, in such a way that AX.11 and AX.12 be satisfied. From these constructive observations we can conclude that

$$PRA(E) + \sum_{k=1}^{8} \text{AX}.k + \text{AX}.10 + \text{AX}.11 + \text{AX}.12$$

is consistent.

B. There remains to look at T_i in the case where the unboundedness axiom AX.10 is replaced by the boundedness axiom AX.9. It is enough to notice that f can be so defined as to have Geo($\mathcal{C}_i$) within a unit disk. This can be accomplished with the following change in the definition of g:

$$g(i,k,\ell) = \text{quot}(1, 2^{2k}5^{\ell}13^{i}) \qquad \text{iff } \psi_\ell(k) = i \text{ and } \sim L\text{-vect-f}(k)$$

$$g(i,k,\ell) = \text{quot}(1, 3^{2k}7^{\ell}11^{i}) \qquad \text{iff } \psi_\ell(k) = I + i \text{ and } \sim L\text{-vect-f}(k)$$

$$g(i,k,\ell) = 0 \qquad \text{iff } \sim L\text{-vect-f}(k),$$

while leaving everything else unchanged. Thus Geo($\mathcal{C}_i$) will be convex and bounded in $\mathbb{Q}^L$. From this one can proceed as before, although the satiation axiom can conflict with the given P_i. This can be overcome with a variant of P_i given by the inverted order of norms, i.e., preferences increase with the smallness of the norm, as one gets nearer to the origin; we can then change (with no substantial effect) the definition of f, so as to erase the terms quot$(0,n)$ from its values. As the origin would be the only satiation point in in this inverted order, we would get consistency with the satiation axiom.

C. We now take up the existence proof for a denumerable infinity of T_i's. This can be obtained by requiring one more prime number in the product appearing in the definition of g, so as to get a denumerable infinity of distinct variants of $g \equiv g_1$. That is,

$$g_2(i,k,\ell) = \text{quot}(2^{2k}5^{\ell}13^{i}\ 17,\ 1) \qquad \text{iff } \psi_\ell(k) = i \text{ and } \sim L\text{-vect-f}(k)$$

$$g_2(i,k,\ell) = \text{quot}(3^{2k}7^{\ell}11^{i}\ 19,\ 1) \qquad \text{iff } \psi_\ell(k) = I + i \text{ and } \sim L\text{-vect-f}(k)$$

$$g_2(i,k,\ell) = 0 \qquad \text{iff } \sim L\text{-vect-f}(k),$$

....

$g_n(i,k,\ell) = \text{quot}(2^{2k}5^{\ell}13^{i}\ 17 \ldots p_n,\ 1)$ iff $\psi_\ell(k) = i$ and $\sim L$-vect-f(k)

$g_n(i,k,\ell) = \text{quot}(3^{2k}7^{\ell}11^{i}\ 19 \ldots q_n,\ 1)$ iff $\psi_\ell(k) = I + i$ and $\sim L$-vect-f(k)

$g_n(i,k,\ell) = 0$ iff $\sim L$-vect-f(k),

....

where (p_n, q_n) are pairs of successive prime numbers starting from $(p_2, q_2) = (17, 19)$. To each different g_j there corresponds a different definition of f, and hence a T_i with a different commodity choice function.□

The above theorem establishes the one can actually build agents obeying our standard axioms. One corollary of the theorem is that agents of such a kind admit of a topological representation:

COROLLARY C.1 *Let* Geo($\mathbb{C}_i$) *be built as in theorem* TH.1. *Then its metric completion* X_i *in the Euclidean topology of* $\mathbb{R}^L$ *satisfies:*

(i) X_i *is convex and closed in* $\mathbb{R}^L$*;*

(ii) *if* Geo($\mathbb{C}_i$) *is bounded, so is* X_i*, which would accordinly be compact;*

(iii) *the completion of the preordering* P_i *yields on* X_i *the same preordering as the Euclidean norms, which is convex and continuous.*

PROOF This follows from the canonical properties of of the metric completion in $\mathbb{R}^L$ with the Euclidean topology of a convex subspace with co-ordinates in $\mathbb{Q}^L$.□

We are now in the position to establish the existence of an economy as defined in D.9, which is done in the following

THEOREM TH.2 (existence of an economy) *For any arbitrarily fixed natural number* $I > 1$ *there exists a denumerable infinity of I-uples of formal systems* $\Gamma_j = \{T_1^j, \ldots, T_I^j\}$, *where* T_i^j *is consistent and satisfies* TH.1, *such that* Γ_j *satisfies definition* D.9, *that is*

(i) *for any pair* $i, k \in \mathbb{I} = \{1, \ldots, I\}, i \neq k$, T_i^j *and* T_k^j *are inconsistent;*

(ii) Γ_j *satisfies MPH (assumption* A.3*);*

(iii) *given* j*, the language of all* T_i^j*'s is the same and it coincides wih* $\mathfrak{L}$.

Accordingly, there exist one economy E_j.

PROOF Consider a formal system T_i^j as it results from theorem TH.1, where the index j identifies the infinite forms of T_i which can be obtained by varying g_j. Notice that, as i

goes from 1 to I, the defining axioms of the function letter ψ_ℓ assign different values to ψ_ℓ, corresponding to the same points in the domain, that is,

for $i = 1$, $\psi_\ell(z) = 1$ iff z is a code of a formula and $\mathrm{Pr}_{V_\ell}(z)$

$\psi_\ell(z) = I + 1$ iff z is a code of a formula and $\sim \mathrm{Pr}_{V_\ell}(z)$

$\psi_\ell(z) = 0$ otherwise

for $i = 2$, $\psi_\ell(z) = 2$ iff z is a code of a formula and $\mathrm{Pr}_{V_\ell}(z)$

$\psi_\ell(z) = I + 2$ iff z is a code of a formula and $\sim \mathrm{Pr}_{V_\ell}(z)$

$\psi_\ell(z) = 0$ otherwise

....

for $i = I$ $\psi_\ell(z) = I$ iff z is a code of a formula and $\mathrm{Pr}_{V_\ell}(z)$

$\psi_\ell(z) = I + I$ iff z is a code of a formula and $\sim \mathrm{Pr}_{V_\ell}(z)$

$\psi_\ell(z) = 0$ otherwise.

For fixed j, this yields I inconsistent and different T_i^j's, since (e.g.) $\psi_\ell(z) = 1 \wedge \psi_\ell(z) = 2$ is obviously a contradiction. Moreover, as j varies (i.e., for different g_j's as in the proof of TH.1), we shall have a denumerable infinity of I-uples $\Gamma_j = \{T_1^j, ..., T_I^j\}$ of agents who will be different and inconsistent with each other. This proves (i).

As to (ii), notice that a consistency proof for T_i, as the latter emerges from TH.1, is independent of the index i: indeed, our consistency proof is an extension (independent of i) of the Gentzen consistency proof for PA (Takeuti, 1987, p.101), the form of which is known. Hence, from a formalized consistency proof for T_s one obtains a formalized consistency proof for T_k, simply by uniformly replacing s with k. But then T_i does not prove $Coer(T_k)$ for any k: should it do so, it could derive $Coer(T_i)$ therefrom – that is, its own consistency, against Gödel's theorems (Theorem IV) (Smorynski, 1977). Moreover, the structure isomorphism among the consistency proofs of the T_i's can be described within Provability Logic demonstrable in PRA (Smoynski, 1985, p.9; 1977, p.837): hence, for every $k \in \mathbb{I}$ it is the case that

$$\vdash_{PRA} (\exists z\, Coer(T_z)) \leftrightarrow Coer(T_k).$$

However, T_i includes PRA and so it cannot prove $\exists z\, Coer(T_z)$: if so, it could derive $Coer(T_i)$, against Gödel's theorems. So the I-uple $\{T_1^j, ..., T_I^j\}$ satisfies MPH.

As to (iii), notice that the language $\mathcal{L}$ is indeed common: the functional letters

ψ_ℓ, g, f occur in the language of all T_i's.□

To conclude, we stress that our necessary conditions for the existence of an economy – concerning both T_i's standard axioms and his choice-set theorems – rely on a consistency proof which is syntactical and quasi-constructive, by induction of the denumerable ordinals. This somehow strengthens our existence statements, giving them a degree of effectiveness which is the maximum compatible with agents being non-recursive.

5. *Concluding remarks*

One of the basic methodological principles of economics is that choice should be looked at as the outcome of rational pondering by purposeful individuals. Historically, this idea has been made precise by the Samuelsonian approach: a rational agent is formalized as maximizing some objective function, subject to (environment-driven) constraints – that is, rationality amounts to a formal criterion which sets rules for choosing among numbers. In this paper we suggest that rationality is best seen as a set of properties of (first-order) formal system in a mathematical-logical sense: accordingly, a rational agent is someone able to draw valid inferences, where the standard of validity is set by basic logical methods. One immediate, yet interesting, consequence of this is having a formal distinction between statements known to economic agents, and statements concerning them.

This approach encompasses the standard geometric description of economic agents as maximizers. Indeed, the connection with the Samuelsonian approach has been emphasized: in particular, we have showed that it is always possible for an external observer to translate the axiomatic description of economic agents into a precise topological framework, and *viceversa*. Also, we are able to show that for any given finite Euclidean commodity space, there exists a denumerable infinity of different consistent formal systems, which can be so translated. An existence proof is also provided for the corresponding formal definition of an economy as a set of first-order formal systems which are inconsistent with each other.

Our formalization of economic agents as consistent formal systems is proposed as a first methodological steps, which should hopefully help our understanding of the nature of economic competition and the relationship among agents. Indeed, our research agenda plays on the formalized notion of truth typical of logical formal systems, in order to study the degree of consistency among agents implied by competitive equilibria.

APPENDIX

In this appendix we develop some definitions and propositions which are used in text. The basic system $PRA(E)$ including $PRA(Z)$ (Forcheri, Gentilini, Molfino, 1996, sec.2) – that is, Arithmetic extended to integers – makes it possible to avail ourselves of terms which may represent all negative integers. We now introduce in $PRA(E)$ recursive terms, which codify in an injective manner the fractions a/b of the set Q of rational numbers, as well as the vectors of fractions in the $\mathbf{Q}^L$ L-dimensional space. We shall also suitably define some recursive operations, reproducing the standard arithmetic operations between fractions in Q and vectors in $\mathbf{Q}^L$. All of which allows us to represent in the language of $PRA(E)$ the geometric part of any agent T_i's choice theorems. We want the latter to be generally vectors in $\mathbf{Q}^L$, and not necessarily of integers, as we believe it important that agent be allowed to choose among fractions – not least, this allows a simpler connection between our logical approach and the standard topological approach. The main instrument we use to this end is gödelization. In particular, we recall that one can attribute a G-number $\#(e_1,...,e_n)$ to any finite sequence of expressions $\{e_j\}_{j=1}^n$ in the language of $PRA(E)$ (smorynski, 1977, p.829).

DEFINITION B.1: Let *quot* be a function letter different from all those in $PRA(Z)$; we extend to it the canonical code #, chosen for the expressions in the language of $PRA(Z)$, and define over the set of integers, as $quot(\cdot,\cdot)$, a recursive binary function letter in the following way.

Let $a,b,p,q,..$ be ground (i.e., terms without variables) $PRA(E)$-terms representing integers, which w.l.o.g. can be read as numerals of $PRA(Z)$. (For ease of exposition, we shall identify the numerals $a,b,p,q,...$ with the numbers they represent; also, we stipulate that whenever a fraction p/q is considered, the fraction is reduced and the denumerator is always positive). Then:

(*i*) quot(0,0) = #(quot,1,1);

(*ii*) if a is a numeral not corresponding to 0, quot(a,0)=#(quot,2^{10000},1);

(*iii*) if b is a numeral not corresponding to 0, quot(0,b)=#(quot, 0 , 1);

(*iv*) let (a,b) be pairs of numerals representing integers, prime between them, different from zero and with b positive. Then

quot(a,b) = #(quot, a,b), if neither a nor b are numerals which codify sequences of the form (quot,m,n), with m and n numerals prime between them and n positive.

quot(a,b) = #(quot,u,v), where u/v is the reduced form of m/nb, if a is #(quot,m,n), with m and n numerals prime between them, n positive, and b is not.

quot(a,b) = #(quot, u,v) where u/v is the reduced form of an/m, if b is #(quot,m,n), with m and n numerals prime between them, n positive, and a is not.

quot$(a,b) = \#$(quot,u,v), where u/v is the reduced form of mt/ns, if a is $\#$(quot,m,n), with m and n numerals prime between them, n positive, and b is $\#$(quot,s,t), s and t prime between them and t positive;

(v) given a pair of numerals (s,t) representing non-nul numbers which are non-prime between them, or else with t negative, quot(s,t) equals quot(a,b), where a/b is the reduced form of the fraction p/q;

(vi) If r,s are ground terms which are not numerals of $PRA(Z)$, then quot(r,s) equals quot(p,q), where p is a numeral which is $PRA(Z)$-equal r, and q is $PRA(Z)$-equal to t;

The values of the function quot$(\cdot,\cdot)$ are also called *terms of the quot form*: they can only have the form $\#$(quot,m,n), with m and n representing integers prime between them, with positive n (except for cases B.1(ii) and B.1(iii) above). The definition of quot$(\cdot,\cdot)$ is such that the function behaves itself as intended; eg, quot(quot(c,d),quot(e,f)) codifies the fraction corresponding to the division between the two fractions represented by quot(c,d) and quot(e,f). The definition quot$(a,b)=\#$(quot,a,b) would be consistent, but less precise. Given the value in $\mathbb{N}$ of the function quot(a,b), an external operator Ext (defined below) who knows $\mathbb{Q}$ is able to associate to it a reduced fraction p/q by means of an effective procedure.

PROPOSITION B.2: *The function Ext: Im* quot$\mapsto\mathbb{Q}$, *which associates to every value of the function quot the reduced fraction p/q corresponding to it, is surjective.* The proof is obvious; notice that an integer m is given by the corresponding term $\#$(quot,m, 1).

DEFINITION B.3: We define as operations O^* in $PRA(E)$ the binary functions $+^*(\cdot,\cdot)$, $\cdot^*(\cdot,\cdot)$, $-^*(\cdot,\cdot)$, $:^*(\cdot,\cdot)$, as follows:

(i) if s is $\#$(quot,m,n) and t is $\#$(quot,r,h) – both of which are values of the function quot –, then $+^*(s,t)$ is $\#$(quot,u,v), with u/v is the reduced fraction $(mh+rn)/hn$; similarly one can define the product $\cdot^*(\cdot,\cdot)$, difference $-^*(\cdot,\cdot)$, and division $:^*(\cdot,\cdot)$, corresponding to the term quot(s,t).

(ii) if either s or t is not of the quot form, then $O^*(s,t) = 0$ for any operation O^*.

(iii) if c and d are ground terms which are not numerals of $PRA(Z)$, then for any operation O^*, $O^*(c,d) = O^*(s,t)$, where s and t are numerals $PRA(Z)$-equal to c and d.

DEFINITION B.4: We define as relations R^* in $PRA(E)$ the binary relations $=^*(\cdot,\cdot)$, $\neq^*(\cdot,\cdot)$, $>^*(\cdot,\cdot)$, $<^*(\cdot,\cdot)$, $\geq^*(\cdot,\cdot)$, as follows:

(i) Let s and t be numerals representing integers: then

if s is $\#$(quot,m,n) and t is $\#$(quot,r,h), both of which are values of the function quot, then $=^*(s,t)\leftrightarrow(m=r\wedge n=h)$; $=^*(s,t)\leftrightarrow\bot$, if either of them is not of the quot form;

$\neq^*(s,t)\leftrightarrow\sim\ =^*(s,t)$;

if s is $\#(\text{quot},m,n)$ and t is $\#(\text{quot},r,h)$, both of which are values of the quot function, then $>^*(s,t) \leftrightarrow (mh > nr)$(recall that n and h are positive by definition); if either of them is not of the quot form, then $>^*(s,t) \leftrightarrow \perp$;

$<^*(s,t) \leftrightarrow >^*(t,s)$;

$\geq^*(s,t) \leftrightarrow >^*(s,t) \vee =^*(s,t)$;

(ii) If c and d are ground terms which are not numerals of $PRA(Z)$, then $R^*(c,d) \leftrightarrow R^*(s,t)$ for any binary relation R^*, where s and t are numerals which are $PRA(Z)$-equal to c and d.

DEFINITION B.5 Given the L-dimensional commodity space, we now want to define terms representing vectors in $\mathbb{Q}^L$. We do so by considering the set of L ground terms of $PRA(E)$ $\{\omega = (u_1, ..., u_L)\}$. We define the following unary recursive function L-vect(·), from $\mathbb{N}$ to $\mathbb{N}$:

(i) if m is not the G-number of a sequence ω of L ground terms which are $PRA(Z)$-equal to numerals of the quot form, then $L\text{-vect}(m) = 0$;

(ii) $L\text{-vect}(m) = \text{quot}(a_1, b_1)$ if $m = \#(\omega)$ with $L = 1$ and $\omega = (\text{quot}(a_1, b_1))$, or $\omega = (u)$ with u ground term $PRA(Z)$-equal to $\text{quot}(a_1, b_1)$;

(iii) $L\text{-vect}(m) = \#(\omega)$ if $m = \#(\omega)$, with $\omega = (u_1, ..., u_L)$, where u_ℓ, $\ell = 1, ..., L$, are ground terms $PRA(Z)$-equal to $(\text{quot}(a_\ell, b_\ell)_{\ell=1}^L)$, $L > 1$.

(Notice that in general u, its code, and the code of the sequence (u) are different numbers). We call *terms of the L-vect form* the non-nul values of the function L-vect. We shall identify these terms, w.l.o.g., with the terms $L\text{-vect}(\#(\text{quot}(a_\ell, b_\ell)_{\ell=1}^L)$.

DEFINITIONS B.6, B.7, B.8, B.9 We now define operations O^{**} among L-vect terms, which represent scalar summation and scalar product between vectors, the scalar-vector product, the projection of the ℓ-component of an L-vector. We do so as follows:

DEFINITION B.6 We define in $PRA(E)$ the binary function $+^{**}(\cdot,\cdot)$, corresponding to vector summation between $\mathbb{Q}^L$-vectors: Let s, t be both terms of the L-vect form, where $s = L\text{-vect}(\#(\text{quot}(a_\ell, b_\ell)_{\ell=1}^L))$ and $t = L\text{-vect}(\#(\text{quot}(c_\ell, d_\ell)_{\ell=1}^L))$; $+^{**}(s,t)$ is then L-vect$(\#(\text{quot}(a_\ell, b_\ell) +^* \text{quot}(c_\ell, d_\ell))_{\ell=1}^L)$, where $+^*$ is defined in B.3; otherwise, $+^{**}$ is zero. (Notice that $+^{**}$ reduces to $+^*$ for $L = 1$).

DEFINITION B.7 We define in $PRA(E)$ the binary function $\cdot^{**}(\cdot,\cdot)$, corresponding to scalar product between $\mathbb{Q}^L$-vectors: Let s, t be both terms of the L-vect form, where $s = L\text{-vect}(\#(\text{quot}(a_\ell, b_\ell)_{\ell=1}^L))$ and $t = L\text{-vect}(\#(\text{quot}(c_\ell, d_\ell)_{\ell=1}^L))$; $\cdot^{**}(s,t)$ is then $\sum_{\ell=1}^{L} {}^*[(\text{quot}(a_\ell, b_\ell) \cdot^* \text{quot}(c_\ell, d_\ell)]$, where $\sum^*$ is summation with respect to operation $+^*$, and $\cdot^*$ is defined in B.3; otherwise $\cdot^{**}(s,t)$ is zero.

DEFINITION B.8 We define in $PRA(E)$ the binary function $\S^{**}(\cdot,\cdot)$, corresponding to scalar-vector multiplication in $\mathbb{Q}^L$: Let s be of the L-vect form, where $s = L$-

vect($\#(\text{quot}(a_\ell, b_\ell)_{\ell=1}^L)$), and t be of the form quot(c,d); $\S^{**}(s,t)$ is then L-vect($\#(\text{quot}(a_\ell, b_\ell) \cdot {}^*\text{quot}(c,d))_{\ell=1}^L$), where $\cdot^*$ is defined in B.3; otherwise, $\cdot^{**}$ is zero.

DEFINITION B.9 We define in $PRA(E)$ the operation π_r^{L**} of projection of the rth component over terms of the L-vect form, corresponding to the projection of the rth component of a vector in $\mathbb{Q}^L$: If t is L-vect($\#(\text{quot}(a_\ell, b_\ell)_{\ell=1}^L)$), then $\pi_r^{L**}(t)$ is quot(a_r, b_r); $\pi_r^{L**}(t)$ is zero otherwise.

DEFINITION B.10 We define relations R^{**} reproducing $=$ and $>$ between L-vectors. To do so, we define in $PRA(E)$ the binary relations $=^{**}(\cdot,\cdot)$, $\neq^{**}(\cdot,\cdot)$, $>^{**}(\cdot,\cdot)$, $<^{**}(\cdot,\cdot)$, $\geq^{**}(\cdot,\cdot)$, as follows:

(i) Let s and t be terms of the L-vect form, $s = L$-vect($\#(\text{quot}(a_\ell, b_\ell)_{\ell=1}^L)$) and $t = L$-vect($\#(\text{quot}(c_\ell, d_\ell)_{\ell=1}^L)$); then

$$=^{**}(s,t) \leftrightarrow \wedge_{\ell=1,\dots,L}(\text{quot}(a_\ell, b_\ell) =^* \text{quot}(c_\ell, d_\ell)), \text{ with } =^* \text{ defined in B.4;}$$

$$\neq^{**}(s,t) \leftrightarrow \sim\ =^{**}(s,t);$$

$$>^{**}(s,t) \leftrightarrow \wedge_{\ell=1,\dots,L}(\text{quot}(a_\ell, b_\ell) >^* \text{quot}(c_\ell, d_\ell)), \text{ with } >^* \text{ defined in B.4;}$$

$$<^{**}(s,t) \leftrightarrow >^{**}(t,s);$$

$$\geq^{**}(s,t) \leftrightarrow >^{**}(s,t) \vee =^{**}(s,t);$$

(ii) $\mathrm{R}^{**}(s,t) \leftrightarrow \perp$ otherwise.

PROPOSITION B.11 *The function Ext: Im L-vect$\mapsto\mathbb{Q}^L$, associating to every value of the function L-vect the corresponding vector w, is surjective.* An external operator *Ext*, which includes $PRA(E)$ and knows $\mathbb{Q}^L$, can associate $w \in \mathbb{Q}^L$ to the value in $\mathbb{N}$ of the function L-vect($\#\omega$), using an effective procedure.

DEFINITIONS B.12 and B.13 We now use the former definitions to select formally within $PRA(E)$ the geometric parts of the commodity choice sentences of our traders:

DEFINITION B.12 Let $A{:}\,f(i,n,1) = r_1 \wedge \dots \wedge f(i,n,L) = r_L$, with $r_\ell = \text{quot}(a_\ell, b_\ell)$, be a commodity choice sentence of trader T_i. Then the recursive term geo($\#A$) $= L$-vect($\#(r_1, \dots, r_L)$) is naturally defined, and indicated as the *geometric part* of the sentence A. (For ease of notation we often write geo(A) instead of geo($\#A$)).

DEFINITION B.13 We define the recursive predicates: (i) quot-f(t)$\leftrightarrow$ "t is the code of a term of the quot form"; (ii) L-vect-f(t)$\leftrightarrow$ "t is the code of a term of form L-vect". (For ease of notation, we often identify quot-f(t) and L-vect-f(t) with t and its code; thus, eg, a formula "quot-f(t) $\wedge A(t)$" will be read as "t is a term of the quot form, and for it the property A holds").

DEFINITION B.14 We would like to have terms of the quot form containing variables, so as to have quantify values of the choice function of agent T_i, and make use of *open* commodity choice formulae. To do so, we briefly recall some standard instruments in Provability Logic (Smorynski, 1985, 1977;

Gentilini, 1997). which are used to define a provability predicate for open formulae: indeed, we know that if A is an open formula, $\Pr(A)$ can be canonically defined, so that there results an open formula with the same free variables. (Smorinski, 1985, 41-3). The following recurisve functions may be defined in $PRA(Z)$:

subst$(x, y, z) =$ "code of what results from substituting the variable codified by y with the term codified by z, within the expression codified by x";

num$(v) =$ "code of the numeral of $PRA(Z)$ representing number v".

These being given, if $A(z)$ is a formula with the free variable z, the open term with respect to the variable z, subst$(\#A(z), \#z, \text{num}(z))$ (which we indicate as $\#A(\hat{z})$ for short), represents the code of the formula obtained from $A(z)$ by replacing the variable z with the numeral of z. Hence, through the instantiation of z by k, we obtain $\#A(\boldsymbol{k})$; $\#A(\hat{z})$ is accordingly an open term, which for any of its ground instances yields the corresponding ground instance of A. This can be extended also to open terms $t(y)$, and we call open codes the terms $\#A(\hat{z})$ and $\#t(\hat{y})$. As a result, the definition of the open term quot$(s(x), t(y))$, with x and y vectors of free variables, can be given as follows:

Take definition B.1 and interpret the codes therein as open codes: instead of, say, "s and t are numeral with the property $\Delta \subset \mathbb{Z} \times \mathbb{Z}$", one has "for all x and y such that $s(x)$ and $t(y)$ have the property $\Delta \subset \mathbb{Z} \times \mathbb{Z}$"; in the various cases, quot$(s(x), t(y))$ will be an open code of the form $\#(\text{quot}, u(\hat{x}), v(\hat{y}))$. Now notice that points (i) to (v) of B.1 give a partition of $\mathbb{Z} \times \mathbb{Z}$ which may be represented with the values from 1 to 5 of a suitable recursive characteristic function, $X: \mathbb{Z} \times \mathbb{Z} \mapsto \{1, 2, 3, 4, 5\}$. Hence B.1 can be so defined within $PRA(E)$:

(i) $X(s(x), t(y)) = 1 \leftrightarrow \text{quot}(s(x), t(y)) = \#(\text{quot}, 1, 1)$;

...

(iv) $X(s(x), t(y)) = 4 \leftrightarrow \text{quot}(s(x), t(y)) = \#(\text{quot}, u(\hat{x}), v(\hat{y}))$, where $u(x)$ and $v(y)$ are terms which are canonically built for every value of x and y, starting from $X(s(x), t(y)) = 4$;

and so on

By virtue of this definition, terms like, say, quot$(2 + 3y, x + z + 5)$, acquire a precise meaning. The open terms L-vect are obtained from open-codifying sequences of open terms of the quot form. Thus, given a sequence $\omega(x, y) = (\text{quot}(a_\ell(x), b_\ell(y))_{\ell=1}^{L})$, with x and y vectors of free variables, L-vect$(\#\omega(\hat{x}, \hat{y}))$ will be $\#(\text{quot}, (a_\ell(x), b_\ell(y))_{\ell=1}^{L})$.

FOOTNOTES

[1] See Devlin (1979, ch.2), Barwise (1977, ch.A1), Takeuti (1987, ch.1).

[2] Provability Logic is the branch of logic which studies the properties of the provability predicate $\mathrm{Pr}_S(\cdot)$, when S is a suitably chosen system within arithmetics. It is a very strong tool to represent self-reference: in our case, the self-reference of an agent to his own inferences (nota bibliografica).

[3] Although in this section we shall be concerned with one agent, indexing allows occasional reference to other agents. Interactions among agents are touched upon in Section 4. The set $\mathbb{I}$ form which i is drawn is a finite subset of $\mathbb{N}$.

[4] A remark on notation: is we extend a formal system T with other proper axioms Ax and inference rules Ru, the resulting formal system is written for short $T + Ax + Ru$.

[5] From now on, the following convention is useful: in writing formulae, $\sim$ link more than $\vee$ and $\wedge$, which in turn link more than $\leftrightarrow$ and $\rightarrow$. Parentheses will be omitted accordingly. E.g., $((\sim A) \wedge B) \rightarrow C$ is written $\sim A \wedge B \rightarrow C$.

[6] On recursively axiomatizable ssystems, see Section 2.1 and Shoenfield (1967, p.125); undecidable systems have been defined in section 2.1 (see also Shoenfield, 1967, p.123); see also the discussion on assumption A.1 for a definition of PRA-undecidable.

[7] Although in principle this should be the starting point for studying strategic interaction among first-order formal systems.

[8] It might perhaps be possible to show that some of these axioms (and hence, geometrical properties) can be derived as theorems starting from other axioms.

[9] In Benassi and Gentilini (1997) we show that for each agent there is at leat one world where the conjunction of all agents' optimality theorems is true: although traders are globally inconsistent, they can choose (optimal) choice-set theorems which, once aggregated, turn out to be consistent with each other and with respect to each agent. Although there exist no model of the union of all agents, there exist at least one non-trivial model of the equilibrium situation. It is worth stressing again that the competitive allocation is in that model.

[10] Euristically, MPH may mean something like "no agent is god for other agents", or "no agent can be sure about another being in a sane or insane state of mind".

[11] What an agent proves is (metatheoretically) what he knows. Thus agent T_i being able to prove theorems where any other T_k's action or predicates appear, together with his knowing only what he can prove, implies no actual knowledge of T_k on T_i'part. Indeed, one can support this statement with a technical note: within E, the union $T_i \cup T_k$ is inconsistent. Hence, it has no model in the formal sense – there is no world where T_i's statements about T_k, Th(T_i), Th(T_k) are simulteaneously true.

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