Equity, Envy, and Independence of Irrelevant Alternatives

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Abstract

We show that any social choice function that always selects envy-free Pareto efficient allocations must violate Arrow's condition of independence of irrelevant alternatives.

INTRODUCTION

In a pure exchange economy, individual i envies individual j if $u_i(x_i) < u_i(x_j)$, where x_i denotes i's consumption bundle and u_i his utility function. According to Foley (1967), an allocation is fair (or envy-free) if no individual envies anybody else. Clearly, this concept of equity does not rest on any kind of interpersonal utility comparison.

In a pure exchange economy, envy-free and Pareto efficient allocations exist under relatively mild conditions (Varian, 1974). It is obvious that the condition of no-envy implies nondictatorship (with monotonic preferences, a dictator would get the whole aggregate endowment and would be envied by anybody else). Thus the fairness approach seems to avoid Arrow's impossibility.

How is this possible? This note shows by example that the condition of no-envy is in fact inconsistent with Arrow's (1963) condition of independence of irrelevant alternatives.

Formally, let X be the universal set of alternatives and S any feasible subset of X. Let $u = (u_1, ..., u_n)$ be a preference profile, that is a list of n utility functions, one for each individual. A social choice function $C(u, S)$ assigns to each admissible profile u and each feasible S a non empty subset $C(u, S)$ of S. Think of $C(u, S)$ as the set of the choosable alternatives from S at u. Arrow's condition of independence of irrelevant alternatives says that if two profiles u and u' have the same restriction to S, then $C(u, S) = C(u', S)$.

Then we have:

Theorem 1. *Any social choice function that always selects fair and Pareto efficient allocations must violate Arrow's independence condition.*

Proof. Consider a pure exchange economy with two individuals and two goods. Figure 1 illustrates. H denotes the midpoint of the Edgeworth box (that is, equal division of the aggregate endowment). Indifference curves are linear. With linear utility functions, it can be easily shown that, for each individual, the indifference curve passing through H coincides with his fairness boundary. For instance, consider 1's indifference curve AA'. Then, any point lying above AA' is fair to 1. And any point below 2's indifference curve BB' is fair to him. Thus the set of fair allocations corresponds to the triangle ABH. The set of fair and Pareto efficient allocations is the line segment AB.

Next suppose that individual preferences change to CC' for individual 1 and DD' for individual 2. The new set of fair and Pareto efficient allocations is now the line segment CD.

Consider a subset of the set of all feasible allocations like the one corresponding to the line segment MN. Clearly, individual preferences on the set MN have not changed (indeed, individual preferences on MN are determined just by the condition that preferences are monotonic). Yet the choice set out of MN must change from some subset of AB to some subset of CD. This contradicts Arrow's condition of independence of irrelevant alternatives. ■

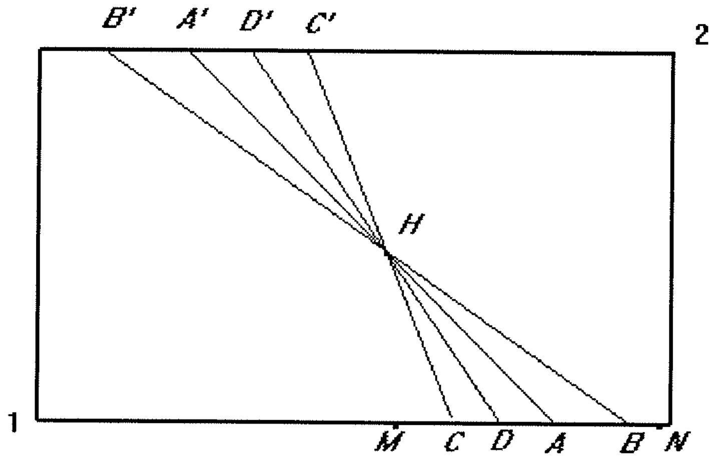

Figure 1

Remark 1. As point H in figure 1 is the only allocation that is envy-free at both profiles, the condition of Pareto optimality could be replaced by the much weaker condition that the equal division allocation cannot be uniquely chosen.

Remark 2. Our example requires two individuals only. With two individuals, many variants of the no-envy condition that have been proposed in the literature collapse to no-envy, and so our example shows that they also violate Arrow's independence of irrelevant alternatives. This applies to such concepts as balanced envy, coalitional envy-freeness, and per-capita fairness (see Thomson and Varian 1985 for definitions).

Remark 3. We have referred Foley's condition of no-envy to social choice functions, i.e. collective choice rules that to any preference profile assign a non empty choice set. However, it could be argued that the Foley criterion is naturally defined on the universal set of all feasible allocations only. Indeed, if one restricts attention to arbitrary subsets of the universal set of alternatives, then fair allocations need not even exist. Thus one is led to interpret the fairness criterion in terms of a social choice correspondence, that is a social choice function the agenda domain of which is restricted so as to contain only the universal set of alternatives.

Clearly, in such a fixed agenda framework Arrow's condition of independence of irrelevant alternatives is always vacuously satisfied. However, one may define intra-agenda conditions that are in the same spirit as Arrow's independence of irrelevant alternatives, and may be considered its fixed agenda relatives. One such condition is the Hansson (1969) independence condition: Given any two alternatives $x, y \in X$, if x is chosen and y is rejected at some profile u, and if the restriction of profile u' to the pair $\{x, y\}$ coincides with that of u, then y cannot be chosen at profile u'. It can be shown that this condition is precisely the fixed agenda counterpart of Arrow's independence of irrelevant alternatives (see Denicolò, 1993).

Our example can be easily shown to imply that any social choice correspondence that always selects Pareto efficient and envy-free allocations must violate Hannson's independence condition. It suffices to consider an allocation (lying on the segment AB) chosen at the first profile and denote it x, and another allocation, denoted y, chosen at the second profile (and therefore lying on the segment CD) and check that Hansson's independence is violated. Remarks 1 and 2 can also be extended to a fixed agenda framework.

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