On the Dichotomy between Horizontal and Vertical Product Differentiation*

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Abstract

In this paper, we show that, contrarily to the conclusions obtained by Cremer and Thisse [1991], the equilibrium emerging from a model of vertical product differentiation does not yield maximum differentiation because of the strategic complementarity between products in the quality space.

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I. Introduction

In a recent paper, Cremer and Thisse [1991] study the relationship between the horizontal product differentiation models *à la* Hotelling and the vertical product differentiation models, showing that the usual specification of the Hotelling model can be conceived as a special case of a vertical differentiation model. The main consequence of this formal equivalence is that the same equilibrium should emerge from both.

Our aim is to show that Cremer and Thisse's result heavily depends on an assumption which is not likely to hold in a model of vertical differentiation. We will show that it is violated in the specific example chosen by the authors. Moreover, an extremely peculiar case in which the equilibrium closely resembles the maximum differentiation principle emerging from the horizontal model can be outlined, if unit production cost is assumed to be constant and low enough to allow for the market to be covered at equilibrium.[1]

II. The models

In this section, we summarize the models adopted by Cremer and Thisse [1991, pp.384-6]. Without loss of generality, let us take into consideration a differentiated duopoly.

II (i) Model *H*: horizontal product differentiation

- Firms 1 and 2 supply a physically homogeneous commodity at constant marginal cost, normalised to zero. Firms noncooperatively choose their locations $q_i \in [0,1]$, where i=1,2, and charge the mill price $p_i \geq 0$. Thus, firm i's profit is $\pi_i^H(p,q) = p_i x_i^H(p,q)$, where $x_i^H(p,q)$ is the

1. This example is due to Tirole [1988, pp.296-7].

market demand for product i.

- Consumers are located at $\theta \in [0, 1]$, according to the density function $f(\theta)$. Each of them has unit demand, and buys if and only if the following condition is met

$$U_0 - t(|\theta - q_i|) - p_i \geq 0 \tag{1}$$

where U_0 is a positive constant and $t(\cdot)$ is strictly incresing in the argument, with $t(0)=0$. Each consumer will obviously patronize the product assuring the greatest value of (1).

II (ii) Model V: vertical product differentiation

- Seller i supplies a good of quality $q_i \in [q_-, q^+]$, at the marginal cost $c(q_i)$, increasing in quality but constant w.r.t. quantity, and charges a price $p_i \geq c(q_i)$. The profit function is then $\pi_i^V(p,q) = [p_i - c(q_i)]x_i^V(p,q)$, where $x_i^V(p,q)$ is the demand for product i.

- Consumers are characterized by their marginal willingness to pay, $\theta \in [\underline{\theta}, \overline{\theta}]$. The density function is $f(\theta)$. Consumers have unit demands, and each consumer buys if

$$u^V(\theta, q_i) - p_i \geq 0 \tag{2}$$

where u^V is strictly increasing in q_i.

In both models, firms play a noncooperative two-stage game, the first stage being played in the product space (location for model H and quality for model V) and the second in the price space. The solution concept is subgame perfect equilibrium.

III. Discussion

Let us recall Cremer and Thisse's proposition:

"Consider any specification of model H such that $t(\cdot)$ is continuously differentiable on [0, 1].

Then, there exists a specification of model V such that $\pi_i^H(p,q) = \pi_i^V(\hat{p},q)$, where $\hat{p}_i \equiv p_i + c(q_i)$, holds for all i, p and q. Therefore, (p^*, q^*) is an equilibrium of model H if and only if $(\hat{p}^*, q^*)$ is an equilibrium of the corresponding specification of model V." [1991, p.386]

The crucial issue is the equivalence of the profit functions, since, provided that any Hotelling game (if played noncooperatively) is symmetric, this entails an analogous simmetry for model V. In other words, if model H must be considered as a special case of model V, then the equilibrium emerging from this particular specification of the latter must be symmetric. This is quite an unusual requirement, since vertical product differentiation does not usually yield symmetric equilibria.[2]

To give model V the required specification, Cremer and Thisse [1991, p.386] first set $q_- = \underline{\theta} = 0$ and $q^+ = \overline{\theta} = 1$. Then they define

$$u^V(\theta, q) = u_1(q) + u_2(\theta) - t(|\,\theta - q\,|) \tag{3}$$

and

$$c(q) = u_1(q) \tag{4}$$

where $u_1(q)$ is defined for u^V to be strictly increasing in q over [0, 1]. Then, a proof of the equivalence between model V and model H is given. Our view is that this proof heavily relies on the identity $\hat{p}_i \equiv p_i + c(q_i)$, which cannot be innocently assumed since it isn't generally true that the equilibrium price vectors of the two models respect it. Through the explicit exposition of the example adopted by Cremer and Thisse [1991, p.388], we are going to show that:

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2. See, *inter alia*, Shaked and Sutton [1982, 1983], where both firms' profit is increasing in the top quality.

i) the equilibrium price (the same for both firms) in model H is higher than in model V, that is, $p^* > \hat{p}_i^*$;

ii) profits differ in the two models, both ex ante and at equilibrium;

iii) consequently, the two models necessarily yield different subgame perfect equilibria and, specifically, model V exhibits a degree of differentiation which is strictly less than the one observed in model H. This outcome can be attributed to the strategic complementarity (substitutability) between products under vertical (horizontal) differentiation.[3]

Let us define[4]

$$u^V(\theta, q) = \theta q \tag{5}$$

and

$$c(q) = \frac{q^2}{2} \tag{6}$$

Since $\theta q = \frac{q^2}{2} + \frac{\theta^2}{2} - \frac{(\theta - q)^2}{2}$, (5) and (6) are special cases of (3) and (4), so that this model should correspond to the Hotelling model with quadratic transportation costs and, according to the above proposition, the two models should be characterized by maximum differentiation, with $q_1=0$ and $q_2=1$. This is surely true for model H, which is simply a modified version of the model introduced by D'Aspremont, Gabszewicz and Thisse [1979]. It is well known that this model yields maximum differentiation. The equilibrium price is $p^* = \frac{1}{2}$, while the equilibrium payoffs

3. This concept is due to Bulow, Geanakoplos and Klemperer [1985].

4. Cfr. Cremer and Thisse [1991, p.388]

are $\pi^{H*} = \frac{1}{4}$ for both sellers.[5]

Let us now turn to model V. Note first that, for the consumer identified by $\underline{\theta} = 0$ to be able to buy the low-quality product, condition (2) must be met, but this implies that the good is being supplied at zero price. As a consequence, the market cannot be expected to be covered at equilibrium. Assume firm 2 supplies the high-quality good. The demand for good 2 is then

$$x_2^V = 1 - \frac{(\hat{p}_2 - \hat{p}_1)}{q_2 - q_1} \tag{7}$$

while the demand for the low-quality good is

$$x_1^V = \frac{(\hat{p}_2 - \hat{p}_1)}{q_2 - q_1} - \frac{\hat{p}_1}{q_1} \tag{8}$$

The profit functions are defined as follows

$$\pi_2^V = \left(\hat{p}_2 - \frac{q_2^2}{2}\right)\left(1 - \frac{(\hat{p}_2 - \hat{p}_1)}{q_2 - q_1}\right) \tag{9}$$

$$\pi_1^V = \left(\hat{p}_1 - \frac{q_1^2}{2}\right)\left(\frac{(\hat{p}_2 - \hat{p}_1)}{q_2 - q_1} - \frac{\hat{p}_1}{q_1}\right) \tag{10}$$

5. Cfr. D'Aspremont, Gabszewicz and Thisse [1979, pp.1148-9] and Tirole [1988, pp.280-1].

Differentiating (9) and (10) w.r.t. $\hat{p}_2$ and $\hat{p}_1$, respectively, we obtain the first order conditions (FOCs) relative to the second stage of the game. Solving the FOCs we get the equilibrium prices

$$\hat{p}_2^* = \frac{q_2(2q_2^2 + 4q_2 + q_1^2 - 4q_1)}{2(4q_2 - q_1)} \tag{11}$$

$$\hat{p}_1^* = \frac{q_1(2q_1q_2 + q_2^2 + 2q_2 - 2q_1)}{2(4q_2 - q_1)} \tag{12}$$

The profit functions can be reformulated as follows

$$\pi_2^V = \frac{q_2^2(q_2 - q_1)(2q_2 + q_1 - 4)^2}{4(q_1 - 4q_2)^2} \tag{13}$$

$$\pi_1^V = \frac{q_1q_2(q_2 - q_1)(q_1 - q_2 + 2)^2}{4(q_1 - 4q_2)^2} \tag{14}$$

Differentiating (13) and (14) w.r.t. qualities, we obtain the FOCs relative to the first stage of the game

$$\frac{\delta\pi_2^V}{\delta q_2} = \frac{q_2(4 - q_1 - 2q_2)(2q_1^3 - 8q_1^2 + 12q_1q_2 + 5q_1^2q_2 - 16q_2^2 - 22q_1q_2^2 + 24q_2^3)}{4(q_1 - 4q_2)^3} \tag{15}$$

$$\frac{\delta\pi_1^V}{\delta q_1} = \frac{q_2(q_1 - q_2 - 2)(4q_2^3 + 8q_2^2 - 19q_1q_2^2 - 2q_1^3 - 14q_1q_2 + 17q_1^2q_2)}{4(q_1 - 4q_2)^3} \tag{16}$$

The above conditions must be solved through numerical computation, yielding $q_1^* = 0.3987$ and $q_2^* = 0.8195$. The equilibrium prices are $\hat{p}_1^* = 0.1500$ and $\hat{p}_2^* = 0.4533$, while profits are $\pi_1^{V*} = 0.0243$ and $\pi_2^{V*} = 0.0328$. As a consequence, this specification of model V does not exhibit maximum differentiation, and the profit vector doesn't coincide with that observed in the horinzontal model, simply because model V is not symmetric. Furthermore, the equilibrium price vector of model V doesn't satisfy the requirement made in the above proposition; more precisely, $p^* > \hat{p}_i^*$. It is quickly verified that equilibrium demands are also different. This outcome is generated by two opposite forces: on the one hand, sellers have a strong incentive to differentiate, in order to soften price competition, exactly as in model H; on the other, since consumers differ w.r.t. their marginal willingness to pay, sellers must take into account that their relative positions in the product space entail relevant consequences on profits. This aspect can be explicited by resorting to the concept of strategic complementarity/substitutability.

In model H,

$$\frac{\delta^2 \pi_i^H}{\delta q_1 \delta q_2} = \frac{1}{18}(q_2 - q_1) \tag{17}$$

which is negative for all $q_1, q_2 \in [0, 1]$, $q_1 \neq q_2$. This implies that both players' reaction functions are downwards sloping in the product space, so that products act as strategic substitutes.

In model V,

$$\frac{\delta^2 \pi_2^V}{\delta q_2 \delta q_1} = \frac{q_2(16q_1^2 - q_1^4 + 80q_1q_2 - 108q_1^2q_2 - 16q_1^3q_2 + 48q_1^2q_2^2 - 68q_1q_2^3 + 32q_2^4)}{2(q_1 - 4q_2)^4} \tag{18}$$

$$\frac{\delta^2\pi_1^V}{\delta q_1 \delta q_2} = (2q_1^4 - q_1^5 + 28q_1^2 q_2 - 32q_1^3 q_2 + 11q_1^4 q_2 + 32q_1 q_2^2 - 6q_1^2 q_2^2 - 16q_1^3 q_2^2 - 32q_1 q_2^3$$

$$+46q_1^2 q_2^3 + 32q_2^4 - 56q_1 q_2^4 + 16q_2^5)/(2(q_1 - 4q_2)^4 \quad (19)$$

Through numerical computation, it can be shown that (18) and (19) are both positive for all $q_i \in [0,1]$. This means that products are everywhere strategic complements, that is, both reaction functions are upwards sloping in the quality space. This prevents sellers from achieving maximum differentiation, because if, say, firm 2 increases her own quality, firm 1 does the same, so that the lower bound of the product range can never be reached. This property is common to a large class of models of vertical product differentiation, in which consumer's utility is given by (5), while $c(q)=q^n$, where n is greater than 1. There exists a neighborhood Φ of the Nash equilibrium in the quality space within which the reaction functions are upwards sloping. The upper bound of Φ is increasing in n, so that when n goes to infinity the two products act as strategic complements over the whole unit interval.

A last remark is now in order. Maximum differentiation can be achieved under two strong assumptions: (i) unit production cost is the same positive constant c for both firms; and (ii) the market is covered, that is $c + \frac{\bar{\theta} - 2\underline{\theta}}{3}(q_2 - q_1) \leq \underline{\theta} q_1$ [Tirole, 1988, p.296].[6] Nevertheless, the equilibrium profits are strictly greater for firm 2. Furthermore, in this example, the mixed derivatives of the profit functions w.r.t. qualities are everywhere nil, so that no strategic complementarity is present to mitigate the incentive to differentiate so as to soften price competition.

IV. Conclusions

Contrarily to Cremer and Thisse's argument, we have shown that a model of spatial

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6. Note that θ cannot be normalized to lie in the interval [0, 1].

product differentiation with convex transportation costs cannot be considered as a special case of a model of vertical product differentiation. This outcome can be interpreted as a consequence of the strategic complementarity (substitutability) existing in the vertical (horizontal) model, yielding a crucial asimmetry between the profit functions, both within the vertical model and between the latter and the horizontal one.