Vertical Product Differentiation:
Some More General Results*

Luca Lambertini
Dipartimento di Scienze Economiche
Università degli Studi di Bologna
Strada Maggiore 45
40125 Bologna

May 1993

Abstract

In this note, it is shown that the "$\frac{4}{7}$ rule" obtained by Choi and Shin [1992] can be derived as the asymptotic result of a more general model in which costs are convex with respect to quality. General solutions to a wider class of models of vertically differentiated duopoly are also given.

JEL classification number: L 13.

**Acknowledgements*

I wish to thank Paolo Garella, Gianpaolo Rossini and Dario Sermasi for helpful suggestions. The responsibility rests however with me only.

1. Introduction

The wide literature on product quality under imperfect competition stresses the role of vertical differentiation in softening price competition, so to avoid Bertrand's paradox and to allow sellers to gain strictly positive profits at equilibrium.[1]

In a recent note, Choi and Shin (1992) propose a model of vertically differentiated duopoly in which, at equilibrium, the lower quality is $\frac{4}{7}$ of the higher one, under the assumptions that i) production costs are zero for both firms and ii) the market is not covered. According to these authors (Choi and Shin, 1992, p.229), their contribution could be considered as "little more than a footnote to Tirole's discussion", in which positive constant unit production costs are assumed (Tirole, 1988, p.296).

Our purpose is to show that Choi and Shin's "$\frac{4}{7}$ rule" may be obtained as the asymptotic result of a general model characterized by a cost function convex w.r.t. quality. More general results are then derived for a large class of uniform income distributions.

2. The model

Two firms supply the market at different quality levels q_L and q_H, with $q_H > q_L$. quality is chosen in the interval $[0,\overline{q})$, where $\overline{q} > 0$. Quality choice is costless, but producing x_i units of the good of quality q_i costs $c_i = q_i^n x_i$, where $n>1$. Consumers are uniformly distributed over the interval $[\underline{\theta},\overline{\theta}]$, where $\underline{\theta} \geq 0$, $\overline{\theta} = \underline{\theta} + \alpha$ and $\alpha > 0$. Total density is α. Parameter θ represents each

1. See, *inter alia*, Gabszewicz and Thisse (1979) and Shaked and Sutton (1982).

consumer's marginal willingness to pay for quality, and can be characterized as the reciprocal of the marginal utility of income. If consumers' utility funcion is concave in income, θ increases as income increases. Each consumer buys one unit of good i if

$$U = \theta q_i - p_i \geq 0 \qquad (1$$

Otherwise she doesn't buy, getting zero utility. Thus, consumers may be divided into three groups: those who buy the high-quality good, those who buy the low-quality good, and finally those who do not buy at all. The marginal willingness to pay for quality of the consumer who is indifferent between the two goods is

$$\theta_H = \frac{p_H - p_L}{q_H - q_L} \qquad (2$$

while that of the consumer which is indifferent between the low-quality good and not to buy is

$$\theta_L = \frac{p_L}{q_L} \qquad (3$$

If firms do not cover the market, their respective demands are defined as follows

$$x_H = \theta_H - \frac{p_H - p_L}{q_H - q_L} \qquad (4$$

$$x_L = \frac{p_H - p_L}{q_H - q_L} - \frac{p_L}{q_L} \qquad (5$$

The profit functions are then

$$\pi_H = (p_H - q_H^n)\left(\theta_H - \frac{p_H - p_L}{q_H - q_L}\right) \tag{6}$$

$$\pi_L = (p_L - q_L^n)\left(\frac{p_H - p_L}{q_H - q_L} - \frac{p_L}{q_L}\right) \tag{7}$$

Firms play a two-stage game. In the first stage, they simultaneously choose quality,[2] while in the second they compete in prices. The solution concept is perfect subgame equilibrium.

The equilibrium prices given by the reaction functions relative to the second stage of the game are

$$p_H = \frac{q_H(2\bar{\theta}q_H - 2\bar{\theta}q_L + 2q_H^n + q_L^n)}{4q_H - q_L} \tag{8}$$

$$p_L = \frac{\bar{\theta}q_H q_L - \bar{\theta}q_L^2 + 2q_H q_L^n + q_H^n q_L}{4q_H - q_L} \tag{9}$$

Substituting (8-9) into (6-7) and differentiating the latter w.r.t. qualities, we obtain the first order conditions relative to the first stage of the game, which are the following

---

2. The sequential choice assumed by Choi and Shin (1992, pp.229-30) is redundant and can be misinterpreted, since no Stackelberg advantage is assumed and, moreover, simultaneous choice is de facto displaied to derive the demand functions in the remainder of their note.

$$\frac{\delta\pi_H}{\delta q_H} = (q_L^n q_H + q_L q_H^n - 2q_H^{n+1} - 2\bar{\theta} q_L q_H + 2\bar{\theta} q_H^2)(2q_L^{n+2} q_H - q_L^{n+1} q_H^2 - 4q_L^n q_H^3 +$$

$$+5q_L^2 q_H^{n+1} - 10q_L q_H^{n+2} + 8q_H^{n+3} + 2nq_L^3 q_H^n - 14nq_L^2 q_H^{n+1} + 28nq_L q_H^{n+2} -$$

$$-16nq_H^{n+3} + 4\bar{\theta} q_L^3 q_H + 10\bar{\theta} q_L^2 q_H^2 - 14\bar{\theta} q_L q_H^3 + 8\bar{\theta} q_H^4)/$$

$$/(q_H (q_H - q_L)^2 (4q_H - q_L)^3) = 0 \qquad (10$$

and

$$\frac{\delta\pi_L}{\delta q_L} = q_H (2q_L^n q_H - q_L^{n+1} - q_L q_H^n + \bar{\theta} q_L^2 - \bar{\theta} q_L q_H)(9q_L^{n+2} q_H - 2q_L^{n+3} - 18q_L^{n+1} q_H^2 +$$

$$+8q_L^n q_H^3 - 2q_L^3 q_H^n + q_L^2 q_H^{n+1} + 4q_L q_H^{n+2} + 2nq_L^{n+3} + 14nq_L^{n+2} q_H + 28nq_L^{n+1} q_H^2 - 16nq_L^n q_H^3 +$$

$$+7\bar{\theta} q_L^3 q_H - 11\bar{\theta} q_L^2 q_H^2 + 4\bar{\theta} q_L q_H^3)/(q_L^2 (q_L - 4q_H)^3 (q_L - q_H)^2) = 0 \qquad (11$$

Conditions (10-11) have no algebraic solutions.[3] Thus, the equilibrium qualities $q_i^*(\bar{\theta}, n)$ must be derived through numerical calculation. Their generalization is then obtained by induction.

Let us first set $\alpha = 1$ and $\bar{q} = 1$. It can be shown that $q_i^*(1, n)$ monotonically increases as

3. The algorithm is part of *Mathematica*, Wolfram Research Inc..

n increases;[4] when n goes to infinity, Choi and Shin's $\frac{4}{7}$ rule is obtained, since

$$\lim_{n \to \infty} q_i^n = 0 \tag{12}$$

so that unit production costs are nil for both commodities. The equilibrium values generated through the numerical simulation can be fitted by the following exponential functions

$$q_H^*(1,n) \approx 1 - \frac{1}{e^{\frac{n}{h}}} \tag{13}$$

$$q_L^*(1,n) \approx \frac{4}{7}\left(1 - \frac{1}{e^{\frac{n}{l}}}\right) \tag{14}$$

The least squares extimations of parameters h and l are respectively $\hat{h} = 5.08\overline{2}$ and $\hat{l} = 4.9920$. Rescaling the problem for $\alpha \neq 1$ and $\overline{q} \neq 1$, we can verify that the general solutions are

$$q_i^*(\overline{\theta}, n) = \overline{\theta}^{\frac{1}{n-1}} q_i^*(1,n), \quad i = H, L \tag{15}$$

as long as the market is not covered, that is, as long as $\overline{\theta} < \frac{8\alpha}{7}$.

---

4. The results of the numerical simulation for $n \in [10/9, 120]$ are shown in the appendix.

3. Conclusions

In this note, we showed that Choi and Shin's result, namely, the "$\frac{4}{7}$ rule", can be considered as the asymptotic result of a model of duopolistic vertical product differentiation in which it is assumed that the cost function is convex in quality, the market is not covered, consumers' marginal willingness to pay is uniformly distributed over a closed interval and consumers' total density is one. General results relative to the case in which consumers' population is otherwise normalized are also given.

However, the present analysis, as well as the whole stream of literature on product differentiation, leaves unanswered the following question: what does it really prevent duopolists from achieving maximum product differentiation within the vertical framework?