Equilibrium Locations in the Unconstrained Hotelling Game

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Abstract

In this paper, we try to analyse the optimal location choice in a standard game of horizontal differentiation in which firms are free to locate outside the city boundaries. It turns out that the unique Nash equilibrium exhibits a finite distance between the sellers, so that the maximum differentiation principle is not confirmed. Moreover, the two symmetric Stackelberg equilibria exhibit the same degree of differentiation observed when the game is noncooperatively played within the city, except that the leader locates at the center.

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1. Introduction

In their pathbreaking paper, D'Aspremont, Gabszewicz and Thisse (1979) criticized the *minimum differentiation principle*, i.e., the main result reached by Hotelling (1929) in his model of horizontal product differentiation along a linear city, showing that Hotelling's linear transportation cost game may fail to reach a pure-strategy Nash Equilibrium in prices.[1] Moreover, resorting a quadratic transportation cost function, they showed that the only Nash equilibrium for the location stage of the game implied *maximum differentiation*.

Our aim is to investigate the nature of the noncooperative equilibrium within the quadratic transportation cost framework, under the assumption that duopolists are free to play also outside the city boundaries, which seems to be consistent with the observed behaviour. We show that the Nash equilibrium for the location stage is still unique but it does not imply maximum differentiation in any meaningful sense, though it implies a degree of differentiation which is greater than the one we observe when the duopolists are compelled to choose their respective locations within the city limits.

2. The Model

Our starting point is the horizontal differentiation model explored by D'Aspremont et al. (1979). The duopolists sell the same physical good. Consumers are uniformly distributed along an interval whose length can be normalized to 1 without loss of generality, and their total

1. While a mixed-strategy Nash equilibrium in prices exists (Dasgupta and Maskin, 1986).

density is 1. Consumers have unit demands, and consumption yields a positive constant surplus s; then, each consumer buys if and only if the net utility derived from consumption is non negative,

$$U = s - tx^2 - p_i \geq 0, \quad 0 \leq x \leq 1, \quad t > 0, \quad i = 1,2; \tag{1}$$

where tx^2 is the transportation cost incurred by a consumer living at distance x from store i, and p_i is the price of good i. We assume that s is large enough for total demand to be always equal to 1. Firm 1 is located at a, while firm 2 is located at $1 - b \geq a$, with $a, b \in R$. Clearly, if a and b are both negative, the firms are located outside the city boundaries. The demand functions are, respectively:

$$y_1 = a + \frac{1 - a - b}{2} + \frac{p_2 - p_1}{2t(1 - a - b)}; \tag{2}$$

$$y_2 = 1 - y_1 = b + \frac{1 - a - b}{2} + \frac{p_1 - p_2}{2t(1 - a - b)}. \tag{3}$$

Clearly, for a=1-b, i.e., when sellers locates at the same point, demands (2) and (3) are not determined and profits are nil as a consequence of the Bertrand paradox. Since unit costs are constant and normalized to zero, the two objective functions are then

$$\pi_1 = p_1 y_1; \tag{4}$$

$$\pi_2 = p_2 y_2. \tag{5}$$

Since we are looking for the perfect subgame equilibrium for the two-stage game in locations and prices, let's proceed by backward induction, maximizing (4) and (5) w.r.t. p_1 and p_2. The equilibrium prices are:[2]

$$p_1^* = t(1-a-b)\left(1+\frac{a-b}{3}\right); \qquad (6$$

$$p_2^* = t(1-a-b)\left(1+\frac{b-a}{3}\right). \qquad (7$$

Following Bulow, Geanakoplos and Klemperer (1985, p.501), we can show that

$$\frac{\delta^2\pi_i}{\delta p_i \delta p_j} = \frac{1}{2t(1-a-b)} > 0, \quad i,j=1,2; \quad i \neq j. \qquad (8$$

This means that the two goods are everywhere strategic complements in prices, due to the fact that demand is completely inelastic. The degree of strategic complementarity obviously increases as both a and b increase, so that when locations coincide, the value of expression (8) goes to infinity. This gives a first intuition of the fact that sellers are incentived to differentiate, so as to soften price competition.

If we substitute (6) and (7) into the profit functions (4) and (5), we obtain:

$$\pi_1 = \frac{t}{18}(a-b+3)^2(1-a-b); \qquad (9$$

2. Cfr. D'Aspremont et al. (1979, p.1149) and Tirole (1988, p.281).

$$\pi_2 = \frac{t}{18}(a-b-3)^2(1-a-b); \tag{10}$$

the first order conditions (FOCs) w.r.t. locations are:

$$\frac{\delta\pi_1}{\delta a} = \frac{t}{18}(b-a-3)(1+3a+b) = 0; \tag{11}$$

$$\frac{\delta\pi_2}{\delta b} = \frac{t}{18}(a-b-3)(1+a+3b) = 0. \tag{12}$$

If sellers are constrained to locate within the city limits, the sign of (11-12) is negative, as noted by D'Aspremont et al. (1979, p.1149) and Tirole (1988, p.281), since the sign of the first parenthesis is negative, while, within the unit interval, the sign of the second is positive. What we are going to show is that, outside the city, the sign of the FOCs changes, due to the fact that outside the unit interval the sign of the second parenthesis may become negative. Besides, the incentive to locate outside the city boundaries can be pointed out by checking the sign of $\frac{\delta\pi_i^2}{\delta p_i \delta z}$, i=1,2, z=a,b; we obtain:

$$\frac{\delta\pi_1^2}{\delta p_1 \delta a} = \frac{a}{a+b-1}; \tag{13}$$

$$\frac{\delta\pi_2^2}{\delta p_2 \delta b} = \frac{b}{a+b-1}. \tag{14}$$

The sign of both (13) and (14) is positive (negative) for $a<0$ ($a>0$) and $b<0$ ($b>0$), respectively.[3] This means that, given the other firm's location, each firm can increase her profit by locating outside the city limits. Moreover, according again to Bulow et al. (1985, p.494), we can see that the two goods are strategic substitutes in locations by inspection of the following mixed derivative:

$$\frac{\delta\pi_i^2}{\delta a\,\delta b}=\frac{t}{9}(a+b-1), \qquad (15$$

which is everywhere negative, except for $a=1-b$, in which it is nil. This yields further evidence of the incentive to differentiate.[4]

Let us now turn to the Nash equilibrium. The system (11-12) has the following solutions: $\left(a=-\frac{1}{4};b=-\frac{1}{4}\right)$; $\left(a=\frac{1}{2};b=-\frac{5}{2}\right)$; $\left(a=-\frac{5}{2};b=\frac{1}{2}\right)$. We can obtain these solutions by graphically representing the two reaction functions (11) and (12), which are hyperbolic functions collapsed to $a=b-3;\quad a=-\frac{(1+b)}{3}$ and $b=a-3;\quad b=-\frac{(1+a)}{3}$, respectively.

It can be easily verified that the only Nash equilibrium is given by $\left(a=b=-\frac{1}{4}\right)$,[5] since

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3. Conditions (13) and (14) are everywhere defined, except for $a=1-b$. Yet, taking the limit of expression (13) as the degree of differentiation shrinks, we have $\lim_{a\to 1-b\_}\frac{\delta\pi_1^2}{\delta p_1\delta a}=\pm\infty$, depending on the sign of a. *Mutatis mutandis*, the same obviously holds for expression (14).

4. Furthermore, if we define $\frac{\delta\pi_i^2}{\delta a\delta b}=S_i$, we obtain:

$$\frac{\delta S_i}{\delta z}=\frac{t}{9}>0, \quad z=a,b;$$

which means that the degree of strategic substitutability given by (15) increases as z increases, i.e., both firms' marginal profit decrease at an increasing rate as distance shrinks.

5. A formal proof is given in the appendix.

the second order conditions (SOCs) are not simultaneously satisfied by the remaining critical points. In order to show this, let us check the second derivatives of (9) and (10) w.r.t. a and b, respectively. It must be that:

$$\frac{\delta^2\pi_1}{\delta a^2} = b - 3a - 5 \leq 0; \qquad (16$$

$$\frac{\delta^2\pi_2}{\delta b^2} = a - 3b - 5 \leq 0. \qquad (17$$

For $\left(a = b = -\frac{1}{4}\right)$ both (16) and (17) are respected; for $\left(a = \frac{1}{2}; b = -\frac{5}{2}\right)$ (16) is satisfied, while (17) is not: this means that in $b = -\frac{5}{2}$, π_2 is being minimized. The reverse is true for $\left(a = -\frac{5}{2}; b = \frac{1}{2}\right)$. Thus, it turns out that the only Nash equilibrium is given by $a = b = -\frac{1}{4}$, which implies that both firms locate outside the city. This is shown in figure 1 below.

INSERT FIGURE 1

The SOCs and the constraint $a \leq 1 - b$ define the region in which the Nash equilibrium lies. It is immediate to verify that the SOCs cut the reaction functions of the sellers in such a way that, given the choice of $\overline{a}$ by seller 1, the best response by seller 2 is univocally determined, and viceversa. The equilibrium profits are $\pi_1 = \pi_2 = \frac{3}{4}t$, while demands are $y_1 = y_2 = \frac{1}{2}$, obviously. The equilibrium can be characterized in the following way: if seller 1 chooses $a = -\frac{1}{4}$, seller 2 can't do any better than choosing the same value for b; otherwise, she would either loose demand by increasing the degree of differentiation (since transportation costs would rise) or intensify price competition by decreasing the degree of differentiation. Moreover, it is quickly shown that this equilibrium is stable: let $r_1(b)$ and $r_2(a)$ be the sellers' reaction functions in the location stage; if we confine ourselves to the region in which both profit functions are concave, we have that a sufficient condition for stability is met:

$$\left|\frac{\delta r_1}{\delta b}\right|\left|\frac{\delta r_2}{\delta a}\right| = \frac{1}{9} < 1. \qquad (18$$

We can now ask ourselves what the equilibrium looks like if one seller acts as a Stackelberg leader in the location stage. By symmetry, we can confine ourselves to the case in which seller 1 is the leader. She aims at maximizing (9) w.r.t. a, under the constraint given by (12). This yields $\left(a = \frac{1}{2}; b = -\frac{1}{2}\right)$; the equilibrium profits are $\pi_1 = \frac{8}{9}t$ and $\pi_2 = \frac{2}{9}t$, while prices are $p_1 = \frac{4}{3}t$ and $p_2 = \frac{2}{3}t$ and quantities $y_1 = \frac{2}{3}$ and $y_2 = \frac{1}{3}$. Notice that, given the sign of the FOCs within the unit interval, to look for a Stackelberg equilibrium inside the city is economically meaningless.

The same conclusions can be reached by synthetically representing the game through the following payoff matrix:

INSERT TABLE 1

The choice of $a = -\frac{5}{2}$, or, equivalently, $b = -\frac{5}{2}$, isn't clearly an equilibrium strategy, since it is at least weakly dominated by the others. The pair (0, 0) has been included to emphasize the fact that the payoffs yielded by the equilibrium solution for this 'non-constrained game' strictly Pareto-dominate the payoffs yielded by the equilibrium which arises from the 'constrained game' played by D'Aspremont et al. (1979) within the city, for both players.

A few remarks are now in order. First, both duopolists' profits increase at a constant rate as they depart from the center of the city, but this isn't sufficient to infer that the equilibrium implies maximum differentiation, as pointed out above. Second, as a consequence, the payoffs ($3t$, $3t$) or any analogous pair along the main diagonal of matrix 1, which Pareto-dominates the payoffs associated to the Nash equilibrium, could be thought of as the result of collusive behaviour in an infinitely repeated game. This is obviously true to a certain extent (and it can be easily shown that in such a case the discount rate applied by each duopolist must satisfy the condition $r \leq \frac{3}{4}$), but we know that the solution of a repeated game is definitely non-unique. More precisely, if we build the objective function of the cartel, we can quickly verify that the incentive to differentiate is nil, and profit is an increasing function of price; thus, in order to identify the optimal price-and-location policy we have to take into account the condition relative to consumer's surplus, (1) above.[6] Third, the optimal locations chosen by the duopolists can be characterized as the mirror image of the optimal ones chosen by a social planner: since the aim of the latter is to minimize social costs, she would choose $p_1 = p_2 = 0$ and $a = b = \frac{1}{4}$.[7] Moreover, as a rationale for the adoption of the unconstrained model we could think of a public authority which doesn't take consumers' welfare into account and let firms optimize, while the constrained model suggests the presence, behind the scenes, of a regulative authority which is partially taking care of consumers. Fourth, the possibility of locating outside the city matters in the non-cooperative case only, given that the duopolists' objective functions are highly sensitive to the reciprocal distance.

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6. Cfr. Lambertini (1992).

7. Cfr. Tirole (1988, p.282).

3. Conclusions

We have investigated the existence of Nash and Stackelberg equilibria in locations in a game of horizontal differentiation within a standard quadratic transportation cost framework, assuming that both firms are free to locate outside the linear city. It turns out that the Nash equilibrium is unique and characterized by a finite distance between sellers, and that it strictly dominates the so-called *maximum differentiation equilibrium* which is observed when firms are compelled to choose their respective locations within the linear city. Besides, the two symmetric Stackelberg equilibria exhibit the same degree of differentiation associated to the non cooperative equilibrium of the constrained game explored by D'Aspremont et al. (1979), although the leader locates in the middle of the city.