A NOTE ON THE COST OF DIVERSIFICATION AND THE OPTIMAL NUMBER OF ACTIVITIES

Gianpaolo Rossini
University of Bologna

May 1985

N° 18

Provisional draft not to be quoted

The author wishes to aknowledge the suggestions of Paolo Onofri and Maurizio Emaldi,keeping total responsability for the content of this paper.

Paper to be distributed at the Symposium of Economies of Scope Brussels, June 3-4, 1985 et the EIASM.

A note on the costs of diversification and the optimal number of activities.

Gianpaolo Rossini

University of Bologna May 1985

1. Introduction

Literature on portfolio selection shows that diversification of a portfolio pays out something whenever systematic risk is present. In less technical terms a return on an asset above the risk free rate of return is warranted by the market whenever the asset has to bear a recurrent risk which is not diversifiable.

Diversification can be thought of as a mutual insurance policy whose premium is been paid by high returns on an asset when another asset has low or even negative returns.

For systematic risk it is meant the risk we face when the variance of a portfolio tends to the average covariance among the assets if each asset enters the portfolio with the same weight and the number of assets tends to infinity.The average covariance is often referred to also as social risk or market risk or undiversifiable risk.It is also the measure of the ability of a market to eliminate risk[1].

In this paper we want to examine market diversification and scope of production of a firm.To this purpose financial models of portfolio theory can be conveniently borrowed.The focus of our analysis will be on the necessary and correct posture of costs of diversification. These costs influence two variables which come out from two related approaches.The first is the number of activities undertaken by a firm, whilst the second is the evaluation of these activities.These two variables will be easily derived by resorting to both a standard mean variance utility model and the capital asset pricing model (CAPM).

These two models are closely related since they are different ways of mirroring the same thing.

2.Assets as lines of production

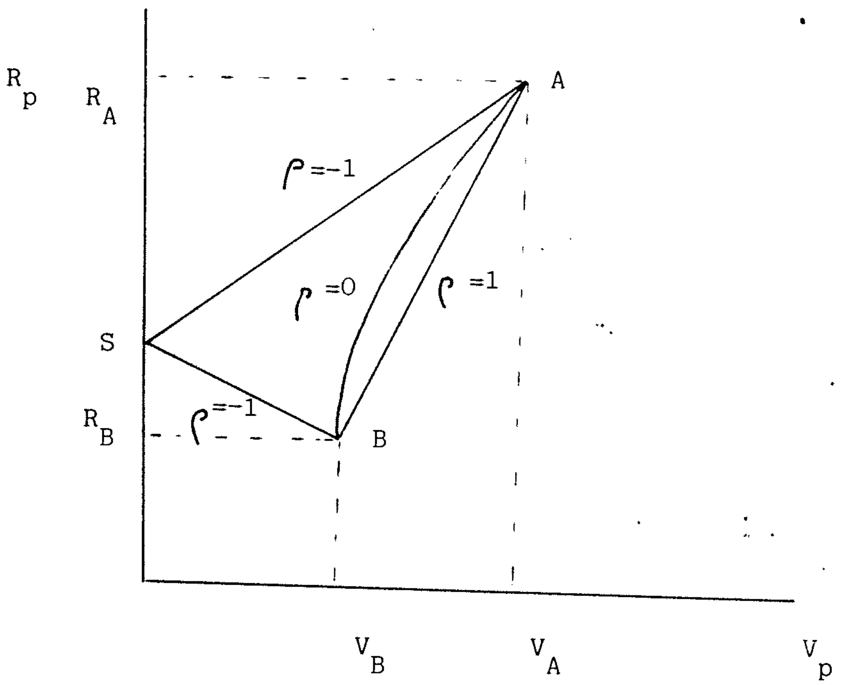
Usually we can consider that some costs of a multiproduct firm are related to the extent of diversification or, in industrial terms, to the scope of production.Management,advertising,commercial networks imply costs which could be linear,convex or concave as the scope of production varies.If there are these costs it follows that there must be some optimal decision as to the number of production lines to be worked out.What are then the relationships between the number of activities,their evaluation in financial terms by a firm and the costs of diversification?This is what we are going to explore in this paper. Yet the scope of firm production is being investigated only in its risk and returns aspects whenever diversifications costs are there.Take the case of a firm that produces cars,trucks,airplanes. We want to investigate the effects of diversification over total risk and profitability of this multiproduct firm which acts as a risk diversifier in presence of either economies or diseconomies of of production scope.No account will be taken of production costs, substitution effects in demand and production among products;only the total effects of the number of production lines on total profitability is being assessed.

The framework is a financial one ,which is justified by the kind of analysis undertaken since we are dealing with the optimal number either of activities or of assets of a general purpose representative firm guided by risk,returns and costs of diversification.In this sense we might think either of a financial investor (a holding firm) or simply of a multiproduct firm.In the former case information,control etc. are the sources of costs;in the latter,as said above,it is a matter of advertising,management,diverse expertise,commercial networks of diverse kind etc.

3.A naive financial framework

Initially we consider the opportunity set for a two asset portfolio according to diverse levels of the coefficient of correlation of asset returns (-1.0;+1.0;intermediate values).

Figure 1



R_p = portfolio return

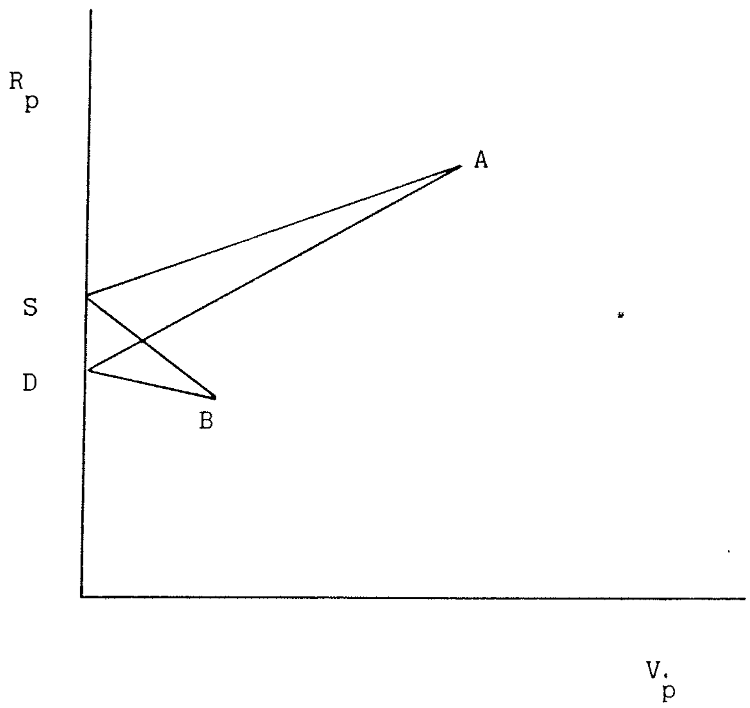
V_p = portfolio standard deviation

Point S in figure 1 corresponds to a no-risk portfolio(which is feasible when ρ = -1).In this case the portfolio made up by two assets(A and B) in a proportion $x_A = (V_B/(V_A + V_B)$ is equivalent to a risk free asset. The return on it has to equal the riskfree (R_f) rate of interest paid on safe assets. Introducing a cost of diversification,means that holding two assets costs more than holding only one.We face then two possibilities.Either none will hold the riskfree portfolio or the return on it is higher than the riskfree rate.Consider for example a that

portfolio worth one thousand ECU of Dupont securities costs 1020 ECU,while a portfolio with 500 ECU of Dupont and 500 ATT costs 1040 ECU,due to transaction costs,information costs etc.

In terms of figure 2(limiting the analysis to the case of ρ =-1) we shall obtain a new opportunity set ADB.The portfolio corresponding to D will be held only if OD $\geq R_f$ (Riskfree rate of return).

Figure 2



SD is then the cost of transaction or the measure of market inefficiency in Fama's terms;yet it could also be thought of as a sort of diversification-opportunity cost.Therefore SD might be deemed to be the cost of keeping money (with no inflation) or any certain balance,when the riskfree rate is OS.

If we were to define an equilibrium condition,we would say that the cost of keeping a money balance yelding a riskfree rate OS has to be equal to the cost of diversification in a riskfree portfolio.

In terms of product lines of a firm we cannot conceive a riskfree venture.Yet we can design costs of diversification,i.e. of changing the scope of production,as far as this is the outcome of a risk minimization-return maximization process.

4.Modeling diversification costs

4.1.The portfolio approach

Let us shortly review some fundamentals of financial analysis of diversification.When the correlation between asset returns is -1,the return on a portfolio of two assets A and B is

$$R_p = x_A R_A + (1 - x_A) R_B \tag{1}$$

while the standard deviation is either

$$V_p = - x_A V_A + (1 - x_A) V_B \tag{2.1}$$

or

$$V_p = x_A V_A - (1 - x_A) V_B \tag{2.2}$$

As above anticipated the point of minimum risk corresponds to the value of the proportion of the asset A in the portfolio which follows

$$x_A = (V_B /(V_B + V_A)) \tag{3}$$

The introduction of diversification costs is going to change (3) as seen in figure 2.This means that diversification costs do not leave the investor with the same portfolio,and he may even be induced to hold a one asset menu,because the opportunity set has changed .We shall see the question more rigorously in a few paragraphs.

The simplest way of modeling costs of diversification is the introduction of linear cost function as

$$C = N z \tag{4}$$

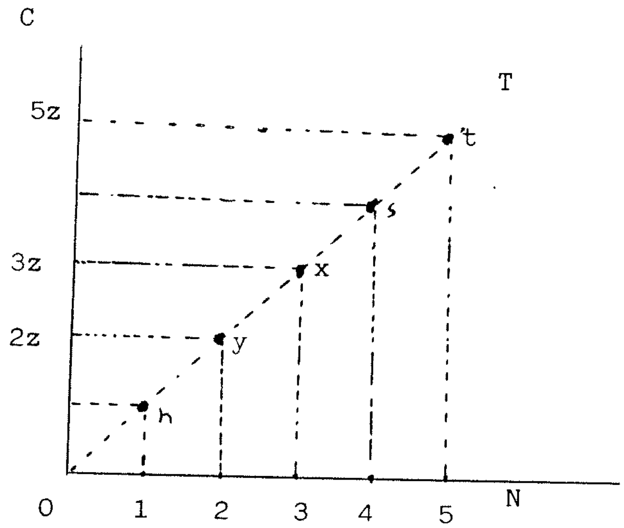
where C is total cost

N is the number of activities

z is a cost parameter

The (4) is a linear function of integers, since N can only assume integer values: it is a point like function, described in figure 3, like the points x,y,t,h,s . If we allow for real values of the number of activities we can join those points, getting OT.

Figure 3



The portfolio return function with diversification costs becomes

$$R_p = x_A R_A + (1 - x_A) R_B - 2z \tag{5}$$

The optimal mix of the two assets is going to change: net portfolio returns decreased by 2z and this has to be somehow compensated by a lower total risk.

We consider now a more general case without any specification of the correlation coefficient of returns among assets and with an undefined number of assets N_i. We already know that a portfolio equally partitioned on N assets randomly chosen has the following variance:

$$V_p^2 = (1/N)\ V_i^2 + ((N-1)/N)\ V_{ik} \tag{6}$$

where $i=1 \ldots\ldots N$

$k=1 \ldots\ldots N$

$k \neq i$

and V_i^2 is the average variance of assets returns, and V_{ik} is the average covariance. As far as portfolio average returns are concerned, we see that they do not vary as the number of assets changes. Yet as N gets large (6) reduces to the well known result of average covariance among assets, called market risk or social or undiversifiable risk. If the average covariance does not vanish as N gets large we cannot form any zero risk portfolio.

Let us consider risk averse individuals with utility functions expressed in mean-standard deviation space

$$U(E_p, V_p) \tag{7}$$

A market equilibrium condition is easily derived. Considering two situations, one with

E_1, V_1 with N_1 assets

and another with

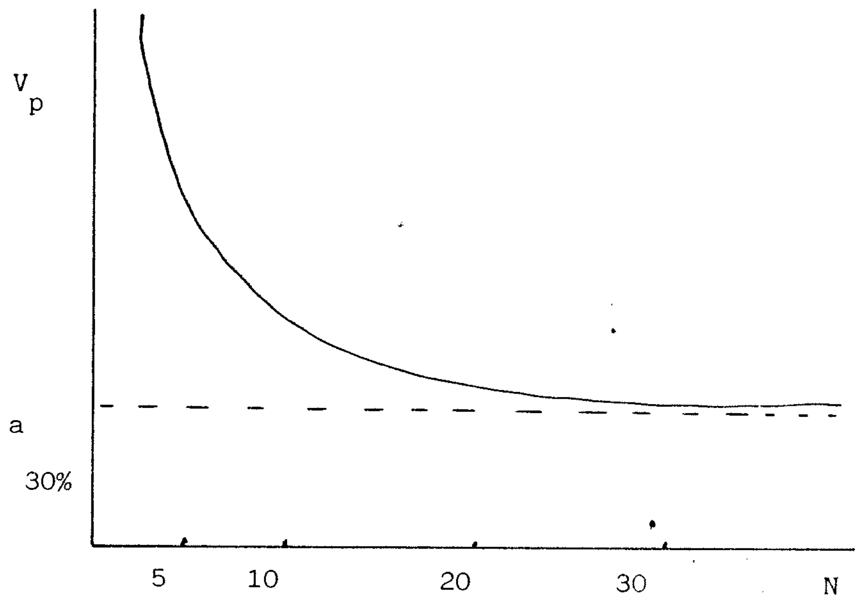
E_2, V_2 with N_2 assets

we should have

$$\frac{dU(E_1,V_1)}{dU(E_2,V_2)} = \frac{dC(N_1)}{dC(N_2)} = \frac{p(N_1)}{p(N_2)} \qquad (8)$$

which means that equilibrium requires costs of diversification to show well behavedness properties according to what the utility function dictates.Otherwise equilibrium cannot be obtained[2].
Let us shift our attention towards the derivation of some types of demand functions for number of activities or in other terms for financial variety.This is indeed the focus of our analysis since we would like to explain some sort of stylized fact .
As observed above the variance of a portfolio of randomly chosen assets equiproportionally distributed tends to the market average risk.Figure 4 describes this phenomenon,which is the representation of (5),empirically observed in comparative studies of national financial markets[3].

Figure 4



As the number of securities approaches 10-15 V_p tends to the average covariance (= a)[4].Most of the times the number of different securities in a portfolio or of product lines of a multiproduct firm is less than 10-15.It has been observed that they are few and that indivisibilities are not able to explain the phenomenon by themselves.[5] This fact is exactly what we aim to explain and to perform this goal we proceed inductively by using a mean - variance approach.Taking into account (5) and its graphical represantation (Figure 5) we can approximate the variance of a portfolio in various ways.Let us consider the <u>first</u> one,which corresponds to

$$V_p = (d/N) + a \tag{9}$$

where

a is the average or market risk

a+d is the average variance of the assets (N=1) which is always bounded if a is bounded

N number of assets in the portfolio with $N \geq 1$

The portfolio returns will be the sum of average returns minus the cost of inclusion of each asset (z)

$$R_p = N R_n - Nz \tag{10}$$

We could alternatively model costs in order to have constant, decreasing or increasing costs.The first set of cost function we propose to this purpose are (11) and (12)

$$C = Nz + f N^2 \quad (11)$$

$$C = Nz - f N^2 \quad (12)$$

where f is a cost parameter.

Consider a utility function homogeneous of degree one in its arguments (mean and standard deviation) and linear costs as from (4):

$$U = U (N R_n - Nz ;(d/N)+a) \quad (13)$$

We want to find the optimal number of assets.

Taking the total differential and equalling it to zero we get

$$dU/dN =(dU/dR)(dR/dN) + (dU/dV)(dV/dN) \quad (14)$$

$$0 = U_2 (- d/N^2) + U_1 (R_n -z)$$

where $U_2 = dU/dV$

$U_1 = dU/dR$

U_2/U_1 = absolute risk aversion = AP

from which we can get

$$N = \pm\sqrt{(U_2/U_1)(d /(R_n -z))} \quad (15)$$

we only consider the positive root,which can then be written as

$$N = \sqrt{AP \quad (d/(R_n -z))}$$

To get some empirical flavour about the solution just found, we sketch out some numerical solution,by attributing some standard values to the parameters:

$AP=1$;$d=.7$;$R_n=.1$;$z=.05$.

With these parameters the optimal number of activities for a price taker is approximately 4 when linear diversification costs are present.

If we evaluate analytically (15)we can see some comparative statics.When the cost z of holding an asset increases

(marginal costs are equal to average costs)we face two possibilities if the number of activities has to remain unchanged.Either R_n increases or AP moves downward to compensate for the decline of net return. As far as parameter d is concerned,it is a measure of average riskiness of assets.If it increases people have to augment the number of securities they randomly buy to decrease the risk of their portfolio. If only z grows an agent has to buy more types (variety) of assets: by doing that he lowers the variance of the portfolio and so compensates for a lower rate of return.[6]

Let us now consider the other two cases where cost elasticity is $\neq 1$,i.e. costs functions are not linear.

Case 1:<u>decreasing costs</u>

$$U = U\ (NR_n - Nz + fN^2 ; (d/N) + a) \tag{16}$$

optimization leads to

$$0 = U_2\ (-d/N^2) + U_1\ (R_n - z + 2fN)$$

from which we get

$$N^3\ 2\ U_1\ f + N^2 (U_1 R_n - U_1 z) - U_2 d = 0 \tag{17}$$

if we set

$$\frac{R_n - z}{2f} = b \qquad (18.1)$$

$$(U_2 / U_1)\,(d/2f) = c \qquad (18.2)$$

we get

$$N^3 + N^2 b - c = 0 \qquad (19)$$

Equation 19 is of 3rd degree of incomplete type.The solution can be found bya trasformation and by Cardano procedures.[7] In the appendix it is possible to trace the solution of (19).

<u>Case 2 :increasing costs</u>

We have to consider

$$U = U(NR_n - Nz - fN^2;\ (d/N)+a\ ) \qquad (20)$$

which can be reduced,by using the same procedure used above,to

$$N^3 - b\,N^2 + c = 0 \qquad (21)$$

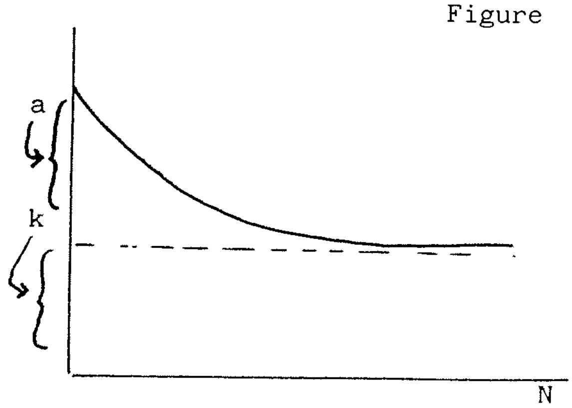
the reader is again sent to the appendix for the algebra,to get the solution of (20).Here we stress that both (19) and (20) have only one real solution(the others are complex-conjugate) which is fairly criptic to be investigated at an analytical level.Hence we decided to get some easier insights by some numerical solution which allows us to elicit some comparative statics consideration.By attributing standard numerical values to the parameters (see Appendix) we end up with a value of the unknown N which is greater when we have increasing costs than when we have decreasing costs.This might seem quite surprising .Yet it is the outcome of a mean-variance approach,which obliges the agent to decrease the total variance of its portfolio whenever costs of diversification increase.Viewing the question under a different perspective this would mean that whenever costs of diversification increase more than proportionally of the menu of assets, a firm (price taker) faces two ways to stay in equilibrium:either it increases returns (becoming more profitable overall)or decreases its riskiness by furtherly diversifying.This might explain why firms sometimes tend to diversify when returns decrease,not only owing to the costs of being multiproduct or multibranch.

A second model introduces new functions of costs and of variance. For the variance a new approximation is being used:

$$V_p = a\, e^{-N} + k \qquad (22)$$

represented below in figure 5 .

Figure 5



Costs equations are of the following type:

case.1

$$C = z\,e^{N} \qquad (23)$$

hence

$$U_2(-ae^{-N}) + U_1(z\,e^{N}) = 0 \qquad (25)$$

$$U_2/U_1\,(-ae^{-N}) + (ze^{N}) = 0$$

by multiplying by e^{N} and rearranging we get

$$z\,e^{2N} = (U_2/U_1)\,a$$

taking logs

$$2N = \lg AP + \lg a - \lg z \qquad (26)$$

case.2

$$C = ze^{-N} \qquad (24)$$

hence

$$U_2(-ae^{-N}) + U_1\,(-ze^{-N}) = 0 \qquad (27)$$

from which wecan get

$$(\,-APa - z\,)\,e^{-N} = 0$$

with logs becomes

$$\lg\,(-APa - z\,) = N \qquad (28)$$

(28)has no solution since we cannot take the log of a negative number.

<u>case .3</u>

$$C = Nz \pm fN^2$$

then we get

$$U_2 \ (-ae^{-N}) + U_1(z \pm 2fN \ ) = 0 \tag{29}$$

$$a \ AP \ e^{-N} = (z \pm 2fN)$$

taking logs

$$\lg a + \lg AP - N = \lg (z \pm 2fN)$$

which can also be written as

$$N = \lg AP + \lg a - \lg (z \pm 2fN \ ) \tag{30}$$

where lg $(z \pm 2fN)$ is the log of marginal costs of diversification. Equation (30) is not the algeabric solution of (29) but it allows us somehow to draw few economic conclusions about the results. As for (26) the results lend themselves to economic interpretation. When diversification costs increase the number of activities decreases. This means that when exponential costs are introduced the decrease of variance cannot compensate lower net returns and hence it is better to keep a smaller number of activities. If costs decrease exponentially no solution exists, therefore we cannot say anything about the optimal N. Case 3 deserves some more comments. There are quadratic costs and exponential utility. There is a solution but still costs enter negatively, which means that when they incerase, N decreases. This result hinges upon utility functions we assumed, which are not able to compensate with lower variance, due to higher N, the growing costs.

4.2 The CAPM approach

Take the case of a firm with two product lines. How can it evaluate a third line given a certain cost of diversification? CAPM [8] can give as an answer and provide a criterion of evaluation of assets and product lines. Whenever we want to refer to a new asset or to a new activity of a firm, we have to see whether it is profitable compared to the existing ones. The CAPM is an equilibrium model of financial activity which can be expressed in terms of either returns or prices of assets.

The CAPM is based on different assumptions, among which some might give rise to consistency problems when dealing with the evaluation of a multiproduct firm. That is the case of the assumptions of no transaction costs (i.e., no costs of diversification or scope), perfect divisibility and perfect competition. In terms of returns the CAPM model can be written as follows:

$$R_n = R_f + (R_M - R_f)/V_M^2 \ \mathrm{cov}(R_n, R_M) \qquad (31)$$

$$= R_f + \beta_n (R_M - R_f) \qquad (32)$$

where R_n is the return on the asset N, R_f is the risk free borrowing-lending rate, R_M is the average return of a portfolio which has the same composition of the entire market, V_M^2 is the variance of the entire market, while

$$\beta_n = \frac{\mathrm{cov}\ (R_n, R_M)}{V_M^2} \qquad (33)$$

is the measure of systematic risk of security n.

Since R_M and R_f are not related to the security n,the return on the nth asset depends only on its β factor.

If we introduce costs of diversification we can write

$$R_n = R_f + \beta_n (R_M - R_f) - f(n) \tag{34}$$

if we have linear costs of diversification with elasticity =1

$$f(n) = z\ N$$

otherwise we shall have (11') or (12).

We can also write the equation in terms of the price of the asset;remembering that[9]

$$R_n = \frac{\text{Ending value of the asset -Beginning value}}{\text{Beginning value}}$$

in symbols

$$= (Y_n - P_n)/P_n \tag{35}$$

where Y_n is the value of an asset + dividends or any other kind of returns

P_n value of the assets at the beginning of a period

$$R_M = (Y_M - P_M)/P_M \tag{36}$$

after some manipulations we get

$$P_n = (1/(1+R_f-f'(N)))(Y_n - (Y_M - (1 + R_f)P_M) \frac{cov(Y_N, Y_M)}{Var\ Y_M}) \tag{37}$$

This amounts to saying that if f(N) increases P_n has to increase as well.That implies that equilibrium requires the return on this activity to increase if its demand has to stay constant.

The explanation is similar to the one given in the previous section, when we used a standard portfolio approach.Take the case of fixed marginal costs (=z);equation (37) says that when z goes up,the price of the Nth activity has to increase to be in equilibrium;this in terms of returns corresponds to higher returns.Hence the only feedback is on price or, indirectly,returns,yet not on variance because CAPM does allow us to look for any optimal N (as we did in the previuos section)but only to variations of internal valuation of an activity according to the costs of adding it to an already existing portfolio.If P_n was in equilibrium (in a market sense) before introducing costs ,the new P_n will no longer be like that. This has an extraordinarily important consequence: "the portfolio of risky assets that any investor will own"[10] is no longer the "market portfolio". Costs crowd out the equilibrium features of CAPM assets.More will be done on CAPM and costs of diversification in a next paper.

5.Interpretations of results in terms of firms diversification and conclusions

Using a mean-variance approach it has been shown that costs of diversification make a difference as far as the number of randomly chosen activities (or product lines of a firm) is concerned.For suitable values of the parameters it is possible to calculate the optimal number of activities.This is usually a function of absolute risk aversion,costs of diversification and a market variable describing the ability of a market to eliminate risk .What emerges seems to be interesting even though the use of integer programming could furnish better information. According to the most standard utility and costs specifications,costs of diversification or scope should be compensated either by a higher average returns on each activity possessed or by a decrease of total risk.High costs of diversification are a quest for more rewarding portfolios

(either in terms of risk or of return)since they are a sign of market imperfections both in structural and Fama terms.
By resorting to a CAPM approach we have a similar answer even though it emerged more easily.Costs of diversification require an increase of the value of an activity,since further diversification is not viable and hence cannot compensate the decrease of net rates of returns by decreasing risk.

Yet the most outstanding result of this paper is that for standard values of parameters an optimal portfolio can be formed by randomly taking few activities(in one case we found 4) equiproportionally without investigating much further the risk and return properties of each activity.This accords well with empirical observation of potfolios and of product specifications of firms.In the latter case industrial considerations might play a more important role,yet as far as the risk and returns of the activities of a firm are concerned,costs of diversifications are going to change in some way the scope of production.

APPENDIX

The equation

$$x^3 + bx^2 - c = 0 \qquad \text{A.1}$$

can be reduced to a normal form by posing

$$x = y - b/3 \qquad \text{A.2}$$

hence we get by substitution

$$y^3 - y\frac{b^2}{3} - (b^3/27 + c) = 0 \qquad \text{A.3}$$

defining

$$p = - b^2/3 \qquad \text{A.4.1}$$

$$q = -(b^3/27 + c) \qquad \text{A.4.2}$$

we write the discriminant

$$D = q^2/4 + p^3/27 \qquad \text{A.5}$$

If A.5 =0 we have 3 real solutions of which two are coincidents.

If A.5 $<$ 0 ....1................and two complex.

If A.5 $>$ 0 .... 1.............. and two complex.

in terms of original parameters

$$D = b^6/16 + c^2/4 + 2b^3c/8 - b^6/3^6$$

in our case $b > 0$ and $c > 0$: hence $D > 0$.

The solution will be S + L where:

$$S = \sqrt[3]{-q/2 + \sqrt{D}} \qquad \text{A.6}$$

$$L = \sqrt[3]{-q/2 - \sqrt{D}} \qquad \text{A.7}$$

In order to study the solutions we attribute some numerical values to the parameters.Suppose that

$$\begin{aligned} R_n &= .I \\ z &= .05 \\ f &= .04 \\ d &= .7 \\ AP &= 1 \end{aligned} \qquad \text{A.8}$$

hence $b = .625$

$c = 8.75$

$q = -8.76$

$p = -.13$

$D = 19.19$

$S = 0$

$L = 2.1$

$S+L = 2.1$

$x = 2.1 - .2 = 1.9$ which can be aproximated to 2

This is the optimal number of assets when we face decreasing costs of diversification.In the case of increasing costs the solution becomes 2.3 which is higher than 1.9. This is the compensation effect seen in the paper, when lower returns have to be balanced by less risk.

Footnotes

1.see Elton-Gruber(1981),Fama-Miller(1972),Arrow(1970).

2.Condition (8)is expressed in its right parts as a function of integers(N_1,N_2).This could make wellbehavedness just partial due to indivisibilities.

3.See Solnick(1975).

4.It has been estimated (Solnick,1975) that a is roughly 30% of the average variance of assets in advanced western countries financial markets.

5.See Elton-Gruber (1981).

6.The problem we examine is one of integer optimization.We leave a solution by integer programming for a next paper.

7.See Kurosh(1971)

8.See Sharpe(1964),Lintner(1965),Fama(1971).

9.See Elton-Gruber(1981) Ch.11.

10.Ibidem pg.284

References

K.J.Arrow ,Essays inthe Theory of Risk Bearing,(1970),New York.

E.J.Elton-M.J.Gruber,Modern Portfolio Theory and Investment Analysis,2nd.ed.(1981) New York.

E.Fama "Risk Return and Equilibrium" Journal of Political Economy (1971),pp.30-55.

E.Fama-M.H.Miller,The Theory of Finance,(1972),New York.

A.Kurosh, Cours d'algèbre supérieure,(1971),Moscow.

J.Lintner,"The Market Price of Risk,Size of Market and Investor's Risk Aversion",Review of Economics and Statistics,(1970), pp.87-99.

W.F.Sharpe,"Capital Assets Prices:a Theory of Market Equilibrium Under Conditions of Risk",Journal of Finance (1964)pp.425-442.

B.Solnick,"The Advantages of Domestic and International Diversification" in Elton-Gruber,International Capital Markets(1975)Amsterdam.