DURABLE CAPITAL INPUTS: CONDITIONS FOR PRICE RATIOS TO BE INVARIANT TO PROFIT-RATE CHANGES

A COMMENT ON PROFESSOR SAMUELSON

by

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Durable Capital Inputs: Conditions for Prices Ratios to be Invariant to Profit-Rate Changes. A Comment on Professor Samuelson.

In a recent issue of this Journal, Professor Samuelson (1983) provided necessary and sufficient conditions for relative prices to be invariant to profit rate changes when production is carried out in a model of the Leontief-Sraffa type using durable capital which depreciates exponentially. According to this criterion, relative prices do not change when the rate of profit changes if and only if capital-labour ratio is uniform in all industries.

Professor Samuelson considers this condition surprisingly simple and new if compared to the assumption of uniform "organic composition of capital" which would assure the invariance of relative prices only in models of circulating capital. Moreover he maintains that conditions for relative price invariance become much more restrictive under depreciation schedules for durable commodities which are age-depending and therefore more complex than the exponential one; and in these general cases, Samuelson's analysis shows that the Sraffian Standard Commodity does not exist.

In this note we intend to prove that the uniform capital intensity assumption is a general condition for relative prices to be invariant both in models with circulating capital only and in fixed capital models irrespective of machine-depreciation schedules.

Moreover we shall prove that in fixed capital models the Standard Commodity exists and that it shows all the properties which Sraffa ascribed to it even if durable capital depreciation schedules are quite general.

1. Durable capital inputs and joint production

"When some inputs are durable, we are in the domain of joint products. Direct labor, working with leather and a new hammer, produces shoes and a used hammer"[1]. It is really surprising that, after such an assertion, Samuelson's analysis eventually eliminates joint production reducing the qualitative process of ageing and wearing of machines to a quantitative process of physical shrinking of durable commodities[2]. Since we think that an old machine is a commodity qualitatively different from a new one and not a new machine of reduced size, in this note we shall follow the Sraffian tradition and treat explicitly the durable instruments, obtained by the production process, as joint products.

Let us therefore consider an n-sector model of the Leontief-Sraffa type and let us suppose that the last industry uses in its own production a durable instrument, for instance, the commodity produced by the first industry[3]. Accordingly, the n.th industry must be regarded as being subdivided into as many separate activities as are the years of the total life of the instrument in question. Each activity will produce a certain quantity of the n.th commodity and, jointly, the instrument one year older than the one which it uses.

To avoid difficulties due to the presence of general forms of joint production, we shall suppose that the aged machines are not marketable commodities, that is to say they are not used in sectors different from those which produce them and no residual scrap survives out of the process using the machine in the last year of its working life.

Let T be the age of the machine in its last year of potential utilization, $a^j = \left[a_i^j\right]$ and $b^j = \left[b_i^j\right]$ $(i=1,2,\ldots,n;\ j=1,2,\ldots,n+T)$ be the vectors whose components measure respectively the inputs and outputs of marketable commodities in the j.th productive activity and let a_{0j} $(j=1,2,\ldots,n+T)$ be the corresponding labour requirements. Obviously the use of the first commodity will be zero everywhere except in the n.th activity -the first of the n.th industry- since only this activity uses the durable instrument when it is new.

If we denote by m_s $(s=0,1,\ldots,T)$ the s-years-old machines, the production structure of the model with durable instruments may be represented as follows:

$$A = \begin{bmatrix} a^1 & a^2 & \cdots & a^{n-1} & a^n & a^{n+1} & \cdots & a^{n+T-1} & a^{n+T} \\ 0 & 0 & \cdots & 0 & 0 & m_1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & m_{T-1} & 0 \\ 0 & 0 & \cdots & 0 & \underbrace{0 & 0 & \cdots & 0 & m_T}_{n.\text{th industry}} \end{bmatrix}$$

$$a_0 = \begin{bmatrix} a_{01} & a_{02} & \cdots & a_{0,n-1} & \underbrace{a_{0n} & a_{0,n+1} & \cdots & a_{0,n+T-1} & a_{0,n+T}} \end{bmatrix} \qquad (1)$$

$$B = \begin{bmatrix} b^1 & b^2 & \cdots & b^{n-1} & b^n & b^{n+1} & \cdots & b^{n+T-1} & b^{n+T} \\ 0 & 0 & \cdots & 0 & m_1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & & & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & \underbrace{0 & 0 & \cdots & m_T & 0} \end{bmatrix}$$

In the pages that follow, we shall suppose, to siplify notations, that the economy is stationary so that $m_0=m_1=\ldots=m_T=m$.

As a matter of fact, A and B are square matrices, since for every additional year of employment of the machine we add one activity and one additional commodity -the used machine of corresponding age-.

The competitive price vector p corresponding to the productive structure (1) is the solution of the following system of equations[4]:

$$pB = (1+r)pA + wa_0 \qquad (2)$$

that is to say

$$p = wa_0\left[B-(1+r)A\right]^{-1} \qquad (3)$$

As it was proved, if technique (1) is viable, then there exists a positive value r* of the profit rate such that for $r\in[0,r^*)$ solution (3) shows positive values for the marketable commodities[5]. The possible presence of negative prices for used machines means that the working life for the machine is not the most profitable..If we assume that it is possible to truncate the working life of the machine before the limit of its technical life without additional costs, then it is profitable to reduce conveniently the working life of the durable instrument obtaining a higher real wage at the prefixed value of the rate of profit. Associated with the most profitable working life of the machine we find positive prices for all commodities.

Hence it is obvious that the most profitable working life of the machine depends, in general, on the value of the rate of profit.

2. Conditions for relative prices to be invariant to profit rate changes

If we assume as numéraire the price, for instance, of the n.th commodity so that prices and the wage rate are expressed as a command on this good, the law of variation in relative prices when the rate of profit changes will be synthesized by the following system of relations

$$\frac{d}{dr} p^{(n)} = \frac{d}{dr} w^{(n)} a_0 \left[B-(1-r)A\right]^{-1} + w^{(n)} a_0 \left[B-(1-r)A\right]^{-1} A \left[B-(1-r)A\right]^{-1} = \tag{4}$$

$$= \left(\frac{d}{dr} w^{(n)} a_0 + p^{(n)} A\right) \left[B-(1-r)A\right]^{-1}$$

Since $\left[B-(1+r)A\right]$ is non singular for $r \in \left[0, r^*\right)$[6], a necessary and sufficient condition for relative prices to be invariant to profit rate changes is that

$$\frac{d}{dr} w^{(n)} a_0 + p^{(n)} A = 0 \tag{5}$$

If the real wage is uniform in all sectors, as we have assumed, then (5) implies that vectors a_0 of labour requirements and $p^{(n)} A$, whose components measure the value of capital in each production activity, are proportional.

Accordingly, relative prices appear to be invariant to profit rate changes if and only if capital intensity is uniform in all productive activities. In this case the slope of the w-r relationship will measure the common value of the sectoral capital-labour ratio. Denoting by A^i

and a_{0i} the i.th column of A and the corresponding component of vector a_0 respectively, from (5) we have

$$\frac{d}{dr} w^{(n)} = - \frac{p(n)A^i}{a_{0i}} \qquad (i=1,2,\dots,n+T)$$

Condition (5) is indipendent of the structure of matrix B and therefore it holds also in the case of circulating capital models.

Obviously, if the distributive variables are uniform, uniform capital-labour condition and uniform organic composition of production are quite equivalent assumptions. If relative prices are invariant, they are proportional to the "quantity of labour" embodied in the different commodities. Hence it follows that capital-labour ratio and dead-to-live labour ratio are equal and, consequently, that uniformity both of capital intensity and of organic composition are equivalent assumptions[7].

If $\alpha > 0$ measures the uniform rate of capital intensity, it will result $p^{(n)}A = \alpha a_0$. From (2) we shall have[8]

$$\left[(1+r)\alpha + w^{(n)}\right] a_0 B^{-1} = p^{(n)} \cdot$$

Then $(1+r^*)\alpha a_0 B^{-1} = p^{(n)}(r^*)$ will satisfy the following set of relations

$$(1-r^*)\ a_0 B^{-1}\left[B-(1+r^*)A\right] = 0$$

or,

$$(1+r^*)\ a_0 B^{-1}\left[I-(1+r^*)AB^{-1}\right] = 0$$

In other words $a_0 B^{-1}$ happens to be the left eigenvector of matrix AB^{-1} associated to r^*.

If we read $a_0 B^{-1}$ as the vector of labour coefficients, the formal analogy of these results with those of the models with circulating capital only is clear.

3. <u>On the depreciation schedule of machines</u>

The price system (2) was formulated without making restrictive assumptions on the dynamics with which the machine gradually exhausts itself from period to period. Such a dynamics may assume very complex forms that in general depend on the age of the durable instruments and are related to non proportional and even non monotonic behaviours in the use of commodities and labour for unit production. In general, therefore, the degree of machine efficiency in different years of use will not be definable in physical terms. In the logic of joint production efficiency and prices of aged machines are in fact obtained as solution of the whole price system. They implicitly define a schedule of the annual allowances for depreciation which allows a uniform price for the commodity produced with the use of durable instruments in their different ages and a uniform rate of profit in these activities. The pattern of changes in value of a machine has, except for very special cases, nothing to do with the idea of a physical law of change in its efficiency. In fact, prices of old machines depend in general on income distribution and they modify themselves when profit rate changes so as to lead even to reversals in their depreciation schedule[9].

As we have already noticed, the law of depreciation of the machine would be <u>a priori</u> identifiable, at least in its direction, if for instance production,coming out from a single machine, should change with constant physical inputs of circulating capital and labour for unit of production. If we want that the depreciation law of the machine be <u>a priori</u> defined in its intensity besides being in its direction, then we have to assume that the working life of the machine is endless. On the contrary the depreciation intensity will depend, in general, on the working life and the rate of profit.

Consider, for instance, the case in which the efficiency of a machine is definible as decreasing in physical terms at a constant rate $0\leq\delta<1$ with a maximum working life of T years. If for convenience we denote by the components of vector $\tilde{p}$ the prices of non durable marketable commodities and by p_{ns} the price of the machine when it is s years old[10], the price equations for the activities using the durable instrument may be written as follows:

$$\tilde{p}b^{n+s}-(1+r)\tilde{p}a^{n+s}-wa_{0,n+s}=\left[rp_{ns}+(p_{ns}-p_{n,s+1})\right]m= \qquad (6)$$

$$=(1-\delta)^{s}\left[rp_{n0}+(p_{n0}-p_{n1})\right]m \qquad (s=0,1,\dots,T;\ p_{n,T+1}=0)$$

From (6), we have

$$p_{ns}/p_{n0}=\left[(1-\delta)^{s}-(1+r)^{s}\left(\frac{1-\delta}{1+r}\right)^{T+1}\right]\Big/\left[1-\left(\frac{1-\delta}{1+r}\right)^{T+1}\right] \quad (s=0,1,\ldots,T) \qquad (7)$$

(7) shows, exactly, how the law of depreciation depends on the rate of profit and on the working life of the machine even in the very simple case we are considering. Only if the working life of the machine is endless, the law of depreciation reproduces the law of change in efficiency. From (7) we have in fact

$$\lim_{T\to\infty} p_{n1}/p_{n0} = (1-\delta)^{s} \qquad (s=0,1,\ldots,T)$$

This is the case of exponential depreciation examined by Professor Samuelson[11].

The case in which the law of depreciation is indipendent of the rate of profit, even though a physical law of change of its efficiency is not necessarily defined, is one in which capital intensity is uniform in all productive activities. As we have in fact seen, in this case, relative prices are invariant to the profit rate and so is, consequently, the law of depreciation of the durable instrument.

Obviously it may happen that, for the maximum value of the rate of profit, it is not profitable to use the durable instrument up to the limit T of its physical life. In this case the prices of the old machines, which should be eliminated, will be negative and, consequently, will be negative also the corresponding components of vector a_0B^{-1}[12]. The adjustment of the most profitable working life of the machine to the rate of profit will modify, in general, the productive structure of the system and, accordingly, the condition of uniform capital intensity will fail.

From this point of view we may assert that conditions of invariance of relative prices in fixed capital models depend on profit rate.

4. Fixed capital and Standard Commodity.

With regard to the fixed capital technique under examination, let us consider the following system of relations:

$$(1+g)Ax=Bx \qquad (8)$$

It was proved that, if all marketable commodities are basic in the

Sraffian sense, (8) possesses, for $g=r^*$, a solution $x^* > 0$.[13] Such a (column) vector can be interpreted as the set of sectoral multipliers which make the output structure equal to the input structure. This is the characteristic of the Sraffian standard commodity and system (8) can be assimilated to the corresponding standard system. Obviously whenever the price of x^* is adopted as numéraire for prices and the wage rate, the relationship between the distributive variables is linear. In fact, setting $p(r)(B-A)x^*=1$ and $a_0x^*=1$, from (2) we have $rpAx^*+w=1$. Since from (8) we obtain $r^*pAx^*=1$, it follows that the relation between r and w is linear.

There is therefore an essential parallelism with the case of production with circulating capital only. As a matter of fact the symmetry would be complete if a working life of the machine extended up to the limit of the technical life -T years- were the most profitable for all admissible values of the rate of profit. But this condition does not hold in general and this fact is pointed out, as we have already seen, by the presence of negative prices for old machines. The consequent possible change in the working life of the machine will lead to alternative technical configurations which will assure positive prices within those specific ranges of the profit rate for which such truncations are the most profitable. The process of profitable choice in the working life of the durable instrument leads therefore to a whole technology which, as usually happens, cannot provide us with a unique standard commodity but with a whole set of standard commodities, one for each productive structure referring to specific working life of the machine itself. For each single productive configuration the corresponding standard commodity will lead to a linear relationship between r and w if its price is used as numéraire.

5. Why results differ

By an operation of "vertical integration" in a temporal sense[14], we can replace the set of activities employing the machine in its various years of age by a fictious industry which is a weighted average of the original activities. The weights for making up this assembled industry are suitable powers of the factor of profit. It allows formally to eliminate the joint production component and bring the analysis back to the forms of single production.

Professor Samuelson uses implicitly this expedient in dealing with durable instruments as it is evident in section 9 of his paper[15]. This device, otherwise useful, is inconvenient for the problem of price invariance. If we impose, as Professor Samuelson does, the condition of uniform capital intensity looking at the system emboding the temporally integrated version of the mechanized industry, the condition becomes generally rate-of-profit dependent in the sense that the common value of the capital intensity changes when the rate of profit changes. This implies that prices cannot be constant and the wage-profit relationship cannot be linear. On the contrary, if we assume that capital intensity is uniform in each separate activity of the mechanized industry, the temporally integrated industry too has the same capital intensity irrespective of the weights used in assembling it.

Similar consequences appear for the Standard Commodity. When it is used in the temporally integrated system it has to face an "integrated technique" which is continuously moving with the rate of profit: no wonder that it cannot keep its characteristics. If we analyze the model in its spread version no substantial trouble erises.

From this point of view, the case of exponential depreciation is very peculiar. If we look at equation (6) in section 3 and remember that in this case we have $p_{ns} = (1-\delta)^s p_{n0}$, we have that all the price equations of the mechanized activities are identical so that the temporally integrated industry reproduces the same characteristics of each single component. Imposing the uniform capital intensity assumption on the integrated or on the spread mechanized industry leads, in this case, to the same result.

6. Some final remarks

We can now say that, for each working life of the machine, the presence of durable means of production does not affect the condition assuring the invariance of relative prices to profit rate changes that is to say the uniformity of the capital-labour ratio in all activities. Such a condition is quite general and it is independent of any assumption on the law of depreciation of the machine or on the law of change in its physical efficiency. Moreover, for each given working life of the machine, a standard commodity exists and it possesses all the characteristics that the Sraffian analysis attributes to it.

Whether any difficulties arise, they are due to the fact that the most profitable working life of the machine is not in general a technical datum but it is endogenously determined by the price mechanism and, like prices, it depends on the rate of profit. So we are faced by a set of technical alternatives characterized by a different working life of the machine, the choice of which depends on the rate of profit.[16] Apart from some more or less interesting cases-including exponential depreciation -the technical life of the machine may not necessarely be a remarkable parameter from an economic point of view.

But we cannot deduce that the assumption of uniform capital intensity is not a general condition for relative prices invariance. What may be asserted is that if in passing from a steady state to another one, characterized by a different profit rate, the most profitable working life of the machine had to change, the corresponding technical change will not in general allow to preserve the conditions for possibly uniform capital intensity.[17] Consequently, the wage-profit relationships corresponding to the most profitable working lives of the machine cannot be all linearized by using a common standard of value.

A similar result is obtained for the Standard Commodity. It changes with the working life of the machine and, therefore, with the rate of profit. The value of one of these standard commodities cannot be assumed as a common numéraire to linearize the wage-profit relationships corresponding to different working lives.

By the way, let us make a further point. It may be worth to point out that in the presence of many means of production linearity in each wage profit relationship is a necessary but in general not a sufficient condition to realize the conditions suitable for global price-ratio invariance.[18] It is in fact necessary that the outermost contour of the wage-profits relationships be an economically efficient locus. Only in this case at the switching points prices in terms of wage of the involved techniques are equal. As the assumption of uniform capital intensity requires prices to be proportional to the vector of unit labor requirements and this one to be left eigenvector of the matrix of technical coefficients, it is necessary that the techniques involved in the construction of the wage-profit frontier have the same left eigenvector and therefore the same structure in the unit employment of labour.[19] This makes the conditions for a correct aggregation of the technology even more difficult.

A numerical example

Let us consider the case in which a machine (commodity 1) is produced by using labor and circulating capital and this one (commodity 2) is produced by using itself, labor and the machine which has a technical life of 3 years. The industry of circulating capital is therefore subdivided into 3 separate activities using the machine in its different ages. Let the input-output structure be as follows:

$$A=\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0.6496 & 0.5 & 0.4 & 0.5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad B=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$a_0=\begin{bmatrix} 0.6496 & 1.312 & 1.04 & 0.8 \end{bmatrix}$$

The efficiency of the machine is certainly increasing from the first to the second year of use but its change when the machine passes from the second to the third year of employment is not determined in physical terms.

It is easy to verify that $\lambda=0.8$ is the only positive root of $|\lambda B-A|=0$ and that $a_0B^{-1}=[0.6496\ 0.8\ 0.512\ 0.24]$ satisfies for $\lambda=0.8$ $a_0B^{-1}[\lambda B-A]=0$.

The conditions assuring that capital intensity is uniform are therefore satisfied. Accordingly, the prices of the machines, in terms of commodity 2, will result to be constant even if the depreciation schedule is not of the exponential type. They are $p_{n0}=0.812$, $p_{n1}=0.64$ and $p_{n2}=0.3$.

Obviously the relationship between the distributive variables is linear. We have in fact

$$w = 0.25 - r$$

The corresponding Standard Commodity is the column vector $x^*=[1\ \ 0.8\ \ 0.64\ \ 0.512]$.

Footnotes

(1) See Samuelson (1983) p. 2.

(2) This is a recurrent attitude in Samuelson's economic view. See, in fact, Dorfman, Samuelson, Solow (1958) p. 383.

(3) We assume that only one industry is mechanized to make exposition simpler.

(4) As a matter of fact Professor Samuelson considers a price system with advanced wage. Following Sraffa analysis we prefer here to consider wage as paid post factum. Conditions about relative price invariance are however the same. If we denote by $\hat{p}$ the price vector with advanced wage, we have

$$\hat{p} = (1+r)wa_0[B-(1+r)A]^{-1} = (1+r)p$$

In terms of the i.th commodity, we have therefore

$$\hat{p}^{(i)} = p^{(i)}$$

(5) See, for instance, Baldone (1980), Schefold (1980) and Varri (1980) in Pasinetti (1980)

(6) r* corresponds to the minimum positive root of $|B-(1+r)A| = 0$. If the technique is viable, then for $r \in [0,r^*)$ there are no other singularities for matrix $[B-(1+r)A]$. Obviously since B is non diagonal, for such values of the rate of profit $[B-(1+r)A]^{-1}$ will show some negative components.

(7) If we denote by v the vector of labour-values and by w=vb the value of the physical wage, the organic composition of capital of the i.th sector may be written as follows:

$$(vA^i + vba_{0i})/vba_{0i} = vA^i/vba_{0i} + 1$$

If vb is uniform and so must be the organic composition, then vA^i/a_{0i} too is uniform. This implies the uniformity of the capital-labour ratio and vice versa.

(8) As it is easy to verify, the technical structure we are examining is such that B is non singular.

(9) See Baldone (1980) §15.

(10) Obviously we have $p=[p_{n0}\ \tilde{p}\ \ p_{n1} \cdots p_{nT}]$.

(11) It may be worth to stress that it is the price of the machine and not the machine to reduce itself, even if, from a quantitative point of view, the two phenomena may be considered equivalent.

(12) If the capital intensity is uniform, we have in fact $p(r^*) \div a_0 B^{-1}$.

(13) See Baldone (1980) §12.

(14) See, for instance, Sraffa (1980) ch. X.

(15) See Samuelson (1983) eq. (9.2) and following.

(16) This problem is quite similar to the choice of the most profitable technique in the case of production with circulating capital only, with all the phenomena associated to it included the possibility of the reswitching of a same truncation.

(17) This may happen if the commodity produced by the durable instrument is a pure consumption good.

(18) See Samuelson (1983) § 6.

(19) On this point, see Baldone (1982).

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