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Abstract

When in proxy-SVARs the covariance matrix of VAR disturbances is subject to exogenous, permanent, nonrecurring breaks that generate target impulse response functions (IRFs) that change across volatility regimes, even strong, exogenous external instruments can result in inconsistent estimates of the dynamic causal effects of interest if the breaks are not properly accounted for. In such cases, it is essential to explicitly incorporate the shifts in unconditional volatility in order to point-identify the target structural shocks and possibly restore consistency. We demonstrate that, under a necessary and sufficient rank condition that leverages moments implied by changes in volatility, the target IRFs can be point-identified and consistently estimated. Importantly, standard asymptotic inference remains valid in this context despite (i) the covariance between the proxies and the instrumented structural shocks being local-to-zero, as in Staiger and Stock (1997), and (ii) the potential failure of instrument exogeneity. We introduce a novel identification strategy that appropriately combines external instruments with "informative" changes in volatility, thus obviating the need to assume proxy relevance and exogeneity in estimation. We illustrate the effectiveness of the suggested method by revisiting a fiscal proxy-SVAR previously estimated in the literature, complementing the fiscal instruments with information derived from the massive reduction in volatility observed in the transition from the Great Inflation to the Great Moderation regimes.

Keywords: External instruments, Fiscal multipliers, Identification, Proxy-SVARs, Structural breaks, Shifts in volatility, Weak instruments.

JEL Classification: C32, C51, C52, E62

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Non-technical summary

This research investigates the extent to which proxy-SVAR (SVAR-IVs) methods are effective. Such methods are used to understand how economic policies affect the economy. For instance, in the empirical application, we study what happens when governments implement polices that change their spending or tax revenues. Historically, however, the parameters of interest change with volatility regimes due to changes in market structures, preference parameters, or policy conduct. Then, standard proxy-SVAR methods fail to pinpoint the effect of such policies. In the presence of volatility regimes, we find that proxy-SVAR methods can yield reliable results if two conditions are met. First, the way policies affect the economy must stay the same, even when the overall economic environment changes. Second, the method must focus on relative effects of these policies, rather than their absolute value. This paper addresses this challenge and establishes that, by meeting a technical necessary and sufficient rank condition, informed by volatility regimes, we can identify and accurately estimate parameters of interest, even when faced with irrelevant or contaminated external instruments, and standard inference applies.

The study demonstrates that properly combining volatility changes with external instruments can significantly improve inference quality. We employ this novel approach to revisit the analysis of a seminal fiscal proxy-SVAR estimated for the US economy, augmenting the set fiscal instruments with the change in the unconditional VAR error covariance matrix. This captures the decline in volatility observed in the transition from the Great Inflation to the Great Moderation period.

Our findings reveal that: (i) the estimated fiscal multipliers are larger than one; (ii) the peak tax multiplier appears smaller than the estimated peak fiscal spending multiplier, albeit not dramatically; and (iii) the uncertainty around the dynamic tax multiplier substantially reduces when we consider the volatility change, as opposed to using the fiscal proxy-SVAR across the entire estimation sample disregarding the shift.

1 Introduction

The last decades have witnessed significant advancements in the development of novel methods for the identification of macroeconomic shocks in Structural Vector Autoregressions (SVARs). Among these, "the identification by external instruments" and "the identification by heteroskedasticity" play an important role, see e.g. Stock and Watson (2017). In the identification by external instruments, the focus is on specific structural shocks which are identified through the use of variables external to the VAR, henceforth referred to as instruments or proxies interchangeably. External instruments address a "partial identification" problem, following the approach proposed by Stock (2008), Stock and Watson (2012, 2018), and Mertens and Ravn (2013). The proxies must be relevant, i.e. correlated with the structural shock(s) of interest, and exogenous, i.e., uncorrelated with the non-instrumented shocks. The fulfillment of both relevance and exogeneity conditions ensures that, under regular conditions, the target impulse response functions (IRFs) can be point-identified, consistently estimated, and standard asymptotic inference applies. Montiel Olea, Stock, and Watson (2021) have extended asymptotic inference to cases where SVARs feature "weak" proxies as in Staiger and Stock (1997). Their contribution emphasizes that even external variables which are poorly correlated with the target structural shocks can offer valuable information for identification. In this paper, we show that external instruments provide valuable information for identification even in seemingly problematic situations, such as the occurrence of shifts in the unconditional volatility of the variables associated with changes in the dynamic causal effects of interest.

As is known, economic relationships are affected by structural breaks, typically induced by changes in underlying structural behavior, market conditions or changes in policy conduct. These breaks typically lead to shifts in the dynamics and volatility of the variables of interest. The identification through heteroskedasticity approach, first introduced in SVARs by Lanne and Lütkepohl (2008) and inspired by the seminal work of Rigobon (2003) (also see Sentana and Fiorentini, 2001), relies on the information present in the data through variations in the unconditional covariance matrix of the VAR. This method is typically based on the assumption that breaks in volatility do not alter the impact and propagation of structural shocks but only shift the variance of these shocks.[1] Bacchiocchi and Fanelli (2015), Bacchiocchi, Castelnuovo, and Fanelli (2018) and Angelini, Bacchiocchi, Caggiano, and Fanelli

[1] This approach is commonly acknowledged as a "statistical" identification method, given that the shocks can only be labeled ex-post, i.e., after the model is estimated, typically by examining the signs of estimated on-impact coefficients or the implied shape of IRFs.

(2019) have shown that when there is a rationale to believe that volatility shifts may be induced by breaks in structural parameters, the heteroskedasticity approach can be extended to scenarios where IRFs change across volatility regimes.[2] In such cases, point identification of the target IRFs can be achieved by incorporating a limited set of theory-driven or institutionally-knowledge-based constraints into the model, while still benefiting from the identification power provided by shifts in volatility. These constraints, referred to as "stability restrictions" by Magnusson and Mavroeidis (2014), involve specifying particular structural parameters to vary across volatility regimes while keeping other structural parameters constant; see also Bacchiocchi and Kitagawa (2022b).

This paper contributes to the literature on SVARs by exploring how changes in unconditional volatility contribute to the identification of the target structural shocks with external instruments. These models will be denoted as proxy-SVARs (SVAR-IVs) throughout. Specifically, we focus on proxy-SVARs where the covariance matrix of VAR errors exhibits exogenous, permanent, nonrecurring breaks, leading to changes in the target IRFs. In such cases, it is essential to explicitly incorporate the shifts in unconditional volatility into the analysis because even strong, exogenous external instruments may produce inconsistent estimates of the target IRFs if the breaks are not properly taken into account. The combination of external instruments with the shifts in volatility ensures the consistency of the estimator of dynamic responses and the use of standard asymptotic inference. This result marks an important difference relative to the scenario in which the target IRFs remain constant across volatility regimes. In that scenario, indeed, we show that relative (normalized) IRFs can be estimated consistently using strong and exogenous instruments even if the breaks in volatility are not accounted for.

We establish that under a necessary and sufficient rank condition derived from moments induced by changes in volatility, identification is achieved even in the presence of proxies that are neither relevant nor exogenous. Notably, standard asymptotic inference continues to hold despite: (i) the covariance between proxies and instrumented structural shocks are local-to-zero as in Staiger

[2] SVARs, whose IRFs change across macroeconomic regimes in correspondence to different levels of unconditional volatility, are denoted as "SVAR-WB" (with WB standing for "with breaks") in Bacchiocchi, Castelnuovo, et al. (2018) and Bacchiocchi and Kitagawa (2022b). These models are intended to reflect shifts in key structural parameters related to the behavior of economic agents, market functioning, and/or policy conduct. Similar phenomena have been studied in the literature, for example, by Lubik and Schorfheide (2004) and Castelnuovo and Fanelli (2015) in the context of solutions generated by monetary DSGE models, and by Clarida, Galí, and Gertler (2000) and Boivin and Giannoni (2006) to explain macroeconomic phenomena such as the Great Inflation and the Great Moderation.

and Stock (1997); and (ii) the potential breakdown of instrument exogeneity, wherein instruments, beyond their correlation with the target shocks, also exhibit correlation with some or all non-target shocks, a phenomenon we refer to as "contamination". In this context, the investigator has the flexibility to deduce the role of external instruments in the identification process directly from the data. In the least favorable scenario, external instruments serve as a labeling device for the target structural shocks.

Based on the above results, we develop a novel identification strategy that integrates external instruments with shifts in volatility regimes, eliminating the necessity to assume proxy relevance and exogeneity prior to estimation. Within this framework, the Classical Minimum Distance (CMD) estimator emerges as a natural choice.[3] We call external variables that are both credibly relevant and exogenous as "valid instruments" or "valid proxies". Conversely, we use the terms "invalid instruments" or "invalid proxies" for external variables where the conditions of relevance and/or exogeneity is not satisfied. A more comprehensive characterization is offered in Definition 1, Section 2. Our analytic results, supported by extensive Monte Carlo simulations, demonstrate that using external instruments in a framework where IRFs change across volatility regimes is generally advantageous. In fact, when strong and exogenous instruments complement identification based on shifts in volatility, there are considerable gains in estimation precision. Notably, even with contaminated yet strong instruments consistency is not affected and there are significant gains in estimation precision relative to the case in which only changes in volatility are leveraged. Remarkably, even in the worst-case scenario where weak, contaminated instruments are included in the analysis, the estimator of the target IRFs remains consistent and asymptotically Gaussian.

Intuitively, the relevance condition is not strictly necessary to meet in our framework as point identification of the target structural shocks can be achieved, under the derived necessary and sufficient rank condition, by the moment conditions implied by the shifts in unconditional volatility. This result parallels the findings of Antoine and Renault (2017) regarding the "relevance of weaker instruments", i.e. the scenario that emerges in Generalized Method of Moments estimation when the moment conditions characterized by local-to-zero instruments as in Staiger and Stock (1997) complement the moment conditions associated with strong instruments. Thus, the changes in the VAR covariance matrix assume a role akin to relevant instruments in the external instruments approach.

[3] An alternative Quasi Maximum Likelihood (QML) approach, where the process governing VAR innovations and proxies is assumed conditionally normal within each volatility regime, is developed in the associated supplementary material.

The exogeneity condition can be relaxed due to the inherent "full identification" nature of the identification through changes in volatility, which delivers also information concerning the non-target shocks. The comprehensive point identification of both target and non-target shocks, a distinctive characteristic of the changes in volatility approach, gives rise to both advantages and limitations when external instruments are employed. The principal benefit lies in the fact that external instruments can be correlated with non-target structural shocks, other than the target shocks. Working with macroeconomic, aggregate data, there is a growing consensus that even when researchers carefully pick instruments that are plausibly exogenous, it is still unlikely that an instrument perfectly satisfies the orthogonality condition. Our approach empowers applied researchers to make reliable inferences in setups where instruments are nearly exogenous, not perfectly exogenous and, in general, correlated with the non-target shocks. In this regard, we note that, similar to Ludvigson, Ma, and Ng (2020, 2021), our analysis does not primarily focus on relaxing exogeneity per se. Instead, our broader objective is to leverage the properties of external variables to facilitate identification

Recently, Schlaak, Rieth, and Podstawski (2023) underscored the advantages that changes in volatility offer, under specific conditions, for testing instrument exogeneity in point-identified proxy-SVARs. In their framework, only the variances of structural shocks change across volatility regimes, while IRFs remain constant. A thorough comparison of our approach with that of Schlaak et al. (2023) will be presented in Section 3. Ludvigson et al. (2021), and Braun and Brüggemann (2023) have demonstrated the possibility, in principle, of handling proxies akin to the concept of "plausibly exogenous" instruments, as discussed by Conley, Hansen, and Rossi (2012).[4] Although we refer to point identification, our framework yields a similar result in the sense that the exogeneity condition need not to be imposed in estimation. Failure of the exogeneity condition as well as the strength of the proxies can be directly inferred from the data when the stability restrictions are correctly specified. Our framework also shares the same flexibility highlighted by Keweloh, Klein, and Prüser (2024) for models identified through the combination of proxy variables and non-Gaussian shocks. Remarkably, a significant advantage of our approach is that, given the stability restrictions, we do not rely on any assumptions regarding the distribution and/or cross-independence of the structural shocks. Our framework operates under the assumption that the structural

[4] In the microeconometric literature on Instrumental Variable (IV) regressions, there is a growing consensus that even when researchers carefully pick instruments that are plausibly exogenous, it is unlikely that an instrument perfectly satisfies the orthogonality condition; see, e.g. Berkowitz, Caner, and Fang (2012) and references therein.

shocks are cross-uncorrelated, see Ramey (2016, Section 2.1). Furthermore, in comparison to the aforementioned contributions, the suggested stability restrictions approach requires the specification of a set of (minimal) restrictions to achieve identification. Therefore, other than combing external instruments with changes in volatility, it addresses the challenges and limitations regarding the causal interpretation of purely statistical approaches to identification based on heteroskedasticity or non-Gaussian shocks, as emphasized by Montiel Olea, Plagborg-Møller, and Qian (2022)

We present the key aspects of our approach by revisiting the fiscal proxy-SVAR estimated in Mertens and Ravn (2014) on US quarterly data covering the period 1950:Q1-2006:Q4. We use two fiscal instruments, one for the tax shock, which coincides with the narrative tax instrument of Mertens and Ravn (2014), and the other for the fiscal spending shock, taken from Angelini, Caggiano, Castelnuovo, and Fanelli (2023). Additionally, we consider a structural break in the VAR covariance matrix, capturing the substantial reduction in volatility observed during the transition from the Great Inflation to the Great Moderation period.

The paper is organized as follows. Section 2 covers our baseline proxy-SVAR specification (Section 2.1), the data generating process (DGP) and assumptions (Section 2.2), and our main results on identification, estimation and check of identifiability (Section 2.3). Section 3 discusses connections and differences with contributions in the recent literature. Section 4 applies the methodology to the identification of US fiscal multipliers. Section 5 concludes. A supplement complements the paper in various dimensions, including formalization, comprehensive Monte Carlo experiments, proofs of propositions, additional empirical results, and the extension of the analysis to the case of multiple volatility regimes and QML estimation.

2 Proxy-SVARs with shifts in unconditional volatility

In this section, we introduce the baseline proxy-SVAR specification within a DGP that incorporates a break ($M = 1$) in the error covariance matrix, resulting in $M + 1 = 2$ volatility regimes in the data. We extend the analysis to more than two structural breaks in the supplementary material. Section 2.1 presents the proxy-SVAR as an augmented SVAR model and defines proxy properties. Section 2.1 outlines the DGP and the assumptions underpinning the analysis. Section 2.3 introduces the stability restrictions approach, covering identification conditions, CMD estimation, informal methods to check identifiability.

2.1 Baseline model

Our reference model is the SVAR:

$$\begin{aligned} Y_t &= \Pi X_t + u_t, \quad t = 1, ..., T \\ u_t &= H\varepsilon_t, \end{aligned} \tag{1}$$

where Y_t is the $n \times 1$ vector of endogenous variables, $X_t := (Y'_{t-1}, ..., Y'_{t-l})'$ is the vector collecting l lags of the variables, T the number of time observations, $\Pi := (\Pi_1, \ldots, \Pi_l)$ is the $n \times nl$ matrix containing the autoregressive (slope) parameters and u_t an $n \times 1$ martingale difference sequence (MDS), $\mathbb{E}(u_t \mid \mathcal{I}_{t-1}) = 0$, $\mathcal{I}_t := (Y_t, Y_{t-1}, ...)$ being the information set available at time t. Throughout the paper, we refer to the term u_t in equation (1) as "VAR innovations" or "VAR disturbances". Deterministic terms have been excluded from the analysis without loss of generality. The initial values $Y_0, ..., Y_{1-l}$ are treated as fixed constants throughout the analysis. In the following we denote with $\mathcal{C}_y$ the VAR companion matrix that depends on the parameters in Π; i.e. $\mathcal{C}_y = \mathcal{C}_y(\Pi)$.

In (1), the system of equations $u_t = H\varepsilon_t$ maps the $n \times 1$ vector of structural shocks ε_t to the reduced-form innovations by the $n \times n$ nonsingular matrix H, that contains the on-impact (instantaneous) coefficients. It is assumed that the structural shocks have normalized covariance matrix $\Sigma_\varepsilon := \mathbb{E}(\varepsilon_t \varepsilon'_t) = I_n$, meaning that in the rest of the paper we consider responses to one-standard deviations shocks, except where indicated. In this framework, model (1) implies the "conventional" SVAR restrictions $\Sigma_u := \mathbb{E}(u_t u'_t) = HH'$. We temporary assume that the parameters (Π, H), hence $\Sigma_u < \infty$, are constant, i.e., time-invariant, over the sample period $Y_1, ..., Y_T$. This hypothesis will be relaxed in Section 2.2.

Let $\varepsilon_{1,t}$ be the $k \times 1$ subvector of elements in ε_t containing the $1 \leq k < n$ *target* structural shocks. We consider the corresponding partition of the structural relationships $u_t = H\varepsilon_t$:

$$u_t = \begin{pmatrix} u_{1,t} \\ u_{2,t} \end{pmatrix} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \begin{pmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \end{pmatrix} = H_{\bullet 1}\varepsilon_{1,t} + H_{\bullet 2}\varepsilon_{2,t} \tag{2}$$

where $\varepsilon_{2,t}$ contains the $(n-k)$ structural shocks that are not of interest, $u_{1,t}$ and $u_{2,t}$ are VAR disturbances and have the same dimensions as $\varepsilon_{1,t}$ and $\varepsilon_{2,t}$, respectively; $H_{\bullet 1} := (H'_{11}\ , H'_{21})'$ is $n \times k$ and collects the on-impact coefficients associated with the target structural shocks. Finally, $H_{\bullet 2}$ is $n \times (n-k)$

and collects the on-impact coefficients associated with the non-target shocks. The objective of the analysis is the identification and estimation of the h-period ahead responses of the variables, Y_{t+h}, to the j-th target shock in $\varepsilon_{1,t}$, corresponding to:

$$IRF_{\bullet j}(h) := \frac{\partial Y_{t+h}}{\partial \varepsilon_{1,j,t}} = (S_n(\mathcal{C}_y)^h S_n')H_{\bullet 1}e_j \ , \quad 1 \leq j \leq k \tag{3}$$

where $S_n := (I_n \ , \ 0_{n\times n(l-1)})$ is a selection matrix and e_j is the $k \times 1$ vector containing '1' in the j-th position and zero elsewhere. Equation (3) refers to responses to one-standard deviation target shocks. Note that we denote the IRFs in (3) as "absolute" IRFs since they do not incorporate "unit effect" normalizations, as discussed below. While the reduced-form parameters in the companion matrix $\mathcal{C}_y$ can be easily estimated from the reduced-form VAR, the identification and estimation of the on-impact coefficients in $H_{\bullet 1}$, or of the relative impacts $H_{2,1}^{rel} := H_{2,1}(H_{1,1})^{-1}$, may be challenging in the absence of auxiliary information. The solution provided by the "external instruments approach" is to consider an $r \times 1$ vector of external variables, say z_t, $r \geq k$, which satisfy the conditions:[5]

$$\mathbb{E}(z_t\varepsilon_{1,t}') = \Phi \ , \ rank[\Phi] = k \quad \text{(relevance)} \tag{4}$$

$$\mathbb{E}(z_t\varepsilon_{2,t}') = 0_{r\times(n-k)} \quad \text{(exogeneity)} \tag{5}$$

where Φ is an $r \times k$ matrix of parameters. By combining (2) with the external instruments yields the moment conditions:

$$\mathbb{E}(u_t z_t') := \Sigma_{u,z} := \begin{pmatrix} \Sigma_{u_1,z} \\ \Sigma_{u_2,z} \end{pmatrix} = H_{\bullet 1}\Phi' := \begin{pmatrix} H_{1,1}\Phi' \\ H_{2,1}\Phi' \end{pmatrix} \quad \begin{matrix} k \times r \\ (n-k) \times r \end{matrix} \tag{6}$$

which represent the key ingredients of the proxy-SVAR approach; see Stock (2008), Mertens and Ravn (2013) and Stock and Watson (2018). Note, in particular, that under fairly general conditions on the processes that generate VAR innovations and proxies, $\eta_t := (u_t', z_t')'$, the estimator $\hat{\Sigma}_{u,z} := \frac{1}{T}\sum_{t=1}^{T} \hat{u}_t z_t'$, where $\hat{u}_t$, $t = 1, ..., T$, are the VAR residuals, is a $\sqrt{T}$-consistent, asymptotically Gaussian estimator of the covariance matrix $\Sigma_{u,z}$ in (6); see supplemen-

[5]In equations (4)-(5) it is maintained that the proxies z_t are expressed in "innovations form", i.e. a serially uncorrelated process, condition that can be relaxed. Actually, we may also consider a vector of $r \times 1$ "raw" proxies Z_t for which one can consider the decomposition $Z_t = Proj(Z_t \mid \mathcal{I}_{t-1}) + z_t$, where $Proj(Z_t \mid \mathcal{I}_{t-1})$ is the linear projection of Z_t onto the space spanned by the variables in the information set at time $t-1$, $\mathcal{I}_{t-1}$, and z_t captures the "unsystematic component" of Z_t, that is, what remains of Z_t after appropriately filtering out the dynamics induced by past information.

tary material for details.

For our purposes, a convenient summary of the conditions (4)-(5) is given by the linear measurement error model:

$$z_t = \Phi\varepsilon_{1,t} + \Omega_\omega \omega_t \tag{7}$$

where ω_t is a normalized measurement error term, meaning that $\mathbb{E}(\omega_t\omega_t') = I_r$, uncorrelated with the structural shocks $\varepsilon_t := (\varepsilon_{1,t}', \varepsilon_{2,t}')'$, and Ω_ω is a symmetric and positive definite matrix, so that $\mathbb{E}(z_t z_t') = \Phi\Phi' + \Omega_\omega\Omega_\omega'$.

The proxy-SVAR can be then compacted in the expression:

$$\underbrace{\begin{pmatrix} Y_t \\ Z_t \end{pmatrix}}_{W_t} = \underbrace{\begin{pmatrix} \Pi \\ * \end{pmatrix}}_{\Gamma} X_t + \underbrace{\begin{pmatrix} u_t \\ z_t \end{pmatrix}}_{\eta_t} \tag{8}$$

$$\underbrace{\begin{pmatrix} u_t \\ z_t \end{pmatrix}}_{\eta_t} = \underbrace{\begin{pmatrix} H & 0 \\ R_\Phi & \Omega_\omega \end{pmatrix}}_{G} \underbrace{\begin{pmatrix} \varepsilon_t \\ \omega_t \end{pmatrix}}_{\xi_t} \tag{9}$$

where Γ collects the VAR autoregressive parameters as well as the parameters "$*$" associated with the potential lags necessary to whiten the raw instruments, R_Φ contains the relevance parameters and the exclusion restrictions arising from the exogeneity conditions, and $\xi_t := (\varepsilon_t', \omega_t')'$, with $\mathbb{E}(\xi_t\xi_t') = I_{n+r}$, collects the structural shocks and the normalized measurement errors (for similar representations, see e.g., Angelini and Fanelli, 2019; Arias, Rubio-Ramírez, and Waggoner, 2021; Giacomini, Kitagawa, and Read, 2022). The vector η_t in (9) incorporates the VAR innovations and the proxies; the implied $(n+r)\times(n+r)$ covariance matrix $\Sigma_\eta := \mathbb{E}(\eta_t\eta_t') = GG'$ is symmetric and positive definite. The matrix G in (9) plays a key role for our analysis. In its more general form, it gives rise to the covariance restrictions:

$$\begin{aligned} \Sigma_\eta := \begin{pmatrix} \Sigma_u & \Sigma_{u,z} \\ \Sigma_{z,u} & \Sigma_z \end{pmatrix} &= GG' = \begin{pmatrix} HH' & HR_\Phi' \\ R_\Phi H' & R_\Phi R_\Phi' + \Omega_\omega\Omega_\omega' \end{pmatrix} \\ &= \begin{pmatrix} H_{\bullet 1}H_{\bullet 1}' + H_{\bullet 2}H_{\bullet 2}' & H_{\bullet 1}\Phi' + H_{\bullet 2}\Upsilon' \\ \Phi H_{\bullet 1}' + \Upsilon H_{\bullet 2}' & \Phi\Phi' + \Upsilon\Upsilon' + \Omega_\omega\Omega_\omega' \end{pmatrix}, \end{aligned} \tag{10}$$

where, for future developments, we have parameterized the matrix R_Φ as $R_\Phi := (\Phi, \Upsilon)$, with Υ is such that $\Upsilon = 0_{r\times(n-r)}$ when proxy exogeneity holds, and $\Upsilon \neq 0_{r\times(n-r)}$ in the presence of contamination effects. In other words, possible non-zero elements in Υ capture proxy contamination, see below.[6] The moment

[6] We opt to use the term "contamination" when exogeneity fails, rather than the term

conditions (10) incorporate four cases of interest for the proxies z_t, addressed in Definition 1.

Definition 1 (Instrument properties) *Given the proxy-SVAR in* (8)-(9) *with* R_Φ *corresponding to the bottom-left block of* G*, consider a drifting DGP characterized by* sequences *of models in which* $\mathbb{E}(z_t\varepsilon_{1,t}') = \Phi_T$*; then the proxies* z_t *are:*

(i) *"strong and exogenous" if* $\Phi_T \to \Phi$ *with* $rank[\Phi] = k$*, and* $\Upsilon = 0_{r\times(n-r)}$*;*

(ii) *"local-to-zero and exogenous" if* $\Phi_T = T^{-1/2}C$*,* C *being an* $r \times k$ *matrix with finite norm,* $\|C\| < \infty$*, and* $\Upsilon = 0_{r\times(n-r)}$*;*

(iii) *"strong and contaminated by the non-target shocks" if* $\Phi_T \to \Phi$ *with* $rank[\Phi] = k$*, and* $\Upsilon \neq 0_{r\times(n-r)}$*.*

(iv) *"local-to-zero and contaminated by the non-target shocks" if* $\Phi_T = T^{-1/2}C$*,* C *being an* $r \times k$ *matrix with finite norm,* $\|C\| < \infty$*, and* $\Upsilon \neq 0_{r\times(n-r)}$*.*

Starting from the tenet that valid external instruments in the sense of Stock and Watson (2018) are those matching the condition in Definition 1.(i), the external instruments in Definitions 1.(ii)-1.(iv) will be deemed invalid. Specifically, Definition 1.(iv) characterizes our most extensive interpretation of the concept of invalid proxies, namely a scenario in which the external instruments z_t exhibit weak correlations in the sense of Staiger and Stock (1997) with the target shocks, while being correlated with some or all non-target shocks. It is worth emphasizing that, in all four cases in Definition 1, the matrix G in (9) maintains its nonsingularity, provided $rank[H] = n$ and $rank[\Omega_\omega] = r$. This particular detail holds significant importance for our developments in Section 2.3.

"endogenous proxies". Our consideration is focused on a scenario in which external instruments are primarily associated with the target shocks. However, their correlation with the target shocks can be altered by potential links with some of the non-target shocks which, possibly, can be explained by economic arguments. To provide a concrete example, Angelini, Caggiano, et al. (2023) utilize the measure of Total Factor Productivity (TFP) from Fernald (2014) as an instrument for the output shock. They find that the TFP instrument is positively correlated with the output shock but also negatively correlated with the tax shock, albeit not to a significant extent. Angelini, Caggiano, et al. (2023) offer comprehensive theoretical and empirical explanations for why such an effect might be observed in the data.

2.2 DGP and main assumptions

In this section, we relax the assumption that the SVAR parameters (Π, Σ_u,H) and the external instruments parameters (R_Φ, Ω_ω, Σ_z) are constant, i.e., time-invariant over the sample $(W_1, ..., W_T)$. The two subsequent assumptions introduce a structural break and establish the regularity conditions under which our analysis is conducted. It is worth noting that the framework described below is consistent with approaches such as Bai (2000) and Qu and Perron (2007). However, unlike these authors, given the scopes of the analysis, our focus does not extend to inference on the break dates.

Hereafter, superscripts or subscripts "(0)" denote vectors/matrices evaluated in a neighborhood of the true parameter values. The indicator function is represented by $\mathbb{I}(\cdot)$.

Assumption 1 (Break in the error covariance matrix) *Let T_B be a break date, $1 < T_B < T$. The reduced form associated with the proxy-SVAR in (8) belongs to the DGP:*

$$W_t = \Gamma(t)X_t + \eta_t\,, \quad \Sigma_\eta(t) := \mathbb{E}(\eta_t\eta_t')\,, \quad t = 1, ..., T \tag{11}$$

where it holds

$$\Gamma(t) := \Gamma_1 \cdot \mathbb{I}(t \leq T_B) + \Gamma_2 \cdot \mathbb{I}(t > T_B)$$
$$\Sigma_\eta(t) := \Sigma_{\eta,1} \cdot \mathbb{I}(t \leq T_B) + \Sigma_{\eta,2} \cdot \mathbb{I}(t > T_B)$$

and

(i) *the process $\{\eta_t\}$, with η_t, is α-mixing on both samples $(W_1, ..., W_{T_B})$ and $(W_{T_B+1}, ..., W_T)$, meaning that it satisfies the conditions in Assumption 2.1 in Brüggemann, Jentsch, and Trenkler (2016); moreover, the process $\{\eta_t\}$ has absolutely summable cumulants up to order eight on both samples $(W_1, ..., W_{T_B})$ and $(W_{T_B+1}, ..., W_T)$;*

(ii) *$\Sigma_{\eta,1} < \infty$ and $\Sigma_{\eta,2} < \infty$ are positive definite;*

(iii) *each true regime parameter $(\Gamma_i^{(0)}, \Sigma_{\eta,i}^{(0)})$, $i = 1, 2$, corresponds to a covariance stationary VAR process for W_t;*

(iv) *$\Sigma_{\eta,2}^{(0)} \neq \Sigma_{\eta,1}^{(0)}$.*

Assumption 2 (Known break date) *The break date T_B is known. Moreover, $T_B = \lfloor \tau_B^{(0)} T \rfloor$, $0 < \tau_B^{(0)} < 1$ being the true fraction of observations in the first volatility regime.*

Assumption 1 postulates that the unconditional error covariance matrix Σ_η changes at the break date T_B; despite this change, the system remains "stable" in the two volatility regimes in the following sense. First, the process that generates the VAR disturbances and the proxies, $\{\eta_t\}$, is α-mixing and has absolutely summable cumulants up to order eight (Assumption 1(i)) in both regimes. The α-mixing condition for η_t means that VAR disturbances and proxies may be driven by conditionally heteroskedastic processes (e.g. GARCH) within the two volatility regimes and/or the proxies may be, e.g., generated by zero-censored processes; see e.g. Jentsch and Lunsford (2022). The hypothesis of absolutely summable cumulants up to order eight for $\{\eta_t\}$ is a technical requirement essential to guarantee Moving Block Bootstrap (MBB) consistency, see Brüggemann et al. (2016) and Assumption 2.4 in Jentsch and Lunsford (2022). It is intended that MBB resampling is conducted separately on $(W_1, ..., W_{T_B})$ and $(W_{T_B+1}, ..., W_T)$. Second, the unconditional covariance matrices $\Sigma_{\eta,1}$ and $\Sigma_{\eta,2}$ are "finite" and positive definite (Assumption 1(ii)), and the VAR for W_t is asymptotically stable in both volatility regimes (Assumption 1(iii)). Finally, Assumption 1(iv) simply establishes that the unconditional covariance matrices $\Sigma_{\eta,1}$ (pre-break period) and $\Sigma_{\eta,2}$ (post-break period) are different. While this implies that our analysis relies on (at least) a change in unconditional volatility, Assumption 1.(iv) is not sufficient to ensure identification. The extent to which $\Sigma_{\eta,1}$ and $\Sigma_{\eta,2}$ need to differ for our approach to be effective is formalized implicitly in terms of a necessary and sufficient rank condition for identification, as discussed below (see Proposition 1). Note that in this framework, the slope VAR parameters may or may not change across the two volatility regimes. Both cases, where $\Gamma_1 \neq \Gamma_2$ and $\Gamma_1 = \Gamma_2$, are allowed and can be considered in our analysis.

Assumption 2 posits that the break date T_B is known to the econometrician, reflecting the empirical macroeconomics scenario where it is acknowledged that there are volatility regimes in the data, often associated with crises and/or significant policy changes linked to distinct macroeconomic regimes. The condition $T_B = \lfloor \tau_B T \rfloor$, see, e.g. Bai (2000), ensures that we can rely on the typical methods of asymptotic inference.[7] In principle, Assumption 2 can be relaxed and the break date can be inferred from the data with some statistical procedure or test. When the break date T_B is inferred from the data, post-test inference should account for Bonferroni-Holm type adjustments.

[7] It is maintained that there are sufficient observations to estimate the model in each regime.

2.3 The stability restrictions approach

The main implication of Assumptions 1-2 is that the subsets of observations $(W_1, ..., W_{T_B})$ and $(W_{T_B+1}, ..., W_T)$ are characterized by two distinct VAR covariance matrices, $\Sigma_{\eta,2}$ and $\Sigma_{\eta,1}$, respectively. The modeling of $\Sigma_{\eta,2} \neq \Sigma_{\eta,1}$ holds critical importance in this framework.

A pertinent, related question is whether the target IRFs (3) can still be estimated consistently by using the instruments z_t alone, despite the shift in volatility. We tackle this issue in Section 3 and the supplementary material (see Section S.3) when we directly compare our approach with Schlaak et al. (2023). Our primary finding is that external instruments can be exclusively employed for inference, even when volatility breaks are disregarded, under two conditions: (i) the target IRFs remain constant across volatility regimes, indicating that the break solely affects the variance of the structural shocks while leaving their impact and propagation unchanged; (ii) "relative", not absolute responses are estimated, i.e., responses obtained by imposing "unit effect" normalizations (see Proposition S.1). Conversely, when the target IRFs change across volatility regimes due to shifts in the impact of the target structural shocks, even strong and exogenous proxies fail to result in consistent estimation if the breaks in volatility are disregarded. Therefore, in general, shifts in volatility need to be incorporated in proxy-SVAR analysis.

Our novel approach to model the change in volatility, based on target IRFs that change across volatility regimes is summarized in the next three sections. The supplementary material, Section S.4, provides a more "conventional" alternative where IRFs are assumed constant across volatility regimes.

2.3.1 Target IRFs and their identifiability

We supplement Assumptions 1-2 with a crucial condition under which the target IRFs deviate from those in equation (3).

Assumption 3 (Regime-dependent IRFs) *The dynamic causal effects produced by the target shocks can be summarized, for $1 \leq j \leq k$, by the IRFs:*

$$IRF_{\bullet j}(t,h) := \begin{cases} (S_n'(\mathcal{C}_{y,1})^h S_n)H_{\bullet 1}e_j, & t \leq T_B, \\ (S_n'(\mathcal{C}_{y,2})^h S_n)(H_{\bullet 1} + \Delta_{H_{\bullet 1}})e_j & t \geq T_B + 1 \end{cases} \tag{12}$$

where $\Delta_{H_{\bullet 1}}$ denotes an $n \times k$ matrix whose non-zero coefficients capture possible changes in the on-impact parameters in $H_{\bullet 1}$ in the shift from the first to the second volatility regime; $\Delta_{H_{\bullet 1}} := H_{\bullet 1}^{(2)} - H_{\bullet 1}$, $H_{\bullet 1}^{(2)}$ being the analogous of the matrix $H_{\bullet 1}$ in the second volatility regime.

Note that for $\Delta_{H_{\bullet 1}} \neq 0_{n\times k}$, Assumption 3 implies $\Sigma_{\eta,2} \neq \Sigma_{\eta,1}$, hence it does not conflict with Assumption 1(iv). Furthermore, the companion matrices $\mathcal{C}_{y,i}$ in (12) depend, for $i = 1, 2$, on the autoregressive (slope) parameters in Γ_i, see (11), hence the slope parameters can remain possibly constant under Assumption 1.[8] Moreover, (12) is formulated to depict responses to one-standard deviation target shocks in both volatility regimes. We elaborate on this concept later in the paper.

Throughout, for any matrix A, the notation Δ_A will refer to a matrix with the same dimensions as A, where non-zero elements represent potential parameter changes from the first to the second volatility regime. Formally, Δ_A is defined as $\Delta_A := A^{(2)} - A$, with $A^{(2)}$ corresponding to A in the second volatility regime. We use the same notation for elements of a matrix, i.e. $\Delta_{a_{i,j}} := a^{(2)}_{i,j} - a_{i,j}$, for $a_{i,j}$ (i,j)-element of A, and $a^{(2)}_{i,j}$ (i,j)-element of $A^{(2)}$. To point-identify and estimate the target IRFs (12) in system (11), we model the relationship between the vector collecting VAR innovations and proxies, η_t, and the vector that includes the structural shocks and the measurement errors associated with the proxies, ξ_t by:

$$\eta_t = G\xi_t + \Delta_G \cdot \mathbb{I}\left(t > T_B\right)\xi_t, \tag{13}$$

where the matrix G has the structure discussed in Section 2.1, see (9), and the term ξ_t here is such that it is respected the condition $\mathbb{E}(\xi_t\xi_t') = I_{n+r}$. In its general form, the structure of the matrix $G + \Delta_G$ reads:

$$\begin{aligned} G + \Delta_G &= \underbrace{\begin{pmatrix} H & 0 \\ R_\Phi & \Omega_\omega \end{pmatrix}}_{G} + \underbrace{\begin{pmatrix} \Delta_H & 0 \\ \Delta_{R_\Phi} & \Delta_{\Omega_\omega} \end{pmatrix}}_{\Delta_G} \\ &:= \underbrace{\begin{pmatrix} H_{\bullet 1} & H_{\bullet 2} & 0 \\ \Phi & \Upsilon & \Omega_\omega \end{pmatrix}}_{G} + \underbrace{\begin{pmatrix} \Delta_{H_{\bullet 1}} & \Delta_{H_{\bullet 2}} & 0 \\ \Delta_\Phi & \Delta_\Upsilon & \Delta_{\Omega_\omega} \end{pmatrix}}_{\Delta_G} \end{aligned} \tag{14}$$

where it is seen that Δ_Φ, Δ_Ψ and Δ_{Ω_ω} capture possible changes in the parameters governing proxy properties, while $\Delta_{H_{\bullet 2}}$ captures possible changes in the on-impact coefficients associated with the non-target shocks. Recall that the top-right blocks of zeros in G and Δ_G pertain to the impact of measurement errors associated with the instruments on the variables, which must be zero in both volatility regimes by construction. Under (14), the dynamics of the

[8] It turns out that the target IRFs may change across the two volatility regimes under Assumption 1 either because $\Delta_{H_{\bullet 1}} \neq 0$, or possibly because both conditions $\Delta_{H_{\bullet 1}} \neq 0$ and $\mathcal{C}_{y,2} \neq \mathcal{C}_{y,1}$ hold.

proxies z_t can be described by the linear measurement error model:

$$
\begin{aligned}
z_t = & \underbrace{R_\Phi \varepsilon_t + \Omega_\omega \omega_t}_{\text{first volatility regime}} + \underbrace{[\Delta_{R_\Phi} \varepsilon_t + \Delta_{\Omega_\omega} \omega_t]\, \mathbb{I}\,(t > T_B)}_{\text{second volatility regime}} \\
= & \underbrace{[\Phi + \Delta_\Phi \mathbb{I}\,(t > T_B)]}_{\tilde{\Phi}_t} \varepsilon_{1,t} + \underbrace{[\Upsilon + \Delta_\Upsilon \mathbb{I}\,(t > T_B)]}_{\tilde{\Upsilon}_t} \varepsilon_{2,t} \\
& + \underbrace{[\Omega_\omega + \Delta_{\Omega_\omega} \mathbb{I}\,(t > T_B)]}_{\tilde{\Omega}_{\omega,t}} \omega_t
\end{aligned}
\tag{15}
$$

where, under drifting DGPs characterized by *sequences* of models in which $\mathbb{E}(z_t \varepsilon_{1,t}') = \tilde{\Phi}_T \to \tilde{\Phi} = (\Phi + \Delta_\Phi)$, and for $\tilde{\Upsilon}_t = 0_{r \times (n-k)}$ or $\tilde{\Upsilon}_t \neq 0_{r \times (n-k)}$, the model accounts for instrument properties as featured in Definition 1.

Equation (15) remarks that the relevance and exogeneity of z_t, as well as the variance of measurement errors, are allowed to change across volatility regimes. It also suggests that in line with Definition 1 above and as in, e.g., Ludvigson et al. (2020, 2021), the external instrument z_t can be correlated with shocks other than the target shocks, via nonzero elements in Υ and/or Δ_Υ. As is well-known, the proxy-SVAR approaches developed by Mertens and Ravn (2013) and Stock and Watson (2018) achieve point identification by assuming that the external variables exhibit zero correlations with the non-target shocks. Conversely, the methodology proposed in the frequentist framework by Ludvigson et al. (2020, 2021) for set-identified SVARs allows the external variables to display departures from exogeneity; see Braun and Brüggemann (2023) for a Bayesian perspective. Ludvigson et al. (2020) note that the focus of their analysis is not on relaxing exogeneity per se, but on the broader objective of using the external instruments to aid in identification. Here, we demonstrate that breaks in unconditional volatility enable us to address a situation akin to Ludvigson et al. (2020), with the crucial distinction being that we achieve point identification. Importantly, this methodology offers the potential for the data to inform us about the extent of the correlations between the proxies z_t and (part of) the non-target shocks.

The moment conditions implied by model (13) are:

$$
\Sigma_\eta(t) = \begin{cases} \Sigma_{\eta,1} = GG' & t \leq T_B, \\ \Sigma_{\eta,2} = (G + \Delta_G)(G + \Delta_G)' & t \geq T_B + 1 \end{cases}
\tag{16}
$$

and in light of the structure of the matrices G and Δ_G in (14), system (16) implies that even when proxy exogeneity holds in both volatility regimes, $\Upsilon = 0_{r \times (n-r)}$ and $\Delta_\Upsilon = 0_{r \times (n-r)}$, the moment conditions supply information not only on the parameters of direct interest $H_{\bullet 1}$ and $\Delta_{H_{\bullet 1}}$, see (12), but also on elements in $H_{\bullet 2}$ and $\Delta_{H_{\bullet 2}}$, that are not of direct interest in the analysis. We discuss the "full/partial" nature of the approach that incorporates volatility changes to external instruments in this and in the next section.

It is important to highlight that the parameterization in (16) is more general than one might expect. Consider, e.g., an alternative parameterization given by:

$$\Sigma_\eta(t) = \begin{cases} \Sigma_{\eta,1} = \breve{G} V_{(1)} \breve{G}' & t \leq T_B, \\ \Sigma_{\eta,2} = (\breve{G} + \breve{\Delta}_G) V_{(2)} (\breve{G} + \breve{\Delta}_G)' & t \geq T_B + 1 \end{cases} \tag{17}$$

where the structural shocks and proxy measurement errors, $\xi_t := (\varepsilon_t', \omega_t')'$, are now intended to have variance $\mathbb{E}(\xi_t \xi_t') = V_{(1)}$ in the first volatility regime, and $\mathbb{E}(\xi_t \xi_t') = V_{(2)}$ in the second volatility regime. Here $V_{(1)}$ and $V_{(2)}$ are diagonal matrices with positive elements on the diagonal, respectively, and $\breve{G}$ and $\breve{G} + \breve{\Delta}_G$ differ from G and $G + \Delta_G$ in (14) only in having "1" on their main diagonals. One interpretation of (17) is that the change in the covariance matrices $\Sigma_{\eta,2}$ and $\Sigma_{\eta,1}$ is now explained by changes in the variances of the structural shocks, captured by $V_{(2)} \neq V_{(1)}$, as well as changes in on-impact coefficients, captured by the nonzero elements in the matrix $\breve{\Delta}_G$. However, taken (17) and $\breve{G}$, $\breve{\Delta}_G$, $V_{(1)}$ and $V_{(2)}$ as DGP, it is always possible to find matrices G and Δ_G such that the following equalities hold:

$$\begin{array}{ll} GG' = \breve{G} V_{(1)} \breve{G}' & t \leq T_B, \\ (G + \Delta_G)(G + \Delta_G)' = (\breve{G} + \breve{\Delta}_G) V_{(2)} (\breve{G} + \breve{\Delta}_G)' & t \geq T_B + 1. \end{array}$$

For example, the equations above hold with $G := \breve{G} V_{(1)}^{1/2}$ and $\Delta_G = (\breve{G} + \breve{\Delta}_G) V_{(2)}^{1/2} - \breve{G} V_{(1)}^{1/2}$. In light of this equivalence, we prefer to rely on the moment conditions in (16) whose implied IRFs refer to one-standard deviation shocks in both volatility regimes.

Defined the vectors $\sigma_{\eta,1} = vech(\Sigma_{\eta,1})$ and $\sigma_{\eta,2} = vech(\Sigma_{\eta,2})$, then the moment conditions can be expressed in the more compact form:

$$\begin{array}{l} \sigma_{\eta,1} = vech(GG') \\ \sigma_{\eta,2} = vech\big((G + \Delta_G)(G + \Delta_G)'\big) \end{array} \tag{18}$$

and, as in Magnusson and Mavroeidis (2014) identification can be attained through the following set of (linear) constraints on G and Δ_G:

$$vec(G) \quad = \quad S_G \gamma, \tag{19}$$

$$vec(\Delta_G) \quad = \quad S_{\Delta_G}\delta. \tag{20}$$

In (19)-(20), S_G is an $(n+r)^2 \times a$ selection matrix of full column rank, $a < (n+r)^2$, $\gamma := (\gamma_H', \gamma_R', \gamma_{\Omega_\omega}')'$ is the a-dimensional vector collecting the free nonzero parameters entering the matrix G, with γ_H, γ_R, and γ_{Ω_ω} containing the nonzero elements in H, R_Φ, and Ω_ω, respectively; S_{Δ_G} is an $(n+r)^2 \times b$ selection matrix of full column rank b, $b < (n+r)^2$, and $\delta := (\delta_H', \delta_R', \delta_{\Omega_\omega}')'$ is the b-dimensional vector of free nonzero parameters in the matrix S_{Δ_G}, with δ_H, δ_R and δ_{Ω_ω} having analogous interpretation as γ_H, γ_R and γ_{Ω_ω}, respectively. Thus, since the elements in the vector δ capture changes from the first to the second volatility regime, the restrictions on Δ_G should be strictly interpreted as stability restrictions.

The stability restrictions in (19)-(20) serve a dual purpose. On the one hand, the constraints imposed on matrix G are instrumental in identifying k target structural shocks, alongside $n-k$ non-target shocks possibly under a parsimonious set of restrictions (more on this below).[9] On the other hand, the constraints on Δ_G determine which of the non-zero and zero proxy-SVAR parameters contained in matrix G undergo changes in the transition from the first to the second volatility regime. As it will be shown in the empirical illustration in Section 4, the specifications of the matrices G (γ), Δ_G (δ), S_G and S_{Δ_G} in (14)-(20) are tailored to the specific problem and the scopes of the analysis. Importantly, besides leveraging the change in volatility, the stability restrictions that the investigator specifies in (19)-(20) are grounded in the underlying theory or knowledge of the phenomenon under study. They do not rely on statistical information such as the distribution of the structural shocks or their cross-independence.

It can be noticed that, jointly equation (18) through (20) jointly characterize a "full identification" problem in the sense that, as they stand, the moment conditions and identification restrictions affect not only the target structural shocks but also the non-target ones. The decision to take a (partial) stance on the non-target shocks arises when $(n-k) > 1$ (see Section 4).[10] In the

[9]It is worth noting that, for $k > 1$, even in a "conventional" proxy-SVAR with no structural breaks, achieving point-identification necessitates at least $\frac{1}{2}k(k-1)$ additional restrictions beyond the instruments in, e.g., Mertens and Ravn (2013) and Angelini and Fanelli (2019).

[10]In principle, our approach can be extended to the case in which identification and estimation are developed by partialing out the influence of non-target shocks from the analysis. Specifically, it is possible to maintain that changes in volatility can be exclusively attributed to parameters related to the impact of the target shocks on the variables. One advantage of this solution is that it relieves the investigator from taking a stance on the non-target shocks. A drawback is that one must assume proxy exogeneity in estimation, leading to the loss of one of the benefits of our suggested approach.

remainder of the paper, we delve into the (local) point identification and estimation of the proxy-SVAR under the general specification defined by equations (18)-(20), emphasizing the strengths of the proposed methodology.

The parameters associated with the target IRFs in (12) are elements of γ and δ, that we collect in the vector θ. More precisely, $\theta := (\gamma'_{H_{\bullet 1}}, \delta'_{H_{\bullet 1}})'$, with $\gamma_{H_{\bullet 1}}$, $\delta_{H_{\bullet 1}}$, being subvectors of γ_H and δ_H, respectively. θ is referred to as the vector of parameters of interest. The moment conditions (18) feature $(n+r)(n+r+1)$ reduced-form coefficients, $\sigma_{\eta,1}$ and $\sigma_{\eta,2}$, and $a+b$ free parameters that we collect in the vector $\varsigma := (\gamma', \delta')'$. We can conveniently summarize these moment conditions by the distance function:

$$m(\sigma_\eta, \varsigma) := \begin{pmatrix} m_1(\sigma_{\eta,1}, \varsigma) \\ m_2(\sigma_{\eta,2}, \varsigma) \end{pmatrix} = \begin{pmatrix} \sigma_{\eta,1} - vech(GG') \\ \sigma_{\eta,2} - vech\big((G+\Delta_G)(G+\Delta_G)'\big) \end{pmatrix}, \quad (21)$$

where it is intended that the matrices G and Δ_G are constrained as in (19)-(20). Equation (21) shows that the point identification problem of θ is equivalent to the problem of uniquely recovering the vector ς, comprising some nuisance parameters, from the reduced-form covariance parameters in $\sigma_{\eta,1}$ and $\sigma_{\eta,2}$, respectively. The next proposition establishes the necessary and sufficient conditions for this to happen. We denote with ς_0 the true value of ς.

PROPOSITION 1 (IDENTIFICATION UNDER CHANGING IRFS) *Given the proxy-SVAR from Assumptions 1-3, consider the moment restrictions in (21) with G and Δ_G restricted as in* (19)-(20)*. Assume ς_0 is a regular point in the parametric space $\mathcal{P}_\varsigma$. Then, irrespective of whether the proxies z_t satisfy one of the conditions in Definition 1:*

(i) *a necessary and sufficient rank condition for the identification of ς in a neighborhood of ς_0 is that $rank[\mathcal{J}(\varsigma_0)] = a+b$, where $\mathcal{J}(\varsigma_0)$ is the $(n+r)(n+r+1)\times(a+b)$ Jacobian evaluated at ς_0, given by:*

$$\mathcal{J}(\varsigma_0) := \left.\frac{\partial m(\sigma_\eta,\varsigma)}{\partial \varsigma'}\right|_{\varsigma=\varsigma_0},$$

$$\frac{\partial m(\sigma_\eta,\varsigma)}{\partial \varsigma'} = 2\left(I_2 \otimes D^+_{n+r}\right)\begin{pmatrix} (G\otimes I_{n+r}) & 0_{(n+r)^2\times(n+r)^2} \\ (G+\Delta_G)\otimes I_{n+r} & (G+\Delta_G)\otimes I_{n+r} \end{pmatrix}\begin{pmatrix} S_G & 0 \\ 0 & S_{\Delta_G} \end{pmatrix}; \quad (22)$$

(ii) *a necessary order condition is:*

$$(a+b) \leq (n+r)(n+r+1). \quad (23)$$

The main message from Proposition 1 is that in the presence of a shift in unconditional volatility, the stability restrictions in (19)-(20) allow to point-identify the parameters in a neighborhood of ς_0, hence the proxy-SVAR parameters, θ, and the target IRFs. The result holds regardless of the properties of the external instruments outlined in Definition 1. The possible breakdown of proxy relevance and exogeneity does not affect the necessary and sufficient rank condition in Proposition 1.[11]

Intuitively, relevance is not strictly necessary because regardless of the local rank properties of Φ, the rank of the Jacobian matrix $\mathcal{J}(\varsigma)$ in (22) remains unaffected by sequences of matrices Φ_T converging to Φ; this implies that even in cases where the proxies satisfy the conditions in Definition 1.(ii) and 1.(iv), identification of the proxy-SVAR can still be achieved through the shift in volatility. On the other hand, the exogeneity condition can be potentially relaxed due to the "full identification" nature of the approach through changes in volatility, which inherently delivers information concerning the non-target shocks, other than the target shocks. Provided the necessary and sufficient rank condition holds, the target structural shocks can be recovered even when the instruments are correlated with some non-target shocks. This flexibility, however, comes at the cost of the investigator needing to take a stance, at least partially, on how non-target shocks impact the variables, which requires a few constraints on $H_{\bullet 2}$ and $\Delta_{H_{\bullet 2}}$, via (19)-(20). As shown in the empirical illustration in Section 4 where we estimate a fiscal proxy-SVAR for the US economy with a shift in unconditional volatility, this stance can often be established by leveraging insights from other studies.[12]

2.3.2 Estimation

To estimate the target IRFs under Proposition 1, we adopt the CMD approach. Assumptions 1-3 suffice to guarantee the consistency and asymptotic normality of the CMD estimator, as indicated in Proposition 2 below. Notably, no distribution assumption is required for η_t.

[11] It is noteworthy that the necessary and sufficient rank condition in Proposition 1 remains valid, as expected, even when the restrictions imposed by the investigator G and Δ_G imply "sub-sample identification". With this term we mean that the identification of parameters γ and δ can be achieved through two SVAR analyses conducted separately on the two volatility regimes; see e.g. Blanchard and Galí (2009). It is evident that the utility of the identification conditions outlined in Proposition 1 extends well beyond the phenomenon of sub-sample identification.

[12] We establish an analogous proposition to Proposition 1 for cases in which the target IRFs are assumed to remain constant across volatility regimes; see supplementary material, Section S.4, Proposition S.3. This proposition provides a theoretical foundation for the results that Schlaak et al. (2023) documented only through simulation studies for the case $r = k = 1$.

To introduce the estimation method, it may be useful to preliminary start from the asymptotic properties of the estimator of the reduced-form covariance matrices of the proxy-SVAR in (11), $\hat{\sigma}_\eta := (\hat{\sigma}'_{\eta,1}, \hat{\sigma}'_{\eta,2})'$. Under Assumptions 1-2, it holds the asymptotic normality result:

$$\sqrt{T}(\hat{\sigma}_\eta - \sigma_{\eta,0}) = \sqrt{T} \begin{pmatrix} \hat{\sigma}_{\eta,1} - \sigma_{\eta,1,0} \\ \hat{\sigma}_{\eta,2} - \sigma_{\eta,2,0} \end{pmatrix} \xrightarrow{d} N(0, V_{\sigma_\eta}) \quad , \quad V_{\sigma_\eta} := \begin{pmatrix} V_{\sigma_{\eta,1}} & 0 \\ 0 & V_{\sigma_{\eta,2}} \end{pmatrix} \tag{24}$$

where $\sigma_{\eta,0} := (\sigma'_{\eta,0,1}, \sigma'_{\eta,0,2})'$ is the true value of σ_η, and the structure of the asymptotic covariance matrices $V_{\sigma_{\eta,i}}$, $i = 1, 2$ is discussed in detail by e.g. Brüggemann et al. (2016); see also references therein. Henceforth, we assume the existence of a consistent estimator for V_{σ_η}, denoted $\hat{V}_{\sigma_\eta}$.[13]

Under the conditions of Proposition 1, the estimator of the parameters ς, hence of θ, is obtained by solving the minimization problem:

$$\hat{\varsigma}_T := \arg\min_{\varsigma \in \mathcal{P}_\varsigma} m_T(\hat{\sigma}_\eta, \varsigma)' \hat{V}_{\sigma_\eta}^{-1} m_T(\hat{\sigma}_\eta, \varsigma) \tag{25}$$

where $m_T(\hat{\sigma}_\eta, \varsigma)' := \left(m_{T,1}(\hat{\sigma}_{\eta,1}, \varsigma)', m_{T,2}(\hat{\sigma}_{\eta,2}, \varsigma)'\right)'$ is the distance function in (21) with σ_η replaced with the estimator $\hat{\sigma}_\eta$.[14] Let $\hat{\theta}_T := (\hat{\gamma}'_{H_\bullet 1,T}, \hat{\delta}'_{H_\bullet 1,T})'$ be the subvector of the CMD estimator $\hat{\varsigma}_T := (\hat{\gamma}'_T, \hat{\delta}'_T)'$. The next proposition establishes asymptotic properties.

Proposition 2 (Asymptotic properties CMD estimator) *Let $\hat{\varsigma}_T$ be the CMD estimator of the parameters ς obtained from (25), and $\hat{\theta}_T := (\hat{\gamma}'_{H_\bullet 1,T}, \hat{\delta}'_{H_\bullet 1,T})'$ be the corresponding subvector of $\hat{\varsigma}_T$. Let ς_0 be an interior of $\mathcal{P}_\varsigma$ (assumed compact), with θ_0 subset of ς_0 ($\theta_0 \in \mathcal{P}_\theta \subseteq \mathcal{P}_\varsigma$). Under the conditions of Proposition 1:*

$$\hat{\varsigma}_T \xrightarrow{p} \varsigma_0 \quad , \quad \sqrt{T}(\hat{\varsigma}_T - \varsigma_0) \xrightarrow{d} N(0, V_\varsigma)$$

$$\hat{\theta}_T \xrightarrow{p} \theta_0 \quad , \quad \sqrt{T}(\hat{\theta}_T - \theta_0) \xrightarrow{d} N(0, V_\theta)$$

where $V_\varsigma := \left(\mathcal{J}(\varsigma_0)' V_{\sigma_\eta}^{-1} \mathcal{J}(\varsigma_0)\right)^{-1}$ and V_θ is the corresponding block of V_ς.

[13] Under Assumptions 1-2 and mild additional auxiliary conditions, the residual-based moving block bootstrap (MBB), as introduced by Brüggemann et al. (2016) (and outlined by Jentsch and Lunsford (2022) in the context of proxy-SVAR models), is consistent. This means that the regime-dependent covariance matrices $V_{\sigma_{\eta,i}}$ that form V_{σ_η} in (24) can be estimated by applying the MBB separately within each volatility regime.

[14] As is known, the optimization problem in (25) can be influenced by multiple local minima, aligning with the fact that the necessary and sufficient rank condition in Proposition 1 holds for local, not global identification; we refer to Bacchiocchi and Kitagawa (2022a) for SVARs in which identification is local and not global. When two or more local minima arise from (25), one reasonable selection criterion is to inspect ex-post the signs of estimated coefficients and compare them with the signs expected from the theory.

Proposition 2 establishes that the target IRFs can be consistently estimated, and standard asymptotic inference can be applied regardless of the proxy properties formalized in Definition 1. The simulation studies, summarized in the supplementary material, emphasize the benefits arising from the results in Propositions 1-2 in terms of gains in efficiency in the estimation of target IRFs relative to not including external instruments in the analysis.

Proposition 2 ensures that when $(n+r)(n+r+1) > (a+b)$, the number of overidentifying restrictions resulting from the estimation problem (25), $(n+r)(n+r+1)-(a+b)$, can be empirically evaluated using the overidentifying restrictions test. Notably, as evidenced by our simulation studies, the test for overidentifying restrictions appropriately rejects when the stability restrictions include incorrect constraints; for instance, when proxy exogeneity is imposed in estimation but the DGP of the instrument belongs to model (15) with Υ or $(\Upsilon + \Delta_{\Upsilon})$ different from zero.

Jointly, Propositions 1-2 provide the foundation for our approach to the identification and estimation of proxy-SVARs with permanent, nonrecurring breaks in the error covariance matrix. Essentially, the suggested approach (i) does not necessitate pre-testing proxy strength and exogeneity; (ii) does not need to rely on weak-instrument robust methods; (iii) does not require imposing proxy exogeneity in estimation.

2.3.3 Checks of identifiability

The identification and estimation approach discussed in the previous section holds if and only if the Jacobian matrix $\mathcal{J}(\varsigma)$ in (22) is full column rank. When the rank condition $rank[\mathcal{J}(\varsigma)] = a+b$ holds locally, the shift in unconditional volatility captured under Assumption 1(iv) by the distance $\Sigma_{\eta,2} - \Sigma_{\eta,1}$, conveys sufficient information to compensate for potential instrument weakness without compromising the inference on the target structural shocks and the validity of standard asymptotics. Even contaminated instruments do not affect estimates' consistency when stability restrictions are correctly specified.

Conversely, when the shift $\Sigma_{\eta,2} - \Sigma_{\eta,1}$ is "weak", the shift in unconditional volatility does not provide sufficient information for identification. We study such phenomenon in Section S.6 of the supplement, focusing in detail on how changes in volatility impact the necessary and sufficient rank condition for identification derived in Proposition 1. In particular, we approximate scenarios characterized by weak changes in volatility with the phenomenon of "shrinking shifts", which arises when Assumption 1(iv) does not hold because the distance $\Sigma_{\eta,2} - \Sigma_{\eta,1}$ tends to vanish as $T \to \infty$. We postpone to future research the elaboration of a comprehensive robust approach to proxy-SVARs that fully integrates potentially weak external instruments in a context in which the identification information stemming from heteroskedasticity is scant.

In practical situations practitioners can, in principle, test the rank of the Jacobian matrix ex-post, meaning after estimating the model and substituting the elements in G and Δ_G in (22) with their estimates, obtaining $\mathcal{J}(\hat{\varsigma}_T)$; see, e.g., Kleibergen and Paap (2006) and Al-Sadoon (2017) and references therein. Alternatively, one can apply the pre-test for the null hypothesis that IRFs do not change, against the alternative that they change across a finite number of volatility regimes, as recently introduced by Lütkepohl and Schlaak (2022) for the one instrument-one shock case, generalized in Bruns and Lütkepohl (2024) to the multiple instruments setup. However, while our approach accommodates invalid proxies, Bruns and Lütkepohl (2024) test requires strong, exogenous instruments. Hence, it can be interpreted as an implicit assessment of the validity of the rank condition of the Jacobian $\mathcal{J}(\varsigma)$ only when the investigator has no doubts about instrument strength and exogeneity. In any case, if a pre-test of the identification conditions is undertaken, post-test inference must be adjusted accordingly.

In the empirical illustration discussed in Section 4, following the indications arising from our Monte Carlo sperimentations, we assess the quality of identification by inspecting the smallest singular value implied by the estimated Jacobian matrix $\mathcal{J}(\hat{\varsigma}_T)$ relative to its associated uncertainty as captured by a bootstrap confidence interval, emphasizing that no rigorous inference is being drawn in this process.[15]

3 Connections and differences with the literature

In this section, we review contributions in the existing literature where external instruments are explicitly combined with changes in volatility, highlighting the main differences with our approach. While the current literature sees a growing number of articles driven by the idea that "blending" different methods enhances the (point- or set-) identification of the structural shocks of interest,

[15]In our Monte Carlo experiments, we evaluate the identifiability of the analyzed proxy-SVARs with changes in volatility by examining the distribution of the smallest eigenvalue of the matrix $\mathcal{J}(\hat{\varsigma}_T)'\mathcal{J}(\hat{\varsigma}_T)$, which corresponds to the smallest singular value of the estimated Jacobian matrix $\mathcal{J}(\hat{\varsigma}_T)$. Results suggest that in situations in which the necessary and sufficient conditions for identification in Proposition 1 holds, the variability surrounding the estimated smallest singular value of the Jacobian matrix $\mathcal{J}(\hat{\varsigma}_T)$ is considerably smaller compared to cases in which the distance between covariance matrices across volatility regimes tends to shrink.

our focus here is specifically on contributions that integrate external instruments and volatility shifts within point-identified SVARs.

Schlaak et al. (2023) concentrate on proxy-SVARs where a single instrument is used for a single target shock ($r = k = 1$), and breaks in the unconditional VAR covariance matrix imply that the target IRFs remain constant across volatility regimes. They demonstrate, primarily through extensive simulation studies, that the identification through the heteroskedasticity approach not only enables refining the inference on the target structural shocks obtained by external instruments, but also facilitates testing the exogeneity condition and the evaluation of the strength of the proxy. Our analysis formally establishes that, in the one shock-one instrument framework they consider, by concentrating on relative IRFs that remain constant across volatility regimes, a valid external instrument ensures consistent estimation even if the break in volatility is disregarded. Conversely, our approach demonstrates that regardless of whether one considers absolute or relative IRFs, the responses (12) cannot be estimated consistently by the instruments alone.

More precisely, in Schlaak et al. (2023), the (absolute) target IRFs are not defined by those in (12), but rather are given by:

$$IRF_{\bullet j}(t,h) := \begin{cases} (S_n(\mathcal{C}_y)^h S_n')H_{\bullet 1}e_j & t \leq T_B, \\ (S_n(\mathcal{C}_y)^h S_n')H_{\bullet 1}P_{\bullet 1}^{1/2}e_j & t \geq T_B+1 \end{cases} \quad , \quad 1 \leq j \leq k$$

where $P_{\bullet 1}$ is a diagonal matrix with distinct, positive elements on the diagonal. It turns out that the relative (normalized) IRFs are given by:

$$\frac{IRF_{\bullet 1}(t,h)}{IRF_{11}(t,0)} = (S_n(\mathcal{C}_y)^h S_n') \begin{pmatrix} 1 \\ H_{2,1}^{rel} \end{pmatrix} \ , \ t = 1, ..., T$$

where $H_{2,1}^{rel} := H_{2,1}(H_{1,1})^{-1} = \Sigma_{u_2,z}(\Sigma_{u_1,z})^{-1}$, see (6). Proposition S.1 in the supplementary material, Section S.3, shows that if the instrument satisfies the relevance and exogeneity conditions in Definition 1.(i), then:

$$\hat{\Sigma}_{u_2,z}\hat{\Sigma}_{u_1,z}^{-1} \xrightarrow{p} H_{2,1}^{rel},$$

where $\hat{\Sigma}_{u_1,z}$ and $\hat{\Sigma}_{u_2,z}$ are the corresponding blocks of the estimator $\hat{\Sigma}_{u,z} := \frac{1}{T}\sum_{t=1}^{T} \hat{u}_t z_t'$. This result implies that in under the assumptions of Schlaak et al. (2023), a valid external instrument potentially suffices alone for the consistent estimation of the relative IRFs without the need to take the break in volatility into account. Obviously, incorporating the moment conditions stemming from the break in volatility in the analysis can only enhance, rather than hinder, the inference. Also, in the framework of Schlaak et al. (2023), an informative change in volatility compensates for the occurrence of weak instruments, a result that we prove analytically in the supplementary material, Section S.4.

A comparable scenario, not exclusively restricted to point identification, is explored by Carriero, Marcellino, and Tornese (2023)), who also consider a framework in which the impact and propagation of the structural shocks remain unchanged across volatility regimes. Carriero et al. (2023) also combine proxy-SVARs and heteroskedasticity (proxy-SVAR-H, see supplement, Section S.4) in a Bayesian framework. Their approach is primarily driven by the idea that heteroskedasticity offers a way to sharpen identification while simultaneously relaxing the zero restrictions that are typically essential in proxy-SVARs when dealing with multiple target shocks. Conversely, our analysis shows that if the hypothesis of invariant target IRFs across volatility regimes cannot be empirically justified, augmenting the external instruments with moment conditions stemming from shifts in unconditional volatility becomes a crucial measure to reestablish consistency and standard inference. Hence, in comparison to both Schlaak et al. (2023) and Carriero et al. (2023), our analysis highlights that, in empirical cases of interest, augmenting external instruments with the breaks in volatility emerges as the sole approach to ensure the consistent estimation of the dynamic causal effects of interest.

Finally, Lütkepohl and Schlaak (2022) and Bruns and Lütkepohl (2024) develop tests for changing IRFs due to shifts in volatility based on using external instruments. In these tests, however, instruments must be both relevant and exogenous to ensure proper size control and power. See e.g., Bruns and Lütkepohl (2023) for an application to the oil market.

4 Empirical illustration: a fiscal proxy-SVAR with a shift in volatility

In this section, we illustrate the effectiveness and potential of our methodology by re-evaluating fiscal proxy-SVAR of Mertens and Ravn (2014) by our approach. Specifically, we estimate US fiscal multipliers by incorporating external fiscal instruments for fiscal shocks and simultaneously considering the (almost abrupt) shift in volatility that occurred during the transition from the Great Inflation to the Great Moderation period. The Great Moderation, observed since the mid-1980s, is characterized by a notable decrease in the standard deviation of GDP, along with other macroeconomic and financial variables. Subsequently, we compare our findings with those in the existing literature.

Since the seminal contribution of Mertens and Ravn (2014), the estimation of US fiscal multipliers using multiple external instruments for the tax and fiscal spending shocks has attracted considerable attention, as evidenced by contributions in works such as Caldara and Kamps (2017) and Angelini, Caggiano, et al. (2023), among many others. Fiscal multipliers have been inferred also leveraging changes in volatility, see, e.g., Lewis (2021) and Fritsche, Klein, and Rieth (2021).[16] Lewis (2021) introduces a nonparametric approach exploiting the volatility of US data, assuming constant IRFs. In contrast, Fritsche et al. (2021), focusing specifically on government spending shocks, acknowledge, within various empirical specifications, the possibility of Markov switching state-dependent IRFs.

We consider a VAR model for the variables $Y_t := (TAX_t, G_t, GDP_t)'$ ($n = 3$), where TAX_t is a measure of per capita real tax revenues, G_t denotes per capita real government spending and GDP_t is per capita real output. Government spending includes federal government expenditure and gross investment; tax revenues are the sum of federal current tax receipts, social insurance contributions, and corporate income taxes.[17] All variables are in logarithms after being deflated by the GDP deflator. The estimation sample covers the period from 1950:Q1 to 2006:Q4, comprising a total of $T = 228$ quarterly observations. The time series are linearly detrended. The reduced-form VAR includes $p = 4$ lags and a constant.

Standard residual-based diagnostic tests show that VAR disturbances are serially uncorrelated but display conditional heteroskedasticity. The graphs in the left panel of Figure 1 plot the VAR residuals on the period 1950:Q1-2006:Q4. As is known, McConnell and Perez-Quiros (2000) identify 1984:Q1 as the break-date of the variance of the US real GDP; see also, among many others, Justiniano and Primiceri (2008). We take $T_B = $ 1984:Q1 as a break point in our sample, corresponding to the vertical lines in Figure 1, which delineates two volatility regimes. The first volatility regime, denoted as the Great Inflation, spans the period from 1950:Q1 to 1984:Q1 and includes 137 quarterly observations. The second volatility regime, denoted as the Great Moderation, covers the period from 1984:Q2 to 2006:Q4 and includes 91 quarterly observations. The graph clearly illustrates that VAR residuals become less volatile during the Great Moderation.

Let $u_t := (u_t^{tax}, u_t^{g}, u_t^{gdp})'$ be the vector of VAR disturbances, and $\varepsilon_{1,t} :=$

[16] In this context, we do not explicitly address contributions that make use of models for conditional heteroskedasticity in the identification of fiscal multipliers, such as the work of Bouakez, Chihi, and Normandin (2014). Similarly, we do not discuss works that solely utilize third and fourth unconditional moments of the reduced-form innovations like the work of Guay (2021).

[17] All variables are taken from Caldara and Kamps (2017), where a more detailed explanation of the dataset can be found.

$(\varepsilon_t^{tax}, \varepsilon_t^g)'$ the vector of (target) fiscal shocks, $\varepsilon_{2,t} := \varepsilon_t^{gdp}$ being the (non-target) output shock. In line with Mertens and Ravn (2014), we use fiscal instruments for the fiscal shocks. Specifically, we consider two fiscal instruments collected in the vector $z_t := (z_t^{tax}, z_t^g)'$, $r = k = 2$. z_t^{tax} represents Mertens and Ravn (2011) series of unanticipated tax shocks, derived from a subset of shocks identified through a narrative analysis of tax policy decisions (Romer and Romer, 2010). On the other hand, z_t^g represents a novel series of unanticipated fiscal spending shocks introduced in Angelini, Caggiano, et al. (2023).[18]The two fiscal instruments are plotted in the right column of Figure 1.

To proceed with a benchmark in mind, we initially estimate the fiscal proxy-SVAR without considering the break in unconditional volatility (Section 4.1). Subsequently, we amend the empirical analysis by incorporating the break at T_B = 1984:Q1 and implementing our stability restrictions approach (Section 4.2).

4.1 Benchmark: no volatility shifts

We start by estimating a proxy-SVAR over the whole sample period 1950:Q1-2006:Q4, maintaining that the fiscal instruments z_t are relevant and exogenous for the target fiscal shocks $\varepsilon_{1,t} := (\varepsilon_t^{tax}, \varepsilon_t^g)'$. We consider the following proxy-SVAR specification:

$$\begin{pmatrix} u_t^{tax} \\ u_t^g \\ u_t^{gdp} \end{pmatrix} = \underbrace{\begin{pmatrix} h_{1,1}^{(1)} & h_{1,2}^{(1)} \\ h_{2,1}^{(1)} & h_{2,2}^{(1)} \\ h_{3,1}^{(1)} & h_{3,2}^{(1)} \end{pmatrix}}_{H_{\bullet 1}} \underbrace{\begin{pmatrix} \varepsilon_t^{tax} \\ \varepsilon_t^g \end{pmatrix}}_{\varepsilon_{1,t}} + H_{\bullet 2} \underbrace{\varepsilon_t^{gdp}}_{\varepsilon_{2,t}} \tag{26}$$

$$\underbrace{\begin{pmatrix} z_t^{tax} \\ z_t^g \end{pmatrix}}_{z_t} = \underbrace{\begin{pmatrix} \varphi_{1,1} & \varphi_{1,2} \\ 0 & \varphi_{2,2} \end{pmatrix}}_{\Phi} \underbrace{\begin{pmatrix} \varepsilon_t^{tax} \\ \varepsilon_t^g \end{pmatrix}}_{\varepsilon_{1,t}} + \underbrace{\begin{pmatrix} \tilde{\omega}_t^{tax} \\ \tilde{\omega}_t^g \end{pmatrix}}_{\tilde{\omega}_t} \tag{27}$$

and estimate the model by the CMD approach introduced in Angelini and Fanelli (2019). In (26)-(27), $\tilde{\omega}_t := (\tilde{\omega}_t^{tax}, \tilde{\omega}_t^g)'$ denotes the vector of (unnormalized) measurement errors associated with the two fiscal proxies. As is known, with $k = 2$ target shocks, it is necessary to impose at least $\frac{1}{2}k(k-1) = 1$ restriction on the parameters in $(H_{\bullet 1}', \Phi')'$ to achieve point identification; see

[18]Hence, the time series z_t^g used in this paper does not coincide precisely with the one used in Mertens and Ravn (2014) for instrumenting fiscal spending shocks. This explains why our results, obtained on the estimation sample from 1950:Q1-2006:Q4 without taking the change in volatility into account (see Section 4.1), do not match precisely those in Mertens and Ravn (2014).

Angelini and Fanelli (2019). The zero in the (2,1) position of the matrix Φ in (27) posits that the fiscal spending proxy solely instruments the fiscal spending shock. In turn, we permit the tax proxy to possibly convey information on the fiscal spending shock other than the tax shock, allowing the data to inform on the significance of the parameter $\varphi_{1,2}$.

The so-estimated dynamic fiscal multipliers are plotted in bold (solid lines) in Figure 2, surrounded by 68% MBB confidence intervals, represented by the dotted thin black lines. Dynamic multipliers are computed from the estimated target fiscal IRFs using the same method as in Mertens and Ravn (2014); see also Caldara and Kamps (2017) and Angelini, Caggiano, et al. (2023).[19] In the upper panel of Figure 2, we present the dynamic tax multipliers. Consistent with the findings in Mertens and Ravn (2014), the peak tax multiplier is 2.7 (3 quarters after the shock), and the associated 68% confidence interval, while broad, does not include zero. Turning our attention to the estimated dynamic fiscal spending multipliers in the lower panel of Figure 2, the peak effect is 1.66 (4 quarters after the shock), with an associated 68% MBB confidence interval of (0.76, 1.91). We summarize the estimated peak tax and fiscal spending multipliers, denoted as $\mathcal{M}_{tax}^{peak}$ and $\mathcal{M}_{g}^{peak}$, respectively, in column (i) of Table 1.

As emphasized in the empirical fiscal literature, a critical parameter influencing the size of the tax multiplier is the output elasticity of tax revenues (automatic stabilizer), denoted as ψ_y^{tax}; see Mertens and Ravn (2014), Caldara and Kamps (2017), Lewis (2021), and Angelini, Caggiano, et al. (2023). Our point estimate of the parameter ψ_y^{tax} in column (i) of Table 1, is 3.26, with a 68% confidence interval of (0.75, 5.84). Overall, these findings for the estimation sample 1950:Q1-2006:Q4 align with the main conclusions in Mertens and Ravn (2014). We complete our calculations computing the correlations

[19]Let P_t represent either the level of fiscal spending or the level of tax revenues (not in logs), and GDP_t^e denote the unlogged level of output. For simplicity, we use β_{y_h} to denote the response of log-output at horizon h to a one-standard deviation fiscal policy shock, and β_{p_0} for the on-impact response of the logged fiscal variable to the corresponding one-standard deviation fiscal policy shock. Then, in our context, dynamic multipliers, defined as the dollar response of output to an effective change in the fiscal variable of 1 dollar occurred h period before, are given by the expression:

$$\mathcal{M}_{p,h} := (\beta_{y_h}/\beta_{p_0}) \times Scaling_p$$

where $Scaling_p$ is a policy shock-specific scaling factor converting elasticities to dollars. We set the scaling factor equal to the sample means of the series (GDP_t^e/P_t) computed over the estimation period. Thus, when dealing with the change in volatility and the stability restriction approach (Section 4.2), the scaling factor is calculated considering observations in the corresponding volatility regime. We refer to Caldara and Kamps (2017) and Angelini, Caggiano, et al. (2023) for a detailed discussion.

between the fiscal proxies and the estimated fiscal shocks, also reported in column (i) of Table 1. The estimated correlation between z_t^g and ε_t^g is 96%, with a relatively narrow 68% MBB confidence interval of (96%, 98%). In contrast, the estimated correlation between z_t^{tax} and ε_t^{tax} is 27%, accompanied by a 68% confidence interval of (12%, 37%), raising concerns about the strength of the tax proxy over the entire estimation sample of 1950:Q1-2006:Q4.[20]

The estimated fiscal multipliers are obtained from normalized IRFs, i.e. relative effects. As demonstrated in Section 3 (see also the supplementary material), when the target IRFs incorporate unit effect normalizations and remain constant across volatility regimes in the data generating process, the dynamic causal effects of interest can be consistently estimated, even ignoring shifts in volatility, if the instruments are both relevant and exogenous. We revisit this consideration later in the text, bearing in mind that the empirical results obtained thus far seem to cast doubt on the relevance used for the tax proxy.

4.2 Shift in unconditional volatility: The stability restrictions approach

Next, we proceed with our stability restrictions approach, which requires explicitly complementing the fiscal instruments z_t with the shift in unconditional volatility occurring at the date T_B = 1984:Q1.[21]

The empirical counterpart of model (13) is specified as follows:

$$\begin{pmatrix} u_t^{tax} \\ u_t^{g} \\ u_t^{gdp} \\ z_t^{tax} \\ z_t^{g} \end{pmatrix} = \underbrace{\begin{pmatrix} h_{1,1}^{(1)} & h_{1,2}^{(1)} & h_{1,1}^{(2)} & 0 & 0 \\ h_{2,1}^{(1)} & h_{2,2}^{(1)} & 0 & 0 & 0 \\ h_{3,1}^{(1)} & h_{3,2}^{(1)} & h_{3,1}^{(2)} & 0 & 0 \\ \varphi_{1,1} & \varphi_{1,2} & \Upsilon_{1,1} & \sigma_{\omega,1} & 0 \\ 0 & \varphi_{2,2} & \Upsilon_{2,1} & \sigma_{\omega,1,2} & \sigma_{\omega,2} \end{pmatrix}}_{G:=\begin{pmatrix} H_{\bullet 1} & H_{\bullet 2} & 0 \\ \Phi & \Upsilon & \Omega_\omega \end{pmatrix}} \underbrace{\begin{pmatrix} \varepsilon_t^{tax} \\ \varepsilon_t^{g} \\ \varepsilon_t^{gdp} \\ \omega_t^{tax} \\ \omega_t^{g} \end{pmatrix}}_{\xi_t}$$

[20] A formal test of relevance of the tax proxy z_t^{tax} on the same sample may be found in the Supplementary Material of Angelini, Cavaliere, and Fanelli (2024).

[21] We also conducted a simple sub-sample estimation exercise, wherein the same (constant parameters) proxy-SVAR in (26)-(27) estimated on the whole sample is re-estimated separately on the Great Inflation and Great Moderation samples, respectively. Due to space constraints, detailed results on this, and graphs of the implied dynamic multipliers, are available upon request.

$$+\underbrace{\begin{pmatrix} \Delta_{h^{(1)}_{1,1}} & \Delta_{h^{(1)}_{1,2}} & 0 & 0 & 0 \\ \Delta_{h^{(1)}_{2,1}} & \Delta_{h^{(1)}_{2,2}} & 0 & 0 & 0 \\ 0 & \Delta_{h^{(1)}_{3,2}} & \Delta_{h^{(2)}_{3,1}} & 0 & 0 \\ \Delta_{\varphi_{1,1}} & \Delta_{\varphi_{1,2}} & 0 & \Delta\sigma_{\omega,1} & 0 \\ 0 & \Delta_{\varphi_{2,2}} & 0 & 0 & \Delta\sigma_{\omega,2} \end{pmatrix}}_{\Delta_G := \begin{pmatrix} \Delta_{H_{\bullet 1}} & \Delta_{H_{\bullet 1}} & 0 \\ \Delta_{\Phi} & \Delta_{\Upsilon} & \Delta_{\Omega_\omega} \end{pmatrix}} \mathbb{I}\left(t > T_B\right) \underbrace{\begin{pmatrix} \varepsilon^{tax}_t \\ \varepsilon^{g}_t \\ \varepsilon^{gdp}_t \\ \omega^{tax}_t \\ \omega^{g}_t \end{pmatrix}}_{\xi_t} \tag{28}$$

and is based on the following set of hypotheses. The first two columns of the matrix G, which pertain to the instantaneous impact of the fiscal shocks on the variables in the first volatility regime, reproduce exactly the structure of the matrix $(H'_{\bullet 1}, \Phi')'$ considered in the specification (26)-(27) for which the proxy-SVAR was estimated on the whole sample period. Given the full identification nature of the changes in volatility approach, the third column of the matrix G refers to the instantaneous impact of the output (non-target) shock on the variables. In this case, we borrow the restriction $h^{(2)}_{2,1} = 0$ from Mertens and Ravn (2014). This restriction, considered an uncontroversial tenet in the US fiscal empirical literature, posits that fiscal spending does not respond instantaneously to the output shock; see also Blanchard and Perotti (2002). Furthermore, we relax a-priori the exogeneity of the two fiscal proxies z_t with respect to the output shock, leaving the contamination parameters $\Upsilon_{1,1}$ and $\Upsilon_{2,1}$ unrestricted in (28), letting the data to inform us about possible violation of the exogeneity condition. Finally, we compensate for the zero restriction in the position (2,1) of the matrix of relevance parameters Φ by allowing the measurement error affecting the tax proxy to potentially influence the variance of fiscal spending instrument through the parameter $\sigma_{\omega,2,1}$.

In the second volatility regime, i.e. for the sample starting from $t \geq T_B + 1$, we specify a single stability restriction pertaining to the impact of the tax shock. In particular, we posit that the instantaneous impact of the tax shock on output remains constant across the two regimes, implying $\Delta_{h^{(1)}_{3,1}} = 0$ in the first column of $\Delta_{H_{\bullet 1}}$ in (28). To motivate this constraint, note that the inspection of the bottom-right graph Figure 1, which plots the "great ratio" $TAX_t - GDP_t$, suggests that the relative dynamics of tax revenues and GDP remains substantially stable over the sample period 1950:Q1–2006:Q4, as it should be expected, e.g., under a sustainable debt policy. It can be therefore argued that the change in volatility occurring at T_B = 1984:Q1, affects both time series simultaneously, which is unsurprising given the strict cyclical relationship connecting tax revenues and output. Also, considering the nature of the proxy used to instrument the latent tax shock which is highly zero-censored and does not display marked changes across the two volatility regimes, the stability restriction $\Delta_{h_{3,1}^{(1)}} = 0$ appears a reasonable one in our context. All other nonzero on-impact coefficients in $H_{\bullet 1}$, as well as the nonzero relevance parameters in Φ are permitted to change across the two volatility regimes. As for the on-impact coefficients associated with the non-target output shock, we specify the matrix $\Delta_{H_{\bullet 2}}$ similar to $H_{\bullet 2}$, i.e., keeping the restriction borrowed from Mertens and Ravn (2014) also valid in the second volatility regime ($\Delta_{h_{2,1}^{(2)}} = 0$) and letting all other coefficients vary. Lastly, we keep the contamination parameters $\Upsilon_{1,1}$ and $\Upsilon_{2,1}$ unchanged relative to the Great Inflation. Finally, we allow for a shift in the variance of the fiscal instrument's measurement error.

The proxy-SVAR specified in (28) involves $(a+b)$=27 parameters, collected in the vector ς, which are spread across the matrices G and Δ_G in (28), and is based on $(n+r)(n+r+1)$=30 moment conditions. The model is therefore overidentified if the necessary and sufficient rank condition in Proposition 1 holds. The CMD estimates $\hat{\varsigma}_T$ resulting from problem (25) are summarized in the upper panel of Table 2 along with 68% MBB confidence intervals. The overidentifying restrictions test reported in the bottom panel of Table 2 strongly supports the estimated model with a p-value of 0.90, highly supportive of the chosen specification. An informal check of the quality of the identification of the estimated proxy-SVAR with the change in volatility is also summarized in the bottom part of Table 2, which displays the estimated smallest singular value of the Jacobian matrix, $\mathcal{J}(\hat{\varsigma}_T)$, with associated 68% MBB confidence interval. As observed above, we refrain from interpreting the fact that the bootstrap confidence interval for the smallest singular value does not contain zero as conclusive statistical evidence that the rank identification condition in Proposition 1 is met for the estimated model. At the same time, however, we do not observe clear-cut signs suggesting a potential lack of identification due to an insufficiently informative shift in volatility. Therefore, we can reasonably maintain that the reduction in volatility of the data observed in the shift from the Great Inflation to the Great Moderation regime suffices to point-identify the model, and standard asymptotic inference can be used in this framework.

The CMD estimates in Table 2 reveal important information about the properties and quality of the instruments used to estimate fiscal proxy-SVAR. Two main considerations arise. First, apparently, the relevance of the fiscal spending instrument z_t^g captured by the parameters $\varphi_{2,2}$ (Great Inflation) and $\varphi_{2,2} + \Delta_{\varphi_{2,2}}$ (Great Moderation), tends to decline because of the negative and significantly estimated parameter $\Delta_{\varphi_{2,2}}$. However, in line with the Great Moderation phenomenon, this decline is compensated by a decrease in the variance of the associated proxy measurement error (due to the significantly and negative estimated parameter $\Delta_{\sigma_{\omega,2}}$). Overall, the estimated correlation between the proxy z_t^g and the identified fiscal spending shock ε_t^g remains high across the two volatility regimes; see the columns (ii) and (iii) of Table 1. Interestingly, the contamination parameter $\Upsilon_{2,1}$ is estimated at a very low magnitude and is not statistically significant, supporting the hypothesis that the instrument used for the fiscal spending shock is exogenous and relevant on both volatility regimes.

Second, the tax instrument z_t^{tax} is poorly correlated with the tax shock ε_t^{tax} in the Great Inflation period, where the relevance parameter $\varphi_{1,1}$ is not statistically significant. However, relevance increases markedly in the Great Moderation regime, where the relevance parameter change, $\Delta_{\varphi_{1,1}}$, is significant and the magnitude and statistical significance of $\varphi_{1,1} + \Delta_{\varphi_{1,1}}$ become substantial. To illustrate, examining columns (ii) and (iii) of Table 1, we observe that the implied correlation between the tax proxy z_t^{tax} and the estimated tax shock ε_t^{tax} swings from 15% to 46% across the two volatility regimes. This marked change in the relevance condition is not surprising given the zero-censored nature of the narrative tax instrument, which, by construction, contains many zeros that inherently tend to weaken strength. A simple count shows that the number of zeros characterizing the tax instrument in the Great Inflation period, where volatility is higher, is considerably higher than the number of zeros in the Great Moderation, where z_t^{tax} seems to more accurately approximate the latent tax shock. Moreover, the 68% MBB confidence interval for the contamination parameter, $\Upsilon_{1,1}$, suggests that the tax proxy is negatively linked, albeit not dramatically, with the output shock. A similar finding is also documented in Keweloh et al. (2024), leveraging the non-normality of structural shocks in a Bayesian approach. The implied "contamination correlations" in Table 1, specifically in columns (ii) and (iii), vary from -11% in the Great Inflation to -9.8% in the Great Moderation. As emphasized, in our framework inference is reliable also when instruments are nearly exogenous and not necessarily perfectly exogenous. Specifically, the potential breakdown of the exogeneity condition does not compromise the consistency of the parameter estimator under correctly specified stability restrictions. Consequently, the dynamic fiscal multipliers derived from the estimates in Table 2, while fully capitalizing on and reflecting instrument properties, can be deemed robust to instruments being weak and/or contaminated.

4.2.1 Fiscal spending multipliers

The dynamic fiscal multipliers resulting from the proxy-SVAR estimated in Table 2 are plotted in Figure 2. Solid red lines refer to the Great Inflation period and are surrounded by red shaded 68% MBB confidence intervals; solid blue lines pertain to the Great Moderation period and are surrounded by blue shaded 68% MBB confidence intervals. While differences in terms of size and uncertainty for dynamic fiscal spending multipliers (lower panel) are mild across the two volatility regimes, they appear more pronounced for dynamic tax multipliers (top panel).

Focusing first on the dynamic spending multipliers in the lower panel of Figure 2, we observe that, despite some statistically significant differences emerging between the blue and red lines at some initial horizons, magnitudes and associated uncertainty captured by 68% MBB confidence intervals appear not dramatically different across the two volatility regimes. The estimated peak fiscal spending multiplier, $\mathcal{M}_g^{peak}$, summarized in the columns (ii) and (iii) of Table 1, is 2.38 in the Great Inflation and remains 2.38 in the Great Moderation period. These point estimates are surrounded by comparable 68% MBB confidence intervals, namely (1.2, 2.5) and (1.3, 2.9), respectively. The only notable difference that emerges between the two volatility regimes is that the peak effect is achieved 4 quarters after the shock in the Great Moderation and 2 quarters after the shock in the Great Inflation. These findings on the US fiscal spending multiplier diverge from those of Lewis (2021) on the one hand and share contact points with Fritsche et al. (2021) on the other hand. Lewis (2021), who considers our same estimation sample, exploits the nonparametric heteroskedasticity in fiscal data while keeping the target IRFs constant across volatility regimes. He detects a fiscal spending multiplier peaking at 0.75 after two quarters, very imprecisely estimated. Instead, among their many specifications, Fritsche et al. (2021) also rely on Markov Switching dynamics across high and low volatility states, allowing IRFs to change across these two states. Considering an estimation sample that partially covers the period after the Global Financial Crisis, Fritsche et al. (2021) confirm changes in the impact of government spending shocks between high and low volatility regimes, with the high volatility state essentially matching our Great Inflation period, and the low volatility state essentially covering our Great Moderation sample. They establish that the fiscal spending multiplier is significantly higher in the low volatility state (where it peaks around 2.5-3) compared to the high volatility state (where it peaks around 1.72-2). Our results align with those in Fritsche et al. (2021) and further strengthen their findings, as we complement the identification arising from the change in volatility with the information stemming from a relevant, exogenous, fiscal spending instrument.

4.2.2 Tax multipliers

Focusing on the dynamic tax multipliers plotted in the upper panel of Figure 2, noticeable differences across the two volatility regimes become apparent. Relative to the case in which the proxy-SVAR is estimated on the entire sample ignoring the break in volatility (black solid line), we observe in both volatility regimes a significant reduction in the magnitude of estimated dynamic multipliers, accompanied by a substantial decline in associated uncertainty. Specifically, our estimates of the tax multiplier in column (ii) of Table 1, $\mathcal{M}_{tax}^{peak}$, peak at 1.99 (6 quarters after the shock) during the Great Inflation and decline to a peak of 1.66 (2 quarters after the shock) during the Great Moderation. In both cases, 68% MBB confidence intervals deliver considerably more precise estimates relative to the case in which the change in volatility is ignored. The estimated output elasticity of tax revenues, ψ_y^{tax}, is 2.28 in the Great Inflation and 4.92, and imprecisely estimated, in the Great Moderation.[22] These results suggest that the peak tax multipliers obtained with the proxy-SVAR approach on the whole estimation sample, approximately 3 in Mertens and Ravn (2014) and 2.62 in our framework, are likely to reflect a bias induced by the narrative tax instrument being weak, in addition to being contaminated by the output shock. Once we account for the shift in volatility, estimate consistency is restored, accompanied by a remarkable increase in precision.

Overall, our analysis reveals four crucial findings: (i) the relevance of the narrative tax instrument of Mertens and Ravn (2014) shifts from "weak-like" to "strong-like" during the transition from the Great Inflation to the Great Moderation period; (ii) the exogeneity condition fails in the sense that we detect some non-negligible correlation between the tax instrument and the output shock; (iii) once instrument properties in (i) and (ii) are accounted for, the estimated peak tax multiplier is smaller than the estimated peak fiscal spending multiplier, though not dramatically so; (iv) the uncertainty surrounding the estimated dynamic tax multipliers reduces substantially compared to applying the fiscal proxy-SVAR on the entire sample without accounting for the change in volatility. Finding (ii) is also documented in Keweloh et al. (2024), who leverage the non-Gaussianity of structural shocks in a Bayesian context.

[22] As is known, there exists a direct link between the magnitude of the parameter ψ_y^{tax} and the on-impact tax multiplier, as discussed in Caldara and Kamps (2017). In our case, the on-impact tax multiplier associated is not significant.

4.2.3 Forcing regime-invariant IRFs: The Proxy-SVAR-H approach

For comparative purposes, we conclude our empirical analysis by forcing the target IRFs to be constant across the two volatility regimes. Hence, we implement Schlaak et al.'s (2023) approach, denoted Proxy-SVAR-H, whose analytic and empirical results have been summarized in the supplementary material, see Sections S.4.1 and S.4.2.

The implied peak fiscal multipliers are summarized in column (iv) of Table 1. Figure 3 summarizes all dynamic fiscal multipliers estimated in this paper, without reporting confidence intervals to improve readability. Colors are the same as in Figure 2. Dynamic multipliers implied by the Proxy-SVAR-H approach are plotted in green.

5 Concluding remarks

We have developed an identification and estimation strategy for proxy-SVARs in cases where changes in economic behavior, market conditions, policy conduct, and institutional mechanisms induce permanent and nonrecurring shifts in the unconditional VAR error covariance matrix, leading to breaks in the target IRFs across volatility regimes. In such settings, even when instruments are relevant and exogenous, they may fail to produce consistent, asymptotically Gaussian estimates of both absolute and relative target IRFs if changes in volatility are not appropriately considered. We have introduced a novel methodology to address inference in proxy-SVARs in these cases. Our results emphasize that if the moment conditions arising from changes in volatility are sufficiently informative and allow for the point identification of target IRFs through stability restrictions, and if these stability restrictions are correctly specified by the econometrician, even invalid external instruments can contribute to identifying the structural shocks of interest. In general, external instruments improve estimation efficiency even when the exogeneity condition fails, and weak instruments can still convey information on the target shocks without the need to rely on weak-instruments robust methods.

The comprehensive review on fiscal multipliers of Ramey (2019) highlights the significant lack of consensus on the tax multiplier, primarily attributed to the inherent difficulty in identifying exogenous tax shocks, a task more challenging than identifying fiscal spending shocks. Our estimator of the U.S. tax multiplier is robust to the tax proxy being weak in one regime and strong in the other, as well as to contamination from the output shock. Our analysis shows that the identification of the effects of fiscal policy can be substantially improved by properly combining external instruments with changes in volatility.

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Table 1: Estimated peak multipliers and elasticities with 68% MBB confidence intervals (in parentheses). Break date T_B = 1984:Q1, CMD estimation method.

	(i) Proxy-SVAR 1950:Q1–2006:Q4	(ii) 1^{st} vol. regime 1950:Q1–1984:Q1	(iii) 2^{nd} vol. regime 1984:Q2–2006:Q4	(iv) Proxy-SVAR-H break at 1984:Q1
ψ_y^{tax}	3.256 (0.747,5.843)	2.282 (1.852,2.509)	4.920 (2.617,7.775)	1.749 (−11.489,6.811)
$\mathcal{M}_{tax}^{peak}$	2.625(3) (0.952,5.970)	1.997(6) (1.277,3.094)	1.659(2) (0.048,2.942)	0.496(8) (−1.092,1.015)
relevance$_{tax}$ (%)	0.271 (0.118,0.376)	0.151 (−0.033,0.284)	0.463 (0.235,0.632)	0.197 (−0.216,0.270)
contamination$_{tax}$ (%)	-	−0.110 (−0.234,−0.034)	−0.098 (−0.208,−0.030)	−0.167 (−0.336,0.136)
ψ_y^{g}	−0.005 (−0.031,0.037)	0.032 (−0.012,0.052)	−0.046 (−0.085,−0.006)	−0.037 (−0.046,0.027)
$\mathcal{M}_{g}^{peak}$	1.671(4) (0.769,1.914)	2.380(4) (1.165,2.529)	2.370(2) (1.300,2.886)	1.529(5) (1.493,1.905)
relevance$_{g}$ (%)	0.965 (0.964,0.980)	0.962 (0.961,0.978)	0.979 (0.969,0.988)	0.971 (0.963,0.982)
contamination$_{g}$ (%)	-	−0.001 (−0.002,0.001)	−0.002 (−0.002,0.001)	0.020 (−0.002,0.038)

NOTES. Column (i): estimates obtained by the proxy-SVAR approach on the whole sample 1950:Q1–2006:Q4 using the two external instruments z_t, without accounting for a break in volatility. Column (ii): estimates obtained by the proxy-SVAR approach and the the two external instruments $z_t := (z_t^{tax}, z_t^{g})'$ on the first volatility regime 1950:Q1–1984:Q1. Column (iii): estimates obtained by the proxy-SVAR approach and the the two external instruments z_t on the second volatility regime 1984:Q2–2006:Q4. Column (iv): estimates obtained by the proxy-SVAR-H approach, i.e. maintaining that IRFs do not change across the two volatility regimes. ψ_y^{tax} and ψ_y^{g} are the elasticity of tax revenues and fiscal spending to output, respectively. $\mathcal{M}_{tax}^{peak}$ and $\mathcal{M}_{g}^{peak}$ are the peak multipliers. "Relevance$_{(\cdot)}$ (%)" denotes the correlation computed between the instrument $z_t^{(\cdot)}$ and the estimated shock $\hat{\varepsilon}_t^{(\cdot)}$. "Contamination$_{(\cdot)}$ (%)" denotes the correlation computed between the instrument $z_t^{(\cdot)}$ and the estimated non-target, output shock $\hat{\varepsilon}_t^{gdp}$.

Table 2: Estimated parameters of the fiscal proxy-SVAR with break in volatility at time $T_B = 1984$:Q1 and associated 68% MBB confidence intervals (in parentheses).

	G		Δ_G
$h_{1,1}^{(1)}$	0.018 (0.010,0.021)	$\Delta_{h_{1,1}^{(1)}}$	−0.008 (−0.012,−0.001)
$h_{2,1}^{(1)}$	−0.000 (−0.001,0.000)	$\Delta_{h_{2,1}^{(1)}}$	0.001 (−0.000,0.001)
$h_{3,1}^{(1)}$	−0.003 (−0.003,−0.002)		
$h_{1,2}^{(1)}$	0.003 (0.001,0.006)	$\Delta_{h_{1,2}^{(1)}}$	−0.003 (−0.005,0.001)
$h_{2,2}^{(1)}$	0.014 (0.012,0.014)	$\Delta_{h_{2,2}^{(1)}}$	−0.007 (−0.007,−0.005)
$h_{3,2}^{(1)}$	0.003 (0.002,0.004)	$\Delta_{h_{3,2}^{(1)}}$	−0.002 (−0.002,−0.001)
$h_{1,1}^{(2)}$	0.020 (0.016,0.021)	$\Delta_{h_{1,1}^{(2)}}$	−0.005 (−0.010,−0.003)
$h_{3,1}^{(2)}$	0.009 (0.008,0.009)	$\Delta_{h_{3,1}^{(2)}}$	−0.006 (−0.007,−0.005)
$\varphi_{1,1}$	0.020 (−0.004,0.036)	$\Delta_{\varphi_{1,1}}$	0.049 (0.001,0.089)
$\varphi_{1,2}$	−0.010 (−0.018,0.004)	$\Delta_{\varphi_{1,2}}$	−0.009 (−0.025,0.008)
$\varphi_{2,2}$	0.015 (0.013,0.015)	$\Delta_{\varphi_{2,2}}$	−0.007 (−0.007,−0.006)
$\Upsilon_{1,1}$	−0.015 (−0.027,−0.004)		
$\Upsilon_{2,1}$	−0.000 (−0.000,0.000)		
$\sigma_{1,1}$	0.131 (0.096,0.133)		
$\sigma_{2,1}$	−0.000 (−0.000,0.000)		
$\sigma_{2,2}$	0.004 (0.003,0.004)	$\Delta_{\sigma_{2,2}}$	−0.002 (−0.002,−0.001)

Overidentifying restrictions: 0.573 [0.903]

Min. eigenvalue: $1.23e\text{-}6$ $(4.35e\text{-}7, 1.14e\text{-}6)$

NOTES. Upper panel: CMD estimates. Lower panel: overidentifying restrictions test with associated p-value (in brackets). Minimum singular value of the estimated Jacobian matrix $\mathcal{J}(\hat{\varsigma}_T)$ with associated 68% MBB confidence interval.

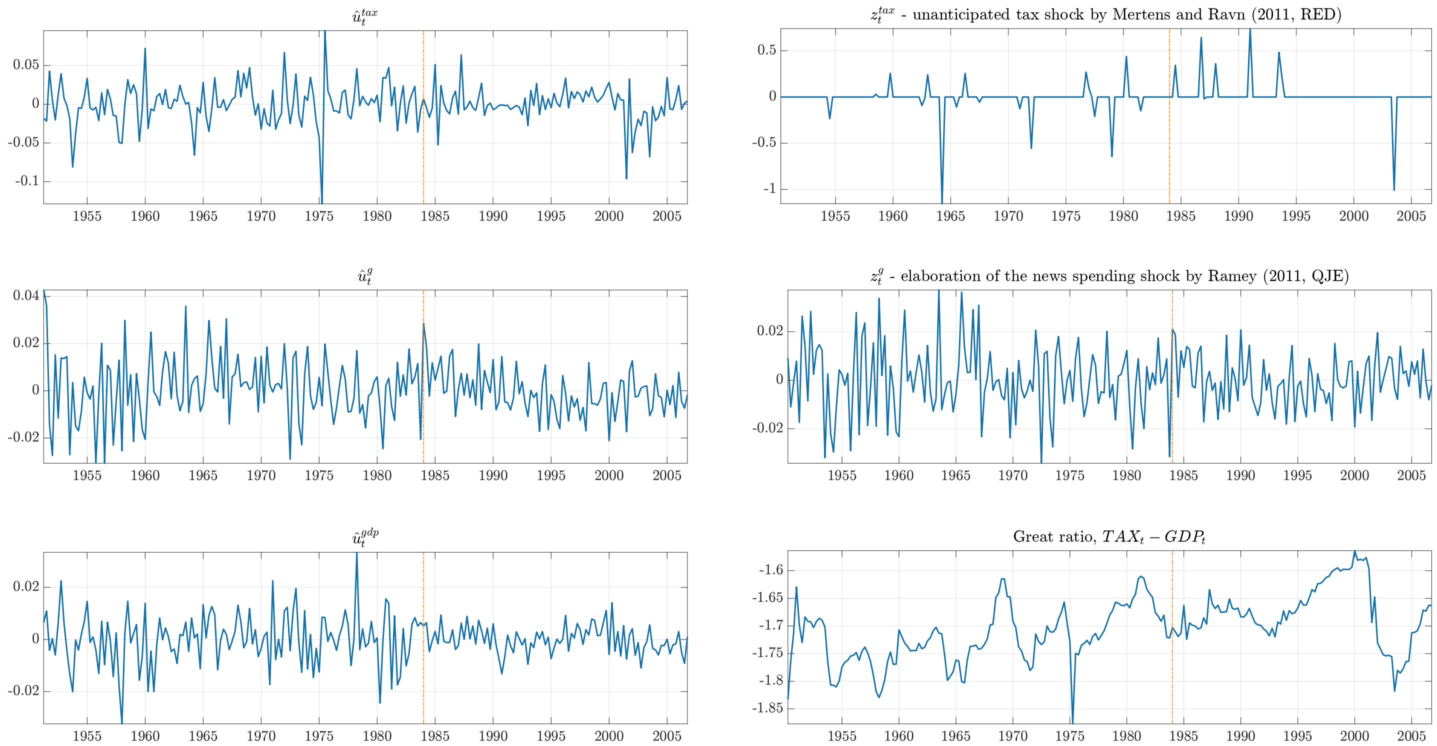

Figure 1: Left panel: reduced-form residuals of the three-equation VAR model for the variables $Y_t := (TAX_t, G_t, GDP_t)'$ estimated on the period 1950:Q1–2006:Q4. The VAR includes $l = 4$ lags, and the variables are linearly detrended. Right panel: series of unanticipated tax shock (z_t^{tax}) from Mertens and Ravn (2011); series of unanticipated fiscal spending shocks (z_t^g); great ratio $TAX_t - GDP_t$.

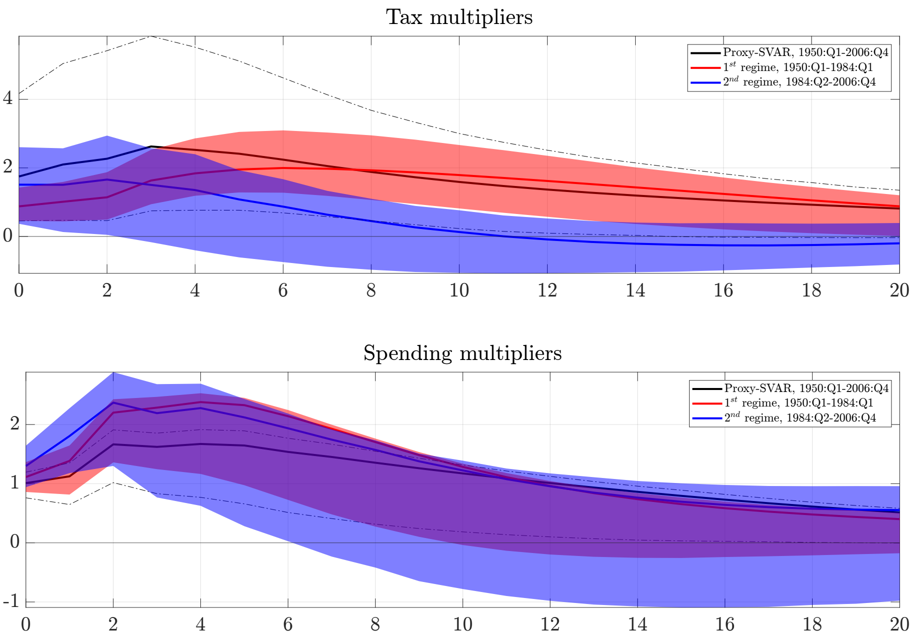

Figure 2: Estimated dynamic fiscal multipliers with 68% MBB (pointwise) confidence intervals at a 20-quarters horizon. Tax multipliers are in the upper panel; fiscal spending multipliers in the lower panel. Black solid lines refer to multipliers estimated on the whole sample 1950:Q1–2006:Q4 without accounting for a break in volatility; dotted thin black lines are the associated 68% MBB confidence intervals. Red solid line refer to multipliers estimated on the first volatility regime 1950:Q1–1984:Q1 (Great Inflation); red shaded areas are the associated 68% MBB confidence intervals. Blue solid lines refer to multipliers estimated on the second volatility regime, 1984:Q2–2006:Q4 (Great Moderation); blue shaded areas are the associated 68% MBB confidence intervals.

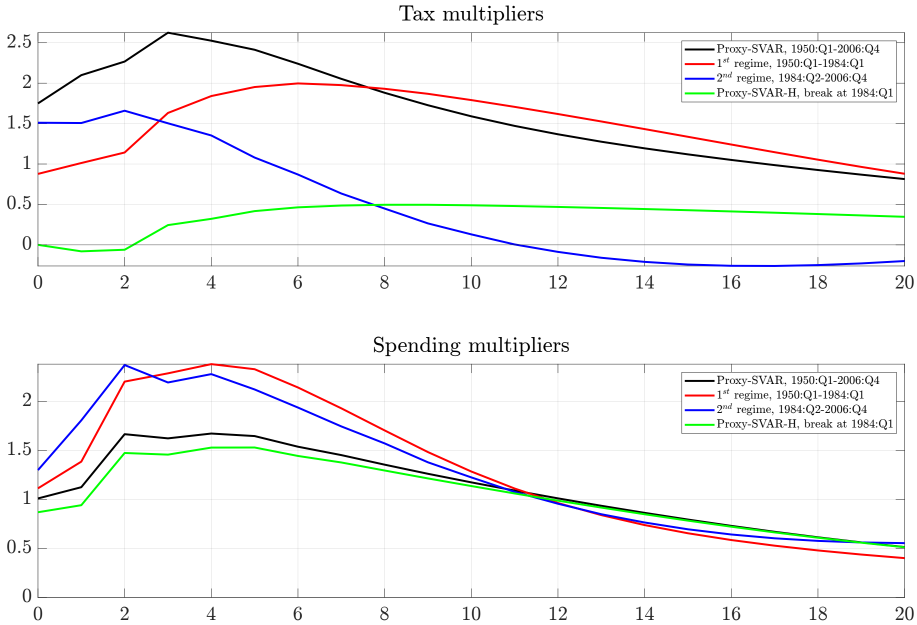

Figure 3: Estimated dynamic fiscal multipliers without confidence intervals at a 20-quarters horizon. Tax multipliers are in the upper panel; fiscal spending multipliers in the lower panel. Black solid lines refer to multipliers estimated on the whole sample 1950:Q1–2006:Q4 without accounting for a break in volatility. Red solid lines refer to multipliers estimated on the first volatility regime 1950:Q1–1984:Q1 (Great Inflation). Blue solid lines refer to multipliers estimated on the second volatility regime, 1984:Q2–2006:Q4 (Great Moderation). Green lines refer to multipliers obtained from the Proxy-SVAR-H approach (see supplement, Section S.4), i.e. accounting for a break in volatility while maintaining that IRFs remain constant across volatility regimes.

SUPPLEMENT TO

INVALID PROXIES AND VOLATILITY CHANGES

By Giovanni Angelini, Luca Fanelli, Luca Neri

March 2024

Abstract

This supplement complements the paper along many dimensions as described in the Introduction below.

S.1 Introduction

In this Supplement, we extend and complete the paper along several dimensions. Section S.2 introduces special matrices utilized in the paper and supplement. Section S.3 discusses the conditions under which the proxy-SVAR approach, which does not explicitly incorporate breaks in the VAR covariance matrices when these breaks are present in the DGP, yields consistent estimates of the target IRFs and when it does not.

Section S.4 focuses on the case in which the shifts in unconditional volatility are assumed to solely affect the variance of the structural shocks, not their impact and transmission mechanisms. This implies constant IRFs across volatility regimes. Subsection S.4.1 formalizes the underlying theory, and Subsection S.4.2 applies it to infer the US fiscal multipliers, complementing the results in the paper.

Section S.5 extends the identification and estimation approach outlined in the paper along two important dimensions. Firstly, it considers the case where the number of breaks in volatility is $M \geq 2$, resulting in $M+1$ volatility regimes. Secondly, it explores QML estimation as an alternative to the CMD estimation considered in the paper.

Section S.6 investigates how the phenomenon of "shrinking shifts" impacts the necessary and sufficient identification rank condition derived in Proposition 1.

Section S.7 summarizes the results of comprehensive Monte Carlo experiments that investigate the finite sample performance of the stability restrictions approach developed in the paper. In particular, we examine the performance of the stability restrictions approach in terms of relative efficiency of estimated target IRFs with respect to other estimation approaches, under different properties of external instruments. Furthermore, we examine the performance of the overidentifying restrictions test when the econometrician incorrectly imposes proxy exogeneity in estimation. We also evaluate methods to indirectly assess the identifiability of the proxy-SVAR when the changes in volatility are "weak".

Unless differently specified, hereafter all references – except those starting with 'S.' – refer to sections, assumptions, equations and results in the main paper.

S.2 Special matrices

In the paper and in what follows, we often make use of the following matrices (Magnus and Neudecker, 1999): D_n is the n-dimensional duplication matrix ($D_n vech(A) = vec(A)$, A being an $n \times n$ matrix) and $D_n^+ := (D_n' D_n)^{-1} D_n$ is the Moore-Penrose generalized inverse of D_n. K_{ns} is the ns-dimensional commutation matrix ($K_{ns} vec(A) = vec(A')$, A being $n \times s$). We simply use K_n in place of K_{nn} when $n = s$. Moreover, $N_n := \frac{1}{2}(I_{n^2} + K_n)$, and note that in the proof of propositions we often exploit the result: $D_n^+ N_n = D_n^+$.

Finally, we denote with $vecd(A)$ the vector containing the diagonal elements of the square matrix A. Then, given the $p \times p$ diagonal matrix A, the $p^2 \times p$ derivative $\mathcal{F}_A := \frac{\partial vec(A)}{\partial vecd(A)'}$ contains by construction '0' and '1'. Specifically, the matrix $\mathcal{F}_A$ is such that $rank[\mathcal{F}_A] = p$ if the diagonal elements of A are distinct. Conversely, $rank[\mathcal{F}_A] = p - c$ when there are c repeated elements on the diagonal of A. Hence, the rank of $\mathcal{F}_A$ depends on whether there are repeated elements on the diagonal of A or not.

S.3 Proxy-SVARs estimation disregarding volatility breaks

The covariance matrix $\Sigma_{u,z} = \mathbb{E}(u_t z_t')$ encountered in (6) plays a crucial role in proxy-SVAR estimation. In the absence of structural breaks, $\Sigma_{u,z}$ can be estimated by its sample analog:

$$\hat{\Sigma}_{u,z} := \frac{1}{T} \sum_{t=1}^{T} \hat{u}_t z_t'$$

where $\hat{u}_t$, $t = 1, ..., T$, are the VAR residuals. As shown by Jentsch and Lunsford (2022) and Angelini, Cavaliere, and Fanelli (2024), under fairly general conditions on the process $\eta_t := (u_t', z_t')'$, which encompasses the α-mixing hypothesis as specified in point (i) of Assumption 1, and regardless of proxy properties, the estimator $\hat{\zeta}_T := vec(\hat{\Sigma}_{u,z})$ is a $\sqrt{T}$-consistent, asymptotically Gaussian estimator of $\zeta := vec(\Sigma_{u,z})$. Hence, subject to standard regularity conditions, proxy-SVAR estimation relies on the following results:

$$\hat{\zeta}_T \overset{p}{\to} \zeta_0 \quad , \quad \sqrt{T}(\hat{\zeta}_T - \zeta_0) \overset{d}{\to} N(0, V_\zeta) \tag{S.1}$$

where ζ_0 is the true value of $\zeta := vec(\Sigma_{u,z})$ and V_ζ is a positive definite covariance matrix.

From equation (6), it follows that in the presence of "well behaved" proxies as captured by Definition 1.(i), $\Sigma_{u_2,z}(\Sigma_{u_1,z})^{-1} = H_{2,1}^{rel} := H_{2,1}(H_{1,1})^{-1}$, which implies that the relative on-impact effects of the target shocks on the variables can be estimated by:

$$\hat{H}_{2,1}^{rel} := \hat{\Sigma}_{u_2,z}(\hat{\Sigma}_{u_1,z})^{-1} \tag{S.2}$$

where $\hat{\Sigma}_{u_2,z}$ and $\hat{\Sigma}_{u_1,z}$ are the corresponding blocks of $\hat{\Sigma}_{u,z}$. It turns out that under relevant and exogenous instruments, $\hat{\Sigma}_{u_2,z} \overset{p}{\to} H_{2,1}\Phi'$ and $\hat{\Sigma}_{u_1,z} \overset{p}{\to} H_{1,1}\Phi'$, $rank[\Phi] = k$, implying that $\hat{H}_{2,1}^{rel}$ in (S.2) is a consistent estimator of the true $H_{2,1}^{rel} := H_{2,1}(H_{1,1})^{-1}$. Consistency, however, is no longer guaranteed when the instruments fail to be relevant and/or exogenous as in Definitions 1(ii)-1(iv).

In the special case where $r = k = 1$ (one instrument is used for one target structural shock), robust asymptotically correct inference on the coefficients in $H_{2,1}^{rel} := H_{2,1}(H_{1,1})^{-1}$ can be grounded on weak-instrument robust techniques as outlined in the test inversion methodologies discussed by Montiel Olea, Stock, and Watson (2021). In these cases, the instrument can possibly be weak according to Definition 1.(ii) yet still informative. However, when $k > 1$, it is not evident how test inversion methods should be handled in the absence of further restrictions; see, e.g., Montiel Olea et al. (2021) and Angelini et al. (2024) for discussions.

In this section, we investigate whether and under which conditions the estimators considered in (S.1) and (S.2) are consistent despite $\Sigma_{u,2} \neq \Sigma_{u,1}$, where $\Sigma_{u,1}$ and $\Sigma_{u,2}$ are the unconditional covariance matrices of VAR disturbances in the two volatility regimes. Assumption 1 implies:

$$\Sigma_u(t) := \Sigma_{u,1} \cdot \mathbb{I}(t \leq T_B) + \Sigma_{u,2} \cdot \mathbb{I}(t \geq T_B + 1) \quad , \quad \Sigma_{u,2} \neq \Sigma_{u,1}. \tag{S.3}$$

To simplify the analysis, we consider the following auxiliary assumptions.

ASSUMPTION 4 *The DGP for the external instruments z_t belongs to model (7), meaning that the parameters in (R_Φ, Ω_ω) are constant across volatility regimes.*

ASSUMPTION 5 *The VAR slope parameters in the companion matrix $\mathcal{C}_y = \mathcal{C}_y(\Pi)$ remain constant across volatility regimes*

With Assumptions 4-5, we define a "favorable" scenario in which the structural break solely impacts the covariance matrix of VAR disturbances. This impact does not affect VAR dynamics, the relevance parameters, the exogeneity condition, and instrument measurement error.

There are two ways by which we can incorporate condition (S.3) in SVAR analysis. A common solution is to exploit the simultaneous factorization (see, e.g., Magnus and Neudecker, 1999, Theorem 23):

$$\begin{array}{ll} \Sigma_{u,1} = HH' = H_{\bullet 1}H_{\bullet 1}' + H_{\bullet 2}H_{\bullet 2}' & t \leq T_B, \\ \Sigma_{u,2} = HPH' = H_{\bullet 1}P_{\bullet 1}H_{\bullet 1} + H_{\bullet 2}P_{\bullet 2}H_{\bullet 2}' & t \geq T_B + 1 \end{array} \tag{S.4}$$

where P is a diagonal matrix with distinct positive elements on the main diagonal, and $P_{\bullet 1}$ and $P_{\bullet 2}$ are diagonal matrices such that:

$$P = \begin{pmatrix} P_{\bullet 1} & \\ & P_{\bullet 2} \end{pmatrix} \quad , \quad HP^{1/2} = \left(H_{\bullet 1}P_{\bullet 1}^{1/2},\ H_{\bullet 2}P_{\bullet 2}^{1/2} \right).$$

This standard modeling of a change in volatility (Lanne and Lütkepohl, 2008) implicitly assumes that the underlying structural specification is defined as follows:

$$u_t = H\varepsilon_t \mathbb{I}\left(t \leq T_B\right) + HP^{1/2}\varepsilon_t \mathbb{I}\left(t > T_B\right)$$

$$= \left\{ H_{\bullet 1}\varepsilon_{1,t} + H_{\bullet 2}\varepsilon_{2,t} \right\} \mathbb{I}\left(t \leq T_B\right) + \left\{ H_{\bullet 1}P_{\bullet 1}^{1/2}\varepsilon_{1,t} + H_{\bullet 2}P_{\bullet 2}^{1/2}\varepsilon_{2,t} \right\} \mathbb{I}\left(t > T_B\right)$$

so that, recalling that $\Sigma_\varepsilon := \mathbb{E}(\varepsilon_t\varepsilon_t') = I_n$, the diagonal elements in P can be interpreted as the variances of the structural shocks in the second volatility regime relative to the first volatility regime (where variances are normalized to 1). It turns out that the on-impact responses to one-standard deviation shocks are captured by the matrix H in the first volatility regime and the matrix $HP^{1/2}$ in the second volatility, simply indicating a proportionate rescaling of IRFs.

In this scenario, the absolute target IRFs are given by the expression:

$$IRF_{\bullet j}(t,h) := \begin{cases} (S_n(\mathcal{C}_y)^h S_n')H_{\bullet 1}e_j & t \leq T_B, \\ (S_n(\mathcal{C}_y)^h S_n')H_{\bullet 1}P_{\bullet 1}^{1/2}e_j & t \geq T_B + 1 \end{cases} \quad , \quad 1 \leq j \leq k \tag{S.5}$$

so that, for e.g. $k = 1$, the relative IRFs are:

$$\frac{IRF_{\bullet 1}(t,h)}{IRF_{1,1}(t,0)} = (S_n(\mathcal{C}_y)^h S_n') \begin{pmatrix} 1 \\ H_{2,1}^{rel} \end{pmatrix} \ , \ \ t = 1, ..., T. \tag{S.6}$$

Equations (S.5) and (S.6) show that while the absolute target IRFs (to one standard deviation shocks) vary between the two volatility regimes due to the re-scaling of the target structural shocks, the relative target IRFs remain unaltered.

The next proposition establishes the conditions under which the target IRFs can be estimated by external instruments ignoring the break in the covariance matrix.

Proposition S.1 (Constant IRFs) *Assume that the DGP belongs to the proxy-SVAR (8) under Assumptions 1-2 and Assumptions 4-5. Consider a drifting DGP characterized by* sequences *of models in which* $\mathbb{E}(z_t\varepsilon_{1,t}') = \Phi_T$, *where the external instruments* z_t *satisfy the condition in Definition 1.(i). Further, assume that the DGP belongs to (S.4). Then*
(i) the estimator $\hat{\Sigma}_{u,z} := \frac{1}{T}\sum_{t=1}^{T}\hat{u}_t z_t'$ *is such that:*

$$\hat{\Sigma}_{u,z} \overset{p}{\to} \tau_B^{(0)} H_{\bullet 1}\Phi' + (1-\tau_B^{(0)})H_{\bullet 1}P_{\bullet 1}^{1/2}\Phi'$$
$$= \left[\tau_B^{(0)} H_{\bullet 1} + (1-\tau_B^{(0)})H_{\bullet 1}P_{\bullet 1}^{1/2}\right]\Phi',$$

$\tau_B^{(0)}$ *being the true fraction of observation in the first volatility regime;*
(ii) for $k=1$ *and rank*$[P]=n$, $\hat{\Sigma}_{u_2,z}\hat{\Sigma}_{u_1,z}^{-1} \overset{p}{\to} H_{2,1}^{rel}$.

Proposition S.1 suggests that while the absolute target IRFs (S.5) cannot be estimated consistently, the relative target IRFs in (S.6) in principle can, despite the break in volatility (but in Σ_u alone). This result, however, is not guaranteed to hold when the external instruments do not satisfy the condition in Definition 1.(i), or Assumptions 4-5 do not hold. It turns out that, in general, correct asymptotic inference on the absolute target IRFs must take the break in volatility into explicit account; see Schlaak, Rieth, and Podstawski (2023).

The alternative method one can exploit to incorporate condition (S.3) in SVAR analysis, is to consider the parameterization:

$$\begin{aligned}\Sigma_{u,1} &= HH' = H_{\bullet 1}H_{\bullet 1}' + H_{\bullet 2}H_{\bullet 2}', \qquad t \le T_B,\\ \Sigma_{u,2} &= (H+\Delta_H)(H+\Delta_H)'\\ &= (H_{\bullet 1}+\Delta_{H_{\bullet 1}})(H_{\bullet 1}+\Delta_{H_{\bullet 1}})' + (H_{\bullet 2}+\Delta_{H_{\bullet 2}})(H_{\bullet 2}+\Delta_{H_{\bullet 2}})', \quad t \ge T_B+1\end{aligned} \tag{S.7}$$

where, as in the paper, $\Delta_H = (\Delta_{H_{\bullet 1}}, \Delta_{H_{\bullet 2}})$ denotes an $n \times n$ matrix whose non-zero coefficients capture possible changes in the on-impact parameters in H in the shift from the first to the second volatility regime. The modeling of the change in unconditional volatility in (S.7) implies that the underlying structural specification is defined as follows (Bacchiocchi and Fanelli, 2015):

$$u_t = H\varepsilon_t \mathbb{I}\left(t \leq T_B\right) + \left\{H + \Delta_H\right\} \varepsilon_t \mathbb{I}\left(t > T_B\right)$$

$$= \left\{H_{\bullet 1}\varepsilon_{1,t} + H_{\bullet 2}\varepsilon_{2,t}\right\} \mathbb{I}\left(t \leq T_B\right) + \left\{\left[H_{\bullet 1} + \Delta_{H_{\bullet 1}}\right]\varepsilon_{1,t} + \left[H_{\bullet 2} + \Delta_{H_{\bullet 2}}\right]\varepsilon_{2,t}\right\} \mathbb{I}\left(t > T_B\right)$$

so that the IRFs change in the shift from the first to the second volatility regime, because the break modifies the magnitude of the responses of the variables to the structural shocks.[1] In this scenario, the (absolute) target IRFs are given in equation (12), here reported for convenience:

$$IRF_{\bullet j}(t,h) := \begin{cases} (S_n(\mathcal{C}_y)^h S_n')H_{\bullet 1}e_j & t \leq T_B \\ (S_n(\mathcal{C}_y)^h S_n')(H_{\bullet 1} + \Delta_{H_{\bullet 1}})e_j & t \geq T_B + 1 \end{cases}, \quad 1 \leq j \leq k. \tag{S.8}$$

Notice that the scenario depicted by the equations in (S.7)-(S.8) coincides with the framework analyzed in the paper but includes Assumptions 4-5. For $k = 1$, the relative target IRFs now are:

$$\frac{IRF_{\bullet 1}(t,h)}{IRF_{1,1}(t,0)} = \begin{cases} (S_n(\mathcal{C}_y)^h S_n')\begin{pmatrix} 1 \\ H_{2,1}(H_{1,1})^{-1} \end{pmatrix} & t \leq T_B, \\ (S_n(\mathcal{C}_y)^h S_n')\begin{pmatrix} 1 \\ (H_{2,1} + \Delta_{H_{2,1}})(H_{1,1} + \Delta_{H_{1,1}})^{-1} \end{pmatrix} & t \geq T_B + 1 \end{cases} \tag{S.9}$$

hence it is evident that for $\Delta_{H_{\bullet 1}} = (\Delta'_{H_{1,1}}, \Delta'_{H_{2,1}})' \neq 0$, both absolute and relative target IRFs change after the break. We have implicitly proved the next proposition.

Proposition S.2 (Regime-varying IRFs) *Assume that the DGP belongs to the proxy-SVAR (8) under Assumptions 1-2 and Assumptions 4-5. Consider a drifting DGP characterized by* sequences *of models in which* $\mathbb{E}(z_t\varepsilon'_{1,t}) = \Phi_T$*, where the external instruments* z_t *satisfy the condition in Definition 1.(i). Further, assume that the DGP belongs to (S.7). Then, both absolute and relative target IRFs can not be estimated consistently if the break in volatility is not taken into account.*

[1] Note that for Δ_H chosen as $\Delta_H := H(P^{1/2} - I_n)$, P being a diagonal matrix with positive (distinct) elements on the diagonal, the moment conditions (S.7) collapse to (S.4), showing that (S.7) nests (S.4).

It is important to note that if in correspondence of the break in volatility the IRFs change, even under the simplifying Assumptions 4-5, the estimators $\hat{\Sigma}_{u,z}$ and $\hat{\Sigma}_{u_2,z}\hat{\Sigma}^{-1}_{u_1,z}$ do not maintain consistency, as explained in detail in the paper.

S.4 Constant IRFs: proxy-SVAR-H approach

In this section, we delve into an alternative modeling approach for the condition $\Sigma_{\eta,1} \neq \Sigma_{\eta,2}$ relative to the stability approach developed in Section 2.3. Specifically, we concentrate on the scenario where it is assumed that the shift in unconditional volatility solely affects the variance of the structural shocks, leaving their impact and transmission mechanisms unchanged; that is, IRFs remain constant across volatility regimes. For simplicity, throughout, we adopt terminology from Carriero, Marcellino, and Tornese (2023) and refer to this approach as the Proxy-SVAR-H approach, with "H" standing for heteroskedasticity.

Section S.4.1 deals with the methodology, and Section S.4.2 applies it to the estimation of US fiscal multipliers, complementing the results in Section 4.2 of the paper.

S.4.1 Methodology

We keep Assumption 1 valid, but focus on the scenario in which the target IRFs are as in (S.5), i.e. regime-invariant. Let G_{un} be the $(n+r)\times(n+r)$ unrestricted nonsingular counterpart of the matrix G in (9), i.e. G_{un} does not incorporate any zero entry. We exploit the simultaneous factorization (see, e.g., Magnus and Neudecker, 1999, Theorem 23):

$$\Sigma_{\eta,1} = G_{un}G'_{un} \tag{S.10}$$

$$\Sigma_{\eta,2} = G_{un}\Lambda G'_{un} = \underbrace{(G_{un}\Lambda^{1/2})}_{G^*_{un}}\underbrace{(G_{un}\Lambda^{1/2})'}_{G^{*\prime}_{un}} \tag{S.11}$$

where Λ is an $(n+r)\times(n+r)$ diagonal matrix with positive elements. A typical interpretation of (S.10)-(S.11) is that the simultaneous factorization captures situations in which, given constant on-impact coefficients, the variances of the elements in $\xi_t := (\varepsilon'_t, \omega'_t)'$ are equal to the identity matrix in the first volatility regime and change in relative terms to Λ in the second volatility regime. Hence, the diagonal elements of Λ can be interpreted as the variances of the elements in ξ_t relative to the first volatility regime, where variances were normalized to one; see, e.g., Lanne and Lütkepohl (2008) and Sims (2021).[2]

A well known result from the conventional literature on identification-through-heteroskedasticity is that the moment conditions (S.10)-(S.11) suffice to point-identify the parameters in G_{un}, up to column permutation and sign normalization. This implies that the identification of structural shocks can occur, at most, ex-post, meaning once the investigator observes the estimated parameters in the columns of G_{un} and the resulting IRFs. Subsequently, based on the significance and sign of these estimates, the investigator can assign labels to the corresponding structural shocks. However, as argued in Schlaak *et al.* (2023), the proxies z_t may help to solve, at least partially, the "up-to-column-permutation" issue implicit in (S.10)-(S.11).

Here we demonstrate analytically what Schlaak *et al.* (2023) show through simulation studies, namely that also in this framework, possibly invalid proxies do not affect the identification achieved by sufficiently strong changes in volatility. Consider the notation $G_{un} = G_{un}(\gamma, \bar{\gamma})$, where γ is the vector of a parameters that enter the matrix G in (9) and $\bar{\gamma}$ is the vector of $(n+r)^2 - a$ parameters that are set to zero in the matrix G but not in G_{un}. In other words, $G = G_{un}(\gamma, 0)$ for $\bar{\gamma} := 0$. Let $\rho = (\gamma', \bar{\gamma}', \lambda')'$, $\lambda = vecd(\Lambda)$, the $(n+r)(n+r+1) \times 1$ vector containing all parameters contained in G_{un} and Λ. The identifiability of ρ ensures that also the parameters of interest and the target IRFs in (S.5) are identifiable. ρ_0 denotes the true value of ρ.

Proposition S.3 *Given the SVAR in (11) and Assumptions 1-2, consider the simultaneous factorization (S.10)-(S.11). Then a necessary and sufficient condition for the identification of ρ in a neighborhood of ρ_0 in the parameter space $\mathcal{P}_\rho$ ($\rho_0 \in \mathcal{P}_\rho$, regular point) is that $\det[\mathcal{J}(\rho_0)] \neq 0$, where $\mathcal{J}(\rho_0)$ is the $(n+r)(n+r+1) \times (n+r)(n+r+1)$ Jacobian matrix evaluated at ρ_0, given by:*

$$\mathcal{J}(\rho) = \left. \frac{\partial m_c(\sigma_\eta, \rho)}{\partial \rho'} \right|_{\rho=\rho_0},$$

$$m_c(\sigma_\eta, \rho) = \begin{pmatrix} \sigma_{\eta,1} - vech(G_{un}G_{un}') \\ \sigma_{\eta,2} - vech(G_{un}\Lambda G_{un}') \end{pmatrix},$$

$$\frac{\partial m_c(\sigma_\eta, \rho)}{\partial \rho'} := 2(I_2 \otimes D^{+}_{(n+r)}) \begin{pmatrix} (G_{un} \otimes I_{(n+r)}) & 0 \\ (G_{un}\Lambda \otimes I_{(n+r)}) & (G_{un} \otimes G_{un}) \end{pmatrix} \begin{pmatrix} I_{n+r} & 0 \\ 0 & \frac{1}{2}\mathcal{F}_F \end{pmatrix}. \tag{S.12}$$

[2]Note that with G_{un} unrestricted, the diagonal elements of Λ coincide with the eigenvalues of the symmetric matrix $\Sigma_{\eta,2}\Sigma_{\eta,1}^{-1}$. Obviously, if the investigator wishes to consider responses to one-standard deviation shocks in both volatility regimes, the on-impact responses to one-standard deviation shock in the second volatility regimes are captured by the matrix $G_{un}\Lambda^{1/2}$, which differs from G_{un} for simple scaling factors.

Some remarks are in order.

First, the stated necessary and sufficient condition for identification in Proposition S.3 demonstrates analytically that identification can be achieved regardless of the proxy properties specified in Definition 1. This result emphasizes that an informative change in volatility tends to enhance identification irrespective of proxy properties. This holds regardless of whether IRFs change or remain constant across volatility regimes. Moreover, Proposition S.3 indirectly rationalizes the findings documented in Schlaak et al. (2023) through simulation methods, i.e. that external instruments positively contribute to identification when the data exhibit changes in unconditional volatility.

Second, equation (S.12) shows that the rank of the Jacobian $\mathcal{J}(\rho)$ in (S.12) may collapse when the matrix $\mathcal{F}_{\Lambda}$ is not full column rank, situation that may occur when some diagonal elements of Λ are not distinct, which in turn implies that the differences in the covariance matrices $\Sigma_{\eta,2}$ and $\Sigma_{\eta,1}$ are not "sufficiently strong" to ensure the point identification of all structural shocks in the system.[3] In their application to the monetary policy framework, Schlaak et al. (2023) estimate the diagonal elements of Λ and verify that confidence intervals constructed for the diagonal elements of Λ, using one standard deviation around point estimates, do not overlap. The checks of identifiability discussed in Section 2.3 can be also applied in this context.

S.4.2 Empirical results: fiscal multipliers revisited

In this section we apply the methodology discussed in the previous section to the estimation of US fiscal multipliers. The general framework is the same discussed in Section 4.

The implied peak fiscal multipliers, summarized in column (iv) of Table 1, are the ones with the lowest magnitude across all estimated models. In particular, the point estimate of the peak fiscal spending multiplier, $\mathcal{M}_g^{peak}$, 1.5, and associated confidence intervals are very close to their counterparts obtained by the fiscal instruments alone, ignoring the break in volatility. Since, in our framework, fiscal multipliers reflect, by construction, the effect of relative IRFs, and considering that the consequences of the change in volatility appear evident on the tax shock and are less marked on the fiscal spending shock, this result is somewhat expected in light of our discussion in Section 3.

[3] Bacchiocchi, Bastianin, Kitagawa, and Mirto (2024) consider a partial identification approach in a Bayesian framework in situations like these, but without considering external instruments.

The scenario is different for the tax shock. The estimated peak tax multiplier is in this case 0.49 with 68% MBB confidence interval equal to (-0.20, 0.75), a marked difference relative to the 2.7 estimated by the "pure" proxy-SVAR approach, see column (i). The point estimated of the tax automatic stabilizer, ψ_y^{tax} at 1.76. These sharp differences in the estimated dynamic causal effects of (relative) tax shocks in the two scenarios represent indirect evidence that a dramatic change in the IRFs of output to the tax shock occurred after the decline in unconditional volatility observed with the Great Moderation phenomenon. Interestingly, the contamination of the tax proxy with the output proxy appears confirmed even when IRFs are kept constant across volatility regimes.

S.5 Multiple volatility regimes and QML estimation

This section extends the identification and estimation approach discussed in the paper along two dimensions: first, considering the case where the are at least two changes in volatility ($M \geq 2$) resulting in $M+1$ volatility regimes (Section S.5.1), and second, QML estimation of the proxy-SVAR under stability restrictions (Section S.5.2).

S.5.1 Multiple volatility regimes

The reduced-form proxy-SVAR model is the same as in equation (11), but now the parameters are allowed to change at the break points $T_{B_1}, \ldots, T_{B_M}$, where $1 < T_{B_1}, \ldots, < T_{B_M} < T$. Conventionally we assume that T_{B_0}:=1 and $T_{B_{M+1}}$:=T. The assumption that follows generalizes Assumptions 1-2 in the paper to a broader framework.

Assumption 6 *Given the proxy-SVAR (11),*
(i) the are M known *break points, $1 < T_{B_1} < \ldots < T_{B_M} < T$, such that $T_{B_1} \geq (n+r)$, $T_{B_i} - T_{B_{i-1}} \geq (n+r)$, $i = 2, \ldots, M+1$, $(n+r) := dim(W_t)$;*
(ii) the law of motion of the autoregressive (slope) parameters $\pi(t)$:=$vec(\Gamma(t))$ and the unconditional covariance matrix $\sigma_\eta(t)$:=$vech(\Sigma_\eta(t))$ are given by:

$$\pi(t) = \sum_{i=1}^{M+1} \pi_i \times \mathbb{I}\left(T_{B_{i-1}} < t \leq T_{B_i}\right), \quad t = 1, ..., T$$

$$\sigma_\eta(t) = \sum_{i=1}^{M+1} \sigma_{\eta,i} \times \mathbb{I}\left(T_{B_{i-1}} < t \leq T_{B_i}\right), \quad t = 1, ..., T \tag{S.13}$$

where $\Sigma_{\eta,i} < \infty$*,* $i = 1, ..., M+1$ *and:*

$$\sigma_{\eta,i} := vech(\Sigma_{\eta,i}) \neq \sigma_{\eta,j} := vech(\Sigma_{\eta,j}) \quad \textit{for} \quad i \neq j.$$

(iii) the process $\{\eta_t\}$*, where* $\eta_t := (u_t', z_t')'$*, is* α*-mixing and has absolutely summable cumulants up to order eight on the* $M+1$ *volatility regimes.*

Similarly to the case of a single break, the relationships between the VAR disturbances and proxies and the structural shocks and measurement errors is given by $u_t = G(t)\xi_t$, where ξ_t is normalized to have unit variance across the $M+1$ volatility regimes, and $G(t)$ is defined by:

$$G(t):=G + \sum_{i=2}^{M+1} \Delta_{G_i} \times \mathbb{I}\left(T_{B_{i-1}} < t \leq T_{B_i}\right) \ , \quad t = 1, ..., T. \tag{S.14}$$

In (S.14), $\Delta_{G_i} := G^{(i)} - G^{(i-1)}$, $i = 2, ..., M+1$ ($G^{(1)} := G$) are $(n+r) \times (n+r)$ matrices. In (S.14), G contains the nonzero structural parameters before any break occurs, while the nonzero elements in the matrices Δ_{G_i} $i = 2, ..., M+1$ describe how and to what extent the instantaneous impact of the structural shocks changes across volatility regimes.

The mapping between the reduced- and structural-form parameters is now given by:

$$\Sigma_{\eta,1} = GG' \, , \qquad \Sigma_{\eta,i} = \left(G + \sum_{j=2}^{i} \Delta_{G_j} \right) \left(G + \sum_{j=2}^{i} \Delta_{G_j} \right)' \, , \quad i = 2, ..., M+1 \tag{S.15}$$

and the linear identifying restrictions characterizing G and Δ_{G_i}, $i = 2, \ldots, M+1$, can be collected in the expression:

$$\begin{pmatrix} vec(G) \\ vec(\Delta_{G_2}) \\ \vdots \\ vec(\Delta_{G_{M+1}}) \end{pmatrix} = \underbrace{\begin{pmatrix} S_G & & \cdots & \\ & S_{\Delta_{G_2}} & \cdots & \\ & & \ddots & \vdots \\ & & & S_{\Delta_{G_{M+1}}} \end{pmatrix}}_{S^*} \begin{pmatrix} \gamma \\ \delta_2 \\ \vdots \\ \delta_{M+1} \end{pmatrix} . \tag{S.16}$$

In (S.16), γ is the $a \times 1$ vector ($a = \dim(\gamma)$) that collects the free (unrestricted) elements in the matrix G, and δ_i is the $b_i \times 1$ vector ($b_i = \dim(\delta_i)$) containing the free elements in the matrices Δ_{G_i}, $i = 2, ..., M+1$. The selection matrices S_G, s_G, $S_{\Delta_{G_2}}$ and $s_{\Delta_{G_2}}$ are of conformable dimensions and have obvious interpretation. To simplify notation, the big selection matrix in system (S.16) is summarized in the $(M+1)(n+r) \times (a+b_2+\ldots+b_{M+1})$ matrix S^*.

Proposition S.4 (Identification under changing IRFs, $M+1$ volatility regimes) *Assume that the DGP belongs to the class of SVARs (11) under Assumption 6, with the law of motion of the covariance matrix $\sigma_\eta(t)$ given as in (S.13). Given the structural specification in (S.14), assume that the matrices G and $(G+\Delta_{G_i})$, $i=2,...,M+1$, are non-singular and that G, Δ_{G_i}, $i=2,...,M+1$, are subject to the restrictions in (S.16).*

(i) A necessary and sufficient rank condition for the parameters $\varsigma := (\gamma', \delta_2', \ldots, \delta_{M+1}')'$ to be locally identified in a neighborhood of ς_0 is that the $\frac{1}{2}(M+1)(n+r)(n+r+1) \times (a+b_2+\ldots+b_{M+1})$ Jacobian matrix $\mathcal{J}(\varsigma)$ defined by the expression:

$$2(I_{M+1} \otimes D_{n+r}^{+}) \begin{pmatrix} (G \otimes I_{n+r}) & & & \\ (G+\Delta_{G_2}) \otimes I_{n+r} & (G+\Delta_{G_3}) \otimes I_{n+r} & & \\ \vdots & \vdots & \ddots & \\ (G+\Delta_{G_{M+1}}) \otimes I_{n+r} & (G+\Delta_{G_{M+1}}) \otimes I_{n+r} & \cdots & (G+\Delta_{G_{M+1}}) \otimes I_{n+r} \end{pmatrix} S^* \tag{S.17}$$

has full column rank in a neighboorood of ς_0.

(ii) A necessary order condition is:

$$(a+b_2+\ldots+b_{M+1}) \leq (M+1)(n+r)(n+r+1)/2.$$

Note that for $M=1$, Proposition S.4 collapses to Proposition 1 in the paper.

When the necessary and sufficient rank condition in Proposition S.4 is satisfied, the proxy-SVAR can be estimated by extending the CMD approach discussed in the paper to the multiple volatility regimes case.

S.5.2 QML estimation

To simplify exposition, in the remainder of this section we consider quasi-maximum likelihood estimation based on observations:

$$W_{-l+1}, W_{-l+2}, ..., W_0, W_1, ..., W_T$$

for the case of a single break ($M=1$) occurring at the known date $t=T_B$, i.e. two volatility regimes in the data. The generalization to the case $M \geq 2$ is tedious but straightforward. Hence, given the reduced-form model (11), the QML estimation for the whole sample, $W_1, ..., W_T$, conditional on $W_{-l+1}, W_{-l+2}..., W_0$, based on the assumption of conditionally Gaussian errors, is given by:

$$\prod_{i=1}^{M+1} \prod_{t=T_{B_{i-1}}+1}^{T_{B_i}} f(W_t \mid W_{t-1}, ..., W_{t-l}; \Gamma_i, \Sigma_{\eta,i})$$

where

$$\begin{aligned} & f(W_t \mid W_{t-1}, ..., W_{t-l}; \Gamma_i \, , \Sigma_{\eta,i}) \\ = \quad & \frac{1}{(2\pi \det(\Sigma_{\eta,i}))^{1/2}} \exp\left\{ -\frac{1}{2} [W_t - \Gamma_i X_t]' \Sigma_{\eta,i}^{-1} [W_t - \Gamma_i X_t] \right\}. \end{aligned}$$

By standard manipulations, and conventionally denoting with $\mathring{G} = G(\gamma)$ and $\mathring{\Delta}_G = \Delta_G(\delta)$ the counterparts of the matrices G and Δ_G that fulfill the identification conditions in Proposition 1, the concentrated, quasi log-likelihood of the proxy-SVAR reduces to:

$$\begin{aligned} \log L_T(\zeta) = const &- \frac{T_B}{2} \log \left| \mathring{G} \right|^2 - \frac{T - T_B + 1}{2} \log \left| \mathring{G} + \mathring{\Delta}_G \right|^2 \\ &- \frac{T_B}{2} tr \left(\mathring{G}^{-1} \left(\mathring{G}^{-1} \right)' \hat{\Sigma}_{\eta,1} \right) \\ &- \frac{T - T_B + 1}{2} tr \left(\left(\left(\mathring{G} + \mathring{\Delta}_G \right)^{-1} \right)' \left(\mathring{G} + \mathring{\Delta}_G \right)^{-1} \hat{\Sigma}_{\eta,2} \right), \end{aligned} \tag{S.18}$$

where $\hat{\Sigma}_{\eta,1}$ and $\hat{\Sigma}_{\eta,2}$ are estimates of the reduced-form covariance matrices obtained from the two volatility regimes. Bacchiocchi and Fanelli (2015) discuss the derivation of the score and associated information matrix for a case analogous to the likelihood function given in (S.18).

S.6 Identification failure under shrinking shifts

To envisage how identification stemming from the change in unconditional volatility may deteriorate and lead to identification failure, we notice that the moment conditions (16) imply that the shift in volatility is entirely due to the nonzero elements in the matrix Δ_G:

$$\Sigma_{\eta,2} - \Sigma_{\eta,1} = G\Delta_G' + \Delta_G G' + \Delta_G \Delta_G'. \tag{S.19}$$

Recall that the nonzero entries in $\Delta_G(\delta)$ capture changes in the parameters in the transition from the first to the second volatility regime. This raises the question of how large the magnitude of shifts in Δ_G (δ) must be in (S.19) for the approach outlined in the previous sections to remain valid.

To characterize the phenomenon of "shrinking covariances", we relax Assumption 1(iv) in the paper (while keeping all the other assumptions valid) and approximate Δ_G in (S.19) by the local-to-zero condition:

$$\Delta_G = \varrho_T \tilde{\Delta}_G \tag{S.20}$$

where ϱ_T is a scalar that converges to zero as the sample size increases, and $\tilde{\Delta}_G$ represents a re-scaled version of the matrix Δ_G that fulfills the same stability restrictions as Δ_G in (20). According to (S.20), the magnitude of the change in volatility in the proxy-SVAR is controlled by the parameter $\varrho_T \to 0$, whose speed of convergence to zero plays a crucial role. Under (S.20), the distance between $\Sigma_{\eta,2}$ and $\Sigma_{\eta,1}$ in (S.19) can be written as:

$$\begin{aligned}\Sigma_{\eta,2} - \Sigma_{\eta,1} &= \varrho_T G\tilde{\Delta}_G' + \varrho_T \tilde{\Delta}_G G' + \varrho_T \tilde{\Delta}_G \varrho_T \tilde{\Delta}_G' \\ &= \varrho_T \underbrace{\left[G\tilde{\Delta}_G' + \tilde{\Delta}_G G' + \tilde{\Delta}_G \varrho_T \tilde{\Delta}_G'\right]}_{\Psi_T}\end{aligned} \tag{S.21}$$

so that it is seen that, as in Bai (2000), $\Psi_T \to \Psi = (G\tilde{\Delta}_G' + \tilde{\Delta}_G G') \neq 0_{(n+r)\times(n+r)}$ and $(\Sigma_{\eta,2} - \Sigma_{\eta,1}) \to 0_{(n+r)\times(n+r)}$, as $\varrho_T \to 0$. Intuitively, given (S.21) and T being large, the moment conditions in system (16) no longer produce $(n+r)(n+r+1)$ independent moment conditions that offer meaningful information on the parameters ς. This could lead to the failure of the necessary and sufficient rank conditions for identification derived in Proposition 1.

The parameterization in (S.21) is a distinctive feature of the literature on structural change detection in VARs and is typically complemented with the following condition on the rate of convergence of ϱ_T to zero, $\varrho_T \to 0$, under which it is possible to prove that $\sqrt{T}(\hat{\sigma}_\eta - \sigma_{\eta,0})$ is still convergent; see, e.g., Bai (2000) and Qu and Perron (2007). However, by combining the conditions $\varrho_T \to 0$ with the stability restrictions (19)-(20), the implied Jacobian matrix now is:

$$\tilde{\mathcal{J}}(\varsigma) := 2\left(I_2 \otimes D_{n+r}^+\right)\begin{pmatrix}(G \otimes I_{n+r}) & 0_{(n+r)^2\times(n+r)^2} \\ (G+\tilde{\Delta}_G)\otimes I_{n+r} & (G+\tilde{\Delta}_G)\otimes I_{n+r}\end{pmatrix}\begin{pmatrix} S_G & 0 \\ 0 & \varrho_T S_{\Delta_G}\end{pmatrix}$$

and demonstrates that, even in cases where $\tilde{\mathcal{J}}(\varsigma)$ has full column rank for nonzero ϱ_T (no shrinking), identification fails as ϱ_T approaches zero.

S.7 Monte Carlo results

In this section, we evaluate the finite sample performance of the proposed stability restrictions approach for proxy-SVARs with breaks in unconditional volatility, considering a series of comprehensive simulation experiments.

Design The design of the experiment is as follows. We generate pseudo-samples of length T from a bivariate ($n = 2$) stable VAR(1) with zero initial values ($Y_0 := 0_{2\times1}$), and a single break in the unconditional covariance matrix occurring at the break date $T_B := \lfloor 0.5T \rfloor$, i.e. located at the middle of the overall sample. The DGP matrix of autoregressive parameters (see (8)-(9)) is given by:

$$\Pi := \begin{pmatrix} 0.8250 & 0.5000 \\ -0.2000 & 0.75 \end{pmatrix}$$

and its largest eigenvalue in modulus is equal to 0.84, a persistence that aligns with the level we observe in empirical analyses. Given the vector of structural shocks, $\varepsilon_t := (\varepsilon_{1,t}, \varepsilon_{2,t})'$, $\varepsilon_{1,t}$ is the target structural shock ($\varepsilon_{2,t}$ the non-target shock), which is instrumented by the proxy z_t ($r = k = 1$).The DGP for the instrument z_t is described by the linear measurement error model:

$$\begin{aligned} z_t \quad = \quad & \left[\varphi + \Delta_{\varphi}\mathbb{I}\left(t > T_B\right)\right]\varepsilon_{1,t} + \left[\Upsilon + \Delta_{\Upsilon}\mathbb{I}\left(t > T_B\right)\right]\varepsilon_{2,t} \\ & + \left[\sigma_{\omega} + \Delta_{\sigma_{\omega}}\mathbb{I}\left(t > T_B\right)\right]\omega_t \,, \qquad t = 1, ..., T \end{aligned}$$

where φ and $\varphi + \Delta_{\varphi}$ are the relevance parameters, Υ, $\Upsilon + \Delta_{\Upsilon}$ the contamination parameters, whose non-zero values capture the connections of the instrument with the non-target shock. Finally, σ_{ω} is the standard deviation of the proxy's measurement error ω_t. In this design, also the variance of the measurement error may changes from σ_{ω}^2 to $(\sigma_{\omega} + \Delta_{\sigma_{\omega}})^2$ in the shift from the first to the second volatility regime. Relevance and (failure of) exogeneity are captured by the correlations:

$$corr(z_t, \varepsilon_{1,t}) = \begin{cases} \frac{\varphi}{(\varphi^2 + \Upsilon^2 + \sigma_{\omega}^2)^{1/2}}\,, & t \leq T_B, \\ \frac{\varphi + \Delta_{\varphi}}{((\varphi + \Delta_{\varphi})^2 + (\Upsilon + \Delta_{\Upsilon})^2 + (\sigma_{\omega} + \Delta\sigma_{\omega})^2)^{1/2}}\,, & t \geq T_B + 1 \end{cases}$$

$$corr(z_t, \varepsilon_{2,t}) = \begin{cases} \frac{\Upsilon}{(\varphi^2 + \Upsilon^2 + \sigma_{\omega}^2)^{1/2}}\,, & t \leq T_B, \\ \frac{\Upsilon + \Delta_{\Upsilon}}{((\varphi + \Delta_{\varphi})^2 + (\Upsilon + \Delta_{\Upsilon})^2 + (\sigma_{\omega} + \Delta\sigma_{\omega})^2)^{1/2}}\,, & t \geq T_B + 1 \end{cases} .$$

We consider scenarios in which the external instrument satisfies the exogeneity condition ($\Upsilon = 0, \Upsilon + \Delta_{\Upsilon} = 0$, implying $corr(z_t, \varepsilon_{2,t}) = 0$ for any t), and scenarios where it does not ($\Upsilon \neq 0, \Upsilon + \Delta_{\Upsilon} \neq 0$). Similarly, we examine situations in which relevance is met, meaning that the correlation with the target shock is strong on the estimation sample, and scenarios in which the external instrument is local-to-zero as in Staiger and Stock (1997), i.e. $\varphi := cT^{-1/2}$, with $|c| < \infty$. In general, the design covers all possible properties for the proxies as per Definition 1. The DGP values for φ, Υ, σ_ω and $\Delta\sigma_\omega$ are specified below.

By combining the VAR with the external instrument for $\varepsilon_{1,t}$, the covariance matrices satisfy, under Assumption 1, the moment conditions:

$$\begin{aligned} \Sigma_{\eta,1} &= GG' \\ \Sigma_{\eta,2} &= (G+\Delta_G)(G+\Delta_G)' \end{aligned}$$

with DGP values for G and Δ_G given by:

$$\begin{aligned} G &= \begin{pmatrix} H_{\bullet 1} & H_{\bullet 2} & 0_{2\times 1} \\ \varphi & \Upsilon & \sigma_\omega \end{pmatrix} = \begin{pmatrix} 1.00 & 0.40 & 0 \\ 0.70 & 0.90 & 0 \\ \varphi & \Upsilon & 1 \end{pmatrix} \\ \Delta_G &= \begin{pmatrix} \Delta_{H_{\bullet 1}} & \Delta_{H_{\bullet 2}} & 0_{2\times 1} \\ \Delta_\varphi & \Delta_\Upsilon & \Delta_{\sigma_\omega} \end{pmatrix} = \begin{pmatrix} -0.50 & 0 & 0 \\ 0 & 0 & 0 \\ \Delta_\varphi & \Delta_\Upsilon & -0.04 \end{pmatrix}. \end{aligned}$$

The true vector of structural parameters, $\varsigma_0 := (\gamma_0', \delta_0')'$, comprises $\gamma_0 := (1, 0.7, 0.40, 0.90, \varphi_0, \Upsilon_0, 1)'$ and $\delta_0 := (-0.5, \Delta_{\varphi,0}, \Delta_{\Upsilon,0}, -0.040)'$.

In this design, the target IRFs in (12) change across the two volatility regimes solely because of changes in the on-impact parameters $H_{\bullet 1} := (1, 0.7)'$, as captured by the elements in $\Delta_{H_{\bullet 1}} := -0.5$. Overall, the total number of structural parameters to estimate when $\Upsilon \neq 0, \Delta_\Upsilon \neq 0$ (exogeneity fails) is 11, while there are $(n+r)(n+r+1)$ =12 moment conditions. Therefore, the proxy-SVAR incorporates d =1 testable overidentifying restriction when $\Upsilon \neq 0, \Delta_\Upsilon \neq 0$ (exogeneity fails), and d =3 testable overidentifying restrictions when $\Upsilon = 0, \Delta_\Upsilon = 0$ (exogeneity holds) and is imposed in estimation. The necessary and sufficient rank condition implied by Proposition 1 is satisfied for the specified values of $(\varphi_0, \Delta_{\varphi,0})$ and $(\Upsilon_0, \Delta_{\Upsilon,0})$ we consider below.

In all experiments, we generate N =10,000 samples of lengths $T = \{250,\ 500,\ 1,000\}$, respectively, under the hypothesis that the structural shocks $\varepsilon_t := (\varepsilon_{1,t}, \varepsilon_{2,t})'$ and the proxy's measurement error ω_t are drawn from iid$N(0,1)$ processes.[4] When dealing with strong proxies, the DGP values of φ and Δ_φ are such that $corr(\varepsilon_{1,t}, z_t) = 0.58$ for the full sample. Instead, when dealing with local-to-

[4] We can relax both Gaussianity and the iid hypothesis provided the process $\eta_t := (u_t', z_t')'$ respects the α-mixing conditions stated in Assumption 1.

zero proxies, the correlations vary with the sample size, namely $corr(\varepsilon_{1,t}, z_t) = \{0.045, 0.0318, 0.0225\}$, depending on whether the sample length T is equal to $250, 500$ or $1,000$ respectively.

Relative performance We start by examining whether there are gains or losses from complementing the identification of the target shock via the change in volatility with the external instrument. We first present the models which are compared and then define the measure of relative efficiency used throughout.

In Table S.1 we call "Model.q", $q = 1, 2, 3, 4, 5$ the five model involved in the comparison. With Model.1 we denote results obtained by our stability restrictions, namely by estimating the proxy-SVAR parameters ς by the CMD approach discussed in Section 2.3, assuming that the econometrician correctly specifies the VAR lag length and knows the break date T_B. Model.1 is used as a benchmark in the comparison; hence, relative efficiency measures in Table S.1 are set to 1 for this model. Model.2 is the same as Model.1 but with the contamination parameters Υ and Δ_Υ set to zero, meaning imposing proxy exogeneity. Model.3 denotes result obtained by the change in volatility approach alone, i.e. without leveraging the instrument for identification. Model.4 denotes results obtained by the proxy-SVAR-H approach, see Section S.4, i.e. assuming that the target IRFs remain constant across the two volatility regimes. Model.5 denotes results obtained by the external instrument, ignoring the break in volatility, i.e. a "conventional" proxy-SVAR approach.

Numbers in Table S.1 correspond to measures of relative efficiency based on Mean Squared Error (MSE) obtained in samples of length $T = 500$ as follows. For $q \geq 2$, we have:

$$rel\text{-}MSE^{Model.q}_{Model.1} := \tau_B \times rel\text{-}MSE^{Model.q}_{Model.1}(t)\mathbb{I}\left(t \leq T_B\right) + (1 - \tau_B) \times rel\text{-}MSE^{Model.q}_{Model.1}(t)\mathbb{I}\left(t > T_B\right) \qquad \text{(S.22)}$$

where $\tau_B := \lfloor T_B/T \rfloor$ is the fraction of the sample covering the first volatility regime and:

$$rel\text{-}MSE^{Model.q}_{Model.1}(t) := \frac{1}{25}\sum_{h=0}^{25-1}\left\{\frac{\frac{1}{N}\sum_{j=1}^{N}\left(IRF^{\widehat{Model.q}}_{i,1,j}(t,h) - IRF^{0}_{i,1}(t,h)\right)^2}{\frac{1}{N}\sum_{j=1}^{N}\left(IRF^{\widehat{Model.1}}_{i,1,j}(t,h) - IRF^{0}_{i,1}(t,h)\right)^2}\right\}. \qquad \text{(S.23)}$$

In (S.22)-(S.23), N =10,000 is the number of Monte Carlo simulations, $i = \{1, 2\}$ denotes the response variable considered in $Y_t = (Y_{1,t}, Y_{2,t})'$, $IRF^{0}_{i,1}(t,h)$

is the true value of the absolute response of $Y_{i,t+h}$ to the target shock $\varepsilon_{1,t}$ (see equation (12)), and $\widehat{IRF_{i,1,j}^{Model.q}}(t,h)$ the corresponding estimate obtained from Model.q on the sample of observation generated at the iteration j. Note that the measures in (S.22)-(S.23) are opportunely adapted to considering the whole sample of length T for Model.4 and Model.5, where the target IRFs are kept constant across the two volatility regimes.

For $q \geq 2$, measures obtained from (S.22)-(S.23) greater than 1 indicate that Model.q performs worse in terms of MSE than the benchmark Model.1. Conversely, values less than 1 indicate that there are relative gains in efficiency. It is worth remarking, however, that in this DGP, the estimators of the target IRFs in the numerator of (S.23) are not consistent for Model 4 and Model.5, as these models maintain that the IRFs do not change across volatility regimes. In these cases, therefore, comparisons drawn from Table S.1 must be interpreted with caution. Panel (a) of Table S.1 focuses on the case where a strong external instrument is used, while panel (b) refers to a local-to-zero instrument. For both cases, the instrument can be exogenous to the non-target shock ($corr(z_t, \varepsilon_{2,t}) = 0$) or can be contaminated to various extents ($corr(z_t, \varepsilon_{2,t}) = \{0.05, 0.15, 0.25\}$).

We notice from Panel (a) of Table S.1 that the incorporation of a strong and exogenous instrument to the identification based on a shift in volatility leads to considerable gains in performance. As expected, only the model which correctly imposes proxy exogeneity in estimation (other than the stability restrictions) performs better than the benchmark. In general, even when the exogeneity condition fails, a strong instrument tends to increase the accuracy with which the target IRFs are estimated, on average. Interestingly and, as expected, Panel (b) of Table S.1 shows that the gains relative to only leveraging the shift in volatility vanish in the presence of local-to-zero instruments. However, even in the scenario characterized by invalid proxies as in Definition 1.(iv), no alternative approach to the stability restrictions approach proves to be better.

Overidentifying restrictions test Next we focus on the rejection frequency of the overidentifying restrictions test resulting from the CMD estimation approach discussed in the paper.

On each of the N =10,000 generated samples of lengths $T = \{250, 500, 1000\}$, we estimate the parameters ς of the proxy-SVAR by the CMD approach discussed in Section 2.3, assuming that the econometrician correctly specifies the VAR lag length and knows the break date T_B. This approach corresponds to Model.1 in the comparisons discussed above. Then, we investigate the rejection frequency of the implied overidentifying restriction test under two main cases.

In one scenario, the econometrician leaves the contamination parameters unrestricted in estimation, with the idea that the significance of Υ and Δ_Υ can be inferred from the data provided the conditions in Proposition 1 hold. The other scenario coincides with the case in which the econometrician imposes the exogeneity restriction in estimation ($\Upsilon = 0, \Delta_\Upsilon = 0$). This implies that the estimated proxy-SVAR is misspecified when contamination ($\Upsilon \neq 0, \Delta_\Upsilon \neq 0$) holds in the DGP, i.e. for $corr(z_t, \varepsilon_{2,t}) = \{0.05,\, 0.15,\, 0.25\}$.

Under the null that the stability restrictions hold, the overidentifying restrictions test statistic (see equation (25)) is asymptotically distributed in the stated DGP as χ^2_d random variable, with degree of freedom d as defined above. Tests are conducted at the 5% nominal significance level and rejection frequencies are summarized in Table S.2.

The right panel of Table S.2 pertains to the case where the econometrician does not impose proxy exogeneity in estimation. The left panel, instead, assumes that exogeneity is imposed. It is observed that, regardless of the correlation between the instrument and the non-target shock, when the external instrument is left free in estimation, the CMD estimation approach provides rejection frequencies compatible with finite sample size control regardless of strength. On the contrary, rejection frequencies tend to increase with the extent of contamination and the increase in sample size regardless of strength, revealing that the test tends to have finite sample power against the failure of instrument exogeneity.

Overall, the results in Table S.2, combined with those in Table S.1, confirm that when identification of the proxy-SVAR is ensured by shifts in volatility, as implied by the necessary and sufficient rank conditions in Proposition 1, it is generally advantageous for the econometrician not to impose exogeneity in estimation. Violations of the exogeneity condition can be detected from the data regardless of whether the instrument is relevant or local-to-zero. Notably, even when the instrument is both relevant and contaminated, it proves to be useful for the inference on the target IRFs. However, the overidentifying restrictions test rejects the model's validity when exogeneity is incorrectly imposed in estimation, and again, this holds irrespective of instrument strength.

We turn on the performance of the overidentifying restrictions test at the end of this section, where we explore how the stability restrictions approach performs when the shifts in volatility provide limited information, resulting in near-rank failure for the Jacobian $\mathcal{J}(\varsigma)$, rendering the results in Propositions 1-2 invalid.

Checks of identifiability and shrinking shifts Results in Table S.1 and Table S.2 are obtained under scenarios in which the proxy-SVAR with a break in unconditional volatility is identified. Identifiability depends on the full column rank condition of the Jacobian matrix $\mathcal{J}(\varsigma)$, as derived in Proposition 1; see equation (22). The validity of the necessary and sufficient rank ensures the use of standard asymptotic inference, as stated in Proposition 2. Here, we investigate to what extent the smallest singular values of $\mathcal{J}(\hat{\varsigma}_T)$, given the CMD estimates $\hat{\varsigma}_T$ and associated measures of uncertainty are informative about the identifiability of the proxy-SVAR.

First, we consider the case in which the change in VAR covariance matrices is sufficient to identify the model, consistent with the DGP considered so far. Table S.3 summarizes the average, across Monte Carlo simulations, of the estimated smallest singular values of the Jacobian matrix along with associated interquartile ranges (IQRs). IQRs are used in this context as broad approximations of confidence intervals. As previously, we explore scenarios with both relevant and local-to-zero instruments, and both exogenous and contaminated instruments. Results in Table S.3 indicate that in situations where the change in volatility is sufficient for identification, the smallest singular values of the estimated Jacobian matrix are far from zero and the associated IQRs tend not to include the zero. Another important finding from Table S.3 is that proxy properties do not affect the identifiability of the model, confirming the analytic results discussed in the paper and the figures in tables S.1 and S.2. The results outlined in Table S.3 also indirectly support the identifiability of the fiscal proxy-SVAR estimated in Section 4 of the paper. In that section, in the bottom part of Table 2, we reported the estimated smallest singular value of the Jacobian matrix along with associated 68% MBB confidence interval, which reassuringly seem to rule out the case of a zero singular value.

Secondly, we reexamine the performance of the stability restrictions approach under a different scenario. Specifically, we address cases where the distance between covariance matrices in the two volatility regimes, $(\Sigma_{\eta,2} - \Sigma_{\eta,1})$, shrinks according to equation (S.21) in the paper, reported here for convenience:

$$\Sigma_{\eta,2} - \Sigma_{\eta,1} = \varrho_T \Psi_T.$$

In this equation, as the scalar ϱ_T converges to zero, $\varrho_T \to 0$, $\Psi_T \to \Psi = (G\tilde{\Delta}_G + \tilde{\Delta}_G G) \neq 0$ determining a near-rank failure setup for the Jacobian $\mathcal{J}(\varsigma)$; see Section 2.3.

Table S.4 summarizes the estimated smallest singular values of the Jacobian matrix and associated IQRs when VAR covariance matrices shrink at the rate $\varrho_T \sim o(T^{-1/2})$. We now observe a departure from the patterns seen in Table S.3. Unlike the scenarios presented there, where the ratio between the estimated average smallest singular values and the average length of IQRs is consistently greater than 2, we now notice a distinct trend. Specifically, the magnitude of the estimated smallest singular values tends to be systematically smaller than 2 times the IQR, indicating a lack of identification resulting from the change in volatility. However, in line with expectations, strong proxies appear to sustain the identifiability of the proxy-SVAR when the exogeneity condition is imposed in estimation, although not when it is not. This phenomenon is explained by the fact that when deviations from exogeneity are allowed, the instruments also provide information about the non-target shocks. Consequently, if the change in volatility poorly identifies the non-target shocks, the failure in identification extends to the entire system. In contrast, when exogeneity is imposed in estimation, whether or not it holds in the DGP, any information originating from the non-target shocks is not transmitted to the target shocks. In this case, the Jacobian matrix maintains full column rank, ensuring identification.

S.8 Proofs of propositions

Proof of Proposition 1: (i) The result follows by deriving the moment conditions in (18)-(21) with respect to the parameter $\varsigma := (\gamma', \delta')'$ and then applying matrix derivative rules; see Bacchiocchi and Fanelli (2015, Proposition 1);
(ii) the necessary order condition follows trivially from the dimensions of the Jacobian matrix:

$$\mathcal{J}(\varsigma) := \underbrace{\frac{\partial m(\sigma_\eta, \varsigma)}{\partial \varsigma'}}_{(n+r)(n+r+1)\times(a+b)} = \begin{pmatrix} \frac{\partial m_1(\sigma_\eta,\varsigma)}{\partial \varsigma'} \\ \frac{\partial m_2(\sigma_\eta,\varsigma)}{\partial \varsigma'} \end{pmatrix}$$

in (22), where both $\partial m_1(\sigma_\eta, \varsigma)/(\partial \varsigma')$ and $\partial m_2(\sigma_\eta, \varsigma)/(\partial \varsigma')$ are of dimension $(\frac{1}{2}(n+r)(n+r+1) \times (a+b))$. ■

Proof of Proposition 2: Let $\hat{Q}_T(\varsigma) := m_T(\hat{\sigma}_\eta, \varsigma)'\hat{V}_{\sigma_\eta}^{-1} m_T(\hat{\sigma}_\eta, \varsigma)$ be the objective function upon which CMD estimation is computed in (25). We observe that: (a) under the conditions of Proposition 1, $Q_0(\varsigma) := m(\sigma_0^+, \varsigma)' V_{\sigma_\eta}^{-1} m(\sigma_0^+, \varsigma)$ is uniquely maximized at ς_0 in the neighborhood $\mathcal{N}_{\varsigma_0}$; (b) $\mathcal{P}_\varsigma$ is compact and $\mathcal{N}_{\varsigma_0} \subseteq \mathcal{P}_\varsigma$; (c) $Q_0(\varsigma)$ is continuous; (d) $\hat{Q}_T(\varsigma)$ converges uniformly in probability to $Q_0(\varsigma)$. To prove that (d) holds, recall that under Assumptions 1-2 $\hat{\sigma}_\eta \overset{p}{\to} \sigma_{\eta,0}$, hence $m_T(\hat{\sigma}_\eta, \varsigma) \overset{p}{\to} m(\sigma_{\eta,0}, \varsigma)$ by the Slutsky Theorem. Also recall that it exists an estimator of the asymptotic covariance matrix V_{σ_η} such that $\hat{V}_{\sigma_\eta} \overset{p}{\to} V_{\sigma_\eta}$, see (24). Then, with $\|\cdot\|$ denoting the Euclidean norm, by the triangle and Cauchy-Schwartz inequalities:

$$\begin{aligned}\left|\hat{Q}_T(\varsigma)-Q_0(\varsigma)\right| &\leq \left|[m_T(\hat{\sigma}_\eta,\varsigma)-m(\sigma_{\eta,0},\varsigma)]'\hat{V}_{\sigma_\eta}^{-1}[m_T(\hat{\sigma}_\eta,\varsigma)-m(\sigma_{\eta,0},\varsigma)]\right| \\ &+\left|m(\sigma_{\eta,0},\varsigma)'[\hat{V}_{\sigma_\eta}^{-1}+\hat{V}_{\sigma_\eta}^{-1\prime}][m_T(\hat{\sigma}_\eta,\varsigma)-m(\sigma_{\eta,0},\varsigma)]\right| \\ &+\left|m(\sigma_{\eta,0},\varsigma)'[\hat{V}_{\sigma_\eta}^{-1}-V_{\sigma_\eta}^{-1}]m(\sigma_{\eta,0},\alpha)\right| \\ &\leq \|m_T(\hat{\sigma}_\eta,\varsigma)-m(\sigma_{\eta,0},\varsigma)\|^2\left\|\hat{V}_{\sigma_\eta}^{-1}\right\| \\ &+2\|m(\sigma_{\eta,0},\varsigma)\|\,\|m_T(\hat{\sigma}_\eta,\varsigma)-m(\sigma_{\eta,0},\varsigma)\|\left\|\hat{V}_{\sigma_\eta}^{-1}\right\| \\ &+\|m(\sigma_{\eta,0},\varsigma)\|^2\left\|\hat{V}_{\sigma_\eta}^{-1}-V_{\sigma_\eta}^{-1}\right\|\end{aligned}$$

so that $\sup_{\varsigma\in\mathcal{P}_\varsigma}\left|\hat{Q}_T(\varsigma)-Q_0(\varsigma)\right|\xrightarrow{p}0$. Given (a), (b), (c), and (d), the consistency result follows from Newey and McFadden (1994, Theorem 2.1).

To prove asymptotic normality, we start from the first-order conditions implied by the problem (25) in the paper:

$$\mathcal{J}(\hat{\varsigma}_T)'\hat{V}_{\sigma_\eta}^{-1}m_T(\hat{\sigma}_\eta,\hat{\varsigma}_T)=0 \tag{S.24}$$

where $\mathcal{J}(\hat{\varsigma}_T)$ denotes the Jacobian matrix $\mathcal{J}(\varsigma):=\frac{\partial m(\sigma_\eta,\varsigma)}{\partial\varsigma'}:=\mathcal{J}(\sigma_\eta,\varsigma)$ evaluated at the estimated parameters $\hat{\sigma}_\eta$ and $\hat{\varsigma}_T$, respectively. By expanding $m_T(\hat{\sigma}_\eta,\hat{\varsigma}_T)$ around ς_0 and solving, yields the expression (valid in $\mathcal{N}_{\varsigma_0}$):

$$\sqrt{T}(\hat{\varsigma}_T-\varsigma_0)=-\left\{\mathcal{J}(\hat{\sigma}_\eta,\hat{\varsigma}_T)'\hat{V}_{\sigma_\eta}^{-1}\mathcal{J}(\hat{\sigma}_\eta,\bar{\varsigma})\right\}^{-1}\mathcal{J}(\hat{\sigma}_\eta,\hat{\varsigma}_T)'\hat{V}_{\sigma_\eta}^{-1}\sqrt{T}m_T(\hat{\sigma}_\eta,\varsigma_0) \tag{S.25}$$

where $\bar{\varsigma}$ is a mean value. From (24) and the delta-method:

$$\sqrt{T}m_T(\hat{\sigma}_\eta,\varsigma_0)\xrightarrow{d}N(0,\mathcal{J}(\varsigma_0)'V_{\sigma_\eta}\mathcal{J}(\varsigma_0)') \tag{S.26}$$

where $\mathcal{J}(\hat{\sigma}_\eta,\varsigma_0)\xrightarrow{p}\mathcal{J}(\sigma_{\eta,0},\varsigma_0):=\mathcal{J}(\varsigma_0)$. From the consistency result in (i), as $T\to\infty$, $\mathcal{J}(\hat{\sigma}_\eta,\hat{\varsigma}_T)\xrightarrow{p}\mathcal{J}(\sigma_{\eta,0},\varsigma_0):=\mathcal{J}(\varsigma_0)$ and $\mathcal{J}(\hat{\sigma}_\eta,\bar{\varsigma})\xrightarrow{p}\mathcal{J}(\sigma_{\eta,0},\varsigma_0):=\mathcal{J}(\varsigma_0)$, respectively. Moreover, the matrix $\mathcal{J}(\varsigma_0)'V_{\sigma_\eta}^{-1}\mathcal{J}(\varsigma_0)$ is nonsingular in $\mathcal{N}_{\varsigma_0}$ because of Proposition 1. It turns out that

$$-\left\{\mathcal{J}(\hat{\sigma}_\eta,\hat{\varsigma}_T)'\hat{V}_{\sigma_\eta}^{-1}\mathcal{J}(\hat{\sigma}_\eta,\bar{\varsigma})\right\}^{-1}\mathcal{J}(\hat{\sigma}_\eta,\hat{\varsigma}_T)\hat{V}_{\sigma_\eta}^{-1}$$

$$\xrightarrow{p}-\left\{\mathcal{J}(\varsigma_0)'V_{\sigma_\eta}^{-1}\mathcal{J}(\varsigma_0)\right\}^{-1}\mathcal{J}(\varsigma_0)'V_{\sigma_\eta}^{-1},$$

so that the conclusion follows from (S.26) and the Slutsky theorem. ■

Proof of Proposition S.1: (i) Consider the sums:

$$\hat{\Sigma}_{u,z} := \frac{1}{T}\sum_{t=1}^{T}\hat{u}_t z_t' = \frac{1}{T}\left\{\sum_{t=1}^{T_B}\hat{u}_t z_t' + \sum_{t=T_B+1}^{T}\hat{u}_t z_t'\right\}$$

$$= \frac{1}{T}\left\{\frac{T_B}{T_B}\sum_{t=1}^{T_B}\hat{u}_t z_t' + \frac{T-T_B}{T-T_B}\sum_{t=T_B+1}^{T}\hat{u}_t z_t'\right\}$$

$$= \frac{T_B}{T}\left\{\frac{1}{T_B}\sum_{t=1}^{T_B}\hat{u}_t z_t'\right\} + \frac{T-T_B}{T}\left\{\frac{1}{T-T_B}\sum_{t=T_B+1}^{T}\hat{u}_t z_t'\right\}.$$

hence, for $T\to\infty$,

$$\frac{T_B}{T}\left\{\frac{1}{T_B}\sum_{t=1}^{T_B}\hat{u}_t z_t'\right\} \xrightarrow{p} \tau_B^{(0)}\mathbb{E}\left[u_t z_t'\mathbb{I}\left(t\le T_B\right)\right]$$

$$= \tau_B^{(0)}\mathbb{E}\left[\left\{H_{\bullet 1}\varepsilon_{1,t} + H_{\bullet 2}\varepsilon_{2,t}\right\}z_t'\right] = \tau_B^{(0)}H_{\bullet 1}\mathbb{E}\left[\varepsilon_{1,t}z_t'\right] = \tau_B^{(0)}H_{\bullet 1}\Phi' \ ;$$

$$\frac{T-T_B}{T}\left\{\frac{1}{T-T_B}\sum_{t=T_B+1}^{T}\hat{u}_t z_t'\right\} \xrightarrow{p} (1-\tau_B^{(0)})\mathbb{E}\left[u_t z_t'\mathbb{I}\left(t> T_B\right)\right]$$

$$= (1-\tau_B^{(0)})\mathbb{E}\left[\left\{H_{\bullet 1}P_{(1)}^{1/2}\varepsilon_{1,t} + H_{\bullet 2}P_{(2)}^{1/2}\varepsilon_{2,t}\right\}z_t'\right] = H_{\bullet 1}P_{(1)}^{1/2}\mathbb{E}\left[\varepsilon_{1,t}z_t'\right] = H_{\bullet 1}P_{(1)}^{1/2}\Phi'.$$

(ii) It is seen that with $\Phi \neq 0$ ($rank[\Phi] = k = 1$) and $rank[P_{(1)}] = 1$:

$$\begin{aligned}
\Sigma_{u,z}\left(\Sigma_{u_1,z}\right)^{-1} &= \begin{pmatrix}\Sigma_{u_1,z}\\ \Sigma_{u2,z}\end{pmatrix}\left(\Sigma_{u_1,z}\right)^{-1}\\
&= \begin{cases}\begin{pmatrix}H_{1,1}\Phi'\\ H_{2,1}\Phi'\end{pmatrix}\left(H_{1,1}\Phi'\right)^{-1} & t\le T_B\\ \begin{pmatrix}H_{1,1}P_{(1)}^{1/2}\Phi'\\ H_{2,1}P_{(1)}^{1/2}\Phi'\end{pmatrix}\left(H_{1,1}P_{(1)}^{1/2}\Phi'\right)^{-1} & t\ge T_B+1\end{cases}\\
&= \begin{cases}\begin{pmatrix}1\\ \left(H_{2,1}\Phi'\right)\left(H_{1,1}\Phi'\right)^{-1}\end{pmatrix}\\ \begin{pmatrix}1\\ \left(H_{2,1}P_{(1)}^{1/2}\Phi'\right)\left(H_{1,1}P_{(1)}^{1/2}\Phi'\right)^{-1}\end{pmatrix}\end{cases} , \ t = 1,...,T
\end{aligned}$$

$$= \begin{pmatrix} 1 \\ H_{2,1}^{rel} \end{pmatrix}, \, t = 1, ..., T.$$

As $\hat{\Sigma}_{u_2,z} \xrightarrow{p} H_{2,1}P_{(1)}^{1/2}\Phi'$ and $\hat{\Sigma}_{u1,z} \xrightarrow{p} H_{1,1}P_{(1)}^{1/2}\Phi'$, it follows that $\hat{\Sigma}_{u_2,z}\left(\hat{\Sigma}_{u_1,z}\right)^{-1}$ is consistent for $H_{2,1}^{rel}$. ■

Proof of Proposition S.3: (i) The nonlinear functional relationship between $\sigma_\eta := (vech(\Sigma_{\eta,1})', vech(\Sigma_{\eta,2})')'$ and ρ is given by $\sigma_\eta = \sigma(\rho)$, where $\sigma(\rho) = (vech[G_{un}G_{un}{}']', vech[G_{un}\Lambda G_{un}{}'])'$. A necessary and sufficient condition for ρ to be uniquely recovered from σ_η is that $rank[\mathcal{J}(\rho)] = \dim(\rho) = (n+r)(n+r+1)$ locally, where $\mathcal{J}(\rho) = \frac{\partial \sigma(\rho)}{\partial \rho'}$ is $(n+r)(n+r+1) \times (n+r)(n+r+1)$. Using matrix algebra derivatives and the properties of duplication matrices:

$$\mathcal{J}(\rho) = \begin{pmatrix} \frac{\partial vech(G_{un}G'_{un})}{\partial \rho'} \\ \frac{\partial vech(G_{un}\Lambda G'_{un})}{\partial \rho'} \end{pmatrix} = \begin{pmatrix} \frac{\partial vech(G_{un}G'_{un})}{\partial \rho'} & \frac{\partial vech(G_{un}G'_{un})}{\partial \lambda'} \\ \frac{\partial vech(G_{un}\Lambda G'_{un})}{\partial \rho'} & \frac{\partial vech(G_{un}\Lambda G'_{un})}{\partial \lambda'} \end{pmatrix}$$

$$= \begin{pmatrix} D^{+}_{(n+r)}\frac{\partial vec(G_{un}G'_{un})}{\partial \rho'} & 0 \\ D^{+}_{(n+r)}\frac{\partial vec(G_{un}\Lambda G'_{un})}{\partial \rho'} & D^{+}_{(n+r)}\frac{\partial vec(G_{un}\Lambda G'_{un})}{\partial \lambda'} \end{pmatrix}$$

$$\begin{pmatrix} D^{+}_{(n+r)}\frac{\partial vec(G_{un}G'_{un})}{\partial vec(G_{un})'} \times \frac{\partial vec(G_{un})}{\partial \rho'} & 0 \\ D^{+}_{(n+r)}\frac{\partial vec(G_{un}\Lambda G'_{un})}{\partial vec(G_{un})'} \times \frac{\partial vec(G_{un})}{\partial \rho'} & D^{+}_{(n+r)}\frac{\partial vec(G_{un}\Lambda G'_{un})}{\partial vec(\Lambda)'} \times \frac{\partial vec(\Lambda)}{\partial \lambda'} \end{pmatrix}.$$

Now, given $N_{(n+r)} := \frac{1}{2}(I_{(n+r)^2} + K_{(n+r)})$ with $K_{(n+r)}$ commutation matrix, and since we have:

$$\begin{aligned} \frac{\partial vec(G_{un}G'_{un})}{\partial vec(G_{un})'} &= 2N_{(n+r)}(G_{un} \otimes I_{(n+r)}); \\ \frac{\partial vec(G_{un}\Lambda G'_{un})}{\partial vec(G_{un})'} &= 2N_{(n+r)}(G_{un}\Lambda \otimes I_{(n+r)}); \\ \frac{\partial vec(G_{un}\Lambda G'_{un})}{\partial vec(\Lambda)'} &= (G_{un} \otimes G_{un}), \end{aligned}$$

and $2D^{+}_{(n+r)}N_{(n+r)} = 2D^{+}_{(n+r)}$, the Jacobian reads:

$$\mathcal{J}(\rho) = 2(I_2 \otimes D^{+}_{(n+r)}) \begin{pmatrix} (G_{un} \otimes I_{(n+r)}) & 0 \\ (G_{un}\Lambda \otimes I_{(n+r)}) & (G_{un} \otimes G_{un})\frac{1}{2}\mathcal{F}_\Lambda \end{pmatrix}$$

$$= 2(I_2 \otimes D^{+}_{(n+r)}) \begin{pmatrix} (G_{un} \otimes I_{(n+r)}) & 0 \\ (G_{un}\Lambda \otimes I_{(n+r)}) & (G_{un} \otimes G_{un}) \end{pmatrix} \begin{pmatrix} S_G & 0 \\ 0 & \frac{1}{2}\mathcal{F}_\Lambda \end{pmatrix}.$$

■

Proof of Proposition S.4: See Bacchiocchi and Fanelli (2015), Supplementary Material. ∎

S.9 Sensitivity of the stability restrictions estimator

Sensitivity analysis consists into measuring the effects of local misspecification of the moments of estimator onto first-order asymptotics of the $\hat{\theta}$. Let us consider misspecification of the moments in the sense of wrong assignment of the parameter τ. Namely, for any $\epsilon \neq 0$, let us consider situations with 2 regimes, where the break date used in the estimation of θ is $T_{B,\epsilon} = \lfloor(\tau_0 + \epsilon)T\rfloor$ as opposed to the true $T_B = \lfloor \tau_0 T \rfloor$, such that as $T \to \infty$, $T_{B,\epsilon} \to T_B$. Let $\hat{\sigma}_\eta$ be a consistent estimator of σ_η.

In the context of the stability restrictions estimator, local misspecifications may realize under the form of estimation error and/or incorrect assignment of a break in the DGP. Without loss of generality, the misspecified moment function (21) can be represented by

$$m(\hat{\sigma}_\eta, =)$$

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Table S.1: Relative efficiency (MSE) of estimators of the target IRFs.

Sample size: $T = 500$	$corr(z_t, \varepsilon_{2t})$							
	0.00		0.05		0.15		0.25	
	$IRF_{1,1}$	$IRF_{2,1}$	$IRF_{1,1}$	$IRF_{2,1}$	$IRF_{1,1}$	$IRF_{2,1}$	$IRF_{1,1}$	$IRF_{2,1}$
Panel a) Strong proxy								
Model.1	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
Model.2	0.95	0.96	1.01	1.02	1.33	1.36	1.95	2.04
Model.3	11.75	7.11	13.34	9.03	16.69	13.36	18.36	15.25
Model.4	5.87	4.10	5.89	4.19	5.94	4.39	6.05	4.54
Model.5	4.62	2.71	5.41	3.37	7.15	5.24	8.51	7.01
Panel b) Local-to-zero proxy								
Model.1	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
Model.2	1.00	1.00	1.00	1.01	1.02	1.03	1.07	1.07
Model.3	1.00	1.00	1.00	1.00	1.00	0.99	1.00	0.99
Model.4	5.80	3.86	5.95	3.93	5.69	3.90	5.72	3.94
Model.5	13.27	11.92	11.12	10.77	7.26	7.86	5.85	7.05

NOTES: Results are based on $N = 10,000$ Monte Carlo simulations, see Section S.7 for details on the design. Model.1 denotes results obtained by the stability restrictions approach discussed in the paper. Model.(2) is the same as Model.1 with the contamination parameters Υ and Δ_Υ set to zero, i.e., imposing proxy exogeneity. Model.3 denotes result obtained by the change in volatility approach alone, i.e., without including the instrument. Model.4 denotes results obtained by the proxy-SVAR-H approach, see Section S.4, i.e., assuming that the target IRFs remain constant across the two volatility regimes. Model.5 denotes results obtained by the external instrument alone, i.e., ignoring the break in volatility. Numbers in the table correspond to measures of relative efficiency in the estimation of target IRFs based on Mean Squared Error (MSE), as discussed in Section S.7. Model.1 is used as a benchmark in the comparison; hence, relative efficiency measures are set to 1 for this model.

Table S.2: Rejection frequencies of the overidentifying restrictions test (5% nominal).

		$corr(z_t, \varepsilon_{2t})$							
		Υ is set to 0				Υ is unrestricted			
		0.00	0.05	0.15	0.25	0.00	0.05	0.15	0.25
Sample size	Relevance	Rejection frequency (5%)							
$T = 250$	Strong	4.06	7.64	40.84	88.30	4.83	4.63	4.42	4.43
	Local-to-zero	4.28	8.22	45.73	91.83	4.22	4.72	4.34	4.67
$T = 500$	Strong	4.68	12.03	75.26	99.80	4.87	4.73	4.55	4.49
	Local-to-zero	4.34	13.26	80.48	99.92	4.67	4.93	4.79	5.10
$T = 1000$	Strong	4.22	21.40	97.64	100.00	5.21	5.04	4.57	5.09
	Local-to-zero	4.64	22.62	98.60	100.00	4.74	4.73	5.04	5.06

NOTES: Rejection frequencies are computed across N =10,000 Monte Carlo simulations, see Section S.7 for details on the design. Estimates of proxy-SVAR parameters are obtained by the CMD approach discussed in Section 3 of the paper.

Table S.3: Estimated smallest singular values of the Jacobian $\mathcal{J}(\varsigma)$ with associated IQR. Full column rank condition holds.

	$corr(z_t, \varepsilon_{2t})$							
	Υ is set to 0				Υ is unrestricted			
	0.00	0.05	0.15	0.25	0.00	0.05	0.15	0.25
Sample size, relevance	Smallest singular value of $J(\varsigma)$							
$T = 250$, Strong	**0.0427** [0.0136]	**0.0418** [0.0128]	**0.0395** [0.0110]	**0.0363** [0.0099]	**0.0020** [0.0008]	**0.0022** [0.0009]	**0.0028** [0.0011]	**0.0033** [0.0012]
$T = 250$, Local-To-Zero	**0.0077** [0.0025]	**0.0077** [0.0024]	**0.0076** [0.0023]	**0.0076** [0.0023]	**0.0061** [0.0017]	**0.0061** [0.0017]	**0.0058** [0.0017]	**0.0052** [0.0017]
$T = 500$, Strong	**0.0443** [0.0099]	**0.0430** [0.0094]	**0.0405** [0.0081]	**0.0374** [0.0071]	**0.0018** [0.0005]	**0.0021** [0.0006]	**0.0026** [0.0007]	**0.0032** [0.0009]
$T = 500$, Local-To-Zero	**0.0071** [0.0015]	**0.0070** [0.0015]	**0.0069** [0.0015]	**0.0066** [0.0014]	**0.0062** [0.0012]	**0.0062** [0.0012]	**0.0058** [0.0012]	**0.0051** [0.0012]
$T = 1000$, Strong	**0.0449** [0.0071]	**0.0437** [0.0069]	**0.0412** [0.0058]	**0.0378** [0.0050]	**0.0018** [0.0004]	**0.0020** [0.0004]	**0.0025** [0.0005]	**0.0031** [0.0006]
$T = 1000$, Local-To-Zero	**0.0067** [0.0010]	**0.0067** [0.0009]	**0.0066** [0.0009]	**0.0063** [0.0009]	**0.0063** [0.0008]	**0.0063** [0.0008]	**0.0058** [0.0009]	**0.0050** [0.0009]

NOTES: Numbers in the table are averages of estimates obtained across N =10,000 Monte Carlo simulations, see Section S.7 for details on the design. Bold entries indicate that the magnitude of the estimated smallest singular value is greater than 2×IQR.

Table S.4: Estimated smallest singular values of the Jacobian $\mathcal{J}(\varsigma)$ with associated IQR. Full column rank condition holds. Shrinking covariance matrices, $\varrho_T \sim o(T^{-1/2})$.

Shrinking shifts	$corr(z_t, \varepsilon_{2t})$							
	Υ is set to 0				Υ is unrestricted			
	0.00	0.05	0.15	0.25	0.00	0.05	0.15	0.25
Sample size, relevance	Smallest singular value of $J(\varsigma)$							
$T = 250$, Strong	**0.0591** [0.0131]	**0.0593** [0.0131]	**0.0638** [0.0139]	**0.0750** [0.0157]	0.0010 [0.0009]	0.0010 [0.0009]	0.0010 [0.0009]	0.0009 [0.0008]
$T = 250$, Local-To-Zero	0.0075 [0.0064]	0.0068 [0.0054]	0.0109 [0.0087]	0.0199 [0.0130]	0.0012 [0.0011]	0.0012 [0.0011]	0.0012 [0.0011]	0.0012 [0.0010]
$T = 500$, Strong	**0.0598** [0.0093]	**0.0599** [0.0093]	**0.0646** [0.0100]	**0.0760** [0.0113]	0.0005 [0.0004]	0.0005 [0.0004]	0.0005 [0.0004]	0.0005 [0.0004]
$T = 500$, Local-To-Zero	0.0040 [0.0035]	0.0037 [0.0028]	0.0086 [0.0062]	0.0185 [0.0095]	0.0006 [0.0005]	0.0006 [0.0005]	0.0006 [0.0005]	0.0006 [0.0005]
$T = 1000$, Strong	**0.0600** [0.0065]	**0.0601** [0.0066]	**0.0648** [0.0071]	**0.0754** [0.0080]	0.0003 [0.0002]	0.0003 [0.0002]	0.0002 [0.0002]	0.0003 [0.0002]
$T = 1000$, Local-To-Zero	0.0020 [0.0017]	0.0022 [0.0016]	0.0073 [0.0044]	**0.0175** [0.0069]	0.0003 [0.0003]	0.0003 [0.0003]	0.0003 [0.0003]	0.0003 [0.0003]

NOTES: Numbers in the table are averages of estimates obtained across N =10,000 Monte Carlo simulations, see Section S.7 for details on the design. Bold entries indicate that the magnitude of the estimated smallest singular value is greater than 2×IQR.

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