



ISSN 2282-6483

Alma Mater Studiorum - Università di Bologna
DEPARTMENT OF ECONOMICS

Dynamic Factor Models and Cointegration

Marco Brianti
Mario Forni
Luca Gambetti
Antonio Granese

Quaderni - Working Paper DSE N°1234



Dynamic Factor Models and Cointegration

Marco Brianti*
Università di Bologna

Mario Forni†
Università di Modena e Reggio Emilia, CEPR and RECent

Luca Gambetti‡
Universitat Autònoma de Barcelona, BSE, Università di Torino and CCA

Antonio Granese§
Università di Modena e Reggio Emilia

October 26, 2026

Abstract

The factors of a stationary Dynamic Factor Model (DFM) are the first differences of *singular* $I(1)$ processes. Being singular, such processes are always cointegrated. This however does not prevent the existence of a VAR for their first differences, so that the standard estimation procedure of the stationary DFM remains valid. Problems arise only when there is a linear combination of the shocks that has transitory effects on all the variables of the panel. We argue that this case is implausible for macroeconomic data and show that in fact it does not occur for two widely used US data sets — the FRED-QD and the FRED-MD. A Monte Carlo exercise shows that the standard procedure outperforms alternative methods based on the factors in levels, especially at long horizons.

JEL classification: C32, E32.

Keywords: dynamic factor models, singular stochastic vectors, cointegration, common trends, impulse–response functions.

*E-mail: marco.brianti@unibo.it.

†E-mail: mario.forni@unimore.it.

‡E-mail: luca.gambetti@uab.cat.

§E-mail: antonio.granese@unimore.it.

We are grateful to Matteo Luciani, Marco Lippi, and Matteo Barigozzi for their valuable comments. Gambetti acknowledges financial support from the Spanish Ministry of Science and Innovation, through the Severo Ochoa Programme for Centres of Excellence in R&D (CEX2019-000915-S), the financial support of the Spanish Ministry of Science, Innovation and Universities through grant PGC2018-094364-B-I00, and the Barcelona Graduate School Research Network.

Non-technical summary

Dynamic Factor Models use large panels of macroeconomic variables to summarize the economy through a relatively small number of common factors and shocks. This approach makes it possible to study the responses of many variables within a unified framework, exploit a richer information set than conventional VARs, and reduce the influence of measurement errors and variable-specific disturbances. A common feature of these models is that the number of factors is larger than the number of shocks driving them. As a result, the factors are singular and, when expressed in levels, necessarily cointegrated. This may appear to invalidate the standard estimation procedure, which estimates a VAR for the factors in first differences. The paper shows that this concern is generally misplaced.

The key insight is that cointegration can arise for two different reasons. First, it may follow mechanically from singularity: several factors are driven by a smaller number of shocks and must therefore share common long-run movements. This type of cointegration does not prevent the first differences of the factors from admitting a VAR representation. Consequently, the Standard Procedure remains valid. A problem arises only when one or more combinations of the common shocks have temporary effects on every variable in the panel. In that case, there are additional cointegrating relationships beyond those implied by singularity, and the Standard Procedure is no longer appropriate. Researchers must instead use methods formulated for the factors in levels, such as a VAR in levels or an error-correction model.

The paper argues that the latter case is unlikely in conventional macroeconomic data sets. This argument can be tested empirically by comparing the estimated number of common shocks with the number of shocks that have permanent effects. If the two numbers coincide, cointegration arises from singularity alone and the Standard Procedure can safely be used. Using the FRED-QD and FRED-MD data sets, the evidence indicates that cointegration is generated by singularity alone.

Finally, Monte Carlo simulations based on a medium-scale DSGE model shows that the Standard Procedure produces more accurate impulse-response estimates than the alternatives, particularly at longer horizons. The simulations also confirm that alternative procedures become necessary when some shocks have a temporary effects on all the variables in the panel.

1 Introduction

Structural analysis with large panels of macroeconomic time series has become standard practice in both academic research and policy institutions. The underlying idea is to exploit the information contained in a large number of macroeconomic variables while allowing their dynamics to be driven by a relatively small number of common shocks. This approach is formalized by the Structural Dynamic Factor Model, introduced by [Stock and Watson \(2005\)](#) and [Forni et al. \(2009\)](#), and is closely related to the Factor Augmented VAR of [Bernanke et al. \(2005\)](#) and the Common Components Structural VAR of [Forni et al. \(2026\)](#).

The use of large panels offers several advantages over traditional structural VAR analysis. First, it allows the impulse–response functions of a large set of macroeconomic variables to be studied within a unified framework. Second, the richer information set can help overcome the non-invertibility problem that may arise when structural shocks are recovered from a limited set of observables ([Hansen and Sargent, 1991](#); [Lippi and Reichlin, 1993, 1994](#)). Third, factor techniques make it possible to estimate the common components of the observed variables and, consequently, to account for measurement errors. These advantages have made factor models an important tool for empirical macroeconomic analysis.¹

An empirical regularity in the factor model literature is that the number of latent factors, say r , is strictly greater than the number of macroeconomic shocks, say q , so that the factors are *dynamically singular*. Now, a few properties of singular time series, including invertibility and cointegration, are rather different from the corresponding properties of nonsingular series, and this is a potential source of misunderstanding. The aim of the present paper is to clarify such properties and to highlight their consequences for the estimation of factor models.

Singular $I(1)$ series are necessarily cointegrated, with cointegration rank at least $r - q$.

¹A partial list of macroeconomic applications includes [Giannone et al. \(2004\)](#); [Eickmeier \(2009\)](#); [Forni and Gambetti \(2010\)](#); [Alessi et al. \(2011\)](#); [Barigozzi et al. \(2014\)](#); [Forni et al. \(2014\)](#); [Luciani \(2015\)](#); [Juvenal and Petrella \(2015\)](#); [Dahlhaus \(2017\)](#); [Alessi and Kerssenfischer \(2019\)](#); [Barigozzi and Luciani \(2023\)](#); [Forni et al. \(2025\)](#); [Brignone and Mazzali \(2025\)](#); [Granese \(2026\)](#); [Brianti et al. \(2026\)](#).

A researcher might therefore conclude that the first differences of the factors do not admit a VAR representation, so that the standard estimation procedure (SP), which relies on fitting such a model, is invalid. This is not necessarily the case. In this paper, we stress that when the cointegration rank is exactly $r - q$, a VAR representation for the differenced factors exists under very mild conditions, and its order is finite.² Cointegration arising from singularity, therefore, does not prevent us from using the standard procedure SP. Problems arise only when the cointegration rank exceeds $r - q$, which happens only if some linear combination of the shocks has transitory effects on *all* the variables of the panel. In this case SP is no longer valid and we must resort to the alternative estimation procedures proposed by [Barigozzi et al. \(2021\)](#) (BLL henceforth): a VAR for the factors in levels (APVAR) and an error correction model (APECM).

We argue that a cointegration rank exceeding $r - q$ is implausible in large macroeconomic data sets including a wide variety of variables, both on theoretical grounds and on the basis of a simulated DSGE economy. A simple way of checking it in practice consists in comparing the number of common shocks with the number of common permanent shocks, as suggested in BLL. Here we apply this test to two authoritative large US macroeconomic data sets, FRED-QD and FRED-MD: in both cases we find two shocks and two permanent shocks, thus ruling out any additional cointegrating relations.

We then perform a Monte Carlo exercise based on the DSGE model of [Justiniano et al. \(2010\)](#) to compare the small-sample performance of the competing procedures: SP, on the one hand, and the alternative methods proposed in BLL, APVAR and APECM, on the other hand. When the cointegration rank is $r - q$, the standard procedure outperforms both alternatives, especially at longer horizons. This is not surprising: APECM and APVAR inherit the features of ECM and level-based VAR: ECM uses generated regressors, which increase estimate uncertainty, whilst the level-based VAR produces inconsistent estimates

²This result is known in the literature, see e.g. [Barigozzi et al. \(2020, Remark 5\)](#), but its implications for empirical research have been overlooked, with the relevant exception of [Forni et al. \(2026\)](#).

of the long-run effects.

Our recommendation is therefore to use SP, unless the test points to a rank reduction at frequency zero, implying that SP is not consistent, whereas APVAR and APECM are.

The above arguments and results demonstrate that the Dynamic Factor Model is essentially immune to estimation issues arising from cointegration, as far as structural macroeconomic analysis is concerned. This is an advantage over structural VARs, for which cointegration precludes estimation using variables in difference.

The paper is organized as follows. Section 2 studies singularity, cointegration and the existence of a VAR representation. Section 3 introduces the Structural Dynamic Factor Model, presents the competing estimation procedures, and discusses the conditions under which each should be preferred. Section 4 describes the Monte Carlo exercise and reports the simulation results. Section 5 illustrates the empirical application. Section 6 concludes.

2 Singularity and cointegration

In this section we introduce the notion of singular vector process and show that singular processes are always cointegrated. Surprisingly, despite this, we may have a VAR for the first differences, unlike what happens for nonsingular vector processes. Singular processes are nothing more than a curiosity in the field of structural VARs, but they arise naturally in DSGE models and structural factor models. We begin by showing a simple example illustrating our point, then we generalize results to the much broader class of vector ARIMA processes.

2.1 An elementary example

Consider the variables

$$\Delta F_{1t} = u_t, \quad \Delta F_{2t} = u_{t-1} + (1 - L)v_t,$$

where u_t and v_t are unit variance white noise processes, mutually orthogonal at all leads and lags. Here ΔF_{1t} and ΔF_{2t} are nonsingular: singularity will be introduced below. Clearly, F_{1t} and F_{2t} are integrated of order one, since their first difference is stationary (whilst they are not stationary). Now let us consider the linear combination $\Delta F_{1t} - \Delta F_{2t}$. We have

$$\Delta(F_{1t} - F_{2t}) = (1 - L)u_t - (1 - L)v_t.$$

Integrating both sides we get

$$F_{1t} - F_{2t} = c + u_t - v_t,$$

where c is an arbitrary constant. Since F_{1t} and F_{2t} are $I(1)$ and the difference $F_{1t} - F_{2t}$ is stationary, F_{1t} and F_{2t} are cointegrated. Cointegration implies (and is implied by) the existence of a common trend representation:

$$\begin{aligned}\Delta F_{1t} &= u_t \\ \Delta F_{2t} &= u_t - (1 - L)u_t + (1 - L)v_t.\end{aligned}$$

where the common trend is $T_t = T_{t-1} + u_t$.³ Put another way, there is just one shock having permanent effects, namely u_t , whereas the other shock, v_t , is transitory (indeed, it affects temporarily ΔF_{2t} , whereas ΔF_{1t} is not affected at all).

Now, the first differences have the vector MA representation

$$\begin{pmatrix} \Delta F_{1t} \\ \Delta F_{2t} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ L & 1 - L \end{pmatrix} \begin{pmatrix} u_t \\ v_t \end{pmatrix}.$$

Indeed, this is the Wold representation. The MA matrix is singular for $L = 1$, since its determinant is $(1 - L)$.⁴ As a consequence, the Wold representation is not invertible, so that a VAR representation for ΔF_{1t} and ΔF_{2t} does not exist. This is a special case of a well-known general result due to [Engle and Granger \(1987\)](#): cointegration implies that we do not have a VAR for the first differences of the variables, so that estimating a VAR in

³This is the bivariate Beveridge and Nelson representation.

⁴This implies that the spectral-density matrix of ΔF_{1t} and ΔF_{2t} is singular at frequency zero (whilst having maximum rank 2 almost everywhere in $(-\pi, \pi]$).

differences is not appropriate and we should instead estimate either a VAR for the levels or a Vector Error Correction Mechanism.

Let us now consider a variant of the above example, where the variables are singular.

Definition of singularity. *A stationary vector process is ‘dynamically singular’ or simply ‘singular’ if the variance-covariance matrix of the Wold residuals is singular. An $I(1)$ vector process is singular if its first difference is singular.*

The factors are only driven by u_t and v_t is no longer there:

$$\Delta F_{1t} = u_t, \quad \Delta F_{2t} = u_{t-1}.$$

In this case, the Wold representation is⁵

$$\begin{pmatrix} \Delta F_{1t} \\ \Delta F_{2t} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ L & 1 \end{pmatrix} \begin{pmatrix} u_t \\ 0 \end{pmatrix}. \quad (1)$$

The variance-covariance matrix of the Wold residuals is singular, implying that the spectral density matrix is singular everywhere in the interval $(-\pi, \pi]$. In brief, the two variables are singular.

Let us observe first that the two variables are still cointegrated. Indeed, they are still $I(1)$, whereas their difference is stationary:

$$F_{1t} - F_{2t} = c + u_t.$$

The common trend representation is now

$$\begin{aligned} \Delta F_{1t} &= u_t \\ \Delta F_{2t} &= u_t - (1 - L)u_t, \end{aligned}$$

where the trend component is the same as before. A simple intuition for cointegration and common trends is that there cannot be more than one permanent shock, since there is just

⁵To see this, consider that, on the one hand, the projection of ΔF_{1t} on past information is zero (since ΔF_{1t} is serially uncorrelated and uncorrelated with the past of ΔF_{2t}) and the residual is $\Delta F_{1t} = u_t$. On the other hand, ΔF_{2t} is predictable without error since it is equal to ΔF_{1t-1} ; hence the projection of ΔF_{2t} on past information is $\Delta F_{1t-1} = u_{t-1}$ and the residual is zero.

one shock in the model. In the next subsection we show that cointegration is a general feature for singular processes.

Surprisingly, the MA matrix in (1) is invertible, since its determinant does not have a unit root (indeed, the determinant is 1)! Inverting the matrix we see that, despite cointegration, ΔF_{1t} and ΔF_{2t} are related by the VAR(1) relation

$$\begin{pmatrix} \Delta F_{1t} \\ \Delta F_{2t} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \Delta F_{1t-1} \\ \Delta F_{2t-1} \end{pmatrix} + \begin{pmatrix} u_t \\ 0 \end{pmatrix},$$

which can be estimated consistently by OLS.⁶

How is this possible? The point is that the result in [Engle and Granger \(1987\)](#) is valid only for non-singular processes. For singular processes, it does not hold.⁷ We have instead a general result by [Anderson and Deistler \(2008\)](#), which is reported below.

2.2 Singularity implies cointegration

Let us now show that the entries of a singular $I(1)$ vector process F_t are always cointegrated. We shall limit ourselves to the case of an ARIMA process, but the argument can be generalized. Let $f_t = \Delta F_t$ be the stationary r -dimensional vector ARMA process

$$H(L)f_t = K(L)u_t, \tag{2}$$

where $H(L)$ is an $r \times r$ polynomial matrix whose determinant vanishes only outside the unit circle, $K(L)$ is an $r \times q$ “tall” polynomial matrix with $r > q$ and u_t is a q -dimensional white noise process. Clearly f_t has the MA representation

$$f_t = H(L)^{-1}K(L)u_t = B(L)u_t, \tag{3}$$

where $B(L)$ is an $r \times q$ matrix of rational impulse-response functions.

Now, singularity of F_t implies cointegration of its entries. A simple intuition for this

⁶For a proof of consistency see [Forni et al. \(2026\)](#).

⁷In particular, point (1) of the Granger representation theorem, as stated in [Engle and Granger \(1987\)](#), namely the presence of a unit root in the determinant of the Wold MA matrix, does not hold. We have seen this for our example.

fact is the one already proposed for the elementary example above: the vector F_t cannot be affected by more than q permanent shocks, since there are only q shocks affecting F_t .⁸ A formal argument is the following: if $B(L)$ is tall as in the present case, there must be a non-zero vector α , belonging to the null space of the columns of $B(1)$, such that $\alpha'B(1) = 0$. This implies that $\alpha'B(L)$ vanishes for $L = 1$ and therefore it admits the factorization $\alpha'B(L) = (1 - L)b(L)$. Hence pre-multiplying (3) by α' and integrating both sides we get

$$\alpha'F_t = \mu + b(L)u_t,$$

where μ is an arbitrary constant. Since $\alpha'F_t$ is stationary, the entries of F_t are cointegrated. Notice that the cointegration rank is at least $r - q$, since the null space of the columns of $B(1)$ has dimension not smaller than $r - q$. We have therefore the following result.

Fact 1. *Singular ARIMA $I(1)$ processes are always cointegrated.*

In principle, some of the q shocks might have transitory effects on all of the entries of F_t ; in this case, the matrix $B(1)$ has reduced rank $\tau < q$; we have just $\tau < q$ permanent shocks, so that the cointegration rank will be $r - \tau > r - q$. We can therefore distinguish between two different sources of cointegration: *cointegration arising from singularity*, on the one hand, and cointegration arising from transitory shocks, on the other hand. The former is necessarily there and is responsible for exactly $r - q$ independent cointegrating relations, whereas the latter may or may not be there and, if it is there, it will be responsible for additional $q - \tau$ cointegrating relations.

2.3 Existence of a VAR representation

To begin, let us introduce the concept of zerolessness.

Definition of zerolessness. *The $r \times q$ matrix $A(L)$, with $r \geq q$, is “zeroless” if the rank of $A(z)$ is q for all complex numbers z .*

⁸The spectral density matrix of f_t has reduced rank q everywhere; therefore, it cannot have rank larger than q at frequency zero.

Note that, if $r = q$, $A(L)$ is zeroless if and only if it has a nonzero constant determinant, a very special case. On the other hand, if $r > q$, as is the case for singular processes like f_t , a sufficient condition for zerolessness of $A(L)$ is that it contains at least two $q \times q$ submatrices whose determinants have no common zeros, a condition which seems very mild. Clearly in equation (3) $B(L) = H(L)^{-1}K(L)$ is zeroless if and only if $K(L)$ is zeroless. To see this, consider that $H(z)^{-1}$ is non singular everywhere in the complex plane,⁹ so that the rank of $B(z)$ is equal to the rank of $K(z)$.

The following crucial result is an immediate application to our setting of an important theorem in the theory of polynomial matrices, see [Anderson and Deistler \(2008\)](#):

Fact 2. *Invertibility of $K(L)$. If $K(L)$ in equation (3) is zeroless, then it has a left inverse, i.e. then there exists an $r \times r$ stable polynomial matrix $\tilde{K}(L)$, with $\tilde{K}(0) = I_r$, such that $\tilde{K}(L)K(L) = K_0 = B_0$.*

Notice that $\tilde{K}(L)$ is a (finite degree) *polynomial* matrix. An immediate consequence of Fact 2 is that setting $C(L) = \tilde{K}(L)H(L)$, f_t has the *finite-order* VAR representation

$$C(L)f_t = B_0u_t = e_t, \quad (4)$$

where $C(0) = I_r$ and $B_0 = K_0$ is the matrix of the impact effects of u_t on f_t .

The existence of representation (4) implies that, for singular variables, global fundamentality (i.e. the existence of a VAR representation in the structural shocks) is a much milder condition than in the square, nonsingular case. This fact has been highlighted in several papers (see the references at the beginning of the following section).

Here, however, we are interested in unit roots and cointegration rather than fundamentality. Clearly, if some linear combination of the shocks in u_t has transitory effects on all entries of F_t , then a linear combination of the columns of $K(z)$ vanishes for $z = 1$ and $K(1)$ has reduced rank $\tau < q$, so that $K(L)$ and $B(L)$ are not zeroless. In this case Fact 2 does not apply. On the other hand, if $K(L)$ is zeroless we have a finite-order VAR representation

⁹The inverse of a polynomial matrix has no finite zeros.

for f_t .¹⁰ This leads us to the first message of the present paper:

Message 1. *Cointegration arising from singularity does not prevent the existence of a VAR for the first differences of the cointegrated variables.*

Put differently, the presence of $r - q$ cointegrating relationships does not prevent the existence of a VAR for the first differences of the cointegrated variables. Problems arise when the number of cointegrating relationships is greater than $r - q$, since, in this case, $K(1)$ has rank less than q and the zerolessness condition does not hold.

In light of Message 1, determining the number of cointegration relationships becomes crucial. A testing procedure is discussed in the next section.

Singularity is a very special case of limited interest in the context of Structural VARs, but it arises quite naturally in the Structural Dynamic Factor Model, presented in the following section.

3 The Structural Dynamic Factor Model

The results of the previous section apply to any singular vector process. They become relevant for empirical macroeconomics because singularity arises naturally in the Structural Dynamic Factor Model, introduced by [Stock and Watson \(2005\)](#) and [Forni et al. \(2009\)](#), where a large panel of time series is driven by a much smaller number of common shocks. In this Section we present the model, the estimation procedures that have been proposed for it, and the test which can be used to choose between them. The treatment here follows [Forni et al. \(2009\)](#) and [Forni et al. \(2026\)](#). While we focus here on structural macroeconomic analysis, our discussion applies to forecasting and financial applications as well.

¹⁰Notice that in the square, nonsingular case, a finite-order MA process cannot have a finite-order VAR representation: invertibility of the MA matrix implies an infinite-order VAR representation. This is not the case for dynamically singular processes.

3.1 Representation

Let x_t be a large, n -dimensional vector of observable macroeconomic variables, transformed to obtain stationarity. The vector x_t is assumed to be part of an infinite-dimensional panel of time series. This is necessary as consistency of the estimators is obtained as both the number of time observations T and the cross-sectional dimension n go to infinity.

Each variable x_{it} , $i = 1, \dots, n$, is decomposed into the sum of two unobservable components, mutually orthogonal at all leads and lags: a common component χ_{it} and an idiosyncratic component ξ_{it} . The idiosyncratic components are specific to individual variables or small groups of variables; they can be interpreted as sector-specific shocks or measurement errors. In contrast, the common components account for most of the co-movement among macroeconomic variables, as they are driven by the same r latent factors, $f_t = (f_{1t} \dots f_{rt})'$.

Formally, x_t admits the representation:

$$x_t = \chi_t + \xi_t = \Lambda f_t + \xi_t \quad (5)$$

where χ_t is the vector of the common components, ξ_t is the vector of the idiosyncratic components and Λ is a $n \times r$ matrix of factor loadings. Equation (5) provides the so called “static-factor” representation, where the latent factors influence the common components contemporaneously. The dynamic nature of the model is introduced by assuming that f_t follows the ARMA process:

$$H(L)f_t = K(L)u_t, \quad (6)$$

where $H(L)$ and $K(L)$ are polynomial matrices of dimension $r \times r$ and $r \times q$, respectively, and u_t is a q -dimensional vector of structural shocks (often referred to as “dynamic factors”), mutually orthogonal at all leads and lags.

Combining equations (5) and (6) we get the structural dynamic representation:

$$x_t = \Lambda H(L)^{-1} K(L) u_t + \xi_t = B(L) u_t + \xi_t. \quad (7)$$

where the macroeconomic variables are represented as being driven by a few pervasive structural shocks, with impulse response functions given by $B(L)$, plus measurement error.

Now let us assume that x_t is the first difference of the $I(1)$ vector process X_t and, similarly, $f_t = \Delta F_t$ and $\xi_t = \Delta \Xi_t$. Clearly we have for the levels in X_t the factor model representation

$$X_t = \mu + \Lambda F_t + \Xi_t, \quad (8)$$

where μ is a vector of constants. Notice that the factor loadings are the same of (5).

There is by now a good deal of evidence that, for large macroeconomic data sets, the number of shocks is smaller than the number of static factors in f_t . Hence in the sequel we assume $q < r$. This implies that $K(L)$ is “tall” and the vector f_t is dynamically singular.¹¹

3.2 Estimation

How can we estimate the above model? Let us begin by considering the case in which $K(L)$ is zeroless. In the sequel we consider the case in which $K(1)$ has reduced rank $\tau < q$, a case which clearly does not meet the zeroless condition.

If $K(L)$ is zeroless, Fact 2 introduced in the previous section applies and we have, for the factors in f_t , the finite-order VAR representation (4). As a consequence, the structural shocks are fundamental for the factors, so that the SDFM, unlike SVARs, is not affected by the nonfundamentality problem highlighted by [Lippi and Reichlin \(1993, 1994\)](#). Moreover, we can consistently estimate the structural shocks and the associated IRFs by means of the following simple estimation procedure, which we call the Standard Procedure (SP):

SP1. Determine r with existing criteria and compute the first r principal components of the standardized observables in x_t , along with the associated eigenvectors, the columns of

$\hat{\Lambda}$. Such principal components, say $\hat{f}_t = \hat{\Lambda}'x_t$, are consistent estimators of the factors

¹¹The idiosyncratic components provide a quite natural explanation for the empirical fact that VAR models for the observables in x_t are not singular.

f_t , while $\hat{\Lambda}$ is a consistent estimator of the factor loading matrix Λ .¹²

SP2. Estimate the VAR (4) with $\hat{f}_t$ to get $\hat{C}(L)$, and get an estimate of B_0 (or a single column of B_0) by imposing suitable identification criteria. Estimate the structural IRFs as $\hat{B}(L) = \hat{\Lambda}\hat{C}(L)^{-1}\hat{B}_0$.

Consistency of the above estimation procedure under the assumption of zerolessness (along with a few minor assumptions) is proved in [Forni et al. \(2026\)](#). Notice that SP does not require estimation of q .¹³ We emphasise once again that consistency of SP holds even if the factors in F_t are cointegrated, provided that $K(L)$ is zeroless.

Now consider the case in which the rank of $K(1)$ is $\tau < q$. This means that the cointegration rank of X_t is $r - \tau > r - q$, or, equivalently, that the factors F_t are driven by τ common trends, with $\tau < q$. In this case existence of a VAR representation for f_t is no longer guaranteed. However, we have the following Alternative Procedure (AP), proposed in BLL:

AP1. Proceed as in SP1 to get an estimate of the factor loadings Λ . Estimate F_t as $\hat{F}_t = \hat{\Lambda}'X_t$ (recall that $\hat{f}_t = \hat{\Lambda}'x_t$).

AP2. Estimate the IRFs either by estimating a VAR for $\hat{F}_t$ (APVAR) or an ECM (APECM).

See [Barigozzi et al. \(2021\)](#) for additional details.

Consistency of AP under suitable assumptions (including the case $\tau = q$) is proved in BLL.

¹²Indeed, the static factors are not identified, so that $\hat{f}_t$ is a consistent estimator for the linear space spanned by f_t . However, $\hat{\Lambda}\hat{f}_t$ is a consistent estimator of Λf_t , which is unique.

¹³The above procedure is a simplification of the one proposed in [Forni et al. \(2009\)](#). The latter involves a further step, consisting of imposing the condition $q < r$ on the variance-covariance matrix of the VAR residuals e_t . This step makes the estimation more efficient if q is estimated correctly, but it complicates the estimation process and entails a risk associated with the possible underestimation of q . For further details, see [Forni et al. \(2026\)](#).

3.3 Testing for the rank of $K(1)$

Clearly, if $K(1)$ has reduced rank, we should use AP in place of SP, since SP is not consistent. We can verify whether $K(1)$ has rank $\tau < q$ as suggested in BLL. First, we estimate q by means of existing criteria, for instance the one of [Hallin and Liška \(2007\)](#) (HL) or the DDR criterion of [Avarucci et al. \(2026\)](#). Then we use the same criteria, evaluated at frequency zero, to estimate τ . Precisely, we estimate the rank of the spectral density matrix of f_t at zero frequency; see BLL for the frequency-zero version of HL. If $\tau < q$ we conclude that $K(1)$ has reduced rank; otherwise we conclude that the rank is maximal.¹⁴

This test refers to the rank of the matrix $K(z)$ for $z = 1$. In fact, zerolessness means that the rank of $K(z)$ is maximum everywhere in the complex plane. However, the case of a rank reduction in $z = 1$ seems the only one where rank reduction may have a compelling economic motivation. On the contrary, the case of a reduced rank $K(z)$ for $z \neq 1$ seems rather implausible by the ‘genericity’ argument below.

3.4 The rank of $K(1)$ for macroeconomic data

In this subsection we argue that the case of a non-zeroless $K(L)$ is rather implausible for macroeconomic data sets, so that the standard procedure can be safely used.

Our first argument arises quite naturally if we assume that the parameters in $K(L)$ are free to vary in $\mathbb{R}^{rps}$, s being the order of $K(L)$. In this case it is easily seen that $K(L)$ is *generically* zeroless, i.e. it is zeroless with the exception of a lower dimensional subset of the parameter space (as already observed above, a sufficient condition is that $K(L)$ contains at least two $q \times q$ submatrices whose determinants have no common zeros). If we assume a diffuse prior for the parameters, the case of $K(1)$ having reduced rank has zero probability. For a more thorough discussion on this point see [Forni et al. \(2026\)](#), Section 3.2.1.

But of course the parameters of $K(L)$ can be functions of the “deep parameters” of a

¹⁴An alternative approach would be the unit root test for factors of [Bai and Ng \(2004\)](#). However, as observed in [Barigozzi et al. \(2021\)](#), footnote 11, that test is designed only for the non-singular case $r = q$.

macroeconomic model: the lower dimensional subset where $K(L)$ is not zeroless could in principle be economically relevant. Since cointegration seems to be the only compelling case for the failure of zerolessness, in the sequel we discuss rank reduction of $K(z)$ for $z = 1$.

A reduced rank of $K(1)$ implies that there is at least one linear combination of the columns of $K(L)$ having the factor $(1 - L)$ for all rows. Put another way, there would be at least one linear combination of the shocks having transitory effects on all entries of F_t ; such a shock would necessarily have transitory effects on all of the $I(1)$ variables in X_t .

Now, it is generally accepted that a monetary policy shock or a demand shock has temporary effects on real activity variables such as consumption, investment or GDP (provided that hysteresis effects are not there). However, there is no theoretical reason to think that such shocks have transitory effects on nominal $I(1)$ variables, as are usually considered prices or monetary aggregates. Moreover, there is no economic reason to think that variables which are regarded as stationary in economic theory, like interest rates, unemployment rates, risk premia or term spreads have unit roots in their IRFs.

Indeed, DSGE models do not produce variables whose $B(1)$ has reduced column rank. For instance, in the following section we simulate a data set based on the model of [Justiniano et al. \(2010\)](#). Our data set includes inflation and the interest rate; for such variables we shall see that the IRFs of the monetary policy shock do not have a unit root.

In [Appendix D](#) we report an example illustrating the above idea.

Finally, we produce an empirical argument. In the application below we apply the testing procedures described in the previous subsection to two widely used macroeconomic datasets, namely, the FRED-QD and the FRED-MD data sets, see [McCracken and Ng \(2016, 2020\)](#). We find that, for both data sets, $K(1)$ has maximum rank.

To carry out the testing procedure we recommend avoiding over-differencing: prices should be taken in first differences; interest rates should be taken in levels. The reason is simple: if we take the first differences of variables which are already stationary we induce

spurious unit roots in the data and this could distort the test result. Simulation A1 in the Appendix A illustrates this point.

Summing up, our second message is the following.

Message 2. *The case of $K(1)$ being of reduced rank is rather implausible for macroeconomic data sets including a wide variety of macroeconomic variables, so that the zeroless condition reasonably holds true and the Standard Procedure can be safely used.*

Message 2 should not be interpreted as suggesting that there are no empirically relevant data sets where $K(1)$ has reduced rank. For instance, in a panel made only of the industrial production indices of different sectors, or of the GDPs of different states, a demand shock might well have only transitory effects on all the variables; Simulation A2 in Appendix B provides an example of this kind. As a further example, a financial data set comprising the share prices of many listed companies could, in principle, be characterised by a number of cointegrating relationships greater than $r - q$. When there is uncertainty about the rank of $K(1)$, the test described above becomes decisive in choosing the estimation method.

3.5 If $K(L)$ is zeroless, should we use SP or AP?

If both SP and AP are consistent, which procedure should be preferred when analyzing macroeconomic data? We argue that the Standard Procedure (SP) is preferable.

Regarding APECM, it requires a preliminary estimate of τ , which may be subject to substantial uncertainty. If $\hat{\tau} \neq \tau$, the resulting estimates may be misleading. Moreover, even if τ is correctly specified, APECM requires the preliminary estimation of the long-run relationships, thereby introducing an additional source of estimation uncertainty.

As for APVAR, it inherits some of the limitations of VAR models with $I(1)$ variables: estimates of long-run effects are not consistent and may exhibit substantial finite-sample bias.

Neither of these issues affects SP.

To assess the small-sample performances of SP, APECM and APVAR, in the next section we carry out a simulation that supports our arguments.

4 Simulation

For our simulation exercise we use the medium-scale DSGE model of [Justiniano et al. \(2010\)](#). The model features seven shocks, i.e., $q = 7$, and is equipped with all frictions that are considered necessary to capture the persistence of macro data.

4.1 Design

We focus on the ten variables Interest rate, GDP, Consumption, Investment, Hours worked, Wage inflation, Price inflation, Potential GDP, Labor productivity, Marginal cost. The five variables GDP, Consumption, Investment, Potential GDP and Labor productivity are $I(1)$; the remaining five are stationary.

To estimate the factor model, we need a large cross section. We build our large dataset as follows. Our $I(1)$ variables are taken in first differences and normalized to have unit variance. The resulting series are used as the first five factors in f_t . We take 55 linear combinations of the above factors, with coefficients uniformly distributed between -1 and +1. These series can be interpreted as additional macroeconomic, financial or survey series conveying information on the growth rate of real macroeconomic activity. The other five stationary variables are then standardized and taken as the final five entries of the vector f_t . Then we take 55 linear combinations of such factors, with coefficients uniformly distributed between -1 and +1. These latter series can be interpreted as additional macroeconomic, financial or survey series conveying information about aspects of the macroeconomy like unemployment, interest rates, inflation and wages.

Our common components χ_{it} are made up by the original ten series produced by the

DSGE model, plus the 110 additional series, so that we have $n = 120$ series. The final observables x_{it} 's are obtained by adding to χ_{it} , $i = 1, \dots, 120$, the independent random normal measurement error ξ_{it} , with mean zero and variance equal to 10% of the variance of χ_{it} . We set $T = 250$ (about sixty years of quarterly data). Notice that we have ten factors, $r = 10$. Since $q = 7$, the factors are dynamically singular. For this data set, $K(L)$ is zeroless and the cointegration rank is $r - q = 3$.

To apply the Alternative Procedures, the $I(1)$ variable X_{it} is simply obtained as the cumulated sum of the corresponding stationary variable x_{it} , even if such variable does not have necessarily an economic interpretation. This ensures that the estimated static factors $\hat{f}_t$ are the same for both SP and AP.¹⁵

We generate 1000 data sets.

Having our artificial data we estimate the IRFs of the monetary policy shock using SP, APECM and APVAR. The last two estimates are obtained by using the codes of BLL, but avoiding preliminary detrending since in our case the X_{it} 's do not contain linear trends. Both r and q , as well as the cointegration rank, are assumed known. For all methods we use four lags of the data. To identify the unit variance monetary shock, say u_t , we assume to have a proxy z_t given by the sum of u_t plus an independent error given by $0.5e_t$, e_t being standard normal. Identification is obtained using the standard external Proxy-SVAR procedure (Mertens and Ravn, 2013; Forni et al., 2026) and normalizing the IRFs to have unit impact effect on the interest rate as suggested in Stock and Watson (2018).

4.2 Results

Figure 1 shows the results of the four variables Interest rate, Output, Hours and Prices. The black solid lines are the true IRFs. Notice that neither the IRF of the Interest rate nor the one of hours and inflation contain the factor $(1 - L)$, since in both cases the sum of the

¹⁵Detrending the $I(1)$ series is not needed here since the x_{it} 's are zero-mean. Applying detrending prior to estimation would reasonably enlarge estimate uncertainty.

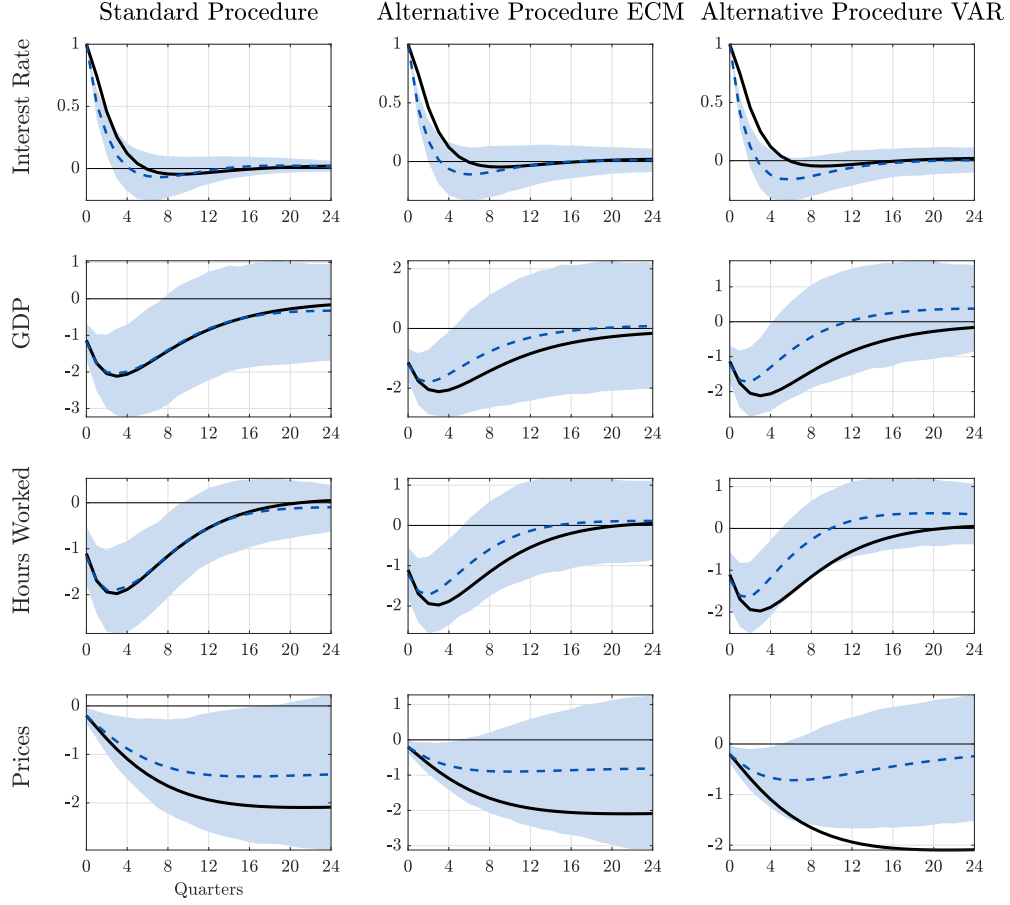


Figure 1: Impulse responses to a monetary policy shock

Note: estimated impulse responses of the Interest rate, GDP, Hours worked and Prices to a monetary policy shock. The first column reports the Standard Procedure (SP), the second the Alternative Procedure based on a VECM (APECM), the third the Alternative Procedure based on a VAR in levels (APVAR). Black solid lines are the true impulse-response functions, blue dashed lines the averages across replications, and the shaded areas the 16th and 84th percentiles of their distribution. Stationary variables are taken in levels and $I(1)$ variables in first differences, so that $K(1)$ is zeroless and the cointegration rank is $r - q$.

effects is not null. This confirms that standard DSGE models do not imply $\tau < q$, as argued in the previous Section.

The gray areas represent the 16-th and the 84-th percentiles of the distribution of the estimated IRFs and therefore provide a measure of the volatility of the estimates. The blue dashed lines are the averages of the estimated IRFs. A comparison with the black lines gives an idea of the small sample bias of the estimates. The first column shows SP estimates; the second column shows APECM estimates and the third column shows the APVAR estimates. A simple inspection should be sufficient to convince the reader that, as argued above, SP outperforms both APECM and APVAR in terms of both bias and volatility.

To better quantify the relative performance of the competing methods, Table 1 reports the Mean Square Errors (MSE) for selected lags of the estimated IRFs. It is seen that SP outperforms the competing methods, the difference being large for horizons 12 to 24.

Table 1: Mean squared errors of the estimated impulse–response functions

Horizon	0	3	6	9	12	15	18	21	24
<i>Interest rate</i>									
SP	0.00	0.06	0.04	0.02	0.01	0.01	0.01	0.00	0.00
APECM	0.00	0.08	0.06	0.04	0.03	0.02	0.01	0.01	0.01
APVAR	0.00	0.13	0.06	0.03	0.02	0.02	0.02	0.01	0.01
<i>GDP</i>									
SP	0.28	1.56	2.24	2.45	2.40	2.28	2.16	2.06	1.99
APECM	0.31	1.80	2.98	3.70	4.08	4.17	4.09	3.94	3.77
APVAR	0.32	1.64	2.85	3.31	3.27	2.99	2.68	2.39	2.13
<i>Hours worked</i>									
SP	0.42	0.97	1.00	0.88	0.73	0.61	0.51	0.42	0.35
APECM	0.46	1.14	1.37	1.34	1.24	1.11	0.96	0.81	0.68
APVAR	0.48	1.10	1.62	1.65	1.41	1.11	0.89	0.74	0.64
<i>Prices</i>									
SP	0.03	0.33	0.82	1.36	1.85	2.25	2.57	2.83	3.05
APECM	0.03	0.40	1.14	2.06	2.96	3.70	4.28	4.75	5.16
APVAR	0.03	0.37	1.14	2.17	3.16	3.93	4.49	4.88	5.15

Note: mean squared errors of the estimated impulse–response functions to the monetary policy shock, at selected horizons, obtained with the Standard Procedure (SP), the Alternative Procedure based on a VECM (APECM) and the Alternative Procedure based on a VAR in levels (APVAR). Simulated data from the model of [Justiniano et al. \(2010\)](#) with $r = 10$, $q = 7$, $T = 250$; 1000 replications; four lags for all methods.

Further work is needed to assess more fully the relative performances of the competing procedures for finite samples. Nevertheless we think that we have enough evidence to reach a provisional recommendation.

Message 3. *Use the Standard Procedure for structural macroeconomic analysis, unless there are special reasons to think that the zeroless condition fails or the testing procedure described above points to a rank reduction of the spectral density of the factors f_t at zero frequency.*

In Appendices A and B we report two additional simulations. The former, Simulation A1, makes use of the same large data set above, but presents a different data treatment: namely, stationary series are taken in first differences. Over-differencing artificially produces a lot of unit roots, so that $K(L)$ is no longer zeroless. In this case, SP is no longer consistent and performs worse than both APECM and APVAR. However, both AP methods have larger errors than the corresponding ones of Table 1. This supports our recommendation to avoid over-differencing.

The second additional simulation, Simulation A2, concerns the effects of the preference shock on GDP, Consumption and Investment. The data set is made up only of trended real activity variables, driven by just one common trend (technology). Now $K(L)$ is not zeroless, since $K(1)$ has rank 1. Here SP does not work properly and we must resort to the Alternative Procedures.

5 Empirical application

In this section we apply the testing procedure described in Section 3.3 to two widely used large macroeconomic data sets: FRED-QD and FRED-MD (see McCracken and Ng, 2016, 2020). For both data sets we consider the sample 1960–2019, so as to exclude the COVID period; this leaves $n = 216$ quarterly series with $T = 239$ and $n = 122$ monthly series with $T = 718$. Following Avarucci et al. (2026), interest rates are taken in levels rather than in first differences, and prices and other nominal variables are taken in log-differences rather

than in double differences of logs. As observed in Section 3.4, differencing variables that are already stationary induces spurious unit roots, which is exactly the kind of artifact the test is meant to detect; it also amplifies high-frequency variation, which are of little interest for macroeconomic analysis.

5.1 Results

Table 2 reports the Dynamic Difference Ratio (DDR), Dynamic Eigenvalue Ratio (DER) and Dynamic Growth Ratio (DGR) criteria of Avarucci et al. (2026). These criteria count the eigenvalues of the spectral density matrix that diverge with n , and can be evaluated either on the whole spectrum or on a frequency band. We apply them to the whole interval $(-\pi, \pi]$, which delivers the total number of shocks q , and to frequency zero, which delivers the number of permanent shocks τ . We also report results for two bands. The first, for quarterly data, is $[0, 2\pi/6]$, which excludes fluctuations of less than six quarters: these are of little interest for macroeconomic analysis and are the frequencies most affected by measurement errors. The second is $[0, 2\pi/80]$, corresponding to waves of periodicity larger than twenty years, sometimes used in the literature to identify long-run components, see Angeletos et al. (2020); it is a natural complement to frequency zero, since Avarucci et al. (2026) document that DDR is somewhat unstable when evaluated at single Fourier frequencies. For the monthly data set the two bands are $[0, 2\pi/18]$ and $[0, 2\pi/240]$. All three criteria select two shocks and two permanent shocks, in both data sets and in every band.

Table 3 reports the criterion of Hallin and Liška (2007) (HL), which estimates q , together with the same criterion evaluated at frequency zero, as proposed by Barigozzi et al. (2021) (BLL), which estimates τ . In both criteria the penalty constant is calibrated as in HL, by selecting the number of factors that remains stable across random subsamples of the cross-section over an interval of values of the constant. Following the implementation of BLL, we report results for a large and a small window, which differ in the minimum length

Table 2: DDR, DER and DGR criteria: number of common shocks and common trends

Frequency band	FRED-QD			FRED-MD		
	DDR	DER	DGR	DDR	DER	DGR
All frequencies	2	2	2	2	2	2
Excluding high frequencies	2	2	2	2	2	2
Frequency zero	2	2	2	2	2	2
Long run	2	2	2	2	2	2

Note: number of diverging eigenvalues of the spectral density matrix selected by the DDR, DER and DGR criteria of [Avarucci et al. \(2026\)](#). “All frequencies” estimates the number of common shocks q ; “Frequency zero” estimates the number of common trends τ . “Excluding high frequencies” is the band $[0, 2\pi/6]$ for FRED-QD and $[0, 2\pi/18]$ for FRED-MD (periodicities longer than 18 months); “Long run” is the band $[0, 2\pi/80]$ for FRED-QD and $[0, 2\pi/240]$ for FRED-MD (periodicities longer than 20 years). Sample 1960Q2–2019Q4 (FRED-QD) and 1960M3–2019M12 (FRED-MD), bandwidth $M = [0.75\sqrt{T}]$, $q_{\max} = 8$. Since $\tau = q = 2$, no linear combination of the shocks has transitory effects on all the variables, and the zeroless condition holds at $z = 1$.

Table 3: BLL and HL information criteria

Penalty	Large window				Small window			
	FRED-QD		FRED-MD		FRED-QD		FRED-MD	
	BLL	HL	BLL	HL	BLL	HL	BLL	HL
(a)	2	2	2	2	7	7	6	6
(b)	2	2	2	2	7	7	6	6
(c)	2	2	2	2	7	7	6	6
(d)	2	2	2	2	7	7	6	6

Note: number of common trends τ (BLL) and of common shocks q (HL) in the FRED-QD and FRED-MD data sets. q is estimated with the criterion of [Hallin and Liška \(2007\)](#) (HL); τ is estimated by evaluating the same criterion at frequency zero, as proposed by [Barigozzi et al. \(2021\)](#) (BLL). Penalties (a)–(d) are those in [Barigozzi et al. \(2021\)](#). Following the implementation of BLL, the large and small windows require the selected number to be stable over an interval of the penalty constant of length at least 0.1 and 0.01, respectively. Each criterion is computed 50 times with different random draws of the cross-sectional subsamples; we report the modal estimate. Bandwidth $M = [0.75\sqrt{T}]$, $q_{\max} = 8$. Under both windows $\tau = q$.

required for this interval; the small window specification tends to select more factors. Since the outcome depends on the random subsamples, each criterion is computed 50 times and we report the modal estimate. With the large window, both criteria select two factors under all four penalties, in line with Table 2. With the small window, the number of selected shocks is larger—seven for the quarterly and six for the monthly data set—but the number of permanent shocks is the same, so that again $\tau = q$.

The results are robust to the choice of the bandwidth M of the lag-window estimator of the spectral density matrix. Our baseline is $M = [0.75\sqrt{T}]$, the value adopted by HL in their empirical application. Tables 6 and 7 in Appendix C report DDR¹⁶ and the BLL and HL criteria over the grid $M = [c\sqrt{T}]$ with $c \in \{0.5, 0.75, 1, 1.25\}$; the estimates are unchanged across the grid. For the BLL and HL criteria we report the large window only.¹⁷

Our estimate of q coincides with the one obtained by Avarucci et al. (2026) on the same data sets, and two shocks are in line with a long-standing body of evidence on the number of sources of aggregate fluctuations. A possible concern is that FRED-QD includes many similar series within some blocks, such as prices, interest rates and industrial production, which may induce cross-correlation among the idiosyncratic components (Boivin and Ng, 2006) and affect the factor-number criteria. However, Brianti et al. (2026) find two shocks and no rank reduction at frequency zero, using DDR and the HL criterion, on the more balanced data set of Granese (2026), which includes 118 quarterly US series.

We conclude that $\tau = q = 2$: no linear combination of the shocks has transitory effects on all the variables of the panel, so that $K(1)$ has full rank and the zeroless condition holds at $z = 1$. The cointegration rank of the factors is therefore $r - \tau = r - q$, that is,

¹⁶We focus on DDR, which is the preferred estimator of Avarucci et al. (2026) because of its small-sample performance—in their Monte Carlo experiments it dominates DER and DGR, and is the most accurate when the idiosyncratic components are large—and is also the only one they employ in their own empirical analysis.

¹⁷The small window, which accepts stability intervals ten times shorter, confirms $\tau = q$ also at $c = 0.5$. At larger bandwidths, however, its estimates vary across penalties and replications, and the number of permanent shocks in most cases differs from the number of shocks, sometimes even exceeding it, which is impossible by construction.

cointegration arises from singularity alone, as argued in Section 3.4. By Message 1, this is precisely the case in which a finite-order VAR for the differenced factors exists, and the Standard Procedure can be safely used on these data sets.

6 Conclusions

Singular $I(1)$ processes are always cointegrated, but, unlike nonsingular ones, they typically admit a finite-order VAR representation in first differences. Cointegration arising from singularity is therefore not an obstacle to the standard estimation of a structural factor model. An obstacle arises only if some linear combination of the shocks has transitory effects on all the variables of the panel, a case that does not arise in standard DSGE models and is ruled out by the large US data sets we examine. Our recommendation is to check the rank of $K(1)$, which only requires comparing the number of common shocks with the number estimated at frequency zero, and, unless the two differ, to use the standard procedure, which our simulations show to be more accurate than the alternatives based on the levels. In either case, over-differencing should be avoided.

References

- Alessi, L., M. Barigozzi, and M. Capasso (2011). Non-fundamentalness in structural econometric models: A review. *International Statistical Review* 79(1), 16–47.
- Alessi, L. and M. Kerssenfischer (2019). The response of asset prices to monetary policy shocks: Stronger than thought. *Journal of Applied Econometrics* 34(5), 661–672.
- Anderson, B. D. O. and M. Deistler (2008). Generalized linear dynamic factor models—A structure theory. *IEEE Conference on Decision and Control*.
- Angeletos, G. M., F. Collard, and H. Dellas (2020). Business cycle anatomy. *American Economic Review* 110(10), 3030–3070.
- Avarucci, M., M. Cavicchioli, M. Forni, and P. Zaffaroni (2026). Frequency-band estimation of the number of factors. *Journal of the American Statistical Association*, 1–13.
- Bai, J. and S. Ng (2004). A panic attack on unit roots and cointegration. *Econometrica* 72(4), 1127–1177.
- Barigozzi, M., A. M. Conti, and M. Luciani (2014). Do euro area countries respond asymmetrically to the common monetary policy? *Oxford Bulletin of Economics and Statistics* 76(5), 693–714.
- Barigozzi, M., M. Lippi, and M. Luciani (2020). Cointegration and error correction mechanisms for singular stochastic vectors. *Econometrics* 8(1).
- Barigozzi, M., M. Lippi, and M. Luciani (2021). Large-dimensional dynamic factor models: Estimation of impulse–response functions with I(1) cointegrated factors. *Journal of Econometrics* 221(2), 455–482.
- Barigozzi, M. and M. Luciani (2023). Measuring the output gap using large datasets. *The Review of Economics and Statistics* 105(6), 1500–1514.
- Bernanke, B. S., J. Boivin, and P. S. Elias (2005). Measuring the effects of monetary policy: A Factor-Augmented Vector Autoregressive (FAVAR) approach. *The Quarterly Journal of Economics* 120, 387–422.
- Boivin, J. and S. Ng (2006). Are more data always better for factor analysis? *Journal of Econometrics* 132(1), 169–194.
- Brianti, M., M. Forni, L. Gambetti, A. Granese, et al. (2026). Nonlinear business-cycle anatomy. *CEPR Discussion Paper* (21333).

- Brignone, D. and M. Mazzali (2025). It is all about demand and supply: a dualistic view of the euro area business cycle. *Bank of England Staff Working Paper Series* (1,124).
- Dahlhaus, T. (2017). Conventional monetary policy transmission during financial crises: An empirical analysis. *Journal of Applied Econometrics* 32(2), 401–421.
- Eickmeier, S. (2009). Comovements and heterogeneity in the euro area analyzed in a non-stationary dynamic factor model. *Journal of Applied Econometrics* 24, 933–959.
- Engle, R. F. and C. W. J. Granger (1987). Cointegration and error correction: Representation, estimation, and testing. *Econometrica* 55, 251–76.
- Forni, M. and L. Gambetti (2010). The dynamic effects of monetary policy: A structural factor model approach. *Journal of Monetary Economics* 57(2), 203–216.
- Forni, M., L. Gambetti, A. Granese, L. Sala, and S. Soccorsi (2025). An american macroeconomic picture: Supply and demand shocks in the frequency domain. *American Economic Journal: Macroeconomics* 17(3), 311–341.
- Forni, M., L. Gambetti, M. Lippi, and L. Sala (2026). Common components structural vars. *Journal of Business & Economic Statistics* 44(1), 39–52.
- Forni, M., L. Gambetti, and L. Sala (2014). No news in business cycles. *The Economic Journal* 124(581), 1168–1191.
- Forni, M., D. Giannone, M. Lippi, and L. Reichlin (2009). Opening the black box: Structural factor models with large cross sections. *Econometric Theory* 25(5), 1319–1347.
- Giannone, D., L. Reichlin, and L. Sala (2004). Monetary policy in real time. *NBER macroeconomics annual* 19, 161–200.
- Granese, A. (2026). Two main business cycle shocks are better than one. *Journal of Economic Dynamics and Control* 182, 105231.
- Hallin, M. and R. Liška (2007). Determining the number of factors in the general dynamic factor model. *Journal of the American Statistical Association* 102, 603–617.
- Hansen, L. P. and T. J. Sargent (1991). Two difficulties in interpreting vector autoregressions. In L. P. Hansen and T. J. Sargent (Eds.), *Rational Expectations Econometrics*, pp. 77–120. Boulder: Westview Press.
- Justiniano, A., G. E. Primiceri, and A. Tambalotti (2010). Investment shocks and business cycles. *Journal of Monetary Economics* 57(2), 132–145.

- Juvenal, L. and I. Petrella (2015). Speculation in the oil market. *Journal of applied econometrics* 30(4), 621–649.
- Lippi, M. and L. Reichlin (1993). The dynamic effects of aggregate demand and supply disturbances: Comment. *American Economic Review* 83(3), 644–652.
- Lippi, M. and L. Reichlin (1994). Var analysis, nonfundamental representations, blaschke matrices. *Journal of Econometrics* 63(1), 307–325.
- Luciani, M. (2015). Monetary policy and the housing market: A structural factor analysis. *Journal of Applied Econometrics* 30(2), 199–218.
- McCracken, M. and S. Ng (2016). FRED-MD: a monthly database for macroeconomic research. *Journal of Business and Economic Statistics*, 574–589.
- McCracken, M. and S. Ng (2020). Fred-qd: A quarterly database for macroeconomic research. Technical report, National Bureau of Economic Research.
- Mertens, K. and M. O. Ravn (2013). The dynamic effects of personal and corporate income tax changes in the united states. *American Economic Review* 103(4), 1212–47.
- Stock, J. H. and M. W. Watson (2005). Implications of dynamic factor models for var analysis.
- Stock, J. H. and M. W. Watson (2018). Identification and estimation of dynamic causal effects in macroeconomics using external instruments. *The Economic Journal* 128(610), 917–948.

Appendix

A Simulation A1

In this Appendix we show the results obtained by using a different treatment of the same data set of the simulation of Section 4. The series that were already stationary in levels and their linear combinations are taken in first differences as far as SP is concerned. Again, X_{it} is the cumulated sum of x_{it} , so that both AP procedures face a mixed data set, including $I(1)$ and $I(0)$ variables. With this data treatment, $K(L)$ is no longer zeroless, since $K(1)$ has reduced rank 1. Indeed, the $I(1)$ variables have just one common trend (given by technology), whereas the stationary variables of course do not have permanent shocks. The cointegration rank is therefore $r - 1$. As in Simulation 1 we use four lags of the factors for all methods.

Figure 2 shows the result. Here SP does not work properly, whereas the Alternative procedures perform reasonably well.

Table 4 reports the MSE of the three methods. The table confirms that SP performs worse than both AP procedures. The numbers in the Table are comparable with the ones of Table 1 in the main text: the MSE is generally much larger not only for SP, but also for both AP methods. Our conclusion is that we should avoid over-differencing, no matter for the estimation procedure used.

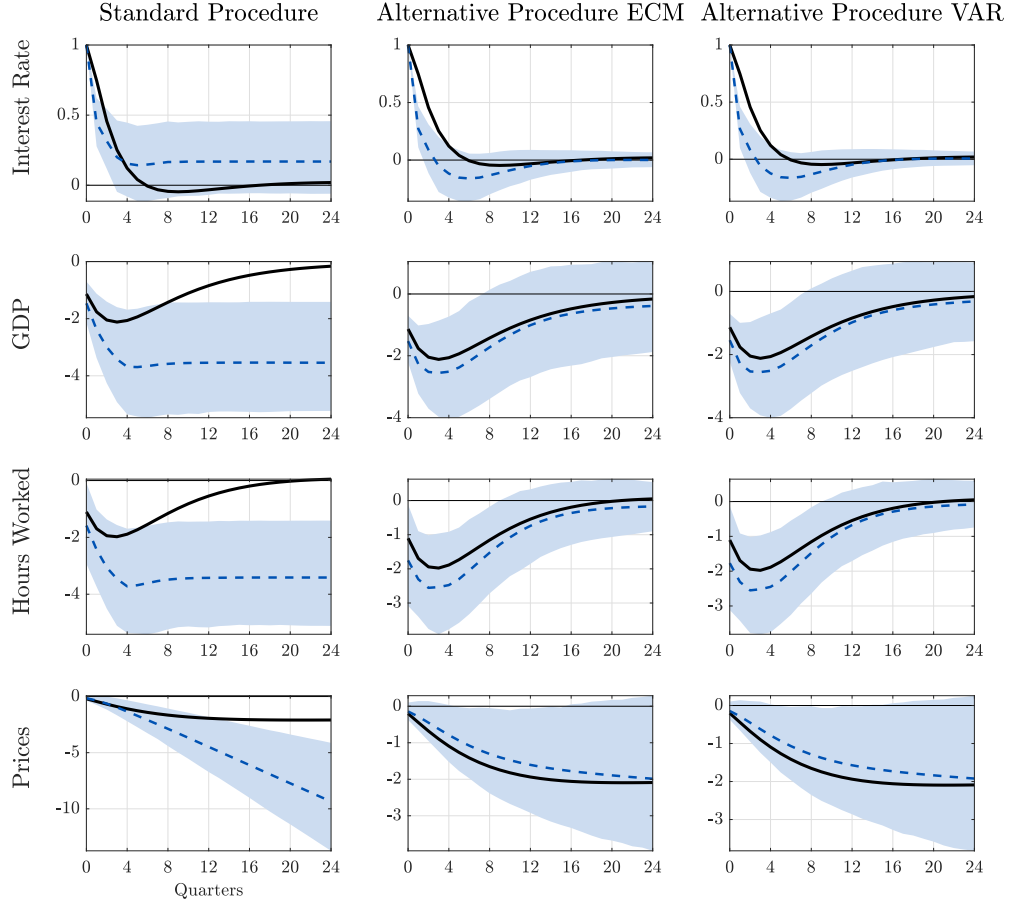


Figure 2: Simulation A1 – impulse responses to a monetary policy shock

Note: estimated impulse responses of the Interest rate, GDP, Hours worked and Prices to a monetary policy shock. The first column reports the Standard Procedure (SP), the second the Alternative Procedure based on a VECM (APECM), the third the Alternative Procedure based on a VAR in levels (APVAR). Black solid lines are the true impulse–response functions, blue dashed lines the averages across replications, and the shaded areas the 16th and 84th percentiles of their distribution. All series are taken in first differences, including those that are stationary in levels. Over-differencing makes $K(L)$ no longer zeroless: $K(1)$ has rank one and the cointegration rank is $r - 1 > r - q$.

Table 4: Simulation A1: mean squared errors of the estimated impulse–response functions

Horizon	0	3	6	9	12	15	18	21	24
<i>Interest rate</i>									
SP	0.00	0.16	0.23	0.27	0.26	0.26	0.25	0.25	0.25
APECM	0.00	0.35	0.23	0.10	0.04	0.02	0.01	0.01	0.01
APVAR	0.00	0.36	0.24	0.10	0.04	0.02	0.01	0.01	0.01
<i>GDP</i>									
SP	2.36	13.59	19.01	21.61	23.65	25.26	26.38	27.14	27.67
APECM	7.35	18.36	12.97	7.69	5.24	4.56	4.30	4.08	3.92
APVAR	7.82	19.61	13.71	7.65	4.84	4.02	3.64	3.22	2.86
<i>Hours worked</i>									
SP	4.21	10.72	13.40	15.33	17.37	19.02	20.06	20.68	21.00
APECM	14.45	9.93	6.90	3.79	2.13	1.54	1.33	1.22	1.16
APVAR	15.62	9.84	6.96	3.65	1.88	1.28	1.07	0.96	0.88
<i>Prices</i>									
SP	0.10	1.28	4.86	12.50	25.15	43.34	67.25	96.96	132.50
APECM	0.30	2.15	4.79	7.49	9.29	10.41	11.29	12.23	13.41
APVAR	0.32	2.28	5.14	8.08	9.98	11.06	11.77	12.46	13.29

Note: mean squared errors of the estimated impulse–response functions to the monetary policy shock, at selected horizons, obtained with the Standard Procedure (SP), the Alternative Procedure based on a VECM (APECM) and the Alternative Procedure based on a VAR in levels (APVAR). Simulated data from the model of [Justiniano et al. \(2010\)](#) with $r = 10$, $q = 7$, $T = 250$; 1000 replications; four lags for all methods. In contrast to the results presented in Table 1, here all series are taken in first differences, so that $K(1)$ has rank one. Notice that the errors are larger than in Table 1 for all three procedures.

B Simulation A2

In our third simulation we estimate the effects of the preference shock. The simulation is based on a data set made up mainly of the first differences of real activity variables. Precisely, we use, as our factors f_t , GDP growth, Consumption growth, Investment growth, Potential GDP growth, Labor productivity growth, the Technology shock, Lagged GDP growth, Lagged consumption growth, and a ninth factor which is the first difference of the preference shock. Hence we have $r = 9$. We standardize our factors and construct 115 linear combinations with coefficients uniformly distributed between -1 and 1. Our common components are then given by the first five variables above: GDP growth, Consumption growth, Investment growth, Potential output growth and Labor productivity growth, plus the 115 additional series. Finally we add the ten percent measurement error as in Section 4. In this way we end up with 120 series. Again, the sample size is $T = 250$. Here the whole data set is driven by just one permanent shock, so that $K(L)$ is not zeroless, $K(1)$ having reduced rank $\tau = 1$, like in Simulation A1. The number of cointegration relations is now $r - \tau = 8$. Again, we use four lags for all estimation methods.

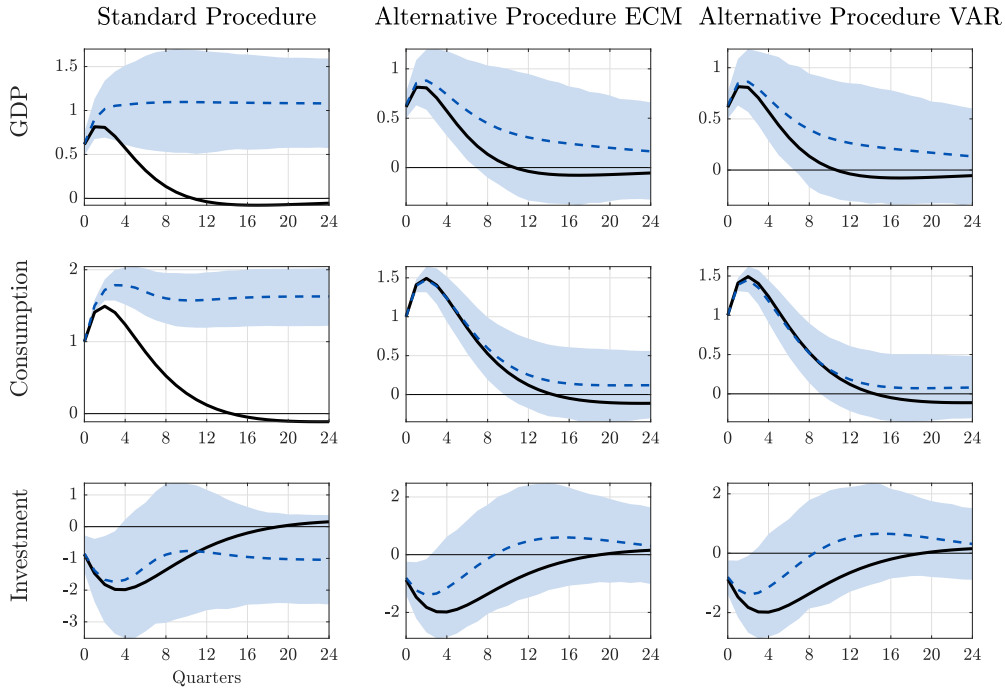


Figure 3: Simulation A2 – impulse responses to a preference shock

Note: estimated impulse responses of GDP, Consumption and Investment to a preference shock. The first column reports the Standard Procedure (SP), the second the Alternative Procedure based on a VECM (APECM), the third the Alternative Procedure based on a VAR in levels (APVAR). Black solid lines are the true impulse-response functions, blue dashed lines the averages across replications, and the shaded areas the 16th and 84th percentiles of their distribution. The panel includes only real activity variables, taken in first differences and driven by a single common trend (technology). $K(1)$ has rank $\tau = 1$, so that $K(L)$ is not zeroless and the cointegration rank is $r - \tau = 8$.

Results for GDP, Consumption and Investment are shown in Figure 3. The figure shows that the Standard Procedure does not work in this case, so that we must resort to the Alternative Procedures, which work reasonably well. Table 5 reports the Mean Square Errors of the three methods for the three variables.

Table 5: Simulation A2: mean squared errors of the estimated impulse–response functions

Horizon	0	3	6	9	12	15	18	21	24
<i>GDP</i>									
SP	0.02	0.28	0.87	1.39	1.60	1.64	1.62	1.58	1.55
APECM	0.02	0.16	0.34	0.47	0.52	0.51	0.45	0.40	0.35
APVAR	0.02	0.15	0.30	0.41	0.46	0.44	0.38	0.33	0.28
<i>Consumption</i>									
SP	0.00	0.19	0.80	1.54	2.28	2.80	3.08	3.18	3.20
APECM	0.00	0.05	0.13	0.18	0.22	0.25	0.27	0.27	0.27
APVAR	0.00	0.05	0.12	0.16	0.19	0.21	0.22	0.22	0.21
<i>Investment</i>									
SP	0.33	2.88	4.68	4.77	3.65	3.11	3.13	3.30	3.46
APECM	0.32	2.86	4.82	5.50	5.23	4.21	3.14	2.43	2.02
APVAR	0.32	2.81	4.65	5.25	4.85	3.74	2.66	2.00	1.64

Note: mean squared errors of the estimated impulse–response functions to the preference shock for GDP, Consumption and Investment. Data generated as described in the text, with $r = 9$, $\tau = 1$, $T = 250$; four lags for all methods.

C Robustness of the empirical results

This Appendix reports the robustness of the empirical results to the bandwidth $M = \lceil c\sqrt{T} \rceil$ of the lag-window estimator of the spectral density matrix, for $c \in \{0.5, 0.75, 1, 1.25\}$; the baseline is $c = 0.75$. Table 6 reports DDR, the preferred criterion of [Avarucci et al. \(2026\)](#), and Table 7 reports the BLL and HL criteria with the large window only (see Section 5). In both cases the number of shocks and the number of permanent shocks are equal to two throughout, so the conclusion $\tau = q$ is unaffected.

Table 6: DDR across bandwidths

Frequency band	$c = 0.50$	$c = 0.75$	$c = 1.00$	$c = 1.25$
<i>Panel A: FRED-QD</i>				
All frequencies	2	2	2	2
Excluding high frequencies	2	2	2	2
Frequency zero	2	2	2	2
Long run	2	2	2	2
<i>Panel B: FRED-MD</i>				
All frequencies	2	2	2	2
Excluding high frequencies	2	2	2	2
Frequency zero	2	2	2	2
Long run	2	2	2	2

Note: number of common shocks and common trends selected by the DDR criterion of [Avarucci et al. \(2026\)](#); frequency bands as in Table 2. Bandwidth $M = \lceil c\sqrt{T} \rceil$, baseline $c = 0.75$, $q_{\max} = 8$. The estimates are invariant across the grid.

Table 7: BLL and HL information criteria across bandwidths

	$c = 0.50$		$c = 0.75$		$c = 1.00$		$c = 1.25$	
Penalty	BLL	HL	BLL	HL	BLL	HL	BLL	HL
<i>Panel A: FRED-QD</i>								
(a)	2	2	2	2	2	2	2	2
(b)	2	2	2	2	2	2	2	2
(c)	2	2	2	2	2	2	2	2
(d)	2	2	2	2	2	2	2	2
<i>Panel B: FRED-MD</i>								
(a)	2	2	2	2	2	2	2	2
(b)	2	2	2	2	2	2	2	2
(c)	2	2	2	2	2	2	2	2
(d)	2	2	2	2	2	2	2	2

Note: number of common trends τ (BLL) and of common shocks q (HL), large window; see Table 3 for details. Each criterion is computed 50 times with different random draws of the cross-sectional subsamples; we report the modal estimate. Bandwidth $M = \lceil c\sqrt{T} \rceil$, baseline $c = 0.75$, $q_{\max} = 8$. The estimates are invariant across the grid.

D An example about transitory effects of demand shocks

In the following example GDP, consumption and prices are driven by both supply and demand shocks. GDP and consumption are cointegrated, since demand shocks have only transitory effects on both variables. The implications of the model vary completely, depending on whether or not there is a measurement error in the prices. In the former case, the model is nonsingular and cointegration is problematic. In the latter case, the model is singular and cointegration is not a problem, unless the demand shock have transitory effects on prices.

Let y_t , c_t and p_t be the logs of GDP, consumption and the price level, and let $\pi_t = \Delta p_t$. Consider first the model

$$\begin{pmatrix} \Delta y_t \\ \Delta c_t \\ \pi_t \end{pmatrix} = \begin{pmatrix} 1 & 1-L & 0 \\ 1 & 0.5(1-L) & 0 \\ -1 & 1-aL & 1 \end{pmatrix} \begin{pmatrix} s_t \\ d_t \\ e_t \end{pmatrix},$$

where s_t is a supply shock, d_t a demand shock and e_t a measurement error in inflation, all unit variance white noise processes, mutually orthogonal at all leads and lags. The demand shock has transitory effects on GDP and consumption, which are therefore cointegrated, since $y_t - c_t = \mu + 0.5d_t$. On the other hand, it has permanent effects on prices, provided that $a \neq 1$. Because of the measurement error, the vector on the left side is nonsingular. The MA matrix is singular for $L = 1$, since its determinant is $-0.5(1-L)$, so that a VAR for the first differences does not exist. The permanent effect of d_t on prices is irrelevant here: cointegration between y_t and c_t is enough to rule out the VAR.

Now let us remove the measurement error:

$$\begin{pmatrix} \Delta y_t \\ \Delta c_t \\ \pi_t \end{pmatrix} = \begin{pmatrix} 1 & 1-L \\ 1 & 0.5(1-L) \\ -1 & 1-aL \end{pmatrix} \begin{pmatrix} s_t \\ d_t \end{pmatrix}.$$

The vector is singular and remains cointegrated, with cointegration rank $r-q = 1$. However, the MA matrix is zeroless, provided that $a \neq 1$, since in this case the determinants of its three 2×2 submatrices, $-0.5(1-L)$, $2 - (1+a)L$ and $1.5 - (0.5+a)L$, have no common zeros.

Now the permanent effect of d_t on prices is crucial. If d_t had transitory effects on prices as well, i.e. $a = 1$, it would have transitory effects on all the variables, the number of permanent shocks would fall to $\tau = 1 < q = 2$, and the VAR would not exist. This would also happen when inflation is overdifferenced, i.e. $(1-L)\pi_t = -(1-L)s_t + (1-L)(1-aL)d_t$.

Summing up, in the nonsingular case the cointegration between GDP and consumption rules out a VAR for the first differences. By contrast, in the singular case, as observed in Section 3.4, cointegration rules it out only if a linear combination of the shocks has transitory effects on all variables.

Quest'opera è soggetta alla licenza Creative Commons



CC BY-NC 4.0 DEED

Attribuzione - Non commerciale 4.0 Internazionale



Alma Mater Studiorum - Università di Bologna
DEPARTMENT OF ECONOMICS

Strada Maggiore 45
40125 Bologna - Italy
Tel. +39 051 2092604
Fax +39 051 2092664
<http://www.dse.unibo.it>