Schauder estimates, Harnack inequality and Gaussian lower bound for Kolmogorov type operators in non-divergence form *

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Abstract

We prove some Schauder type estimates and an invariant Harnack inequality for a class of degenerate evolution operators of Kolmogorov type. We also prove a Gaussian lower bound for the fundamental solution of the operator and a uniqueness result for the Cauchy problem. The proof of the lower bound is obtained by solving a suitable optimal control problem and using the invariant Harnack inequality.

1 Introduction

We consider second order operators of the non-divergence form

$$Lu := \sum_{i,j=1}^{p_0} a_{i,j}(z)\partial_{x_i x_j} u + \sum_{i=1}^{p_0} a_i(z)\partial_{x_i} u + \sum_{i,j=1}^{N} b_{i,j} x_i \partial_{x_j} u - \partial_t u, \tag{1.1}$$

where $z = (x,t) \in \mathbb{R}^N \times \mathbb{R}$, $1 \le p_0 \le N$ and the coefficients $a_{i,j}$ and a_i are bounded continuous functions. The matrix $B = (b_{i,j})_{i,j=1,\dots,N}$ has real, constant entries, $A_0(z) = (a_{i,j}(z))_{i,j=1,\dots,p_0}$ is a symmetric and positive, for every $z \in \mathbb{R}^{N+1}$. In order to state our assumptions on the operator L, it is convenient to introduce the analogous *constant coefficients* operator

$$Ku := \sum_{i,j=1}^{p_0} a_{i,j}\partial_{x_i x_j} u + \sum_{i,j=1}^{N} b_{i,j} x_i \partial_{x_j} u - \partial_t u, \tag{1.2}$$

with the constant matrix $A_0 = (a_{i,j})_{i,j=1,\dots,p_0}$ symmetric and positive. Our assumptions are:

H1 the operator K is hypoelliptic i.e. every distributional solution to $Ku = f$ is a smooth classical solution, whenever f is smooth,

H2 There exists a positive constant Λ such that

$$\Lambda^{-1}|\zeta|^2 \le \langle A_0(z)\zeta, \zeta\rangle \le \Lambda|\zeta|^2, \qquad \forall\, \zeta \in \mathbb{R}^{p_0}, \forall\, z \in \mathbb{R}^{N+1}. \tag{1.3}$$

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H3 The coefficients $a_{i,j}$ and a_j are bounded and Hölder continuous of exponent $\alpha \le 1$ (in the sense of the Definition 1.1 below).

Some comments about our assumptions are in order. We first note that, in the case $p_0 = N$ and $B = 0$, conditions **[H1]-[H2]-[H3]** are verified by every uniformly parabolic operator in non-divergence form, with Hölder continuous coefficients (in that case, K is the heat operator). On the other hand, several examples of degenerate operators (i.e. with $p_0 < N$) appear in kinetic theory and in finance. Consider for instance the linear Fokker-Planck equation

$$\partial_t f - \langle v, \nabla_x f \rangle = \mathrm{div}_v(\nabla_v f + v f),$$

where f is the density of particles at point $x \in \mathbb{R}^n$ with velocity $v \in \mathbb{R}^n$ at time t (see [9] and [30]). It can be written in the form (1.1) by choosing $p_0 = n$, $N = 2n$ and

$$B = \begin{pmatrix} I_n & I_n \\ 0 & 0 \end{pmatrix}$$

where I_n is the identity $n \times n$ matrix. We also recall that in the Boltzmann-Landau equation

$$\partial_t f - \langle v, \nabla_x f \rangle = \sum_{i,j=1}^{n} \partial_{v_i} \left(a_{i,j}(\cdot, f) \partial_{v_j} f \right),$$

the coefficients $a_{i,j}$ depend on the unknown function through some integral expression (see, [20], [7] and [21]). Equations of the form (1.1) arise in mathematical finance as well. More specifically, the following linear equation

$$S^2 \partial_{SS} V + f(S) \partial_M V - \partial_t V = 0, \qquad S, t > 0, \, M \in \mathbb{R}$$

with either $f(S) = \log(S)$ or $f(S) = S$, arises in the Black & Scholes theory when considering the problem of the pricing Asian option (see [3]). Moreover, in the stochastic volatility model by Hobson & Rogers, the price of an European option is given by a solution of the equation

$$\frac{1}{2} \sigma^2 (S - M)(\partial_{SS} V - \partial_S V) + (S - M) \partial_M V - \partial_t V = 0,$$

for some positive continuous function σ (see [16] and [10]). We refer to the paper by Di Francesco and Pascucci [11] for an extensive survey of the financial motivations to the study of operators as above.

With the aim to discuss our assumptions and the regularity properties of the operators K, we introduce some notations. Here and in the sequel, we will denote by $A_0^{\frac{1}{2}} = (\bar{a}_{ij})_{i,j=1,\dots,p_0}$ the unique positive $p_0 \times p_0$ matrix such that $A_0^{\frac{1}{2}} \cdot A_0^{\frac{1}{2}} = A_0$, and by A and $A^{\frac{1}{2}}$ the $N \times N$ matrices

$$A = \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad A^{\frac{1}{2}} = \begin{pmatrix} A_0^{\frac{1}{2}} & 0 \\ 0 & 0 \end{pmatrix}. \tag{1.4}$$

Note that the operator K can be written as

$$K = \sum_{j=1}^{p_0} X_i^2 + Y, \tag{1.5}$$

where

$$X_i = \sum_{j=1}^{p_0} \bar{a}_{ij} \partial_{x_j}, \; i = 1, \dots, p_0, \qquad Y = \langle x, B \nabla \rangle - \partial_t. \tag{1.6}$$

$\nabla = (\partial_{x_1}, \dots, \partial_{x_N})$ and $\langle \cdot, \cdot \rangle$ are, respectively, the gradient and the inner product in $\mathbb{R}^N$.

The following statements are equivalent to hypothesis **[H1]**:

H_1 $\mathrm{Ker}(A^{\frac{1}{2}})$ does not contain non-trivial subspaces which are invariant for B;

H_2 there exists a basis of $\mathbb{R}^N$ such that B has the form

$$\begin{pmatrix} * & B_1 & 0 & \dots & 0 \\ * & * & B_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \dots & B_r \\ * & * & * & \dots & * \end{pmatrix} \tag{1.7}$$

where B_j is a matrix $p_{j-1} \times p_j$ of rank p_j, with

$$p_0 \geq p_1 \geq \ldots \geq p_r \geq 1, \qquad\qquad p_0 + p_1 + \ldots + p_r = N,$$

while $*$ are constant and arbitrary blocks;

H_3 if we set

$$E(s) = \exp(-sB^T), \qquad \mathcal{C}(t) = \int_0^t E(s)AE^T(s)ds, \tag{1.8}$$

then $\mathcal{C}(t)$ is positive, for every $t > 0$;

H_4 the Hörmarder condition is satisfied:

$$\text{rank Lie}\,(X_1, \ldots, X_{p_0}, Y) = N+1, \quad \text{at every point of } \mathbb{R}^{N+1}. \tag{1.9}$$

For the equivalence of the above conditions we refer to [18]. In the sequel, we assume that the basis of $\mathbb{R}^N$ is as in H_2, so that B has the form (1.7). Under the assumption **[H1]**, Hörmarder constructed in [17] the fundamental solution of K:

$$\Gamma(x,t,\xi,\tau) = \frac{(4\pi)^{-\frac{N}{2}}}{\sqrt{\det\mathcal{C}(t-\tau)}} \exp\left(-\frac{1}{4}\langle \mathcal{C}^{-1}(t-\tau)(x - E(t-\tau)\xi), x - E(t-\tau)\xi\rangle - (t-\tau)\mathrm{tr}B \right), \tag{1.10}$$

if $t > \tau$, and $\Gamma(x,t,\xi,\tau) = 0$ if $t \leq \tau$.

Since the works by Folland [14], Rotschild and Stein [31], Nagel, Stein and Wainger [26] concerning operators satisfying the Hörmarder condition, it is known that the natural framework for the regularity of that operators is the analysis on Lie groups. The first study of the group related to the operator (1.2) has been done by Lanconelli and Polidoro in [18]. The group law is defined as follows: for every $(x,t), (\xi,\tau) \in \mathbb{R}^{N+1}$ we set

$$(x,t) \circ (\xi,\tau) = (\xi + E(\tau)x, t+\tau), \tag{1.11}$$

where $E(\tau)$, is the matrix in (1.8). Let $f \in C(\Omega)$, for some open set $\Omega \in \mathbb{R}^{N+1}$. We say that a function $u : \Omega \to \mathbb{R}$ is a classical solution to $Lu = f$ with if $\partial_{x_i}u, \partial_{x_i,x_j}u (i,j = 1, \ldots, p_0)$ and the *Lie derivative*

$$Yu(x,t) = \lim_{h\to 0} \frac{u(E(-h)x, t-h) - u(x,t)}{h}$$

are continuous functions, and the equation $Lu = f$ is satisfied at any point of Ω.

We recall that Γ is invariant with respect to the translations defined in (1.11):

$$\Gamma(x,t,\xi,\tau) = \Gamma\big((\xi,\tau)^{-1} \circ (x,t)\big) := \Gamma\big((\xi,\tau)^{-1} \circ (x,t), 0, 0\big). \tag{1.12}$$

Moreover, if (and only if) all the $*$-block in (1.7) are null, then K is homogeneous of degree two with respect the family of following dilatations,

$$\delta(\lambda) := \big(D(\lambda), \lambda^2\big) = \mathrm{diag}\left(\lambda I_{p_0}, \lambda^3 I_{p_1}, \ldots, \lambda^{2r+1} I_{p_r}, \lambda^2\right), \tag{1.13}$$

(I_{p_j} denotes the $p_j \times p_j$ identity matrix), *i.e.*

$$K \circ \delta(\lambda) = \lambda^2(\delta(\lambda) \circ K), \qquad \forall \lambda > 0 \tag{1.14}$$

(see Proposition 2.2 in [18]), and Γ is a $\delta(\lambda)$-homogeneous function:

$$\Gamma(\delta(\lambda)z) = \lambda^{-Q}\Gamma(z), \qquad \forall\, z \in \mathbb{R}^{N+1} \setminus \{0\},\ \lambda > 0,$$

where

$$Q = p_0 + 3p_1 + \ldots, (2r+1)p_r.$$

Since

$$\det\big(\delta(\lambda)\big) = \det\big(\mathrm{diag}(\lambda I_{p_0}, \lambda^3 I_{p_3}, \ldots, \lambda^{2r+1} I_{p_r}, \lambda^2)\big) = \lambda^{Q+2}, \tag{1.15}$$

the number $Q+2$ is said *homogeneous dimension* of $\mathbb{R}^{N+1}$ with respect to the dilation group $(\delta(\lambda))_{\lambda>0}$ and Q is said *spatial homogeneous dimension* of $\mathbb{R}^N$ with respect to $(\delta(\lambda))_{\lambda>0}$. For every $z = (x,t) \in \mathbb{R}^{N+1}$ we set

$$\|z\| = \sum_{j=1}^{N} |x_j|^{\frac{1}{q_j}} + |t|^{\frac{1}{2}}, \tag{1.16}$$

where q_j are positive integers such that $\delta(\lambda) = \mathrm{diag}(\lambda^{q_1}, \ldots, \lambda^{q_N})$. It is easy to check that $\|\cdot\|$ is a homogeneous function of degree 1 with respect the dilation $\delta(\lambda)$, *i.e.*

$$\|\delta(\lambda)z\| = \lambda\|z\|, \qquad \text{for every } \ \lambda > 0, \text{ and } \ z \in \mathbb{R}^{N+1}. \tag{1.17}$$

Definition 1.1. *Let $\alpha \in]0,1]$. We say that a function $f : \mathbb{R}^{N+1} \to \mathbb{R}$ is Hölder continuous of exponent α, in short $f \in C^\alpha$, if there exists a positive constant c such that*

$$|f(z) - f(\zeta)| \le c\,\|\zeta^{-1} \circ z\|^\alpha, \qquad \textit{for every } \ z, \zeta \in \mathbb{R}^{N+1}.$$

The first main result of this paper is an uniform Harnack inequality for non-negative solution u of $Lu = 0$. We consider a suitable bounded open subset S of $\mathbb{R}^N$, that will be specified at the beginning of Section 5, and we define the *unit cylinder* as $H(T) = S \times]0,T[$, for any positive T. Moreover, we set for every $(\xi,\tau) \in \mathbb{R}^{N+1}$, $R > 0$, $\delta \in]0,1[$, and α, β, γ such that $0 < \alpha < \beta < \gamma < 1$,

$$H_R(\xi,\tau,R^2T) = (\xi,\tau) \circ \delta(R)\big(H(T)\big),$$

$$H^- = \{(x,t) \in H_{\delta R}(\xi,\tau,TR^2) : \tau + \alpha R^2 T \le t \le \tau + \beta R^2 T\},$$

$$H^+ = \{(x,t) \in H_{\delta R}(\xi,\tau,TR^2) : \tau + \gamma R^2 T \le t \le \tau + R^2 T\}.$$

We have

Theorem 1.2. *Assume that L satisfies conditions* **[H1]-[H2]-[H3]**. *Let α, β, γ be such that $0 < \alpha < \beta$ and $\beta + \frac{1}{2} < \gamma < 1$. Then there exist three positive constants M, δ and T, with $\delta < 1$, depending only on α, β, γ and on the operator L, such that*

$$\sup_{H^-} u \le M \inf_{H^+} u.$$

for every positive solution u of $Lu = 0$ in $H_R(\xi,\tau,TR^2)$ and for any $R \in]0,1]$.

The proof of Theorem 1.2 is based on a suitable adaptation of the method introduced by Krylov and Safanov, also used by Fabes and Strook [13] in the study of uniformly parabolic operators. We recall that a Harnack inequality for the positive solutions to $Lu = 0$ has been proved by Polidoro in [27], in the case of homogeneous Kolmogorov operators, and by Morbidelli in [25] for non-homogeneous Kolmogorov operators, by using mean value formulas. In [27] and [25] divergence form operators are considered, under the assumption that the coefficients $a_{i,j}$ and the derivatives $\partial_{x_i} a_{i,j}$ are Hölder continuous for $i, j = 1, \dots, p_0$. Moreover in [27] and [25] the coefficients a_i do not appear in the operator L. We acknowledge that the fact that the Krylov-Safanov-Fabes-Strook approach, combined with the parametrix method, improves the Harnack inequality for operators with Hölder continuous coefficients was pointed out by Bonfiglioli and Uguzzoni in [4].

In order to use the Krylov-Safanov-Fabes-Strook method, we prove a Schauder type estimate that extends the analogous result proved by Manfredini in [24] for homogeneous operators (we refer to Section 3 for the definition of the function spaces and the norms appearing in the following statement).

Theorem 1.3. *Let Ω be a bounded open set, $f \in C^{\alpha}_{d}(\Omega)$, and let u be a bounded function belonging to $C^{2+\alpha}_{loc}(\Omega)$ such that $Lu = f$ in Ω. Then $u \in C^{2+\alpha}_{d}(\Omega)$ and there exist a positive constant c, depending only on the constant Λ, on the Hölder-norm of the coefficients of L and on the diameter of Ω, such that*

$$|u|_{2+\alpha,d,\Omega} \le c \,(\sup_{\Omega} |u| + [f]_{2+\alpha,d,\Omega}). \tag{1.18}$$

We recall that optimal Schauder estimates for the Cauchy problem

$$\begin{cases} Lu & = 0 \qquad \text{in } \mathbb{R}^N \times]0,T], \\ u(x,0) & = \varphi(x) \quad x \in \mathbb{R}^N. \end{cases} \tag{1.19}$$

have been obtained by many authors in the framework of the semigroup theory. In Theorems 1.2 and 8.2 of [23] Lunardi proves an optimal Hölder regularity result for the solution u to (1.19), under the assumption that the initial data φ has Hölder continuous derivatives $\partial_{x_i}\varphi$ and $\partial_{x_i,x_j}\varphi$, $i, j = 1 \dots p_0$. It is also assumed that the matrix $(a_{i,j})$ satisfies our Hypothesis [**H2**] and that the coefficients $a_{i,j}$ are Hölder continuous function of the space variable x that converge as $|x|$ goes to $+\infty$. Lorenzi in [22] improves the results by Lunardi in that the coefficients $a_{i,j}$ are not assumed to be bounded functions. On the other hand, in [22] the coefficients $a_{i,j}$ have Hölder continuous derivatives up to third order and the Lie Algebra related to the constant coefficient operator has step 2. Priola in [29] considers operator with unbounded coefficients $a_i, i = 1, \dots, p_0$. Lunardi states in [23] an interior estimate for the Cauchy problem (1.19) with bounded continuous initial data φ:

$$\|u(\cdot,t)\|_{C^{2+\alpha}_{d}(\mathbb{R}^N)} \le \frac{Ce^{\omega t}}{t^{1+\frac{\alpha}{2}}} \|\varphi\|_{C^{0}_{d}(\mathbb{R}^N)}, \quad 0 \le \alpha < 1.$$

Here C and ω are suitable positive constants and the space $C^{\alpha}_{d}(\mathbb{R}^N)$ is defined in terms of a homogeneous norm analogous to the norm used in our Definition 1.1 (see formula (1.16) in [23] for the details). In order to compare the above estimate with our Theorem 1.3 we give a simple consequence of it.

Corollary 1.4. *Let $\varphi : \mathbb{R}^N \to \mathbb{R}$ be a bounded continuous function, and let u be the (unique) bounded solution to the Cauchy problem* (1.19). *Then, for every positive T, there exists a constant c_T, only depending on T and on the operator L, such that the solution u to* (1.19) *satisfies*

$$\begin{aligned} \sup_{x\in\mathbb{R}^N} |u(x,t)| + \sum_{i=1}^{p_0} \sqrt{t} \sup_{x\in\mathbb{R}^N} |\partial_{x_i} u(x,t)| + \sum_{i,j=1}^{p_0} t \sup_{x\in\mathbb{R}^N} |\partial_{x_i,x_j} u(x,t)| + \\ + t^{1+\frac{\alpha}{2}} \left(\sup_{\substack{x,y\in\mathbb{R}^N \\ x\neq y}} \frac{|u(x,t)-u(y,t)|}{\|(x-y,0)\|^{\alpha}} + \sum_{i=1}^{p_0} \sup_{\substack{x,y\in\mathbb{R}^N \\ x\neq y}} \frac{|\partial_{x_i} u(x,t) - \partial_{x_i} u(y,t)|}{\|(x-y,0)\|^{\alpha}} \right) + \\ + t^{1+\frac{\alpha}{2}} \left(\sum_{i,j=1}^{p_0} \sup_{\substack{x,y\in\mathbb{R}^N \\ x\neq y}} \frac{|\partial_{x_i,x_j} u(x,t) - \partial_{x_i,x_j} u(y,t)|}{\|(x-y,0)\|^{\alpha}} \right) \le c_T \sup_{x\in\mathbb{R}^N} |\varphi(x)|, \end{aligned} \tag{1.20}$$

for every $t \in]0,T]$.

We finally recall the recent paper by Di Francesco and Pascucci in [12], where the estimates (1.20) are improved assuming that the initial condition φ of the Cauchy problem is a Hölder continuous bounded function. More specifically, in Proposition 3.3 in [12] it is shown that

$$|\partial_{x_i} u(x,t) - \partial_{x_i} u(y,t)| \le c_T \, \frac{\|(x-y,0)\|^{\alpha/2}}{t^{1/2-\alpha/4}} \sup_{\substack{\xi,\eta\in\mathbb{R}^N \\ \xi\neq\eta}} \frac{|\varphi(\xi)-\varphi(\eta)|}{\|(\xi-\eta,0)\|^{\alpha}},$$

$$|\partial_{x_i x_j} u(x,t) - \partial_{x_i x_j} u(y,t)| \le c_T \, \frac{\|(x-y,0)\|^{\alpha/2}}{t^{1-\alpha/4}} \sup_{\substack{\xi,\eta\in\mathbb{R}^N \\ \xi\neq\eta}} \frac{|\varphi(\xi)-\varphi(\eta)|}{\|(\xi-\eta,0)\|^{\alpha}},$$

for every $(x,t),(y,t) \in \mathbb{R}^N \times]0,T]$, $i,j = 1,\dots,p_0$, and $\alpha \in]0,1[$.

Our next main result is a pointwise lower bound of the fundamental solution of L satisfying conditions **[H1]-[H2]-[H3]**. We recall that Morbidelli in [25], and Di Francesco and Pascucci in [11], prove the existence of a fundamental solution Γ of L by the Levi parametrix method and that Γ satisfies the pointwise estimate

$$\Gamma(z,\zeta) \le c_T^+ \, \Gamma^+(z,\zeta), \qquad \forall\, z=(x,t), \zeta=(\xi,\tau) \in \mathbb{R}^{N+1}, \text{ such that } 0 < t-\tau < T, \tag{1.21}$$

where Γ^+ is the fundamental solution of the operator

$$K_\mu = \mu \sum_{i=1}^{p_0} \partial_i^2 + \langle x, B\nabla\rangle - \partial_t, \tag{1.22}$$

μ and T are any positive constants, $\mu > \Lambda$ in **[H2]**, c_T^+ is a positive constant only depending on μ, T and on the constants appearing in **[H1]-[H2]-[H3]**. Here we prove an analogous lower bound.

Theorem 1.5. *Assume that L satisfies condition* **[H1]-[H2]-[H3]**. *Then there exist a positive constant μ such that, for every positive T, it holds*

$$\Gamma(x,t) \ge c_T^- \, \Gamma^-(x,t), \qquad \forall\, x\in\mathbb{R}^N,\ 0<t<T.$$

Here Γ^- is the fundamental solution of the operator K_μ in (1.22)*, μ and c_T^- are two positive constants depending on L, $\mu < \Lambda^{-1}$ and c_T^- also depends on T.*

In order to state our last result, we recall that Di Francesco and Pascucci prove in [11] a Tychonoff-type uniqueness result: the Cauchy problem (1.19) has a unique solution u satisfying the growth condition

$$\int_0^T \int_{\mathbb{R}^N} |u(x,t)| e^{-C|x|^2} dx\, dt < +\infty, \tag{1.23}$$

for some positive C (see Theorem 1.6 in [11]). Here we prove a Widder-type uniqueness theorem

Theorem 1.6. *Assume that L satisfied condition* **[H1]-[H2]-[H3]**, *and let u,v be two solution of the Cauchy problem* (1.19)*. If both u and v are non negative, then $u \equiv v$ in $\mathbb{R}^N \times [0,T[$.*

This paper is organized as follows. In Section 2 we recall some known facts about Kolmogorov operators and we give some preliminary results. Specifically, we prove some accurate bounds of the fundamental solution of K and of its derivatives, then we prove a representation formula for the derivatives of the solutions to $Lu = f$, in terms of the function f. In Section 3 we prove the Schauder type estimates stated in Theorem 1.3. In Section 4 we consider the Dirichlet problem related to the cylinder $H_R(\xi,\tau,R^2)$, and we give some pointwise lower bound of the relevant Green function. That lower bound is the key point of the Krylov-Safanov-Fabes-Strook method for the Harnack inequality and is a direct consequence of some pointwise estimates provided by the parametrix method (see Remark 2.3 in Section 2). Then, in Section 5 we give the proof of Theorem 1.2. In Section 6 we prove a non-local Harnack inequality by using repeatedly the invariant (local) Harnack inequality stated in Theorem 1.2 and a method introduced in a recent work by Boscain and Polidoro [6], that is based on the optimal control theory (see Theorem 6.1). We finally give the proof of Theorem 1.5 and Theorem 1.6.

2 Some known and preliminary results

We first discuss some geometric features of the Lie group $(\mathbb{R}^{N+1}, \circ)$, and the related dilations $\delta(\lambda)$. Then we recall some know results about the constant coefficients Kolmogorov-Fokker-Planck operators K.

Lemma 2.1. *For every positive T and for every compact set $H \subset \mathbb{R}^N$ there exists a constant $C_{T,H} \geq 1$ such that*

i) $\|z^{-1}\| \leq C_{T,H}\|z\|$, *for every $z \in H \times [-T, T]$;*

ii) $\|z \circ \zeta\| \leq C_{T,H}\big(\|z\| + \|\zeta\|\big)$, *for every $\zeta \in \mathbb{R}^N \times [-T, T]$ and $z \in H \times \mathbb{R}$.*

Moreover the constant $C_{T,H}$ can be chosen arbitrarily close to 1 provided that T is sufficiently small.

Proof. We decompose the matrix E defined in (1.8) according to (1.7):

$$E(s) = \begin{pmatrix} E_{0,0}(s) & E_{0,1}(s) & \dots & E_{0,r}(s) \\ E_{1,0}(s) & E_{1,1}(s) & \dots & E_{1,r}(s) \\ \vdots & \vdots & \ddots & \vdots \\ E_{r,0}(s) & E_{r,1}(s) & \dots & E_{r,r}(s) \end{pmatrix} \tag{2.1}$$

and we denote

$$x = \big(x^{(0)}, \dots, x^{(r)}\big). \tag{2.2}$$

for every $x \in \mathbb{R}^N$, where $x^{(j)} \in \mathbb{R}^{p_j}$ for $j = 0, \dots, r$. We also denote

$$|x^{(j)}|_\delta = \sum_{k=1}^{p_j} |x_k^{(j)}|^{\frac{1}{2j+1}}, \qquad |x|_\delta = |x^{(0)}|_\delta + \dots + |x^{(r)}|_\delta = \sum_{k=1}^{N} |x_k|^{\frac{1}{q_k}}, \tag{2.3}$$

so that we can write the norm defined in (1.16) as $\|(x,t)\| = |x|_\delta + |t|^{\frac{1}{2}}$.

As a preliminary result, we show that there exists a constant $C_{T,H}$, only depending by T and H, such that

$$|E(s)y|_\delta \leq C_{T,H}\|(y,s)\|, \qquad \forall\, y \in H,\ s \in [-T, T]. \tag{2.4}$$

From condition H_2 it follows that $E_{0,0}(s) = I_{p_0} + s\, O_{0,0}(s)$,

$$\begin{aligned} E_{j,j}(s) =& \big(I_{p_j} + s\, O_{j,j}(s)\big), \qquad j = 1, \dots, r, \\ E_{j,k}(s) =& \frac{(-s)^{j-k}}{(j-k)!}\big(I_{p_j} + s\, O_{j,j}(s)\big) B_j^T \dots B_{k+1}^T, \qquad j = 1, \dots, r,\ k = 0, \dots, j-1 \end{aligned}$$

and $E_{j,k}(s) = s\, O_{j,k}(s)$, for $k > j$. Here $O_{j,k}$ denotes a $p_j \times p_k$ matrix whose coefficients continuously depend on s. We then have

$$\begin{aligned} |E(s)y|_\delta \leq \sum_{j,k=0}^{r} |E_{j,k}(s)y^{(k)}|_\delta = \sum_{j=0}^{r}\sum_{k=0}^{j-1} |E_{j,k}(s)y^{(k)}|_\delta + \\ \sum_{j=0}^{r} |E_{j,j}(s)y^{(j)}|_\delta + \sum_{j=0}^{r}\sum_{k=j+1}^{r} |E_{j,k}(s)y^{(k)}|_\delta \leq \\ \sum_{j=0}^{r}\sum_{k=0}^{j-1} c_T\|(y,s)\|^{\frac{2j-2k}{2j+1}}\|(y,s)\|^{\frac{2k+1}{2j+1}} + c_T\|(y,s)\| + \sum_{j=0}^{r}\sum_{k=j+1}^{r} c_T\|(y,s)\|^{\frac{2k+3}{2j+1}} \end{aligned}$$

where c_T is a constant only depending on T. This proves (2.4).

We are now in position to conclude the proof of Lemma 2.1. The assertion **i)** directly follows from the fact that $(x,t)^{-1} = (-E(-t)x, -t)$, whereas **ii)** is an immediate consequence of

$$\|(x,t)\circ(\xi,\tau)\| = \|(\xi + E(\tau)x, t+\tau)\| \le \|(\xi,\tau)\| + |E(\tau)x|_\delta + |t|^{\frac{1}{2}}$$

and of the fact that $|E(\tau)x|_\delta \le C_{T,H}\|(x,\tau)\| \le C_{T,H}\left(\|(x,t)\| + \|(\xi,\tau)\|\right)$.

We next prove that $C_{T,H}$ can be chosen arbitrarily close to 1 provided that T is sufficiently small. We consider only **ii)**. Since

$$\|(x,t)\circ(\xi,\tau)\| \le \|(\xi + E(\tau)x, t+\tau)\| \le \|(\xi,\tau)\| + \|(x,t)\| + |(E(\tau)-I_N)x|_\delta$$

where I_N denotes the $N\times N$ identity matrix, it is sufficient to show that

$$|(E(s)-I_N)y|_\delta \to 0, \quad \text{as } s\to 0, \tag{2.5}$$

uniformly on $y\in H$. We proceed as above

$$|(E(s)-I_N)y|_\delta \le = \sum_{j=0}^{r}\sum_{k=0}^{j-1}|E_{j,k}(s)y^{(k)}|_\delta + \sum_{j=0}^{r}|(E_{j,j}(s)-I_{p_j})y^{(j)}|_\delta + \sum_{j=0}^{r}\sum_{k=j+1}^{r}|E_{j,k}(s)y^{(k)}|_\delta \le$$
$$\sum_{j=0}^{r}\sum_{k=0}^{j-1} c_T|s|^{\frac{j-k}{2j+1}}\|(y,0)\|^{\frac{2k+1}{2j+1}} + c_T|s|^{\frac{1}{2j+1}}\|(y,0)\| + \sum_{j=0}^{r}\sum_{k=j+1}^{r} c_T|s|^{\frac{1}{2j+1}}\|(y,0)\|^{\frac{2k+1}{2j+1}}.$$

This proves the claim (2.5) and accomplishes the proof. □

In the sequel, in order to simplify the notation and to emphasize the last assertion of the lemma, we will write C_T instead of $C_{T,H}$.

Remark 2.2. *As a direct consequence of Lemma 2.1, we get the following assertion. Let $C_T = C_{T,H}$ and let M be any constant in $]0, C_T^{-2}[$. Then, for every $z,\zeta \in H\times[-T,T]$ we have*

$$\|\zeta\| \le M\|z\| \quad \Rightarrow \quad \frac{1-M\,C_T^2}{C_T}\|z\| \le \|z\circ\zeta\| \le C_T(1+M)\|z\|. \tag{2.6}$$

We finally note that

$$\|(\xi,\tau)^{-1}\circ(x,t)\| \le C_T\big(\|(x,t)\| + \|(\xi,\tau)\|\big), \tag{2.7}$$

for every $(\xi,\tau)\in H\times\mathbb{R}, (x,t)\in\mathbb{R}^{N+1}$ such that $|t-\tau|\le T$, where the constant C_T can be chosen arbitrarily close to 1 provided that T is sufficiently small.

For every operator K of the form (1.2), we define the homogeneous operator K_0 by setting

$$K_0 u := \sum_{i,j=1}^{p_0} a_{i,j}\partial_{x_i x_j}u + Y_0 u, \qquad Y_0 = \langle B_0, \nabla\rangle - \partial_t \tag{2.8}$$

where

$$B_0 = \begin{pmatrix} 0 & B_1 & 0 & \cdots & 0 \\ 0 & 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & B_r \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{2.9}$$

It is known that the corresponding matrices E_0, $\mathcal{C}_0$ and $\mathcal{C}_0^{-1}$ satisfy

$$E_0(\lambda^2 s) = D(\lambda)E_0(s)D(\frac{1}{\lambda}), \quad \mathcal{C}_0(\lambda^2 t) = D\left(\frac{1}{\lambda}\right)\mathcal{C}_0(t)D\left(\frac{1}{\lambda}\right), \quad \mathcal{C}_0^{-1}(\lambda^{-2}t) = D(\lambda)\mathcal{C}_0^{-1}(t)D(\lambda) \tag{2.10}$$

for any $s, t \in \mathbb{R}$ and $\lambda > 0$ (see Remark 2.1 and Proposition 2.3 in [18]). Moreover, for every given $T > 0$, there exists a positive constant c_T such that

$$\begin{aligned} \left\| D\left(\frac{1}{\sqrt{t}}\right)(\mathcal{C}(t) - \mathcal{C}_0(t))\, D\left(\frac{1}{\sqrt{t}}\right) \right\| &\le c_T t\, \|\mathcal{C}_0(1)\| \\ \left\| D(\sqrt{t})\left(\mathcal{C}^{-1}(t) - \mathcal{C}_0^{-1}(t)\right) D(\sqrt{t}) \right\| &\le c_T t\, \|\mathcal{C}_0^{-1}(1)\| \end{aligned} \tag{2.11}$$

for every $t \in]0, T]$, and

$$\begin{aligned} \langle \mathcal{C}_0(t)x, x\rangle (1 - c_T t) \le \langle \mathcal{C}(t)x, x\rangle &\le \langle \mathcal{C}_0(t)x, x\rangle (1 + c_T t), \\ \left\langle \mathcal{C}_0^{-1}(t)y, y\right\rangle (1 - c_T t) \le \left\langle \mathcal{C}^{-1}(t)y, y\right\rangle &\le \left\langle \mathcal{C}_0^{-1}(t)y, y\right\rangle (1 + c_T t); \end{aligned} \tag{2.12}$$

for every $x, y \in \mathbb{R}^N, t \in]0, T]$ (see Lemma 3.3 in [18]). As a direct consequence, there exist two positive constants c'_T, c''_T such that

$$c'_T t^Q (1 - c_T t) \le \det \mathcal{C}(t) \le c''_T t^Q (1 + c_T t), \tag{2.13}$$

for every $(x, t) \in \mathbb{R}^N \times]0, T]$ such that $t < \frac{1}{c_T}$ (see formula (3.14) in [18]).

In order to give a preliminary estimate useful in the proof of the Harnack inequality, we next recall the parametrix method, used in [25] and [11]. For any given z_0 we consider the *frozen* operator

$$K_{z_0} = \sum_{i,j=1}^{p_0} a_{i,j}(z_0)\partial_{i,j} + \langle x, B\nabla\rangle - \partial_t, \tag{2.14}$$

we denote by Γ_{z_0} its fundamental solution and we define the *parametrix* as

$$Z(z, \zeta) = \Gamma_\zeta(z, \zeta). \tag{2.15}$$

We also recall (1.10), so that the fundamental solution of (1.22) with singularity at the origin, is

$$\Gamma_\mu(x, t) = \frac{(4\pi\mu)^{-\frac{N}{2}}}{\sqrt{\det\widetilde{\mathcal{C}}(t)}} \exp\Big(-\frac{1}{4\mu}\langle \widetilde{\mathcal{C}}^{-1}(t)x, x\rangle - t\,\mathrm{tr}B \Big), \qquad \widetilde{\mathcal{C}}(t) = \int_0^t E(s)\begin{pmatrix} I_{p_0} & 0 \\ 0 & 0 \end{pmatrix} E^T(s)ds. \tag{2.16}$$

We look for the fundamental solution Γ of L as a function in the form

$$\Gamma(z, \zeta) = Z(z, \zeta) + J(z, \zeta), \tag{2.17}$$

where J is an unknown function which is determined by the requirement that $L\Gamma(z, \zeta) = 0$, for $z \ne \zeta$. Let Γ_Λ^- and Γ_Λ^+ denote, respectively, the fundamental solution of the operators $K_{\frac{1}{\Lambda}}$ and K_Λ defined in (1.22) (Λ is the constant in hypothesis **[H2]**). Then the following inequalities hold:

$$\Lambda^{-N}\Gamma_\Lambda^-(z, \zeta) \le Z(z, \zeta) \le \Lambda^N \Gamma_\Lambda^+(z, \zeta) \quad \forall\, z, \zeta \in \mathbb{R}^{N+1}. \tag{2.18}$$

Moreover, consider any $\mu > \Lambda$ and denote by Γ^+ the fundamental solution of the operator defined in (1.22). Then, for any positive T, there exists a positive constant C, depending on T, μ and on the matrix B, such that

$$|J(z, \zeta)| \le C\,(t - \tau)^{\frac{\alpha}{2}}\Gamma^+(z, \zeta), \tag{2.19}$$

for every $x, \xi \in \mathbb{R}^N$ and t, τ with $0 < t - \tau < T$ (see [11], Corollary 4.4). Thus, from (2.16), (2.18) and (2.19), it follows that the fundamental solution Γ satisfies the estimate (1.21). We point out that (2.18) and (2.19) also give a lower bound that will be used in the proof of the Harnack inequality.

Remark 2.3. *From (2.18) and (2.19), it follows that*

$$\Gamma(z,\zeta) \geq \Lambda^{-N}\,\Gamma^-_\Lambda(z,\zeta) - C\,(t-\tau)^{\frac{\alpha}{2}}\Gamma^+(z,\zeta)$$

for every $z=(x,t), \zeta=(\xi,\tau)\in\mathbb{R}^{N+1}$ *such that* $0<t-\tau<T$.

We finally recall the usual property of the fundamental solution

$$\Gamma(x,t) = \int_{\mathbb{R}^N}\Gamma(x,t,y,s)\Gamma(y,s)dy, \quad \text{for every } s\in]0,t[, \tag{2.20}$$

and some pointwise bounds of Γ and of its derivatives that have been proved by Di Francesco and Pascucci (see Proposition 3.5 in [11]). For every $T>0$ and for any $k\in\mathbb{N}$, there exists a positive c_k, depending on $T,\Lambda,\lambda,k,$ and B, such that, if we set $\eta = D\left(\frac{1}{\sqrt{t-\tau}}\right)(x-E(t-\tau)\xi)$, then we have

$$\begin{aligned}
\left(1+|\eta|^2\right)^{\frac{k}{2}}\,\Gamma(x,t,\xi,\tau) &\leq c_k\,\Gamma^+(x,t,\xi,t),\\
\left(1+|\eta|^2\right)^{\frac{k}{2}}\,|\partial_{x_i}\Gamma(x,t,\xi,\tau)| &\leq c_k\,\frac{\Gamma^+(x,t,\xi,t)}{\sqrt{t-\tau}}, \quad \text{for } i=1,\dots,p_0\\
\left(1+|\eta|^2\right)^{\frac{k}{2}}\,|\partial_{x_i,x_j}\Gamma(x,t,\xi,\tau)| &\leq c_k\,\frac{\Gamma^+(x,t,\xi,t)}{t-\tau}, \quad \text{for } i,j=1,\dots,p_0\\
\left(1+|\eta|^2\right)^{\frac{k}{2}}\,|Y\Gamma(x,t,\xi,\tau)| &\leq c_k\,\frac{\Gamma^+(x,t,\xi,t)}{t-\tau}.
\end{aligned} \tag{2.21}$$

We consider the operator K_λ defined as

$$K_\lambda := \lambda^2\left(\delta(\lambda)\circ K\circ\delta(\frac{1}{\lambda})\right), \qquad \lambda\in]0,1], \tag{2.22}$$

and we prove some uniform-in-λ estimates of its fundamental solution, and of its derivatives. Then we prove a representation formula for $u\in C_0^\infty$ solution of $Ku=g$.

We first remark that, K is homogeneous (i.e. K satisfy condition (1.14)) if, and only if, $K=K_\lambda$, for every $\lambda>0$. In order to explicitly write K_λ and its fundamental solution, we note that, if

$$B = \begin{pmatrix} B_{0,0} & B_1 & 0 & \cdots & 0\\ B_{1,0} & B_{1,1} & B_2 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ B_{r-1,0} & B_{r-1,1} & B_{r-1,2} & \cdots & B_r\\ B_{r,0} & B_{r,1} & B_{r,2} & \cdots & B_{r,r}\end{pmatrix}$$

where $B_{i,j}$ are the $p_i\times p_j$ blocks denoted by "$*$" in (1.7), then $K_\lambda = \sum_{i,j=1}^{p_0} a_{ij}\partial_{x_ix_j} + Y_\lambda$, where $Y_\lambda := \langle x, B_\lambda\nabla\rangle - \partial_t$, and

$$B_\lambda = \begin{pmatrix} \lambda^2 B_{0,0} & B_1 & 0 & \cdots & 0\\ \lambda^4 B_{1,0} & \lambda^2 B_{1,1} & B_2 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \lambda^{2r}B_{r-1,0} & \lambda^{2r-2}B_{r-1,1} & \lambda^{2r-4}B_{r-1,2} & \cdots & B_r\\ \lambda^{2r+2}B_{r,0} & \lambda^{2r}B_{r,1} & \lambda^{2r-2}B_{r,2} & \cdots & \lambda^2 B_{r,r}\end{pmatrix} \tag{2.23}$$

The fundamental solution Γ_λ of K_λ reads

$$\Gamma_\lambda(x,t,0,0) = \begin{cases} \frac{(4\pi)^{-\frac{N}{2}}}{\sqrt{\det\mathcal{C}_\lambda(t)}}\exp\left(-\frac{1}{4}\langle\mathcal{C}_\lambda^{-1}(t)x,x\rangle - t\,\mathrm{tr}(B_\lambda)\right) & \text{if } t>0,\\ 0 & \text{if } t\leq 0,\end{cases} \tag{2.24}$$

with

$$E_\lambda(s) = \exp(-sB_\lambda^T), \qquad \mathcal{C}_\lambda(t) = \int\limits_0^t E_\lambda(s) \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix} E_\lambda^T(s) ds. \tag{2.25}$$

The translation group "$\circ_\lambda$" related to K_λ is

$$(x,t) \circ_\lambda (\xi,\tau) = (\xi + E_\lambda(\tau)x, t+\tau), \qquad (x,t),(\xi,\tau) \in \mathbb{R}^{N+1}. \tag{2.26}$$

We remark that

$$\delta(\lambda)(\zeta \circ_\lambda z) = \big(\delta(\lambda)\zeta\big) \circ \big(\delta(\lambda)z\big), \quad \forall z,\zeta \in \mathbb{R}^{N+1} \text{ and } \lambda > 0. \tag{2.27}$$

The above identity is a direct consequence of the following result, which relates the matrices E_λ, C_λ and C_λ^{-1} with the dilations.

Lemma 2.4. *For every $t, R > 0$, $\lambda \in]0,1]$, we have*

$$E_{\frac{R}{\lambda}}(\lambda^2 t) = D(\lambda) E_R(t) D\left(\lambda^{-1}\right). \tag{2.28}$$

$$D(\lambda)\mathcal{C}_R(t)D(\lambda) = \mathcal{C}_{\frac{R}{\lambda}}(\lambda^2 t), \tag{2.29}$$

$$D\left(\lambda^{-1}\right)\mathcal{C}_R^{-1}(t) D\left(\lambda^{-1}\right) = \mathcal{C}_{\frac{R}{\lambda}}^{-1}(\lambda^2 t) \tag{2.30}$$

Proof. We use the Taylor expansion of $E_R(t)$ and the fact that $D(\lambda)B_R^T D\left(\frac{1}{\lambda}\right) = \lambda^2 B_{\frac{R}{\lambda}}^T$. We have

$$D(\lambda)E_R(t)D\left(\lambda^{-1}\right) = D(\lambda)\bigg(\sum_{k=0}^{+\infty} \frac{(-t)^k}{k!}(B_R^T)^k\bigg)D\left(\lambda^{-1}\right) =$$

$$\sum_{k=0}^{+\infty} \frac{(-t)^k}{k!}\left(D(\lambda)B_R^T D\left(\lambda^{-1}\right)\right)^k = \sum_{k=0}^{+\infty} \frac{(-t)^k}{k!}\left(\lambda^2 B_{\frac{R}{\lambda}}^T\right)^k = E_{\frac{R}{\lambda}}(\lambda^2 t)$$

This proves (2.28). We next consider (2.29). We have

$$D(\lambda)\mathcal{C}_R(t)D(\lambda) = D(\lambda)\bigg(\int_0^t E_R(s)AE_R^T(s)ds\bigg)D(\lambda) =$$

$$\lambda^2 \int_0^t D(\lambda)E_R(s)D\left(\lambda^{-1}\right)AD\left(\lambda^{-1}\right)E_R(s)^T D(\lambda)ds =$$

(by (2.28))

$$\int_0^t E_{\frac{R}{\lambda}}(\lambda^2 s)AE_{\frac{R}{\lambda}}^T(\lambda^2 s)\lambda^2 ds = \int_0^{\lambda^2 t} E_{\frac{R}{\lambda}}(\tau)AE_{\frac{R}{\lambda}}^T(\tau)d\tau = \mathcal{C}_{\frac{R}{\lambda}}(\lambda^2 t).$$

The proof of (2.30) is an immediate consequence of (2.29). □

The following inequalities analogous to (2.11) and (2.12) hold: for every $T > 0$, there exists a positive constant c_T such that

$$\begin{aligned} \left\| D\left(\frac{1}{\sqrt{t}}\right)\left(\mathcal{C}_\lambda(t) - \mathcal{C}_0(t)\right) D\left(\frac{1}{\sqrt{t}}\right)\right\| &\le c_T \lambda t \left\|\mathcal{C}_0(1)\right\| \\ \left\| D(\sqrt{t})\left(\mathcal{C}_\lambda^{-1}(t) - \mathcal{C}_0^{-1}(t)\right) D(\sqrt{t})\right\| &\le c_T \lambda t \left\|\mathcal{C}_0^{-1}(1)\right\| \end{aligned} \tag{2.31}$$

for every $t \in]0,T]$. Moreover

$$\begin{aligned} \langle \mathcal{C}_0(t)x, x\rangle (1 - c_T\lambda t) \le \langle \mathcal{C}_\lambda(t)x, x\rangle \le \langle \mathcal{C}_0(t)x, x\rangle (1 + c_T\lambda t), \\ \left\langle \mathcal{C}_0^{-1}(t)y, y\right\rangle (1 - c_T\lambda t) \le \left\langle \mathcal{C}_\lambda^{-1}(t)y, y\right\rangle \le \left\langle \mathcal{C}_0^{-1}(t)y, y\right\rangle (1 + c_T\lambda t); \end{aligned} \tag{2.32}$$

for every $x, y \in \mathbb{R}^N, t \in]0,T]$. We omit the proof since it follows the same lines of the proof of (2.11) and (2.12), respectively, and relies on the application of the identities stated in Lemma 2.4. In the sequel we shall need the following result, that is an improvement of the estimate (2.13).

Lemma 2.5. *For every positive T, there exists a constant $c_T > 0$ such that*

$$\frac{|\det \mathcal{C}_\lambda(t) - \det \mathcal{C}_0(t)|}{\det \mathcal{C}_0(t)} \le c_T\lambda t,$$

for every $t \in]0,T]$ and for any $\lambda \in]0,1]$.

Proof. We first note that, from (2.10) and (1.13) it follows that

$$\det \mathcal{C}_0(t) = \det D(\sqrt{t}) \det \mathcal{C}_0(1) \det D(\sqrt{t}), \quad \text{and} \quad \det D(s) = s^Q,$$

then we have

$$\begin{aligned} \frac{\det \mathcal{C}_\lambda(t) - \det \mathcal{C}_0(t)}{\det \mathcal{C}_0(t)} &= \frac{\det \mathcal{C}_\lambda(t) - \det \mathcal{C}_0(t)}{t^Q \det \mathcal{C}_0(1)} = \\ &\frac{\det D\left(\frac{1}{\sqrt{t}}\right)(\det \mathcal{C}_\lambda(t) - \det \mathcal{C}_0(t)) \det D\left(\frac{1}{\sqrt{t}}\right)}{\det \mathcal{C}_0(1)} = \\ &\frac{\det\left(D\left(\frac{1}{\sqrt{t}}\right)\mathcal{C}_\lambda(t) D\left(\frac{1}{\sqrt{t}}\right)\right) - \det \mathcal{C}_0(1)}{\det \mathcal{C}_0(1)} \end{aligned}$$

We recall that, if A and B are two $n \times n$ matrix with $\|A\| \le M$ and $\|B\| \le M$, then $|\det A - \det B| \le C(n,M)\|A - B\|$, for some positive constant $C(n,M)$ only depending on n and M. The first inequality in (2.31) implies that

$$\left\| D\left(\frac{1}{\sqrt{t}}\right)\mathcal{C}_\lambda(t) D\left(\frac{1}{\sqrt{t}}\right)\right\| \le M$$

for some positive constant M only depending on T and on the matrix B, as a consequence we have

$$\frac{\left|\det\left(D\left(\frac{1}{\sqrt{t}}\right)\mathcal{C}_\lambda(t) D\left(\frac{1}{\sqrt{t}}\right)\right) - \det \mathcal{C}_0(1)\right|}{\det \mathcal{C}_0(1)} \le \frac{C(N,M)}{\det \mathcal{C}_0(1)} \left\| D\left(\frac{1}{\sqrt{t}}\right)(\mathcal{C}_\lambda(t) - \mathcal{C}_0(t)) D\left(\frac{1}{\sqrt{t}}\right)\right\|.$$

The thesis then follows from the first inequality in (2.31). □

Proposition 2.6. *For every $R \in]0,1]$, and any $z \in \mathbb{R}^{N+1}$, $z \neq 0$, we have*

$$\Gamma_R\left(\delta(\frac{1}{R})(x,t)\right) = R^Q\Gamma(x,t) \tag{2.33}$$

$$\partial_{x_i}\Gamma_R\left(\delta(\frac{1}{R})(x,t)\right) = R^{Q+1}\partial_{x_i}\Gamma(x,t), \quad \forall\, i = 1, \dots, p_0. \tag{2.34}$$

Proof. The explicit expression of Γ_R is

$$\Gamma_R\left(\delta(R^{-1})(x,t)\right)=\frac{(4\pi)^{-\frac{N}{2}}}{\sqrt{\det\mathcal{C}_R(\frac{t}{R^2})}}\exp\left(-\frac{1}{4}\langle D(R^{-1})\mathcal{C}_R^{-1}(tR^{-2})D(R^{-1})x,x\rangle\right)\exp\left(-tR^{-2}\mathrm{tr}B_R\right).$$

From (2.30) we get

$$D(R^{-1})\mathcal{C}_R^{-1}(tR^{-2})D(R^{-1})=\mathcal{C}^{-1}(t),$$

moreover, from (2.29) it follows that

$$\det\left(\mathcal{C}_R(tR^{-2})\right)=\det\left(D(R^{-1})\right)\det\left(D(R)\mathcal{C}_R(tR^{-2})D(R)\right)\det\left(D(R^{-1})\right)=R^{-2Q}\det(\mathcal{C}(t)).$$

The thesis then follows from the fact that $\mathrm{tr}B_R=R^2\mathrm{tr}B$. The proof of (2.34) is analogous. □

Proposition 2.7. *Let Γ be a fundamental solution of $Ku=0$. For every $T>0$ there exists a positive constant C'_T such that:*

$$\Gamma(z,w)\le\frac{C'_T}{\|w^{-1}\circ z\|^Q},\tag{2.35}$$

$$\left|\partial_{x_j}\Gamma(z,w)\right|\le\frac{C'_T}{\|w^{-1}\circ z\|^{Q+1}},\qquad j=1,\dots p_0,\tag{2.36}$$

$$\left|\partial_{x_ix_j}\Gamma(z,w)\right|\le\frac{C'_T}{\|w^{-1}\circ z\|^{Q+2}},\qquad i,j=1,\dots p_0,\tag{2.37}$$

$$Y\Gamma(z,w)\le\frac{C'_T}{\|w^{-1}\circ z\|^{Q+2}},\tag{2.38}$$

for every $z,w\in\mathbb{R}^N\times[-T,T]$. Moreover, if $H\subset\mathbb{R}^N$ is a compact set and M is as in Remark 2.2, then there exists a positive constant C''_T such that

$$|\Gamma(z,w)-\Gamma(\bar z,w)|\le C''_T\frac{\|z^{-1}\circ\bar z\|}{\|w^{-1}\circ z\|^{Q+1}},\tag{2.39}$$

$$\left|\partial_{x_j}\Gamma(z,w)-\partial_{x_j}\Gamma(\bar z,w)\right|\le C''_T\frac{\|z^{-1}\circ\bar z\|}{\|w^{-1}\circ z\|^{Q+2}},\qquad j=1,\dots p_0,\tag{2.40}$$

$$\left|\partial_{x_ix_j}\Gamma(z,w)-\partial_{x_ix_j}\Gamma(\bar z,w)\right|\le C''_T\frac{\|z^{-1}\circ\bar z\|}{\|w^{-1}\circ z\|^{Q+3}},\qquad i,j=1,\dots p_0,\tag{2.41}$$

$$|Y\Gamma(z,w)-Y\Gamma(\bar z,w)|\le C''_T\frac{\|z^{-1}\circ\bar z\|}{\|w^{-1}\circ z\|^{Q+3}},\tag{2.42}$$

for every $z,\bar z,w\in\mathbb{R}^N\times[-T,T]$ such that $\|z^{-1}\circ\bar z\|\le M\|w^{-1}\circ z\|$ and $w^{-1}\circ z\in H\times[-T,T]$.

The proof is postponed at the end of this section.

In the sequel we will consider the analogous of the operators K_λ with non-constant coefficients a_{ij}:

$$L_\lambda:=\lambda^2\left(\delta(\lambda)\circ L\circ\delta\left(\frac{1}{\lambda}\right)\right).\tag{2.43}$$

If (a_{ij}) is the matrix of the coefficients of the second order part of L, then the coefficients of L_λ are $(a_\lambda)_{ij}(z):=a_{ij}(\delta(\lambda)(z))$. As a direct consequence of (2.27), it is possible to relate the module of continuity of the $(a_\lambda)_{ij}$'s with the module of continuity of the a_{ij}'s, as the following remark states

Remark 2.8. *If the coefficients a_{ij} of L are Hölder continuous in the sense of Definition 1.1, then the coefficients $(a_\lambda)_{ij}$ of L_λ are Hölder continuous with respect to the translation group $\circ_\lambda$. Indeed, we have*

$$|(a_\lambda)_{ij}(z) - (a_\lambda)_{ij}(\zeta)| = |a_{ij}(\delta(\lambda)(z)) - a_{ij}(\delta(\lambda)(\zeta))| \le c\,\|\delta(\lambda)(\zeta)^{-1} \circ \delta(\lambda)(z)\|^\alpha = c\,\lambda^\alpha \|\zeta^{-1} \circ_\lambda z\|^\alpha.$$

Proposition 2.9. *For every $T > 0$ there exist two positive constants μ and c, with $\mu > \Lambda$ depending only on T, on the matrix B and on Λ, such that*

$$|\Gamma_\lambda(x,t) - \Gamma_0(x,t)| \le c\lambda \Gamma_0^+(x,t), \qquad \forall\, x \in \mathbb{R}^N, t \in]0,T[\tag{2.44}$$

$$|\partial_{x_i}\Gamma_\lambda(x,t) - \partial_{x_i}\Gamma_0(x,t)| \le c\frac{\lambda}{\sqrt{t}}\Gamma_0^+(x,t), \quad \forall\, x \in \mathbb{R}^N, t \in]0,T[, i = 1,\dots,p_0 \tag{2.45}$$

In (2.44) and in (2.45), Γ_0 denotes the fundamental solution of K_0 defined in (2.8), *and Γ_0^+ is the fundamental solution of*

$$K_0^+ = \mu \sum_{i=1}^{p_0} \partial_i^2 + \langle x, B_0 \nabla\rangle - \partial_t.$$

Proof. From the explicit expression of Γ_λ and Γ_0, and from the second inequality (2.32), we get

$$|\Gamma_\lambda(x,t) - \Gamma_0(x,t)| \le c_N \left| \frac{1}{\sqrt{\det \mathcal{C}_\lambda(t)}} - \frac{1}{\sqrt{\det \mathcal{C}_0(t)}} \right| e^{-\frac{1}{4}\langle \mathcal{C}_0^{-1}(t)x,x\rangle(1-c_t\lambda t)} +$$

$$+ \frac{c_N}{\sqrt{\det \mathcal{C}_0(t)}} \left| e^{-\frac{1}{4}\langle \mathcal{C}_0^{-1}(t)x,x\rangle(1-c_t\lambda t)} - e^{-\frac{1}{4}\langle \mathcal{C}_0^{-1}(t)x,x\rangle} \right|$$

As a direct consequence of Lemma 2.5 we have that there exist a constant c such that

$$\left| \frac{1}{\sqrt{\det \mathcal{C}_\lambda(t)}} - \frac{1}{\sqrt{\det \mathcal{C}_0(t)}} \right| \le \lambda \frac{c\,T}{\sqrt{\det \mathcal{C}_0(t)}}. \tag{2.46}$$

We next fix a positive T_0 such that $T_0 c_T < 1$. We recall (2.16), and note that it is possible to choose μ such that the function Γ_0^+ satisfies

$$\frac{1}{\sqrt{\det \mathcal{C}_0(t)}} e^{-\frac{1}{4}\langle \mathcal{C}_0^{-1}(t)x,x\rangle(1-c_T t)} \le c_0\, \Gamma_0^+(x,t), \qquad \text{for every } (x,t) \in \mathbb{R}^N \times]0,T_0[,$$

for some positive constant c_0 depending on T_0, μ, and on the matrix B. On the other hand we have

$$\left| 1 - e^{-c_T \lambda t \langle \mathcal{C}_0^{-1}(t)x,x\rangle} \right| \le c_T\, \lambda\, t \langle \mathcal{C}_0^{-1}(t)x,x\rangle \le c_T'\, \lambda\, T \left| D\left(\frac{1}{\sqrt{t}}\right) x \right|^2$$

by the mean value theorem and (2.10). Thus, from (2.21), it follows that

$$\left| 1 - e^{-c_T \lambda t \langle \mathcal{C}_0^{-1}(t)x,x\rangle(1-c_T t)} \right| \frac{1}{\sqrt{\det \mathcal{C}_0(t)}} e^{-\frac{1}{4}\langle \mathcal{C}_0^{-1}(t)x,x\rangle} \le c_1 \lambda\, \Gamma_0^+(x,t),$$

for some positive constant c_1. Summarizing the above inequalities we finally find that there exists a positive constant c_2 such that

$$|\Gamma_\lambda(x,t) - \Gamma_0(x,t)| \le c_2 \lambda\, \Gamma_0^+(x,t)$$

for any $x \in \mathbb{R}^N$ and $0 < t < T_0$. This concludes the proof in the case that $T c_T < 1$. If $T c_T \ge 1$, we use repeatedly the identity (2.20) and we conclude the proof after a finite number of iteration.

In order to prove (2.45), we claim that:

$$|((\mathcal{C}_\lambda^{-1}(t) - \mathcal{C}_0^{-1}(t))x)_i| \le c\frac{\lambda}{\sqrt{t}}\left|D\left(\frac{1}{\sqrt{t}}\right)x\right| \tag{2.47}$$

for every $i = 1, \dots, p_0$. We first observe that

$$|((\mathcal{C}_\lambda^{-1}(t) - \mathcal{C}_0^{-1}(t))x)_i| = \frac{1}{\sqrt{t}}|((D(\sqrt{t})[\mathcal{C}_\lambda^{-1}(t) - \mathcal{C}_0^{-1}(t)]D(\sqrt{t})D(\frac{1}{\sqrt{t}}))x)_i|$$

Thanks to (2.31) and (2.10), we have

$$\begin{aligned}
\|D(\sqrt{t})[\mathcal{C}_\lambda^{-1}(t) - \mathcal{C}_0^{-1}(t)]D(\sqrt{t})\| &= \sup_{|v|=1}|\langle D(\sqrt{t})[\mathcal{C}_\lambda^{-1}(t) - \mathcal{C}_0^{-1}(t)]D(\sqrt{t})v, v\rangle| = \\
&= \sup_{|v|=1}|\langle[\mathcal{C}_\lambda^{-1}(t) - \mathcal{C}_0^{-1}(t)]D(\sqrt{t})v, D(\sqrt{t})v\rangle| \le \\
&\le t\lambda\, c_T \sup_{|v|=1}|\langle \mathcal{C}_0^{-1}(t)D(\sqrt{t})v, D(\sqrt{t})v\rangle| = \\
&= t\lambda\, c_T \sup_{|v|=1}|\langle \mathcal{C}_0^{-1}(1)v, v\rangle| \le T\lambda\, c_T\|\mathcal{C}_0^{-1}(1)\|
\end{aligned}$$

This implies (2.47). The thesis follows from the same argument as above. We omit the details. □

Lemma 2.10. *Let Γ and Γ_0 be the fundamental solution of K and K_0, respectively. Then for every $i, j = 1, \dots, p_0$, we have*

$$\int_{\|z\|=\varepsilon} \partial_{x_i}\Gamma(z)\nu_j d\sigma(z) \to \int_{\|z\|=1} \partial_{x_i}\Gamma_0(z)\nu_j d\sigma(z), \qquad \text{as } \varepsilon \to 0^+$$

where ν_j is the j-th component of the outer normal to the surface $\{z \in \mathbb{R}^{N+1} : \|z\| = 1\}$.

Proof. We split the set $\{z \in \mathbb{R}^{N+1} : \|z\| = \varepsilon\}$ as $B_\varepsilon^+ \cup B_\varepsilon^-$, where

$$B_\varepsilon^+ = \{(x,t) \in \mathbb{R}^{N+1} : \|(x,t)\| = \varepsilon, t \ge 0\}, \quad B_\varepsilon^- = \{(x,t) \in \mathbb{R}^{N+1} : \|(x,t)\| = \varepsilon, t < 0\},$$

and we describe B_ε^+, and B_ε^- as the graph of the function $\Phi_\varepsilon : A_\varepsilon \to \mathbb{R}$, where

$$\Phi_\varepsilon(x) = \left(\varepsilon - |x|_\delta\right)^2, \quad A_\varepsilon = \{x \in \mathbb{R}^N : |x|_\delta \le \varepsilon\}.$$

We have

$$\begin{aligned}
&\int_{B_\varepsilon^+} \partial_{x_i}\Gamma(x,t)\nu_j(x,t)d\sigma(x,t) = -\int_{A_\varepsilon} \partial_{x_i}\Gamma(x, \Phi_\varepsilon(x))\frac{\partial\Phi_\varepsilon(x)}{\partial x_j}dx = \\
&\qquad \text{(by the change of variable } x = D(\varepsilon)y, \text{ since } \Phi_\varepsilon(D(\varepsilon)y) = \varepsilon^2\Phi_1(y)) \\
&-\int_{A_1} \partial_{x_i}\Gamma(\delta_\varepsilon(y, \Phi_1(y)))\frac{\partial\Phi_1(y)}{\partial y_j}\varepsilon^{Q+1}dy = -\int_{A_1} \partial_{x_i}\Gamma_\varepsilon(y, \Phi_1(y))\frac{\partial\Phi_1(y)}{\partial y_j}dy = \\
&\int_{B_1^+} \partial_{x_i}\Gamma_\varepsilon(x,t)\nu_j(x,t)d\sigma(x,t),
\end{aligned}$$

by (2.34). The same argument applied to the set B_ε^- shows that

$$\int_{\|z\|=\varepsilon} \partial_{x_i}\Gamma(z)\nu_j d\sigma(z) = \int_{\|z\|=1} \partial_{x_i}\Gamma_\varepsilon(z)\nu_j d\sigma(z).$$

Then,

$$\left| \int_{\|z\|=\varepsilon} \partial_{x_i}\Gamma(z)\nu_j d\sigma(z) - \int_{\|\zeta\|=1} \partial_{x_i}\Gamma_0(\zeta)\nu_j d\sigma(\zeta) \right|$$

$$\leq \int_{\|\zeta\|=1} |\partial_{x_i}\Gamma_\varepsilon(\zeta) - \partial_{x_i}\Gamma_0(\zeta)|\nu_j d\sigma(\zeta)$$

$$(\text{by } (2.45)) \ \leq c\,\varepsilon \int_{\|(\xi,\tau)\|=1} \frac{\Gamma_0^+(\xi,\tau)}{\sqrt{\tau}}\nu_j d\sigma(\xi,\tau) \leq c_1\varepsilon$$

for some positive constant c_1 only depending on the operator K. This concludes the proof. □

In order to prove a representation formula for the second order derivatives of a solution u to $Ku = g$, we consider a function $\eta \in C^\infty(\mathbb{R}^{N+1})$ such that $0 \leq \eta \leq 1$, $\eta(z) = 1$ if $\|z\| \geq 1$ and $\eta(z) = 0$ if $\|z\| \leq \frac{1}{2}$. For every $\varepsilon > 0$ we set

$$\eta_\varepsilon(z) = \eta\left(\delta\left(1/\varepsilon\right) z\right), \tag{2.48}$$

and we note that there exists a positive constant c, only depending on η, such that

$$|\partial_{x_i}\eta_\varepsilon(w^{-1}\circ z)| \leq \frac{c}{\varepsilon}, \quad |\partial_{x_ix_j}\eta_\varepsilon(w^{-1}\circ z)| \leq \frac{c}{\varepsilon^2}, \quad |Y\eta_\varepsilon(w^{-1}\circ z)| \leq \frac{c}{\varepsilon^2}, \tag{2.49}$$

for every $z, w \in \mathbb{R}^{N+1}, i, j = 1, \ldots, p_0$ and $\varepsilon \in]0,1]$. Besides $\partial_{x_i}\eta_\varepsilon(w^{-1}\circ z) = 0, \partial_{x_ix_j}\eta_\varepsilon(w^{-1}\circ z) = 0$ and $Y\eta_\varepsilon(w^{-1}\circ z) = 0$ whenever $\|w^{-1}\circ z\| \leq \frac{\varepsilon}{2}$.

Proposition 2.11. *Let $u \in C_0(\mathbb{R}^{N+1})$ be such that $u, \partial_{x_i}u, \partial_{x_ix_j}u$ and Yu belong to $C^\alpha(\mathbb{R}^{N+1})$ for $i, j = 1, \ldots, p_0$, and let denote $g = Ku$. Then, for every $z \in \mathbb{R}^{N+1}$, for every $i, j = 1, \ldots, p_0$, we have*

$$\partial_{x_ix_j}u(z) = -\lim_{\varepsilon\to 0}\int_{\|w^{-1}\circ z\|\geq\varepsilon} \partial_{x_ix_j}\Gamma(z,w)g(w)dw - g(z)\int_{\|\zeta\|=1} \partial_{x_i}\Gamma_0(\zeta)\nu_j d\sigma(\zeta) \tag{2.50}$$

where Γ and Γ_0 denote respectively the fundamental solution of K and K_0, and ν_j is the j-th component of the outer normal to the surface $\{\zeta \in \mathbb{R}^{N+1} : \|\zeta\| = 1\}$.

Proof. For convenience, we set $z = (x,t)$, $w = (y,s)$, $\zeta = (\xi,\tau)$. From the very definition of fundamental solution, we have that

$$u(z) = -\int_{\mathbb{R}^{N+1}} \Gamma(z,w)g(w)dw, \qquad z \in \mathbb{R}^{N+1}$$

for every $u \in C_0^\infty(\mathbb{R}^{N+1})$. By our assumptions on u we have $g \in C^\alpha(\mathbb{R}^{N+1})$, then a standard density argument extends the above identity to any function u such that $u, \partial_{x_i}u, \partial_{x_ix_j}u$ and Yu belong to $C^\alpha(\mathbb{R}^{N+1})$. We next prove that

$$\partial_{x_i}u(z) = -\int_{\mathbb{R}^{N+1}} \partial_{x_i}\Gamma(z,w)g(w)dw, \qquad i = 1, \ldots, p_0. \tag{2.51}$$

If η_ε is the function defined in (2.48), we set

$$u_\varepsilon(z) = -\int_{\mathbb{R}^{N+1}} \eta_\varepsilon(w^{-1}\circ z)\Gamma(w^{-1}\circ z)g(w)dw.$$

By using (2.35), we get

$$|u_\varepsilon(z)-u(z)| \le \|g\|_\infty \int_{\|w^{-1}\circ z\|\le\varepsilon} \Gamma(w^{-1}\circ z)dw \le C_T'\|g\|_\infty \int_{\|w^{-1}\circ z\|\le\varepsilon} \frac{1}{\|w^{-1}\circ z\|^Q}dw$$

$$\text{(by the change of variable } \zeta=\delta(\tfrac{1}{\varepsilon})(w^{-1}\circ z),\ \text{note that } \det E_\varepsilon(\tau)=e^{-\varepsilon^2\tau \mathrm{tr}B})$$

$$= C_T'\|g\|_\infty \int_{\|\zeta\|\le 1} \frac{\varepsilon^{Q+2}e^{-\varepsilon^2\tau\mathrm{tr}B}}{\varepsilon^Q\|\zeta\|^Q}d\zeta \le C_T''\varepsilon^2\|g\|_\infty \int_{\|\zeta\|\le 1}\frac{d\zeta}{\|\zeta\|^Q}.$$

Then u_ε uniformly converges to u, as $\varepsilon\to 0$. Note that, for every $i=1,\dots,p_0$, we have

$$\partial_{x_i}u_\varepsilon(z) = -\int_{\mathbb{R}^{N+1}}\Big(\partial_{x_i}\eta_\varepsilon(w^{-1}\circ z)\Gamma(w^{-1}\circ z)+\eta_\varepsilon(w^{-1}\circ z)\partial_{x_i}\Gamma(w^{-1}\circ z)\Big)g(w)dw$$

so that, by using (2.49) and (2.36), we find

$$\left|\partial_{x_i}u_\varepsilon(z)-\int_{\mathbb{R}^{N+1}}\partial_{x_i}\Gamma(w^{-1}\circ z)g(w)dw\right| \le \|g\|_\infty\int_{\|w^{-1}\circ z\|\le\varepsilon}|\partial_{x_i}\Gamma(w^{-1}\circ z)|dw$$

$$+\|g\|_\infty\frac{c}{\varepsilon}\int_{\frac{\varepsilon}{2}\le\|w^{-1}\circ z\|\le\varepsilon}\Gamma(w^{-1}\circ z)dw \le c_1\|g\|_\infty\varepsilon$$

for a positive constant c_1 only depending on K. This proves that $\partial_{x_i}u_\varepsilon$ uniformly converges to the right hand side of (2.51), then (2.51) holds.

In order to conclude the proof, we set

$$v_\varepsilon(z) = -\int_{\mathbb{R}^{N+1}}\eta_\varepsilon(w^{-1}\circ z)\partial_{x_i}\Gamma(w^{-1}\circ z)g(w)dw.$$

Since $\eta_\varepsilon(w^{-1}\circ z)=1$ in the set $\{w\in\mathbb{R}^{N+1}:\|w^{-1}\circ z\|\ge\varepsilon\}$, we have

$$\partial_{x_j}v_\varepsilon(z) = -\int_{\|w^{-1}\circ z\|\ge\varepsilon}\partial_{x_jx_i}\Gamma(w^{-1}\circ z)g(w)dw$$

$$-\int_{\|w^{-1}\circ z\|\le\varepsilon}\partial_{x_j}[\eta_\varepsilon(w^{-1}\circ z)\partial_{x_i}\Gamma(w^{-1}\circ z)]g(w)dw = -I_1(\varepsilon,z)-I_2(\varepsilon,z)$$

for every $j=1,\dots,p_0$. We next show that $I_1(\varepsilon,z)$ uniformly converges on any compact subset of $\mathbb{R}^{N+1}$ as $\varepsilon\to 0^+$. For every $0<\varepsilon'<\varepsilon''$, we have

$$I_1(\varepsilon',z)-I_1(\varepsilon'',z) = \int_{\varepsilon'\le\|w^{-1}\circ z\|\le\varepsilon''}\partial_{x_ix_j}\Gamma(w^{-1}\circ z)(g(w)-g(z))dw$$

$$+g(z)\int_{\varepsilon'\le\|w^{-1}\circ z\|\le\varepsilon''}\partial_{x_ix_j}\Gamma(w^{-1}\circ z)dw = I_1'(\varepsilon',\varepsilon'',z)+g(z)I_1''(\varepsilon',\varepsilon'')$$

Since $g\in C^\alpha(\mathbb{R}^{N+1})$, and (2.37) holds, we find

$$|I_1'(\varepsilon',\varepsilon'',z)| \le c\,C_T'\int_{\varepsilon'\le\|w^{-1}\circ z\|\le\varepsilon''}\frac{dw}{\|w^{-1}\circ z\|^{Q+2-\alpha}} \le c_T'|\varepsilon''-\varepsilon'|^\alpha, \qquad (2.52)$$

for some positive constant c'_T that does not depend on z. By the change of variable in the integral appearing in I''_1, we find

$$\begin{aligned}\int_{\varepsilon'\le\|w^{-1}\circ z\|\le\varepsilon''}\partial_{x_ix_j}\Gamma(w^{-1}\circ z)dw &= \int_{\varepsilon'\le\|\zeta\|\le\varepsilon''}\partial_{\xi_i\xi_j}\Gamma(\zeta)e^{-\tau\,\mathrm{tr}\,B}d\zeta\\ &= \int_{\|\zeta\|=\varepsilon''}\partial_{\xi_i}\Gamma(\zeta)e^{-\tau\,\mathrm{tr}\,B}\nu_j d\sigma(\zeta)-\int_{\|\zeta\|=\varepsilon'}\partial_{\xi_i}\Gamma(\zeta)e^{-\tau\,\mathrm{tr}\,B}\nu_j d\sigma(\zeta),\end{aligned}\tag{2.53}$$

by the divergence Theorem. Hence Lemma 2.10 and (2.52) imply that $I_1(\varepsilon,\cdot)$ uniformly converges as $\varepsilon\to 0$.

We next consider $I_2(\varepsilon,z)$.

$$\begin{aligned}I_2(\varepsilon,z) &= \int_{\|w^{-1}\circ z\|\le\varepsilon}\partial_{x_j}[\eta_\varepsilon(w^{-1}\circ z)\partial_{x_i}\Gamma(w^{-1}\circ z)](g(w)-g(z))dw\\ &\quad +g(z)\int_{\|w^{-1}\circ z\|\le\varepsilon}\partial_{x_j}[\eta_\varepsilon(w^{-1}\circ z)\partial_{x_i}\Gamma(w^{-1}\circ z)]dw = I_2'(\varepsilon,z)+g(z)I_2''(\varepsilon)\end{aligned}$$

We have

$$\begin{aligned}|I_2'(\varepsilon,z)| &\le \int_{\frac{\varepsilon}{2}\le\|w^{-1}\circ z\|\le\varepsilon}|\partial_{x_j}\eta_\varepsilon(w^{-1}\circ z)||\partial_{x_i}\Gamma(w^{-1}\circ z)||g(w)-g(z)|dw\\ &\quad+\int_{\|w^{-1}\circ z\|\le\varepsilon}\eta_\varepsilon(w^{-1}\circ z)|\partial_{x_jx_i}\Gamma(w^{-1}\circ z)||g(w)-g(z)|dw.\end{aligned}$$

Using the fact that $g\in C^\alpha(\mathbb{R}^{N+1})$, (2.36) and (2.37) as before, we easily find that $|I_2'(\varepsilon,z)|\le c\,\varepsilon^\alpha$, for some positive constant c only depending on K. Moreover, by the change of variable $\zeta=w^{-1}\circ z$ and the divergence theorem, we get

$$I_2''(\varepsilon)=\int_{\|\zeta\|=\varepsilon}\eta_\varepsilon(\zeta)\partial_{\xi_i}\Gamma(\zeta)\nu_j(\zeta)e^{-\tau\,\mathrm{tr}\,B}d\sigma(\zeta)=\int_{\|\zeta\|=\varepsilon}\partial_{\xi_i}\Gamma(\zeta)\nu_j(\zeta)e^{-\tau\,\mathrm{tr}\,B}d\sigma(\zeta),$$

thus, by Lemma 2.10, we find

$$I_2''(\varepsilon)\to\int_{\|\zeta\|=1}\partial_{x_i}\Gamma_0(\zeta)\nu_j d\sigma(\zeta),\qquad\text{as }\varepsilon\to 0.$$

This proves that

$$\partial_{x_j}v_\varepsilon(z)\rightrightarrows-\lim_{\varepsilon\to 0}\int_{\|w^{-1}\circ z\|\ge\varepsilon}\partial_{x_ix_j}\Gamma(w^{-1}\circ z)g(w)dw-g(z)\int_{\|\zeta\|=1}\partial_{x_i}\Gamma_0(\zeta)\nu_j d\sigma(\zeta).$$

This completes the proof, since $v_\varepsilon(z)$ converges to $\partial_{x_i}u(z)$ as $\varepsilon\to 0$. □

Proof of Proposition 2.7. We first recall a result from [8]: for every $T>0$ there exists a positive constant C_T such that:

$$\Gamma_\lambda(z,\zeta)\le\frac{C_T}{\|\zeta^{-1}\circ z\|^Q},\qquad\forall z,\zeta\in\mathbb{R}^N\times[-T,T]\text{ and }\lambda\in]0,1].\tag{2.54}$$

Inequality (2.35) is a plain consequence of the above bound (with $\lambda=1$).

In order to prove (2.36), (2.37) and (2.38) we set $\zeta=(\xi,\tau)=w^{-1}\circ z$, so that $\Gamma(z,w)=\Gamma(\xi,\tau)$. Then

$$\begin{aligned}\partial_{x_j}\Gamma(\xi,\tau) &= -\frac{1}{2}\left(\mathcal{C}^{-1}(\tau)\xi\right)_j\Gamma(\xi,\tau)\quad\text{for }j=1,\dots,N,\\ \partial_{x_ix_j}\Gamma(\xi,\tau) &= \left(\frac{1}{4}\left(\mathcal{C}^{-1}(\tau)\xi\right)_i\left(\mathcal{C}^{-1}(\tau)\xi\right)_j-\frac{1}{2}\mathcal{C}^{-1}(\tau)_{i,j}\right)\Gamma(\xi,\tau)\quad\text{for }i,j=1,\dots,N.\end{aligned}\tag{2.55}$$

We next claim that

$$\begin{aligned} \left|\left(\mathcal{C}^{-1}(\tau)\xi\right)_j\right| &\le \frac{c_0}{\tau^{q_j/2}}\left|D(\frac{1}{\sqrt{\tau}})\xi\right| \quad \text{for } j=1,\dots,N,\\ \left|\mathcal{C}^{-1}(\tau)_{i,j}\right| &\le \frac{c_0}{\tau^{\frac{q_i+q_j}{2}}} \quad \text{for } i,j=1,\dots,N, \end{aligned} \tag{2.56}$$

for every $(\xi,\tau)\in\mathbb{R}^N\times[0,T[$, where the q_j's are as in the definition of the norm (1.16), and the constant c_0 only depends on T and on the matrix B. Indeed,

$$\begin{aligned} \left|\left(\mathcal{C}^{-1}(\tau)\xi\right)_j\right| \le & \left|\left(\left(\mathcal{C}^{-1}(\tau)-\mathcal{C}_0^{-1}(\tau)\right)\xi\right)_j\right| + \left|\left(\mathcal{C}_0^{-1}(\tau)\xi\right)_j\right| = \\ & \frac{1}{\tau^{q_j/2}}\left|\left(D(\sqrt{\tau})\left(\mathcal{C}^{-1}(\tau)-\mathcal{C}_0^{-1}(\tau)\right)D(\sqrt{\tau})D(\frac{1}{\sqrt{\tau}})\xi\right)_j\right| + \\ & \frac{1}{\tau^{q_j/2}}\left|\left(D(\sqrt{\tau})\mathcal{C}_0^{-1}(\tau)D(\sqrt{\tau})D(\frac{1}{\sqrt{\tau}})\xi\right)_j\right| \le \\ & \frac{1}{\tau^{q_j/2}}\left\|D(\sqrt{\tau})\left(\mathcal{C}^{-1}(\tau)-\mathcal{C}_0^{-1}(\tau)\right)D(\sqrt{\tau})\right\|\cdot\left|D(\frac{1}{\sqrt{\tau}})\xi\right| + \frac{1}{\tau^{q_j/2}}\left|\mathcal{C}_0^{-1}(1)D(\frac{1}{\sqrt{\tau}})\xi\right|, \end{aligned}$$

by (2.10). From (2.11) it then follows that

$$\left|\left(\mathcal{C}^{-1}(\tau)\xi\right)_j\right| \le \frac{1+c_T\tau}{\tau^{q_j/2}}\left\|\mathcal{C}_0^{-1}(1)\right\|\left|D(\frac{1}{\sqrt{\tau}})\xi\right|.$$

This proves the first assertion in (2.56). The proof of the second one is analogous, it is sufficient to note that

$$\mathcal{C}^{-1}(\tau)_{i,j} = \langle\mathcal{C}^{-1}(\tau)e_i, e_j\rangle, \qquad i,j=1,\dots,N,$$

where e_j denotes the j–th vector of the canonical basis of $\mathbb{R}^N$. By the homogeneity of the norm, we also have that

$$\|(\xi,\tau)\| = \left\|\left(D(\sqrt{\tau})D(\frac{1}{\sqrt{\tau}})\xi,\tau\right)\right\| = \sqrt{\tau}\left\|\left(D(\frac{1}{\sqrt{\tau}})\xi,1\right)\right\| \le c_1\sqrt{\tau}\left(\left|D(\frac{1}{\sqrt{\tau}})\xi\right|+1\right), \tag{2.57}$$

for a constant c_1 only dependent on B. This inequality, together with the first one in (2.56), gives

$$\|(\xi,\tau)\|^{q_j}\cdot\left|\left(\mathcal{C}^{-1}(\tau)\xi\right)_j\right| \le c_2\left(\left|D(\frac{1}{\sqrt{\tau}})\xi\right|+1\right)^{q_j+1}, \qquad j=1,\dots,N, \tag{2.58}$$

for some positive constant c_2, then, by (2.21), we find

$$\|(\xi,\tau)\|^{q_j}\left|\partial_{x_j}\Gamma(\xi,\tau)\right| \le c_3\,\Gamma^+(\xi,\tau), \qquad j=1,\dots,N, \tag{2.59}$$

and inequality (2.36) directly follows from (2.54). The same argument leads to the following inequality

$$\|(\xi,\tau)\|^{q_i+q_j}\left|\partial_{x_ix_j}\Gamma(\xi,\tau)\right| \le c_4\,\Gamma^+(\xi,\tau), \qquad i,j=1,\dots,N, \tag{2.60}$$

which gives (2.37). The bound (2.38) is a straightforward consequence of (2.37) and of the fact that Γ is a solution to $Ku=0$.

Before proceeding with the second set of inequalities, we state a further result that will needed in the sequel

$$\begin{aligned} \|(\xi,\tau)\|^{q_i+q_j+q_m}\left|\partial_{x_ix_jx_m}\Gamma(\xi,\tau)\right| &\le c_5\,\Gamma^+(\xi,\tau), \qquad i,j,m=1,\dots,N,\\ \|(\xi,\tau)\|^{q_i+q_j+q_m+q_n}\left|\partial_{x_ix_jx_mx_n}\Gamma(\xi,\tau)\right| &\le c_6\,\Gamma^+(\xi,\tau), \qquad i,j,m,n=1,\dots,N, \end{aligned} \tag{2.61}$$

for some positive constants c_5, c_6. We omit the proof, since it is analogous to the previous one.

We are now concerned with the proof of (2.39). As before, we set $(\xi,\tau) = w^{-1} \circ z$, $(\eta,\sigma) = z^{-1} \circ \bar{z}$, and we recall that M is as in Remark 2.2. We have

$$\begin{aligned}\Gamma(\xi,\tau) - \Gamma\big((\xi,\tau)\circ(\eta,\sigma)\big) = \Gamma(\xi,\tau) - \Gamma\big((\xi,\tau)\circ(0,\sigma)\big) + \\ \Gamma\big((\xi,\tau)\circ(0,\sigma)\big) - \Gamma\big((\xi,\tau)\circ(\eta,\sigma)\big) = \\ \sigma\, Y\Gamma\big((\xi,\tau)\circ(0,\theta_1\sigma)\big) + \sum_{i=1}^{N} \eta_j \partial_{x_j}\Gamma\big((\xi,\tau)\circ(\theta_2\eta,\sigma)\big),\end{aligned} \tag{2.62}$$

for some $\theta_1, \theta_2 \in]0,1[$. Note that $\|(0,\theta_1\sigma)\| \le \|(\eta,\sigma)\|$, and that $\|(\theta_2\eta,\sigma)\| \le \|(\eta,\sigma)\|$, so that both the inequalities $\|(0,\theta_1\sigma)\| \le M\|(\xi,\tau)\|$, and $\|(\theta_2\eta,\sigma)\| \le M\|(\xi,\tau)\|$ hold true. Then, by (2.6),

$$\|(\xi,\tau)\circ(0,\theta_1\sigma)\| \ge \frac{1 - M\,C_T^2}{C_T}\|(\xi,\tau)\|, \qquad \|(\xi,\tau)\circ(\theta_2\eta,\sigma)\| \ge \frac{1 - M\,C_T^2}{C_T}\|(\xi,\tau)\|. \tag{2.63}$$

Thus, we obtain from (2.59) that

$$\begin{aligned}\left|\sum_{i=1}^{N} \eta_j \partial_{x_j}\Gamma\big((\xi,\tau)\circ(\theta_2\eta,\sigma)\big)\right| \le c_3 \sum_{j=1}^{N} |\eta_j| \cdot \|(\xi,\tau)\circ(\theta_2\eta,\sigma)\|^{-q_j}\Gamma^+\big((\xi,\tau)\circ(\theta_2\eta,\sigma)\big) \le \\ c_3 \sum_{j=1}^{N} \|(\eta,\sigma)\|^{q_j} \cdot \|(\xi,\tau)\circ(\theta_2\eta,\sigma)\|^{-q_j}\Gamma^+\big((\xi,\tau)\circ(\theta_2\eta,\sigma)\big) \le \\ c_T' \frac{\|(\eta,\sigma)\|}{\|(\xi,\tau)\|^{Q+1}},\end{aligned} \tag{2.64}$$

by (2.54) and (2.63). Analogously, we obtain from (2.38) and (2.63)

$$\big|\sigma Y\Gamma\big((\xi,\tau)\circ(0,\theta_1\sigma)\big)\big| \le \frac{C_T''|\sigma|}{\|(\xi,\tau)\circ(0,\theta_1\sigma)\|^{Q+2}} \le c_T'' \frac{\|(\eta,\sigma)\|}{\|(\xi,\tau)\|^{Q+1}}, \tag{2.65}$$

By substituting (2.64) and (2.65) in (2.62), we obtain (2.39).

The proof of (2.40) is analogous: for any $j = 1, \dots, p_0$ we have

$$\partial_{x_j}\Gamma(\xi,\tau) - \partial_{x_j}\Gamma\big((\xi,\tau)\circ(\eta,\sigma)\big) = \sigma\, Y\partial_{x_j}\Gamma\big((\xi,\tau)\circ(0,\theta_1\sigma)\big) + \sum_{i=1}^{N} \eta_i \partial_{x_i x_j}\Gamma\big((\xi,\tau)\circ(\theta_2\eta,\sigma)\big), \tag{2.66}$$

for some $\theta_1, \theta_2 \in]0,1[$. In order to estimate the first term in the right hand side we rely on the very definition of the commutator of ∂_{x_j} and Y and on the fact that Γ is a solution to $Ku = 0$: we find

$$Y\partial_{x_j}\Gamma(x,t) = -\sum_{i,m=1}^{p_0} a_{i,m}\partial_{x_i x_j x_m}\Gamma(x,t) - \sum_{k=1}^{N} b_{j,k}\partial_{x_k}\Gamma(x,t), \qquad \forall (x,t) \ne (0,0). \tag{2.67}$$

Recall that B has the form (1.7), and $j \le p_0$, then $b_{j,k} = 0$ for every $k \ge p_0 + p_1$. Hence, it follows from the firs set of inequalities in (2.56) that

$$\left|\sum_{k=1}^{N} b_{j,k}\partial_{x_k}\Gamma(x,t)\right| = \left|\sum_{k=1}^{p_0+p_1} b_{j,k}\partial_{x_k}\Gamma(x,t)\right| \le c_0' \left(t^{-1/2} + t^{-3/2}\right) \left|D(\frac{1}{\sqrt{t}})x\right|\Gamma(x,t),$$

where the constant c_0' only depend on T and B. Thus, by (2.57) and (2.21), we obtain

$$\left|\sum_{k=1}^{N} b_{j,k}\partial_{x_k}\Gamma\big((\xi,\tau)\circ(0,\theta_1\sigma)\big)\right| \le \frac{c_0'''}{\big\|(\xi,\tau)\circ(0,\theta_1\sigma)\big\|^3}\,\Gamma^+\big((\xi,\tau)\circ(0,\theta_1\sigma)\big). \tag{2.68}$$

The above inequality, the first line in (2.61), and (2.63) then give

$$\left|\sigma Y\partial_{x_j}\Gamma\big((\xi,\tau)\circ(0,\theta_1\sigma)\big)\right| \le c_0'''' \frac{|\sigma|}{\big\|(\xi,\tau)\circ(0,\theta_1\sigma)\big\|^{Q+3}} \le c_T''' \frac{\|(\eta,\sigma)\|}{\big\|(\xi,\tau)\circ(0,\theta_1\sigma)\big\|^{Q+2}} \tag{2.69}$$

for a positive constant c_T''' depending on T, B and Λ in **[H2]**. The last sum in (2.66) can be estimate as (2.64), by using the first set of inequalities in (2.61). We find

$$\left|\sum_{i=1}^{N}\eta_j\partial_{x_ix_j}\Gamma\big((\xi,\tau)\circ(\theta_2\eta,\sigma)\big)\right| \le \; c_T''\frac{\|(\eta,\sigma)\|}{\|(\xi,\tau)\|^{Q+2}},$$

which, together with (2.69), gives (2.40).

The same argument gives the proof of (2.41): in this case we have to use the second set of inequalities in (2.61) and the analogous of (2.67):

$$Y\partial_{x_ix_j}\Gamma(x,t) = \sum_{m,n=1}^{p_0} a_{m,n}\partial_{x_ix_jx_mx_n}\Gamma(x,t) - \sum_{k=1}^{N}\big(b_{j,k}\partial_{x_kx_i}\Gamma(x,t) + b_{i,k}\partial_{x_kx_j}\Gamma(x,t)\big),$$

For all $(x,t)\neq(0,0)$. We omit the other details.

Finally, as in the proof of (2.38), we simply note that (2.42) is an immediate consequence of the fact that Γ is a solution to $Ku=0$. □

3 Schauder estimates

Let Ω be an open subset of $\mathbb{R}^{N+1}$, and $\alpha\in]0,1]$. We will say that $f\in C^\alpha(\Omega)$ if

$$|f|_{\alpha,\Omega} = \sup_{\Omega}|f| + \sup_{\substack{z,\zeta\in\Omega\\ z\neq\zeta}}\frac{|f(z)-f(\zeta)|}{\|\zeta^{-1}\circ z\|^\alpha} \tag{3.1}$$

is finite. Note that $|\cdot|_{\alpha,\Omega}$ is a norm and that

$$|fg|_{\alpha,\Omega} \le 2\,|f|_{\alpha,\Omega}\,|g|_{\alpha,\Omega}, \tag{3.2}$$

for every $f,g\in C^\alpha(\Omega)$. We say that $f\in C^{2+\alpha}(\Omega)$ if

$$|f|_{2+\alpha,\Omega} = |f|_{\alpha,\Omega} + \sum_{i=1}^{p_0}|\partial_{x_i}f|_{\alpha,\Omega} + \sum_{i,j=1}^{p_0}|\partial_{x_ix_j}f|_{\alpha,\Omega} + |Yf|_{\alpha,\Omega} < \infty. \tag{3.3}$$

Moreover, we say that f is locally Hölder-continuous function, and we write $f\in C^\alpha_{\rm loc}(\Omega)$ if $f\in C^\alpha(\Omega')$ for every compact subset Ω' of Ω. For every $z,\zeta\in\Omega$, we set

$$d_{z,\zeta} = \min\{d_z, d_\zeta\}, \qquad\qquad d_z = \inf_{w\in\partial\Omega}\|w^{-1}\circ z\|.$$

We say that a function $f:\Omega\to\mathbb{R}$ belongs to $C^\alpha_d(\Omega)$, if

$$|f|_{\alpha,d,\Omega} = \sup_{\Omega}|f| + \sup_{\substack{z,\zeta\in\Omega\\ z\neq\zeta}} d^\alpha_{z,\zeta}\frac{|f(z)-f(\zeta)|}{\|\zeta^{-1}\circ z\|^\alpha}, \tag{3.4}$$

is finite. We also consider the following norm:

$$[f]_{2+\alpha,d,\Omega} = \sup_{z\in\Omega} d_z^2|f(z)| + \sup_{z,\bar z\in\Omega} d^{2+\alpha}_{z,\bar z}\frac{|f(z)-f(\bar z)|}{\|\bar z^{-1}\circ z\|^\alpha}$$

and we observe that, for every $f, g \in C_d^\alpha(\Omega)$, we have

$$[fg]_{2+\alpha,d,\Omega} \leq 2|g|_{\alpha,d,\Omega}[f]_{2+\alpha,d,\Omega}, \tag{3.5}$$

We say that $f \in C_d^{2+\alpha}(\Omega)$ if

$$\begin{aligned}|f|_{2+\alpha,d,\Omega} := \sup_{z\in\Omega}|f| + \sup_{\substack{z,\zeta\in\Omega\\ z\neq\zeta}} d_{z,\zeta}^{2+\alpha}\frac{|f(z)-f(\zeta)|}{\|\zeta^{-1}\circ z\|^\alpha} + \sum_{i=1}^{p_0}\sup_{z\in\Omega} d_z|\partial_{x_i}f| + \\ \sum_{i=1}^{p_0}\sup_{\substack{z,\bar z\in\Omega\\ z\neq\bar z}} d_{z,\bar z}^{2+\alpha}\frac{|\partial_{x_i}f(z)-\partial_{x_i}f(\bar z)|}{\|\bar z\circ z\|^\alpha} + \sum_{i,j=1}^{p_0}[\partial_{x_ix_j}f]_{2+\alpha,d,\Omega} + [Yf]_{2+\alpha,d,\Omega}\end{aligned} \tag{3.6}$$

is finite. In order to prove our Schauder-type estimate we recall some interpolation inequalities for functions u in the space $C_d^{2+\alpha}(\Omega)$. For every $\varepsilon > 0$ there exist a positive constant C_ε such that

$$\sup_\Omega d_z|\partial_{x_i}u| \leq C_\varepsilon \sup_\Omega |u| + \varepsilon \sup_\Omega d_z^2|\partial_{x_ix_j}u| \tag{3.7}$$

$$\sup_\Omega d_z^2|\partial_{x_ix_j}u| \leq C_\varepsilon \sup_\Omega |u| + \varepsilon \sup_{\substack{z,\bar z\in\Omega\\ z\neq\bar z}} d_{z,\bar z}^{2+\alpha}\frac{|\partial_{x_ix_j}u(z)-\partial_{x_ix_j}u(\bar z)|}{\|\bar z^{-1}\circ z\|^\alpha} \tag{3.8}$$

$$\sup_\Omega |u| + \sum_{i=1}^{p_0}\sup_\Omega d_z|\partial_{x_i}u| + \sum_{i,j=1}^{p_0}\sup_\Omega d_z^2|\partial_{x_ix_j}u| \leq C_\varepsilon \sup_\Omega|u| + \varepsilon\sum_{i,j=1}^{p_0}\sup_{\substack{z,\bar z\in\Omega\\ z\neq\bar z}} d_{z,\bar z}^{2+\alpha}\frac{|\partial_{x_ix_j}u(z)-\partial_{x_ix_j}u(\bar z)|}{\|\bar z^{-1}\circ z\|^\alpha} \tag{3.9}$$

for every $i, j = 1, \dots, p_0$. The above inequalities have been proved by Manfredini (see the statement "Interpolation inequalities" p. 846, in [24]).

We will prove our interior estimate of Schauder type by a classical argument, based on the representation formulas of the solution in terms of the fundamental solution of the frozen operator

$$\widetilde{K}_{z_0}u := \sum_{i,j=1}^{p_0} a_{i,j}(z_0)\partial_{x_ix_j}u + \sum_{i=1}^{p_0} a_i(z_0)\partial_{x_i}u + Y. \tag{3.10}$$

Remark 3.1. *Denote $a = (a_1(z_0), \dots, a_{p_0}(z_0), 0, \dots, 0) \in \mathbb{R}^N$. Then the fundamental solution $\widetilde{\Gamma}_{z_0}$ of $\widetilde{K}_{z_0}$ is*

$$\widetilde{\Gamma}_{z_0}(x,t) = \Gamma_{z_0}(x - at, t)$$

where Γ_{z_0} is the fundamental solution of

$$K_{z_0}u := \sum_{i,j=1}^{p_0} a_{i,j}(z_0)\partial_{x_ix_j}u + Y.$$

As a consequence, the representation formula stated in Proposition 2.11 also holds for $\widetilde{\Gamma}_{z_0}$.

Besides, since the coefficients $a_1, \dots, a_{p_0}$ are bounded functions, the estimates of Proposition 2.7 extend to $\widetilde{\Gamma}_{z_0}$. Indeed, there exist two positive constants c' and c'', depending on $\sup_{i=1,\dots,p_0}|a_i(z)|$, such that

$$c'\|(x,t)\| \leq \|(x - at, t)\| \leq c''\|(x,t)\|.$$

Then we have, for instance

$$\widetilde{\Gamma}_{z_0}(x,t) = \Gamma_{z_0}(x - at, t) \leq \frac{C'_T}{\|(x - at, t)\|^Q} \leq \frac{C'_T}{(c'\|(x,t)\|)^Q}.$$

The other bounds extend to $\widetilde{\Gamma}_{z_0}$ analogously.

In order to avoid cumbersome notations, in the sequel we denote by Γ_{z_0} the fundamental solution of $\widetilde{K}_{z_0}$. We also recall that the function

$$\Gamma^*_{z_0}(\zeta, z) = \Gamma_{z_0}(z, \zeta), \qquad \forall\, z, \zeta \in \mathbb{R}^{N+1}, z \neq \zeta$$

is the fundamental solution of

$$K^*_{z_0} = \sum_{i,j=1}^{p_0} a_{ij}(z_0)\partial_{x_i x_j} - \sum_{i=1}^{p_0} a_i(z_0)\partial_{x_i} - Y - \mathrm{tr}B$$

(see [11], Theorem 1.5). Note that the function $\widetilde{\Gamma}^*_{z_0}(x,t,\xi,\tau) = e^{(t-\tau)\mathrm{tr}B}\Gamma^*_{z_0}(x,t,\xi,\tau)$ is the fundamental solution of

$$\sum_{i,j=1}^{p_0} a_{ij}(z_0)\partial_{x_i x_j} - \sum_{i=1}^{p_0} a_i(z_0)\partial_{x_i} - Y$$

then the results proved in the pervious section apply to $\Gamma^*_{z_0}$.

Proof of Theorem 1.3. We first remark that it suffices to prove inequality (1.18) for compact subsets of Ω. Indeed, let $(\Omega_k)_{k\in\mathbb{N}}$ be a sequence of open bounded subsets of Ω, such that $\Omega_k \subset \Omega_{k+1}$ for all k and $\bigcup_{k\in\mathbb{N}} \Omega_k = \Omega$. Assume that (1.18) holds on every Ω_k, with the same constant c, then every norm $|u|_{2+\alpha,d,\Omega_k}$ is finite. We then fix $z, \zeta \in \Omega$, with $z \neq \zeta$. For sufficiently large k, we have

$$|u(z)| + d^{2+\alpha}(\Omega_k)\frac{|u(z)-u(\zeta)|}{\|\zeta^{-1}\circ z\|^\alpha} + d_z(\Omega_k)|\partial_{x_i}u(z)| + d^{2+\alpha}(\Omega_k)\frac{|\partial_{x_i}u(z)-\partial_{x_i}u(\zeta)|}{\|\zeta^{-1}\circ z\|^\alpha} + d_z^2(\Omega_k)|\partial_{x_i x_j}u(z)|$$

$$+ d^{2+\alpha}_{z,\zeta}(\Omega_k)\frac{|\partial_{x_i x_j}u(z)-\partial_{x_i x_j}u(\zeta)|}{\|\zeta^{-1}\circ z\|^\alpha} + d_z^2(\Omega_k)|Yu(z)| + d^{2+\alpha}_{z,\zeta}(\Omega_k)\frac{|Yu(z)-Yu(\zeta)|}{\|\zeta^{-1}\circ z\|^\alpha}$$

$$\leq c(\sup\nolimits_{\Omega_k}|u| + [f]_{2+\alpha,d,\Omega_k}) \leq c(\sup\nolimits_{\Omega}|u| + [f]_{2+\alpha,d,\Omega})$$

for every $i, j = 1, \ldots, p_0$. Hence, by letting k to infinity, we obtain the inequality

$$|u(z)| + d^{2+\alpha}_{z,\zeta}\frac{|u(z)-u(\zeta)|}{\|\zeta^{-1}\circ z\|^\alpha} + d_z|\partial_{x_i}u(z)| + d^{2+\alpha}_{z,\zeta}\frac{|\partial_{x_i}u(z)-\partial_{x_i}u(\zeta)|}{\|\zeta^{-1}\circ z\|^\alpha} + d_z^2|\partial_{x_i x_j}u(z)|$$

$$+d^{2+\alpha}_{z,\zeta}\frac{|\partial_{x_i x_j}u(z)-\partial_{x_i x_j}u(\zeta)|}{\|\zeta^{-1}\circ z\|^\alpha} + d_z^2|Yu(z)| + d^{2+\alpha}_{z,\zeta}\frac{|Yu(z)-Yu(\zeta)|}{\|\zeta^{-1}\circ z\|^\alpha} \leq c(\sup_\Omega |u| + [f]_{2+\alpha,d,\Omega}),$$

so that (1.18) holds in the set Ω.

We next split the proof into three steps. We first prove a bound of the derivatives of u in the space $C^\alpha_d(\Omega)$, when u is compactly supported, then we extend the bounds to more general solutions u and, finally we conclude the proof by using some interpolation inequalities.

First step We first prove that, if u has compact support, then there exist a positive constant c_Ω such that

$$\begin{aligned} d^\alpha_{z,\bar z}\,|u(z)-u(\bar z)| &\leq c_\Omega|g|_{\alpha,d,\Omega}\,\|z^{-1}\circ\bar z\|^\alpha, \\ d^{1+\alpha}_{z,\bar z}\,|\partial_{x_i}u(z)-\partial_{x_i}u(\bar z)| &\leq c_\Omega|g|_{\alpha,d,\Omega}\,\|z^{-1}\circ\bar z\|^\alpha, \\ d^{2+\alpha}_{z,\bar z}\,|\partial_{x_i x_j}u(z)-\partial_{x_i x_j}u(\bar z)| &\leq c_\Omega|g|_{\alpha,d,\Omega}\,\|z^{-1}\circ\bar z\|^\alpha, \end{aligned} \tag{3.11}$$

for every $z, \bar z \in \Omega$, and for any $i, j = 1, \ldots, p_0$.

Fix $z_0 \in \Omega$ and let K_{z_0} be the frozen operator introduced in (3.10). Let u be compactly supported and set $g = K_{z_0} u$. Since Ω is bounded, we have that $R = \sup_{w,z\in\Omega} \|w^{-1} \circ z\|$ is finite. To prove the first inequality in (3.11), we observe that

$$u(z) - u(\bar{z}) = \int_{B_R(z_0)} \Big(\Gamma_{z_0}(w^{-1} \circ z) - \Gamma_{z_0}(w^{-1} \circ \bar{z})\Big) g(w) dw$$

Let M be the positive constant in the inequality (2.41), we split the above integral as follow:

$$\begin{aligned} u(z) - u(\bar{z}) = & \int\limits_{B_R(z_0)\cap\{\|z^{-1}\circ\bar{z}\|\le M\|w^{-1}\circ z\|\}} \Big(\Gamma_{z_0}(w^{-1} \circ z) - \Gamma_{z_0}(w^{-1} \circ \bar{z})\Big) g(w) dw \\ & + \int\limits_{B_R(z_0)\cap\{\|z^{-1}\circ\bar{z}\|\ge M\|w^{-1}\circ z\|\}} \Big(\Gamma_{z_0}(w^{-1} \circ z) - \Gamma_{z_0}(w^{-1} \circ \bar{z})\Big) g(w) dw \end{aligned}$$

So, by (2.35) and (2.39), we get

$$\begin{aligned} |u(z) - u(\bar{z})| \le & C_T \|g\|_\infty \int\limits_{B_R(z_0)\cap\{\|z^{-1}\circ\bar{z}\|\le M\|w^{-1}\circ z\|\}} \Big|\Gamma_{z_0}(w^{-1} \circ z) - \Gamma_{z_0}(w^{-1} \circ \bar{z})\Big| dw \\ & + C_T \|g\|_\infty \int\limits_{B_R(z_0)\cap\{\|z^{-1}\circ\bar{z}\|\ge M\|w^{-1}\circ z\|\}} \Big|\Gamma_{z_0}(w^{-1} \circ z) - \Gamma_{z_0}(w^{-1} \circ \bar{z})\Big| dw \\ & \le C_T \|g\|_\infty \Bigg(\|z^{-1} \circ \bar{z}\| \int\limits_{B_R(z_0)\cap\{\|z^{-1}\circ\bar{z}\|\le M\|w^{-1}\circ z\|\}} \frac{1}{\|w^{-1} \circ z\|^{Q+1}} dw \\ & + \int\limits_{B_R(z_0)\cap\{\|z^{-1}\circ\bar{z}\|\ge M\|w^{-1}\circ z\|\}} \frac{1}{\|w^{-1} \circ z\|^{Q}} + \frac{1}{\|w^{-1} \circ \bar{z}\|^{Q}} dw \Bigg) \\ & \le C_T' \|g\|_\infty \|z^{-1} \circ \bar{z}\| + C_T'' \|g\|_\infty \|z^{-1} \circ \bar{z}\|^2 \le C''' |g|_{\alpha,d,\Omega} \|z^{-1} \circ \bar{z}\|^\alpha \end{aligned}$$

for some positive constant C''' that depends on Ω and L. The proof of the second inequality in (3.11) is similar and will be omitted. We next prove the third one. By Proposition 2.11, we have

$$\partial_{x_i x_j} u(z) - \partial_{x_i x_j} u(\bar{z}) =$$

$$- \lim_{\varepsilon\to 0} \int\limits_{\varepsilon\le\|w^{-1}\circ z\|\le R} \partial_{x_i x_j} \Gamma_{z_0}(w^{-1} \circ z)[g(w) - g(z)] dw + \lim_{\varepsilon\to 0} \int\limits_{\varepsilon\le\|w^{-1}\circ\bar{z}\|\le R} \partial_{x_i x_j} \Gamma_{z_0}(w^{-1} \circ \bar{z})[g(w) - g(\bar{z})] dw$$

$$-(g(z) - g(\bar{z})) \int_{\|\zeta\|=1} \partial_{x_i} \Gamma_{z_0,0}(\zeta) \nu_j \, \sigma(\zeta)$$

$$-g(z) \lim_{\varepsilon\to 0} \int\limits_{\varepsilon\le\|w^{-1}\circ z\|\le R} \partial_{x_i x_j} \Gamma_{z_0}(w^{-1} \circ z) dw + g(\bar{z}) \lim_{\varepsilon\to 0} \int\limits_{\varepsilon\le\|w^{-1}\circ\bar{z}\|\le R} \partial_{x_i x_j} \Gamma_{z_0}(w^{-1} \circ \bar{z}) dw$$

where $\Gamma_{z_0,0}$ denotes the fundamental solution of

$$\sum_{i,j=1}^{p_0} a_{i,j}(z_0) \partial_{x_i x_j} + \sum_{j=1}^{p_0} a_j(z_0) \partial_{x_j} + Y_0$$

and Y_0 is defined in (2.8). Since $g \in C_d^\alpha$, the first two integrals in the above formula converge as $\varepsilon \to 0$. Then, in the following, we shall omit the limit.

In order to give a bound for the last two integrals in the above formula, we use (2.53) and Lemma 2.10. We find

$$\lim_{\varepsilon\to 0}\int_{\varepsilon\le\|w^{-1}\circ z\|\le R}\partial_{x_ix_j}\Gamma_{z_0}(w^{-1}\circ z)dw =$$
$$-\lim_{\varepsilon\to 0}\int_{\|\zeta\|=\varepsilon}\partial_{\xi_i}\Gamma(\zeta)e^{-\tau\operatorname{tr}B}\nu_j d\sigma(\zeta)+\int_{\|\zeta\|=R}\partial_{\xi_i}\Gamma(\zeta)e^{-\tau\operatorname{tr}B}\nu_j d\sigma(\zeta) =$$
$$-\int_{\|\zeta\|=1}\partial_{\xi_i}\Gamma_\varepsilon(\zeta)e^{-\tau\operatorname{tr}B}\nu_j d\sigma(\zeta)+\int_{\|\zeta\|=R}\partial_{\xi_i}\Gamma(\zeta)e^{-\tau\operatorname{tr}B}\nu_j d\sigma(\zeta)=\widetilde{c}$$

and, analogously,

$$\lim_{\varepsilon\to 0}\int_{\varepsilon\le\|w^{-1}\circ\bar z\|\le R}\partial_{x_ix_j}\Gamma_{z_0}(w^{-1}\circ\bar z)=\widetilde{c}.$$

We summarize the above results in the following formula

$$\partial_{x_ix_j}u(z)-\partial_{x_ix_j}u(\bar z)=$$
$$-\int\limits_{M\|w^{-1}\circ z\|\le\|z^{-1}\circ\bar z\|}\partial_{x_ix_j}\Gamma_{z_0}(w^{-1}\circ z)[g(w)-g(z)]dw-\int\limits_{\|z^{-1}\circ\bar z\|\le M\|w^{-1}\circ z\|\le MR}\partial_{x_ix_j}\Gamma_{z_0}(w^{-1}\circ z)[g(w)-g(z)]dw$$
$$+\int\limits_{M\|w^{-1}\circ\bar z\|\le\|z^{-1}\circ\bar z\|}\partial_{x_ix_j}\Gamma_{z_0}(w^{-1}\circ\bar z)[g(w)-g(\bar z)]dw+\int\limits_{\|z^{-1}\circ\bar z\|\le M\|w^{-1}\circ\bar z\|\le MR}\partial_{x_ix_j}\Gamma_{z_0}(w^{-1}\circ\bar z)[g(w)-g(\bar z)]dw$$
$$-\widetilde{c}_1\left(g(z)-g(\bar z)\right)=I_1+I_2+I_3+I_4-\widetilde{c}_1\left(g(z)-g(\bar z)\right)$$

We estimate separately I_1, I_2, I_3 and I_4. Choose a positive T such that $\Omega\subset\mathbb{R}^N\times]-T,T[$ and apply Proposition 2.7. If Ω' denotes the support of u, then the inequality (2.37) yields

$$|I_1|\quad\le c_1|g|_{\alpha,\Omega'}\int_{M\|w^{-1}\circ z\|\le\|z^{-1}\circ\bar z\|}\frac{\|w^{-1}\circ z\|^\alpha}{\|w^{-1}\circ z\|^{Q+2}}dw=c|g|_{\alpha,\Omega'}\|z^{-1}\circ\bar z\|^\alpha.$$

An analogous procedure can be used to estimate I_3. It is sufficient to observe that, by **(ii)** of Lemma 2.1,

$$\|w^{-1}\circ\bar z\|\le C_T(\|w^{-1}\circ z\|+\|z^{-1}\circ\bar z\|)\le C_T\left(1+\frac{1}{M}\right)\|z^{-1}\circ\bar z\|,$$

for any w such that $M\|w^{-1}\circ z\|\le\|z^{-1}\circ\bar z\|$. Concerning I_2 and I_4, we have

$$\begin{aligned}I_2+I_4\quad=&\int\limits_{\|z^{-1}\circ\bar z\|\le M\|w^{-1}\circ z\|\le MR}\big(\partial_{x_ix_j}\Gamma_{z_0}(\bar z,w)-\partial_{x_ix_j}\Gamma_{z_0}(z,w)\big)\big(g(w)-g(z)\big)dw-\\&\int\limits_{\|z^{-1}\circ\bar z\|\le M\|w^{-1}\circ z\|\le MR}\partial_{x_ix_j}\Gamma_{z_0}(\bar z,w)\big(g(w)-g(z)\big)dw+\\&\int\limits_{\|z^{-1}\circ\bar z\|\le M\|w^{-1}\circ z\|\le\bar{M}R}\partial_{x_ix_j}\Gamma_{z_0}(\bar z,w)\big(g(w)-g(\bar z))\big)dw=\\&\int\limits_{\|z^{-1}\circ\bar z\|\le M\|w^{-1}\circ z\|\le MR}\big(\partial_{x_ix_j}\Gamma_{z_0}(\bar z,w)-\partial_{x_ix_j}\Gamma_{z_0}(z,w)\big)\big(g(w)-g(z)\big)dw+\\&\big(g(z)-g(\bar z)\big)\int_{A_0}\partial_{x_ix_j}\Gamma_{z_0}(\bar z,w)dw+\int_{A_1}\partial_{x_ix_j}\Gamma_{z_0}(\bar z,w)\big(g(w)-g(z)\big)dw+\\&\int_{A_2}\partial_{x_ix_j}\Gamma_{z_0}(\bar z,w)\big(g(w)-g(\bar z)dw=J_1+\big(g(z)-g(\bar z)\big)J_2+J_3+J_4,\end{aligned}$$

where

$$A_0 = \big\{ w \in \mathbb{R}^{N+1} : \|z^{-1} \circ \bar{z}\| \le M \|w^{-1} \circ z\|, \|z^{-1} \circ \bar{z}\| \le M \|w^{-1} \circ \bar{z}\| \le MR \big\},$$
$$A_1 = \big\{ w \in \mathbb{R}^{N+1} : M\|w^{-1} \circ \bar{z}\| < \|z^{-1} \circ \bar{z}\| \le M \|w^{-1} \circ z\| \big\},$$
$$A_2 = \big\{ w \in \mathbb{R}^{N+1} : M\|w^{-1} \circ z\| < \|z^{-1} \circ \bar{z}\| \le M \|w^{-1} \circ \bar{z}\| \big\}.$$

We first consider J_1. From (2.41), we get

$$|J_1| \le c_1 |g|_{\alpha,\Omega'} \|z^{-1} \circ \bar{z}\| \int_{\|z^{-1}\circ\bar{z}\| \le M\|w^{-1}\circ z\|} \frac{1}{\|w^{-1} \circ z\|^{Q+3-a}} dw = c|g|_{\alpha,\Omega'} \|z^{-1} \circ \bar{z}\|^{\alpha}.$$

We now estimate J_2. By the divergence Theorem, we obtain

$$J_2 = \int_{\partial \bar{A}} \partial_{x_i} \Gamma_{z_0}(\bar{\zeta}) \nu_j e^{-\bar{\tau} \operatorname{tr} B} d\sigma(\bar{\zeta})$$

where $\bar{A} = \big\{ \bar{\zeta} \in \mathbb{R}^{N+1} : \|z^{-1} \circ \bar{z}\| \le M \|\bar{\zeta} \circ \bar{z}^{-1} \circ z\|, \|z^{-1} \circ \bar{z}\| \le M\|\bar{\zeta}\| \le MR \big\}$, thus, by (2.36), we find

$$|J_2| \le C'_T \int_{\partial \bar{A}} \frac{1}{\|\bar{\zeta}\|^{Q+1}} d\sigma(\bar{\zeta}) \le C'_T \Bigg(\int_{\|z^{-1}\circ\bar{z}\| = M\|\bar{\zeta}\|} \frac{1}{\|\bar{\zeta}\|^{Q+1}} d\sigma(w) +$$
$$\int_{\|z^{-1}\circ\bar{z}\| = M\|\bar{\zeta}\circ\bar{z}^{-1}\circ z\|} \frac{1}{\|\bar{\zeta} \circ \bar{z}^{-1} \circ z\|^{Q+1}} d\sigma(w) + \int_{\|\bar{\zeta}\| = R} \frac{1}{\|\bar{\zeta}\|^{Q+1}} d\sigma(\bar{\zeta}) \Bigg) = C'',$$

for a suitable positive constant C'', depending on the operator K and Ω.

In order to find a bound for J_3 and J_4 we note that from (2.6) it follows that

$$\frac{1 - M\, C_T^2}{C_T} \|w^{-1} \circ z\| \le \|w^{-1} \circ \bar{z}\| \le C_T(1+M)\|w^{-1} \circ z\|.$$

Then

$$|J_3|, |J_4| \le \bar{C} |g|_{\alpha,\Omega'} \int_{M\|w^{-1}\circ\bar{z}\| < \|z^{-1}\circ\bar{z}\|} \frac{\|w^{-1} \circ \bar{z}\|^{\alpha}}{\|w^{-1} \circ \bar{z}\|^{Q+2}} dw \le \bar{C}' |g|_{\alpha,\Omega'} \|z^{-1} \circ \bar{z}\|^{\alpha},$$

for a suitable positive constant $\bar{C}'$.

Summarizing the above inequalities we conclude that

$$|I_1 + \cdots + I_4| \le C'''\, |g|_{\alpha,\Omega'} \|z^{-1} \circ \bar{z}\|^{\alpha}, \qquad \forall\, z, \bar{z} \in \Omega. \tag{3.12}$$

This accomplishes the proof of (3.11).

Second step We next remove the assumption that u has a compact support in Ω. Denote by $B_r(\bar{z})$ the metric ball with center at $\bar{z}$ and radius r:

$$B_r(\bar{z}) = \Big\{ \zeta \in \mathbb{R}^{N+1} : \|\bar{z}^{-1} \circ \zeta\| \le r \Big\},$$

and suppose that $B_r(\bar{z}) \subset \Omega$. If T is a positive constant such that $\Omega \subseteq \mathbb{R}^N \times]-T,T[$ and C_T is the constant appearing in Remark 2.2, then we choose any $m > C_T^3$. Note that, being $C_T \ge 1$, we have that there exists a positive constant c such that $\|\zeta^{-1} \circ z\| \ge cr$, for every $z \in B_{\frac{r}{m}}(\bar{z})$, $\zeta \in \partial B_r(\bar{z})$. We claim that there exists a positive constant C_Ω, only depending on the operator L and Ω, such that

$$\begin{aligned} r^{2+\alpha} \frac{|u(z) - u(\bar{z})|}{\|\bar{z}^{-1} \circ z\|^{\alpha}} &\le C_\Omega([g]_{2+\alpha,d,B_r(\bar{z})} + \sup_{B_r(\bar{z})} |u|), \\ r^{2+\alpha} \frac{|\partial_{x_i} u(z) - \partial_{x_i} u(\bar{z})|}{[\bar{z}^{-1} \circ z]^{\alpha}} &\le C_\Omega([g]_{2+\alpha,d,B_r(\bar{z})} + \sup_{B_r(\bar{z})} |u|), \\ r^{2+\alpha} \frac{|\partial_{x_i x_j} u(z) - \partial_{x_i x_j} u(\bar{z})|}{\|\bar{z}^{-1} \circ z\|^{\alpha}} &\le C_\Omega([g]_{2+\alpha,d,B_r(\bar{z})} + \sup_{B_r(\bar{z})} |u|), \end{aligned} \tag{3.13}$$

for any $\bar{z}, z$ and $r > 0$ such that $B_r(\bar{z}) \subset \Omega$ and $z \in B_{\frac{r}{2m^3}}(\bar{z})$.

We only prove the last inequality since the first two are simpler than that one. We consider a function $\varphi \in C_0^\infty(B_r(\bar{z}))$ such that $\varphi(z) = 1$ whenever $z \in B_{\frac{r}{2m^2}}(\bar{z})$ and $\varphi(z) = 0$ when $z \in B_r(\bar{z}) \setminus B_{\frac{r}{m^2}}(\bar{z})$. We also require that $|\partial_{x_i}\varphi(z)| \le \frac{c}{r}$, $|\partial_{x_i x_j}\varphi(z)| \le \frac{c}{r^2}$ (for $i, j = 1, \dots, p_0$), and $|Y\varphi(z)| \le \frac{c}{r^2}$. A such function can be defined as η_ε in (2.48) (also recall (2.49)).

We integrate the function $vg - uK_{z_0}^* v$ on $B_r(\bar{z})$, where $g = K_{z_0}u$ and $v(\zeta) = \varphi(\zeta)\Gamma_{z_0}^*(\zeta, z)$:

$$u(z) = -\int_{B_r(\bar{z})} \varphi(\zeta)g(\zeta)\Gamma_{z_0}^*(\zeta, z)d\zeta + \int_{B_{\frac{r}{m^2}}(\bar{z})\setminus B_{\frac{r}{2m^2}}(\bar{z})} u(\zeta)K_{z_0}^*(\varphi(\zeta)\Gamma_{z_0}^*(\zeta, z))d\zeta, \tag{3.14}$$

consequently, we have

$$\begin{aligned}
\partial_{x_i x_j} u(z) &= -\partial_{x_i x_j} \int_{B_r(\bar{z})} \varphi(\zeta)g(\zeta)\Gamma_{z_0}^*(\zeta, z)d\zeta + \partial_{x_i x_j} \int_{B_{\frac{r}{m^2}}(\bar{z})\setminus B_{\frac{r}{2m^2}}(\bar{z})} u(\zeta)K_{z_0}^*(\varphi(\zeta)\Gamma_{z_0}^*(\zeta, z))d\zeta \\
&= -\partial_{x_i x_j} \int_{B_r(\bar{z})} \varphi(\zeta)g(\zeta)e^{-(t-\tau)\mathrm{tr}B}\Gamma_{z_0}(z, \zeta)d\zeta + \\
&\quad \partial_{x_i x_j} \int_{B_{\frac{r}{m^2}}(\bar{z})\setminus B_{\frac{r}{2m^2}}(\bar{z})} u(\zeta)K_{z_0}^*(\varphi(\zeta)e^{-(t-\tau)\mathrm{tr}B}\Gamma_{z_0}(z, \zeta))d\zeta = v_{i,j}(z) + w_{i,j}(z),
\end{aligned} \tag{3.15}$$

for $i, j = 1, \dots, p_0$, where $z = (x, t)$ and $\zeta = (\xi, \tau)$. Consider the second term in (3.15). We first note that,

$$w_{i,j}(z) = \int_{B_{\frac{r}{m^2}}(\bar{z})\setminus B_{\frac{r}{2m^2}}(\bar{z})} \partial_{x_i x_j} u(\zeta)K_{z_0}^*(\varphi(\zeta)e^{-(t-\tau)\mathrm{tr}B}\Gamma_{z_0}(z, \zeta))d\zeta, \tag{3.16}$$

for every $z \in B_{\frac{r}{2m^3}}(\bar{z})$. Then

$$\begin{aligned}
r^\alpha \frac{|w_{i,j}(z) - w_{i,j}(\bar{z})|}{\|\bar{z}^{-1} \circ z\|^\alpha} &\le \frac{c\, r^\alpha}{\|\bar{z}^{-1} \circ z\|^\alpha} \sup_{B_r(\bar{z})} |u| \\
&\quad \Bigg(\sup_{B_r(\bar{z})} |K_{z_0}^*\varphi| \int_{B_{\frac{r}{m^2}}(\bar{z})\setminus B_{\frac{r}{2m^2}}(\bar{z})} |\partial_{x_i x_j}\Gamma_{z_0}(z, \zeta) - \partial_{x_i x_j}\Gamma_{z_0}(\bar{z}, \zeta)|d\zeta \\
&\quad + \sum_{h,k=1}^{p_0} \sup_{B_r(\bar{z})} |\partial_{\xi_h}\varphi| \int_{B_{\frac{r}{m^2}}(\bar{z})\setminus B_{\frac{r}{2m^2}}(\bar{z})} |\partial_{x_i x_j}\partial_{\xi_k}\Gamma_{z_0}(z, \zeta) - \partial_{x_i x_j}\partial_{\xi_k}\Gamma_{z_0}(\bar{z}, \zeta)|d\zeta \Bigg) \\
&= \frac{c\, r^\alpha}{\|\bar{z}^{-1} \circ z\|^\alpha} \sup_{B_r(\bar{z})} |u|(I_1^* + I_2^*),
\end{aligned} \tag{3.17}$$

for some positive constant c only depending on the operator K. We first consider I_1^*:

$$I_1^* = \sup_{B_r((\bar{z}))} |L_{z_0}\varphi|(J_1^* + J_2^*) \tag{3.18}$$

where

$$J_1^* = \int_{B_{\frac{r}{m^2}}(\bar z)\setminus B_{\frac{r}{2m^2}}(\bar z)\cap\{\|\bar z^{-1}\circ z\|\le M\|\zeta^{-1}\circ\bar z\|\}} |\partial_{x_ix_j}\Gamma_{z_0}(z,\zeta) - \partial_{x_ix_j}\Gamma_{z_0}(\bar z,\zeta)|d\zeta,$$

$$J_2^* = \int_{B_{\frac{r}{m^2}}(\bar z)\setminus B_{\frac{r}{2m^2}}(\bar z)\cap\{\|\bar z^{-1}\circ z\|\ge M\|\zeta^{-1}\circ\bar z\|\}} |\partial_{x_ix_j}\Gamma_{z_0}(z,\zeta) - \partial_{x_ix_j}\Gamma_{z_0}(\bar z,\zeta)|d\zeta,$$

and M is the constant in (2.41). Aiming to estimate J_1^*, we note that,

$$\|\zeta^{-1}\circ z\| \ge \frac{1}{C_T}\|\zeta^{-1}\circ\bar z\| - C_T\|\bar z^{-1}\circ z\| \ge \frac{1}{C_T^2}\|\bar z^{-1}\circ\zeta\| - \frac{C_T\, r}{2m^3} \ge \frac{m - C_T^3}{m\, C_T^2}\,\frac{r}{2m^2} > 0, \tag{3.19}$$

for every $\zeta\in B_{\frac{r}{m^2}}(\bar z)\setminus B_{\frac{r}{2m^2}}(\bar z)$ and $z\in B_{\frac{r}{2m^3}}(\bar z)$. Then, by using (2.41) we obtain

$$J_1^* \le c\int_{B_{\frac{r}{m^2}}(\bar z)\setminus B_{\frac{r}{2m^2}}(\bar z)} \frac{\|\bar z^{-1}\circ z\|}{\|\zeta^{-1}\circ\bar z\|^{Q+3}}d\zeta \le c_T'\frac{\|\bar z^{-1}\circ z\|}{r^{Q+3}}\int_{B_{\frac{r}{m^2}}(\bar z)\setminus B_{\frac{r}{2m^2}}(\bar z)} d\zeta \le c_T''\frac{\|\bar z^{-1}\circ z\|}{r}, \tag{3.20}$$

since $\|\zeta^{-1}\circ\bar z\|\ge\frac{r}{2m^2C_T}$, for every ζ out of the ball $B_{\frac{r}{2m^2}}(\bar z)$. On the other hand, by (2.37), we have

$$J_2^* \le c\int_{B_{\frac{r}{m^2}}(\bar z)\setminus B_{\frac{r}{2m^2}}(\bar z)\cap\{\|\bar z^{-1}\circ z\|\ge M\|\zeta^{-1}\circ\bar z\|\}} \left(\frac{1}{\|\zeta^{-1}\circ z\|^{Q+2}} + \frac{1}{\|\zeta^{-1}\circ\bar z\|^{Q+2}}\right)d\zeta,$$

thus, by using again (3.19), and the fact that $\|\zeta^{-1}\circ\bar z\|\ge\frac{r}{2m^2C_T}$ for every ζ out of the ball $B_{\frac{r}{2m^2}}(\bar z)$, we find

$$J_2^* \le \int_{B_{\frac{r}{m^2}}(\bar z)\setminus B_{\frac{r}{2m^2}}(\bar z)\cap\{\|\bar z^{-1}\circ z\|\ge M\|\zeta^{-1}\circ\bar z\|\}} \frac{c'}{r^{Q+2}}d\zeta \le \frac{c'}{r^{Q+2}}\int_{\{\|\bar z^{-1}\circ z\|\ge M\|\zeta^{-1}\circ\bar z\|\}} d\zeta = c''\left(\frac{\|\bar z^{-1}\circ z\|}{r}\right)^{Q+2}.$$

From the above inequality and (3.20), recalling that $\|\bar z^{-1}\circ z\|\le\frac{r}{2\,m^3}$, we finally get

$$I_1^* \le c_\alpha'\sup_{B_r(\bar z)}|K_{z_0}^*\varphi|\left(\frac{\|\bar z^{-1}\circ z\|}{r}\right)^\alpha, \tag{3.21}$$

for some positive constant c_α'. To prove an analogous estimate for I_2^* we state that the function $\partial_{x_ix_j}\partial_{\xi_k}\Gamma_{z_0}$ satisfies the following estimates analogous to Proposition 2.7:

$$|\partial_{x_ix_j}\partial_{\xi_k}\Gamma_{z_0}(z,\zeta) - \partial_{x_ix_j}\partial_{\xi_k}\Gamma_{z_0}(\bar z,\zeta)| \le c\frac{\|\bar z^{-1}\circ z\|}{\|\zeta^{-1}\circ\bar z\|^{Q+4}},$$

for $i,j,k = 1,\dots,p_0$ provided that $\|\bar z^{-1}\circ z\|\le M\|\zeta^{-1}\circ\bar z\|$, for some positive constant M. We omit the proof since it the same as that of Proposition 2.7. The argument used in the estimate of I_1^* gives in this case

$$I_2^* \le c_\alpha''\max_{h=1,\dots,p_0}\frac{\sup_{B_r(\bar z)}|\partial_{x_h}\varphi|}{r}\left(\frac{\|\bar z^{-1}\circ z\|}{r}\right)^\alpha \tag{3.22}$$

If we use (3.21), and (3.22) in (3.17), and we recall that $|\partial_{x_h}\varphi(z)|\le\frac{c}{r}$, for $h=1,\dots,p_0$ and $|L_{z_0}\varphi(z)|\le\frac{c}{r^2}$, we then find

$$r^\alpha\frac{|w_{i,j}(z) - w_{i,j}(\bar z)|}{\|\bar z^{-1}\circ z\|^\alpha} \le c_\alpha\frac{1}{r^2}\sup_{B_r(\bar z)}|u|, \tag{3.23}$$

for some positive constant c_α only depending on L, Ω and α.

Since φg is Hölder continuous and compactly supported in Ω, the estimate (3.12) holds for the first term in (3.15), then

$$|v_{i,j}(z) - v_{i,j}(\bar z)| \le c|\varphi g|_{\alpha, B_{\frac{r}{m^2}}(\bar z)} \|\bar z^{-1} \circ z\|^\alpha$$

Since $r^\alpha |\varphi g|_{\alpha, B_{\frac{r}{m^2}}(\bar z)} \le c|\varphi g|_{\alpha,d,B_{\frac{r}{m}}(\bar z)}$ for a constant c that does not depend on r, we have

$$r^\alpha |v_{i,j}(z) - v_{i,j}(\bar z)| \le c|\varphi g|_{\alpha,d,B_{\frac{r}{m}}(\bar z)} \|\bar z^{-1} \circ z\|^\alpha \le 2\, c|\varphi|_{\alpha,d,B_{\frac{r}{m}}(\bar z)} |g|_{\alpha,d,B_{\frac{r}{m}}(\bar z)} \|\bar z^{-1} \circ z\|^\alpha,$$

Since $|\varphi|_{\alpha,d,B_{\frac{r}{m}}(\bar z)} \le c_\varphi$, and $r^2 |g|_{\alpha,d,B_{\frac{r}{m}}(\bar z)} \le c' [g]_{2+\alpha,d,B_r(\bar z)}$, for some constant c', c_φ that do not depend on r, we finally get

$$r^{2+\alpha} \frac{|v_{i,j}(z) - v_{i,j}(\bar z)|}{\|\bar z^{-1} \circ z\|^\alpha} \le c'' [g]_{2+\alpha,d,B_r(\bar z)}.$$

Combining the above estimate, (3.23), and (3.11), we thus find

$$\begin{aligned} r^{2+\alpha} \frac{|\partial_{x_i x_j} u(z) - \partial_{x_i x_j} u(\bar z)|}{\|\bar z^{-1} \circ z\|^\alpha} &\le r^{2+\alpha} \frac{|v_{i,j}(z) - v_{i,j}(\bar z)|}{\|\bar z^{-1} \circ z\|^\alpha} + r^{2+\alpha} \frac{|w_{i,j}(z) - w_{i,j}(\bar z)|}{\|\bar z^{-1} \circ z\|^\alpha} \\ &\le c([g]_{2+\alpha,d,B_r(\bar z)} + \sup_{B_r(\bar z)} |u|). \end{aligned}$$

This accomplishes the proof of (3.13).

Third step We prove (1.18). Let z_0 and ζ_0 be any two distinct points of Ω, such that $d_{z_0} \le d_{\zeta_0}$. We define the function F as

$$F(z) := f(z) + \sum_{i,j=1}^{p_0} (a_{i,j}(z_0) - a_{i,j}(z)) \partial_{x_i x_j} u(z) + \sum_{i=1}^{p_0} (a_i(z_0) - a_i(z)) \partial_{x_i} u(z)$$

so that $K_{z_0} u = F$, and we consider a constant $\mu < \frac{1}{2C_T^2}$, that will be specified later (C_T is the constant in Lemma 2.1. We observe that, by our choice of μ, we have $d_z \ge \frac{1}{2\, C_T} d_{z_0}$ for every $z \in B_r(z_0)$, with $r = \mu d_{z_0}$. Indeed,

$$\|\zeta^{-1} \circ z\| \ge \frac{1}{C_T} \|\zeta^{-1} \circ z_0\| - \|z^{-1} \circ z_0\| \ge \left(\frac{1}{C_T} - C_T \mu \right) d_{z_0} \ge \frac{1}{2\, C_T} d_{z_0},$$

for every $\zeta \in \partial\Omega$. As a direct consequence $B_r(z_0) \subset \Omega$.

Let m be the positive constant fixed in the previous step. If $\zeta_0 \in B_{\frac{r}{2\, m^2}}(z_0)$, then (3.13) yields

$$\begin{aligned} (\mu d_{z_0})^{2+\alpha} \frac{|u(z_0) - u(\zeta_0)|}{\|\zeta_0^{-1} \circ z_0\|^\alpha} + (\mu d_{z_0})^{2+\alpha} \frac{|\partial_{x_i} u(z_0) - \partial_{x_i} u(\zeta_0)|}{\|\zeta_0^{-1} \circ z_0\|^\alpha} + (\mu d_{z_0})^{2+\alpha} \frac{|\partial_{x_i x_j} u(z_0) - \partial_{x_i x_j} u(\zeta_0)|}{\|\zeta_0^{-1} \circ z_0\|^\alpha} \\ \le c \left([F]_{2+\alpha,d,B_r(z_0)} + \sup_{B_r(z_0)} |u| \right). \end{aligned} \tag{3.24}$$

On the other hand, if $\zeta_0 \notin B_{\frac{r}{2\, m^2}}(z_0)$ we have

$$\begin{aligned} d_{z_0}^{2+\alpha} \frac{|u(z_0) - u(\zeta_0)|}{\|\zeta_0^{-1} \circ z_0\|^\alpha} + d_{z_0}^{2+\alpha} \frac{|\partial_{x_i} u(z_0) - \partial_{x_i} u(\zeta_0)|}{\|\zeta_0^{-1} \circ z_0\|^\alpha} + d_{z_0}^{2+\alpha} \frac{|\partial_{x_i x_j} u(z_0) - \partial_{x_i x_j} u(\zeta_0)|}{\|\zeta_0^{-1} \circ z_0\|^\alpha} \\ \le \frac{c}{\mu^{2+\alpha}} \left(\sup_{z \in \Omega} |u(z)| + \sup_{z \in \Omega} d_z |\partial_{x_i} u(z)| + \sup_{z \in \Omega} d_z^2 |\partial_{x_i x_j} u(z)| \right). \end{aligned} \tag{3.25}$$

Hence, by combining (3.24) and (3.25), we obtain

$$
\begin{aligned}
&d_{z_0}^{2+\alpha}\frac{|u(z_0)-u(\zeta_0)|}{\|\zeta_0^{-1}\circ z_0\|^\alpha}+d_{z_0}^{2+\alpha}\frac{|\partial_{x_i}u(z_0)-\partial_{x_i}u(\zeta_0)|}{\|\zeta_0^{-1}\circ z_0\|^\alpha}+d_{z_0}^{2+\alpha}\frac{|\partial_{x_ix_j}u(z_0)-\partial_{x_ix_j}u(\zeta_0)|}{\|\zeta_0^{-1}\circ z_0\|^\alpha}\\
&\le \frac{c}{\mu^{2+\alpha}}\left([F]_{2+\alpha,d,B_r(z_0)}+\sup_\Omega|u|\right)+\frac{c}{\mu^{2+\alpha}}\left(\sup_{z\in\Omega}|u(z)|+\sup_{z\in\Omega}d_z|\partial_{x_i}u(z)|+\sup_{z\in\Omega}d_z^2|\partial_{x_ix_j}u(z)|\right).
\end{aligned}
\tag{3.26}
$$

We next provide an estimate of $[F]_{2+\alpha,d,B_r(z_0)}$ in terms of $|\partial_{x_ix_j}u|_{\alpha,d,B_r(z_0)}$. We have

$$
\begin{aligned}
[F]_{2+\alpha,d,B_r(z_0)} &\le \sum_{i,j=1}^{p_0}[(a_{i,j}(z_0)-a_{i,j})\partial_{x_ix_j}u]_{2+\alpha,d,B_r(z_0)}+[f]_{2+\alpha,d,B_r(z_0)}\\
&\quad+\sum_{i=1}^{p_0}[(a_i(z_0)-a_i(z))\partial_{x_i}u]_{2+\alpha,d,B_r(z_0)}
\end{aligned}
$$

By (3.5), we have

$$
[(a_{i,j}(z_0)-a_{i,j})\partial_{x_ix_j}u]_{2+\alpha,d,B_r(z_0)}\le 2\,|a_{i,j}(z_0)-a_{i,j}|_{\alpha,d,B_r(z_0)}[\partial_{x_ix_j}u]_{2+\alpha,d,B_r(z_0)},\tag{3.27}
$$

for all $i,j=1,\dots,p_0$. Then, since $d_z\ge\frac{1}{2C_T}d_{z_0}$ for any $z\in B_r(z_0)$, we have $\frac{r}{d_{z,\bar z}}\le 2C_T\mu$, for every $z,\bar z\in B_r(z_0)$, thus

$$
\begin{aligned}
|a_{i,j}(z_0)-a_{i,j}|_{\alpha,d,B_r(z_0)} &\le \sup_{B_r(z_0)}|a_{i,j}(z_0)-a_{i,j}(z)|+r^\alpha\sup_{\substack{z,\bar z\in B_r(z_0)\\ z\neq\bar z}}\frac{|a_{i,j}(z)-a_{i,j}(\bar z)|}{\|\bar z^{-1}\circ z\|^\alpha}\\
&\le r^\alpha\,|a_{i,j}|_{\alpha,d,\Omega}+\sup_{\substack{z,\bar z\in B_r(z_0)\\ z\neq\bar z}}\frac{r^\alpha}{d_{z,\bar z}^\alpha}d_{z,\bar z}^\alpha\frac{|a_{i,j}(z)-a_{i,j}(\bar z)|}{\|\bar z^{-1}\circ z\|^\alpha}\le c_\alpha\,\mu^\alpha\,|a_{i,j}|_{\alpha,d,\Omega},
\end{aligned}
\tag{3.28}
$$

for a positive constant c_α only depending on L and Ω. Analogously,

$$
\begin{aligned}
[\partial_{x_ix_j}u]_{2+\alpha,d,B_r(z_0)} &= \sup_{B_r(z_0)}d_z^2|\partial_{x_ix_j}u|+\sup_{B_r(z_0)}d_{z,\bar z}^{2+\alpha}\frac{|\partial_{x_ix_j}u(z)-\partial_{x_ix_j}u(\bar z)|}{\|\bar z\circ z\|^\alpha}\\
&\le r^2\sup_{B_r(z_0)}|\partial_{x_ix_j}u|+r^{2+\alpha}\sup_{B_r(z_0)}\frac{|\partial_{x_ix_j}u(z)-\partial_{x_ix_j}u(\bar z)|}{\|\bar z^{-1}\circ z\|^\alpha}\\
&\le \sup_{B_r(z_0)}\frac{r^2}{d_z^2}d_z^2|\partial_{x_ix_j}u|+\sup_{B_r(z_0)}\frac{r^{2+\alpha}}{d_{z,\bar z}^{2+\alpha}}d_{z,\bar z}^{2+\alpha}\frac{|\partial_{x_ix_j}u(z)-\partial_{x_ix_j}u(\bar z)|}{\|\bar z^{-1}\circ z\|^\alpha}\\
&\le (2C_T)^{2+\alpha}\left(\mu^2\sup_\Omega d_z^2|\partial_{x_ix_j}u|+\mu^{2+\alpha}\sup_\Omega d_{z,\bar z}^{2+\alpha}\frac{|\partial_{x_ix_j}u(z)-\partial_{x_ix_j}u(\bar z)|}{\|\bar z^{-1}\circ z\|^\alpha}\right).
\end{aligned}
\tag{3.29}
$$

Then, by using the above inequality and (3.28) in (3.27), we obtain

$$
\begin{aligned}
[(a_{i,j}(z_0)-a_{i,j})\partial_{x_ix_j}u]_{2+\alpha,d,B_r(z_0)} &\le c''_\alpha\,|a_{i,j}|_{\alpha,d,\Omega}\mu^{2+\alpha}\Bigg(\sup_{z\in\Omega}d_z^2|\partial_{x_ix_j}u|\\
&\quad+\mu^\alpha\sup_{\substack{z,\bar z\in\Omega\\ z\neq\bar z}}d_{z,\bar z}^{2+\alpha}\frac{|\partial_{x_ix_j}u(z)-\partial_{x_ix_j}u(\bar z)|}{\|\bar z^{-1}\circ z\|^\alpha}\Bigg),
\end{aligned}
\tag{3.30}
$$

for a positive constant c''_α only depending on L and Ω.

We next provide an analogous estimate for the term $(a_i(z_0) - a_i)\partial_{x_i} u$. By (3.5), we have

$$[(a_i(z_0) - a_i)\partial_{x_i} u]_{2+\alpha,d,B_r(z_0)} \leq |a_i(z_0) - a_i|_{\alpha,d,B_r(z_0)} [\partial_{x_i} u]_{2+\alpha,d,B_r(z_0)}, \tag{3.31}$$

for every $i = 1, \dots, p_0$. The same arguments used in the proof of (3.28), and (3.29) give

$$[(a_i(z_0) - a_i)\partial_{x_i} u]_{2+\alpha,d,B_r(z_0)} \leq k_\alpha \, |a_i|_{\alpha,d,\Omega} \mu^{2+\alpha} \left(\sup_{z\in\Omega} d_z |\partial_{x_i} u| + \mu^\alpha \sup_{\substack{z,\bar z\in\Omega\\ z\neq \bar z}} d_{z,\bar z}^{2+\alpha} \frac{|\partial_{x_i} u(z) - \partial_{x_i} u(\bar z)|}{\|\bar z^{-1} \circ z\|^\alpha} \right).$$

where k_α is a positive constant only depending on L and Ω. Note that the norms $|a_{i,j}|_{\alpha,d,\Omega}$ and $|a_i|_{\alpha,d,\Omega}$ are bounded by a constant depending on Ω and on the quantities involved in hypothesis **[H3]**, hence the above inequality and (3.30) give the following estimate for F:

$$\begin{aligned}
[F]_{2+\alpha,d,B_r(z_0)} \quad &\leq C_\alpha \, \mu^{2+\alpha} \Bigg(\sup_{z\in\Omega} d_z^2 |\partial_{x_i x_j} u| + \mu^\alpha \sup_{\substack{z,\bar z\in\Omega\\ z\neq \bar z}} d_{z,\bar z}^{2+\alpha} \frac{|\partial_{x_i x_j} u(z) - \partial_{x_i x_j} u(\bar z)|}{\|\bar z^{-1} \circ z\|^\alpha} \\
&+ \sup_{z\in\Omega} d_z |\partial_{x_i} u| + \mu^\alpha \sup_{\substack{z,\bar z\in\Omega\\ z\neq \bar z}} d_{z,\bar z}^{2+\alpha} \frac{|\partial_{x_i} u(z) - \partial_{x_j} u(\bar z)|}{\|\bar z^{-1} \circ z\|^\alpha} \Bigg) + [f]_{2+\alpha,d,\Omega},
\end{aligned} \tag{3.32}$$

where the constant C_α only depends on α, on Ω and on the operator L.

We next remove the terms $d_z|\partial_{x_i} u|$ and $d_z^2|\partial_{x_i x_j} u|$ from the right hand side of (3.32) by using first the inequality (3.7) with $\varepsilon = 1$ and then the inequality (3.8) with $\varepsilon = \mu^\alpha$. We obtain

$$\begin{aligned}
[F]_{2+\alpha,d,B_r(z_0)} \quad &\leq c' \mu^{2+2\alpha} \sup_{z,\bar z\in\Omega} d_{z,\bar z}^{2+\alpha} \frac{|\partial_{x_i x_j} u(z) - \partial_{x_i x_j} u(\bar z)|}{\|\bar z^{-1} \circ z\|^\alpha} + C'_\mu \sup_\Omega |u| \\
&+ c' \mu^{2+2\alpha} \sup_{z,\bar z\in\Omega} d_{z,\bar z}^{2+\alpha} \frac{|\partial_{x_i} u(z) - \partial_{x_j} u(\bar z)|}{\|\bar z \circ z\|^\alpha} + [f]_{2+\alpha,d,\Omega},
\end{aligned}$$

where c' and C'_μ are suitable positive constants. Hence, by using the above estimate together with (3.9) with $\varepsilon = \mu^{2+2\alpha}$ in (3.26), we find

$$\begin{aligned}
&d_{z_0}^{2+\alpha} \frac{|u(z_0) - u(\zeta_0)|}{\|\zeta_0^{-1} \circ z_0\|^\alpha} + d_{z_0}^{2+\alpha} \frac{|\partial_{x_i} u(z_0) - \partial_{x_i} u(\zeta_0)|}{\|\zeta_0^{-1} \circ z_0\|^\alpha} + d_{z_0}^{2+\alpha} \frac{|\partial_{x_i x_j} u(z_0) - \partial_{x_i x_j} u(\zeta_0)|}{\|\zeta_0^{-1} \circ z_0\|^\alpha} \\
&\quad \leq C''_\mu \left(\sup_\Omega |u| + [f]_{2+\alpha,d,\Omega} \right) + \bar c \mu^\alpha \sup_{z,\bar z\in\Omega} d_{z,\bar z}^{2+\alpha} \frac{|\partial_{x_i x_j} u(z) - \partial_{x_i x_j} u(\bar z)|}{\|\bar z^{-1} \circ z\|^\alpha} \\
&\quad + \bar c \mu^\alpha \sup_{z,\bar z\in\Omega} d_{z,\bar z}^{2+\alpha} \frac{|\partial_{x_i} u(z) - \partial_{x_j} u(\bar z)|}{\|\bar z \circ z\|^\alpha}
\end{aligned} \tag{3.33}$$

for every z_0, $\zeta_0 \in \Omega$, where the constant $\bar c$ does not depend on μ. Thus, if μ is sufficiently small, we have

$$\begin{aligned}
d_{z_0}^{2+\alpha} \frac{|u(z_0) - u(\zeta_0)|}{\|\zeta_0^{-1} \circ z_0\|^\alpha} + \sup_{z,\bar z\in\Omega} d_{z,\bar z}^{2+\alpha} \frac{|\partial_{x_i} u(z) - \partial_{x_j} u(\bar z)|}{\|\bar z^{-1} \circ z\|^\alpha} + \sup_{z,\bar z\in\Omega} d_{z,\bar z}^{2+\alpha} \frac{|\partial_{x_i x_j} u(z) - \partial_{x_i x_j} u(\bar z)|}{\|\bar z^{-1} \circ z\|^\alpha} \\
\leq c\left(\sup_\Omega |u| + [f]_{2+\alpha,d,\Omega}\right).
\end{aligned} \tag{3.34}$$

We next recall that $d_{z_0} \le d_{\zeta_0}$ and use again (3.9), with $\varepsilon = 1$. We find

$$\sup_\Omega |u| + \sum_{i=1}^{p_0} \sup_\Omega d_z |\partial_{x_i} u| + \sum_{i,j=1}^{p_0} \sup_\Omega d_z^2 |\partial_{x_i x_j} u| + \sup_{z,\bar z \in \Omega} d_{z,\bar z}^{2+\alpha} \frac{|u(z) - u(\bar z)|}{\|\bar z^{-1} \circ z\|^\alpha}$$

$$+ \sum_{i=1}^{p_0} \sup_{z,\bar z \in \Omega} d_{z,\bar z}^{2+\alpha} \frac{|\partial_{x_i} u(z) - \partial_{x_i} u(\bar z)|}{\|\bar z^{-1} \circ z\|^\alpha} + \sum_{i,j=1}^{p_0} \sup_{z,\bar z \in \Omega} d_{z,\bar z}^{2+\alpha} \frac{|\partial_{x_i x_j} u(z) - \partial_{x_i x_j} u(\bar z)|}{\|\bar z^{-1} \circ z\|^\alpha}$$

$$\le c(\sup_\Omega |u| + [f]_{2+\alpha,d,\Omega}).$$

As a final step, we observe that $Yu(z) = f(z) - \sum_{i,j=1}^{p_0} a_{i,j}(z) \partial_{x_i x_j} u(z) - \sum_{i=1}^{p_0} a_i(z) \partial_{x_i} u(z)$, so that we get from (3.5) the following inequality

$$[Yu]_{2+\alpha,d,\Omega} \le c \left([f]_{2+\alpha,d,\Omega} + \sum_{i,j=1}^{p_0} [a_{i,j}]_{\alpha,d,\Omega} [\partial_{x_i x_j} u]_{2+\alpha,d,\Omega} + \sum_{i=1}^{p_0} [a_i]_{\alpha,d,\Omega} [\partial_{x_i} u]_{2+\alpha,d,\Omega} \right).$$

The thesis follows from the last two estimates. □

Corollary 3.2. *If $f \in C_d^\alpha(\Omega)$ and Γ_{z_0} is the fundamental solution of the operator K_{z_0}, then the function*

$$v(z) = \int_\Omega \Gamma_{z_0}(z,w) f(w) dw$$

is a classical solution of $K_{z_0} v = -f$ in Ω and belongs to $C_d^{2+\alpha}(\Omega)$.

Proof. By Theorem 1.4 in [11], v is solution of $K_{z_0} v = -f$ in Ω. The conclusion directly follows by Theorem 1.3. □

Proof of Corollary 1.4. Let u be the unique solution of the Cauchy problem (1.19) on the set $\mathbb{R}^N \times]0, 2T[$. By the uniqueness result stated in Theorem 1.4 in [11], we have

$$u(x,t) = \int_{\mathbb{R}^N} \Gamma(x,t,y,0) \varphi(y) dy, \qquad (x,t) \in \mathbb{R}^N \times]0, 2T[,$$

then, from the first estimate in (2.21) it follows that

$$|u(x,t)| \le c_0 \|\varphi\|_\infty \int_{\mathbb{R}^N} \Gamma^+(x,t,y,0) dy,$$

so that

$$\sup_{(x,t) \in \mathbb{R}^N \times]0,2T[} |u(x,t)| \le c_0 \|\varphi\|_\infty \tag{3.35}$$

We next prove the L^∞ bound for u and for its derivatives. Consider any point $x \in \mathbb{R}^N$. By using the invariance with respect to the translation, it is not restrictive to assume that $x = 0$. We set $r = C_T T^2$ and $\Omega = B_{2r}(0) \times]0, 2T[$, where C_T is the constant appearing in Remark 2.2 and

$$B_\rho(0) = \left\{ y \in \mathbb{R}^N : \|(y,0)\| < \rho \right\}$$

is the metric ball with center at the origin and radius ρ. We explicitly note that $d_{(y,t)} = \sqrt{t}$, for every $(y,t) \in B_r(0) \times]0, T[$. Then, by applying Theorem 1.3 to the set $B_r(0) \times]0, T[$, and by (3.35), there exist a positive constant c_T, depending on T, such that

$$\sqrt{t} \sum_{i=1}^{p_0} \sup_{x, \in \mathbb{R}^N} |\partial_{x_i} u(x,t)| + t \sum_{i,j=1}^{p_0} \sup_{x \in \mathbb{R}^N} |\partial_{x_i x_j} u(x,t)| \le c_T \|\varphi\|_\infty \tag{3.36}$$

In order to conclude the proof, we consider the second order derivatives $\partial_{x_i x_j} u$ of u for $i, j = 1 \dots, p_0$. If $y \in B_r(0), y \neq 0$, then from Theorem 1.3 we get

$$t^{1+\frac{\alpha}{2}} \frac{|\partial_{x_i x_j} u(0,t) - \partial_{x_i x_j} u(y,t)|}{\|(y,0)\|^\alpha} \le c_T \|\varphi\|_\infty. \tag{3.37}$$

On the other hand, if $y \notin B_r(0) \times]0,T[$, we have

$$t^{1+\frac{\alpha}{2}} \frac{|\partial_{x_i x_j} u(0,t) - \partial_{x_i x_j} u(y,t)|}{\|(y,0)\|^\alpha} \le 2 \frac{T^{\frac{\alpha}{2}}}{r^\alpha} \sup_{x \in \mathbb{R}^N} t |\partial_{x_i x_j} u(x,t)| \le c_T \|\varphi\|_\infty, \tag{3.38}$$

thanks to (3.36). We then obtain the desired bound of the last term in the left hand side of (1.20). The bound of u and of its first order derivatives can be obtained in the same manner, then the proof is accomplished. □

4 Dirichlet Problem and Green function

In this section, we construct the Green function G related to the Dirichlet problem for L, for a suitable family of cylindrical sets $H_R(z_0, T)$, then we prove an uniformly lower bound for G.

Denote by $e_1 = (1, 0, \dots, 0)$ the first vector of the canonical basis of $\mathbb{R}^N$ and by $B_r(x_0)$ the Euclidean ball in $\mathbb{R}^N$, centered in x_0 with radius r. We fix any $\varepsilon \in]0,1[$, we set

$$S = B_1(\varepsilon\, e_1) \cap B_1(-\varepsilon\, e_1),$$

and we define for every $T > 0$ the *unit cylinder* and its *parabolic boundary* as

$$H(T) = S \times]0,T[, \qquad \partial_r H(T) = (S \times \{0\}) \cup (\partial S \times [0,T]).$$

Moreover, we set for every $z_0 = (x_0, t_0) \in \mathbb{R}^{N+1}$, and $R > 0$,

$$H_R(z_0, T) = z_0 \circ \delta(R)\big(H(T\, R^{-2})\big), \qquad \partial_r H_R(z_0, T) = z_0 \circ \delta(R)\big(\partial_r H(T\, R^{-2})\big). \tag{4.1}$$

Note that T is the true height of the set $H_R(z_0, T)$, by the definition (1.13) of $\delta(R)$. Besides

$$\text{meas}\big(H_R(z_0, T)\big) = T\, R^Q \text{meas}\big(S\big) \tag{4.2}$$

(since $H_R(z_0, T) \subset \mathbb{R}^{N+1}$ and $S \subset \mathbb{R}^N$, $\text{meas}\big(H_R(z_0, T)\big)$ denotes the Lebesgue measure in $\mathbb{R}^{N+1}$ while $\text{meas}\big(S\big)$ is the Lebesgue measure in $\mathbb{R}^N$). Indeed, it is sufficient to use the change of variable related to the translation (1.11) $\Phi(y, s) = (y + E(s)x_0, s + t_0)$ and note that $\det J_\Phi = 1$. Then we use the dilation defined in (1.13), and we find

$$\text{meas}\big(H_R(z_0, T)\big) = \text{meas}\big(H_R(0, T)\big) = R^{Q+2} \text{meas}\big(H(T\, R^{-2})\big) = T\, R^Q \text{meas}\big(H(1)\big).$$

Analogously, if we set

$$S_R(z_0, s) = z_0 \circ \delta(R)\big(S \times \{s\, R^{-2}\}\big), \tag{4.3}$$

with $s \in [0, T]$, we find

$$\text{meas}\big(S_R(z_0, s)\big) = R^Q \text{meas}\big(S\big). \tag{4.4}$$

In the sequel we will denote $S_R(z_0) = S_R(z_0, 0)$.

We say that a function $u : H_R(z_0, T) \cup \partial_r H_R(z_0, T) \to \mathbb{R}$ is a classical solution to the Dirichlet problem

$$\begin{cases} Lu & = -f \quad \text{in } H_R(z_0, T), \\ u & = g \quad \text{in } \partial_r H_R(z_0, T), \end{cases} \tag{4.5}$$

where $f \in C(H_R(z_0,T)$ and $g \in C(\partial_r H_R(z_0,T))$ if it is a classical solution to $Lu = -f$ in $H_R(z_0,T)$ and the boundary datum is attained by continuity. We next prove that the Dirichlet problem (4.5) has a unique classical solution. The uniqueness is an immediate consequence of the following Picone's maximum principle: *if $u \in C(H_R(z_0,T))$ is such that $\partial_{x_i} u, \partial_{x_i,x_j} u$ and Yu belong to $C(H_R(z_0,T))$, for $i,j = 1,\dots,p_0$ and satisfy*

$$\begin{cases} Lu & \geq 0 \quad \text{in } H_R(z_0,T) \\ \limsup_{z\to\zeta} u(z) & \leq 0 \quad \text{for every } \zeta \in \partial_r H_R(z_0,T), \end{cases}$$

then $u \leq 0$ on $H_R(z_0,T)$.

In order to prove the existence of the solution of the Cauchy-Dirichlet problem (4.5), we construct a barrier function at any points of $\partial_r H_R(z_0,T)$. Consider an open set $\Omega \subset \mathbb{R}^{N+1}$, a point $z_0 \in \partial\Omega$, and denote by $\widetilde{A}(z)$ the $(N+1)\times(N+1)$ matrix

$$\widetilde{A}(z) = \begin{pmatrix} A_0(z) & 0 \\ 0 & 0 \end{pmatrix}.$$

We say that a vector $\nu \in \mathbb{R}^{N+1}$ is a *L-non-characteristic outer normal to Ω at z* if $B(z+\nu,|\nu|)\cap\Omega = \emptyset$ and $\langle \widetilde{A}(z)\nu,\nu\rangle > 0$ (here $B(\zeta,\rho)$ is the Euclidean ball of $\mathbb{R}^{N+1}$ with center at ζ and radius ρ).

Lemma 4.1. *For every point $z = (x,t) \in \partial_r H(T)$, with $t \neq 0$ there exist a vector $\nu \in \mathbb{R}^{N+1}$ such that ν is an L-non-characteristic outer normal to $H(T)$ at z.*

Proof. For every $(y,t) \in \partial_r H(T)$, with $t > 0$, we set $y = y' + y''$, where $y' = (y_1,\dots,y_{p_0},0,\dots,0)$ and $y'' = y - y'$. We distinguish two cases: if $y_1 \geq 0$, we set $\nu = y + \varepsilon\, e_1$. Clearly, ν is an outer normal at (y,t). Besides, we have

$$\langle \widetilde{A}(y,t)\nu,\nu\rangle = \langle A_0(y,t)\nu',\nu'\rangle \geq \Lambda^{-1}\|\nu'\|^2 \geq \Lambda^{-1}(\varepsilon + y_1)^2 > 0$$

since $\varepsilon > 0$ and $y_1 \geq 0$. If otherwise $y_1 < 0$, we set $\nu = y - \varepsilon\, e_1$, and we conclude the proof as above. □

Summarizing, we are able to construct a barrier function ω to every point of the parabolic boundary of $H(T)$ as follow:

- if $z = (x,t) \in \partial_r H(T)$, with $t > 0$, we set

$$\omega_z(y,s) = e^{-\lambda|\nu|^2} - e^{-\lambda|(y,s)-(x,t)-\nu|^2}, \tag{4.6}$$

  where ν is an outer normal at (x,t) and λ is a positive constant only depending on the matrix B, on the constant Λ and on the L^∞ norm of the coefficients a_j of L.

- if $z = (x,0) \in \partial_r H(T)$, we set

$$\omega_z(y,s) = s. \tag{4.7}$$

Note that it is possible to choose the constant λ such that ω_z in (4.6) is a barrier for every operator L_R, with $R \in]0,1]$. As a consequence the function $\omega(y,s) = \omega_\zeta\left((\delta(1/R)\left(z_0^{-1}\circ(y,s)\right)\right)$ is a barrier at any point $z \in \partial_r H_R(z_0,T)$ (ω_ζ is the function defined in (4.6) or (4.7), with $\zeta = \delta(1/R)\left(z_0^{-1}\circ z\right)$).

Theorem 4.2. *Let $f \in C^\alpha_d(H_R(z_0,T))$ and $g \in C(\partial_r H_R(z_0,T))$. There exist a unique classical solution $u \in C^{2+\alpha}_d(H_R(z_0,T)) \cap C(\overline{H_R(z_0,T)})$ of the Dirichlet problem*

$$\text{(PD)} \quad \begin{cases} Lu & = -f \\ u|_{\partial_r H_R(z_0,T)} & = g \end{cases}$$

Proof. The uniqueness of the solution is an immediate consequence of the Picone's maximum principle. To prove the existence, we use the *continuity method*, as in the classical study of uniformly parabolic equations (see, for instance [15]).

As a first step, we consider the problem (PD) with homogeneous boundary condition ($g \equiv 0$). Let K_{z_0} be the frozen operator defined in (2.14). For every $\lambda \in [0,1]$, we define the operator $\mathcal{L}_\lambda$ by

$$\mathcal{L}_\lambda = \lambda L + (1-\lambda) K_{z_0}.$$

In the sequel, we shall indicate by $(P_{\lambda,f})$ the Dirichlet problem

$$(P_{\lambda,f}) \quad \begin{cases} \mathcal{L}_\lambda u & = -f \\ u|_{\partial_r H_R(z_0,T)} & = 0 \end{cases}$$

and by Λ the set

$$\Lambda = \Big\{ \lambda \in [0,1] : \text{the problem } (P_{\lambda,f}) \text{ has a solution}$$
$$u \in C_d^{2+\alpha}(H_R(z_0,T)) \cap C(\overline{H_R(z_0,T)}) \text{ for every } f \in C_d^\alpha(H_R(z_0,T)) \Big\}.$$

We clam that Λ contains $\lambda = 0$, and that Λ is at once an open and closed subset of $[0,1]$. It will follow that $\Lambda = [0,1]$, hence, the problem (PD) with $g \equiv 0$ has a solution.

In order to prove that $0 \in \Lambda$, we consider a function $f \in C_d^\alpha(H_R(z_0,T))$ and we denote by Γ_{z_0} the fundamental solution of the frozen operator K_{z_0} defined in (3.10). If we set

$$v(z) = \int_{H_R(z_0,T)} \Gamma_{z_0}(z,w) f(w) dw$$

then $v \in C_d^{2+\alpha}(H_R(z_0,T))$ and $K_{z_0} v = -f$ in $H_R(z_0,T)$, by Corollary 3.2. Since f is bounded, we also have $v \in C(\overline{H_R(z_0,T)})$. On the other hand, a result by Bony (Theorem 5.2 [5]) states that there exist a unique solution $\omega \in C^\infty(H_R(z_0,T)) \cap C(\overline{H_R(z_0,T)})$ to the Dirichlet problem

$$\begin{cases} K_{z_0}\omega & = 0 \\ \omega|_{\partial_r H_R(z_0,T)} & = -v \end{cases}$$

Hence $u = \omega + v \in C_d^{2+\alpha}(H_R(z_0,T)) \cap C(\overline{H_R(z_0,T)})$ and it is the solution of $(P_{0,f})$.

The proof of the fact that Λ is open and closed is analogous to the case of homogeneous Kolmogorov equations, i.e. in the case that all $*$-blocks of the matrix B are null. We refer to [24] for the details of the proof.

We next study the problem (PD) with $f \equiv 0$. Let $(g_n)_{n\in\mathbb{N}}$ be a sequence of functions belonging to $C^\infty(H_R(z_0,T)) \cap C(\overline{H_R(z_0,T)})$, uniformly convergent to g in $\overline{H_R(z_0,T)}$. Denote $f_n = Lg_n$ and let $u_n \in C_d^{2+\alpha}(H_R(z_0,T)) \cap C(\overline{H_R(z_0,T)})$ be the solution of the problem

$$\begin{cases} Lu & = f_n \\ u|_{\partial_r H_R(z_0,T)} & = 0 \end{cases}$$

(the existence of u_n has been proved before). If we define $v_n = u_n - g_n$, then

$$\begin{cases} Lv_n & = 0 \\ v_n|_{\partial_r H_R(z_0,T)} & = -g_n \end{cases}$$

and from the maximum principle it follows that

$$\sup_{\overline{H_R(z_0,T)}} |v_n - v_m| \le \sup_{\partial_r H_R(z_0,T)} |g_n - g_m| \qquad \forall\, n,m \in \mathbb{N}$$

Then (v_n) uniformly converges to some function v, and by the estimate (1.18), we deduce that v belongs to $C_d^{2+\alpha}(H_R(z_0,T)) \cap C(\overline{H_R(z_0,T)})$, it satisfies $Lv = 0$ in $H_R(z_0,T)$ and $v = g$ on $\partial_r H_R(z_0,T)$. This concludes the proof. $\square$

We next construct a Green function for the cylinder $H_R(z_0,T)$ by setting

$$G(z,\zeta) = \Gamma(z,\zeta) - h(z,\zeta), \tag{4.8}$$

where Γ is the fundamental solution of (1.1), and, for any $\zeta \in H_R(z_0,T)$, $h(\,\cdot\,,\zeta)$ is the solution to the follow Dirichlet problem

$$\begin{cases} Lh & = 0 \\ h|_{\partial_r H_R(z_0,T)} & = \Gamma(\cdot,\zeta). \end{cases} \tag{4.9}$$

From the properties of Γ and from the maximum principle it follows that, for every $f \in C(\overline{H_R(z_0,T)})$,

$$u(z) = \int_{H_R(z_0,T)} G(z,\zeta) f(\zeta) d\zeta, \qquad\qquad z \in H_R(z_0,T) \cup \partial_r H_R(z_0,T),$$

is the solution of the Dirichlet problem

$$\begin{cases} Lu & = -f \\ u|_{\partial_r H_R(z_0,T)} & = 0 \end{cases}$$

We next list some basic properties of G.

i) $G(z,\zeta) \geq 0$ for every $z,\zeta \in H_R(z_0,T)$ with $z \neq \zeta$.

ii) $G(\cdot,\zeta)|_{\partial_r H_R(z_0,T)} = 0$, for every $\zeta \in H_R(z_0,T)$.

iii) if the derivatives $\partial_{x_i x_j} a_{i,j}$ and $\partial_{x_i} a_i$ are Hölder continuous of exponent α, for $i,j = 1,\dots,p_0$, then the adjoint operator L^* satisfies hypotheses **[H1]-[H2]-[H3]**. Then $G^*(z,\zeta) = G(\zeta,z)$ (G^* denotes Green function of L^*).

For every $T>0$, $\delta \in]0,1]$, $(\xi,\tau) \in \mathbb{R}^{N+1}$ we set

$$H_\delta^+(\xi,\tau,T) = H_\delta(\xi,\tau,T) \cap \left\{ (x,t) \in \mathbb{R}^{N+1} : t \geq \frac{\tau+T}{2} \right\}.$$

Theorem 4.3. *Consider the Green function G related to any cylinder $H_R(\xi,\tau,R^2T)$, with $R \in]0,1]$. There exist three constants $\delta_0, T \in]0,1]$, and $\kappa > 0$, only depending on the operator L, such that*

$$G(x,t,y,\tau) \geq \frac{\kappa}{R^Q}, \qquad \text{for every } (x,t) \in H^+_{\delta_0 R}(\xi,\tau,R^2T),\ y \in S_{\delta_0 R}(\xi,\tau). \tag{4.10}$$

Proof. We can set, without loss of generality, $(\xi,\tau) = (0,0)$. Let G be the Green function related to $H_R(0,R^2T)$, defined by (4.8)-(4.9). The function $h_R(z,\zeta) = R^Q h(\delta(R)z, \delta(R)\zeta)$ is the solution of the problem

$$\begin{cases} L_R u & = 0 \\ u|_{\partial_r H(T)} & = \Gamma_R(\cdot, \delta(R)\zeta) \end{cases}$$

in the unit cylinder $H(T)$. Let $\alpha \in \left]0, \frac{1}{2}\right]$ be a fixed constant. Since $\overline{S_\alpha(0,0)}$ is a compact subset of the lower basis of $H(T)$, and Γ_R continuously depends on $R \in [0,1]$, we have

$$\max_{(z,(\xi,0),R) \in \partial_r H(T) \times \overline{S_\alpha(0,0)} \times [0,1]} h_R(z,(\xi,0)) \leq \max_{(z,(\xi,0),R) \in M(T) \times \overline{S_\alpha(0,0)} \times [0,1]} \Gamma_R(z,(\xi,0)) \equiv \kappa,$$

where $M(T) = \overline{\partial_r H(T) \cap \{0<t<T\}}$. We stress that the above inequity is uniform in $R \in [0,1]$. Thus, by the maximum principle,

$$h_R(z,\xi,0) \leq \kappa,$$

for every $z \in \overline{H(T)}, (\xi,0) \in \overline{S_\alpha(0,0)}$ and $R \in]0,1]$. On the other hand, by (2.18) and (2.19) we get

$$\Gamma_R(z,\zeta) = Z_R(z,\zeta) + J_R(z,\zeta) \geq F_R(z,\zeta),$$

for any $z, \zeta \in \mathbb{R}^{N+1}$, where

$$F_R(z,\zeta) := \Lambda^{-N}\Gamma^-_{\Lambda,R}(z,\zeta) - C(t-\tau)^{\frac{\alpha}{2}}\Gamma^+_R(z,\zeta)$$

and $\Gamma^-_{\Lambda,R}$ and Γ^+_R are the fundamental solution to

$$\frac{1}{\Lambda}\sum_{i=1}^{p_0}\partial_i^2 + Y_R, \qquad \mu\sum_{i=1}^{p_0}\partial_i^2 + Y_R,$$

respectively, with $\mu > \Lambda$. By the explicit expression (2.16) of the functions $\Gamma^-_{\Lambda,R}$ and Γ^+_R, we find

$$F_R(0,t,0,0) = \frac{e^{-tR^2 \mathrm{tr}(B)}}{(4\pi)^{N/2}\sqrt{\det \mathcal{C}_R(t)}}\left(\Lambda^{-\frac{3}{2}N} - C\, t^{\frac{\alpha}{2}}\mu^{-\frac{N}{2}}\right) \to \infty$$

as $t \to 0+$, uniformly with respect to $R \in [0,1]$. Thus, there exists $T \in]0,1]$ such that $F_R(0,t,0,0) \geq 3\kappa$ for any $t \in]0,T]$ and $R \in [0,1]$. Since F_R is a continuous function, there exists $\beta > 0$ such that

$$\Gamma_R(x,t,y,0) \geq F_R(x,t,y,0) \geq 2\kappa$$

for every $(x,t,y) \in H^+_\beta(0,T) \times S_\beta(0,0)$ and $R \in [0,1]$. Hence, if we set $\delta_0 = \min\{\alpha, \beta\}$, we have

$$\Gamma(x,t,y,0) \geq \frac{2\kappa}{R^Q}, \quad \text{and} \quad h(x,t,y,0) \leq \frac{\kappa}{R^Q}$$

for every $(x,t,y) \in H^+_{\delta_0}(0,T) \times S_{\delta_0}(0,0)$ and $R \in [0,1]$. The thesis then follows from the fact that $G(z,\zeta) = \Gamma(z,\zeta) - h(z,\zeta)$. □

5 Harnack inequality

Fix a positive T as in Theorem 4.3. For every $(\xi,\tau) \in \mathbb{R}^{N+1}$, $R \in]0,1]$ we set $(\xi^*,\tau^*) = (E(-R^2T)\xi, \tau - TR^2)$, $H^*(\xi,\tau,R) = H_R(\xi^*,\tau^*,TR^2)$, and we define

$$\mathrm{Osc}(u,\xi,\tau,R) = \sup_{H^*(\xi,\tau,R)} u - \inf_{H^*(\xi,\tau,R)} u \tag{5.1}$$

Lemma 5.1. *There exist two constants $\rho, \delta_1 \in]0,1[$, only depending on the operator L, such that*

$$\mathrm{Osc}(u,\xi,\tau,\delta R) \leq \rho\, \mathrm{Osc}(u,\xi,\tau,R),$$

for every positive solution u to $Lu = 0$ in $H^*(\xi,\tau,R)$ and for every $\delta \in]0,\delta_1]$.

Proof. Denote

$$m(R) = \inf_{H^*(\xi,\tau,R)} u \qquad M(R) = \sup_{H^*(\xi,\tau,R)} u$$

and, analogously, $m(\delta R)$ and $M(\delta R)$ be respectively the infimum and the supremum of u in $H^*(\xi,\tau,\delta R)$. We set

$$\widetilde{S} = \left\{(x,\tau^*) \in S_{\delta R}(\xi^*,\tau^*) : u(x,\tau^*) \geq \frac{M(R)+m(R)}{2}\right\}.$$

We examine the following two cases.

Case 1 Suppose that $\mathrm{meas}(\widetilde{S}) \geq \dfrac{\mathrm{meas}(S_R(\xi^*,\tau^*))}{2}$. We consider the function $u - m(R)$. We obviously have $u - m(R) \geq 0$ and $L(u - m(R)) = 0$ in $H^*(\xi, \tau, R)$. As a consequence of the comparison principle we find

$$u(z) - m(R) \geq \int_{S_R(\xi^*,\tau^*)} G(z, y, \tau^*)(u(y, \tau^*) - m(R))dy,$$

for every $z \in H^*(\xi, \tau, R)$. On the other hand, by Theorem 4.3, there exist a positive κ such that

$$\int_{S_R(\xi^*,\tau^*)} G(z, y, \tau - R)(u(y, \tau^*) - m(R))dy \geq \int_{\widetilde{S}} G(z, y, \tau^*)(u(y, \tau^*) - m(R))dy$$

$$\geq \frac{M(R) - m(R)}{2} \int_{\widetilde{S}} G(z, y, \tau^*)dy \geq \frac{M(R) - m(R)}{2} \int_{\widetilde{S}} \frac{\kappa}{R^Q} \geq \frac{\kappa\, \mathrm{meas}(\widetilde{S})}{2\, R^Q}(M(R) - m(R)),$$

for every $z \in H^*(\xi, \tau, \delta R)$. Thus,

$$m(\delta R) - m(R) \geq \frac{\kappa\, \mathrm{meas}(S_R(\xi^*, \tau^*))}{4\, R^Q}(M(R) - m(R)).$$

Since it is not restrictive to assume that $\kappa \leq \frac{4}{\mathrm{meas}(S)}$, by (4.4), we have

$$M(\delta R) - m(\delta R) \leq M(R) - m(\delta R) \leq \left(1 - \frac{\kappa\, \mathrm{meas}(S)}{4}\right)(M(R) - m(R)).$$

This proves the claim, with $\rho = 1 - \dfrac{\kappa\, \mathrm{meas}(S)}{4}$.

Case 2 Suppose that $\mathrm{meas}(\widetilde{S}) < \dfrac{\mathrm{meas}(S_R(\xi^*, \tau^*))}{2}$. We set

$$\widetilde{S}' = \left\{(x, \tau^*) \in S_{\delta R}(\xi^*) : u(x, \tau^*) < \frac{M(R) + m(R)}{2}\right\}$$

and we note that $\mathrm{meas}(S') \geq \frac{\mathrm{meas}(S_R(\xi^*,\tau^*))}{2}$. We now consider the function $M(R) - u$, which is non-negative and satisfies $L(M(R) - u) = 0$. As in the previous case, we find

$$M(R) - u(z) \geq \frac{\kappa\, \mathrm{meas}(S_R(\xi^*, \tau^*))}{4\, R^Q}(M(R) - m(R))$$

for every $z \in H^*(\xi, \tau, \delta R)$. Hence

$$M(\delta R) \leq \frac{\kappa\, \mathrm{meas}(S)}{4} m(R) + (1 - \frac{\kappa\, \mathrm{meas}(S)}{4})M(R),$$

and

$$M(\delta R) - m(\delta R) \leq M(\delta R) - m(R) \leq \left(1 - \frac{\kappa\, \mathrm{meas}(S)}{4}\right)(M(R) - m(R)).$$

This completes the proof. □

Proof of Theorem 1.2. We follow the line of the proof of Theorem 5.4 in [13]. We fix any $\delta \in]0, \delta_0[$, (δ_0 is the constant in Proposition 4.3) and three positive constants α, β, γ such that $\alpha < \beta < \gamma < 1$ and that $\gamma \geq \beta + 1/2$. There exists $(\bar{x}, \bar{t}) \in \overline{H^+}$ such that $u(\bar{x}, \bar{t}) = \min_{\overline{H^+}} u$. It is not restrictive to assume $u(\bar{x}, \bar{t}) = 1$. We consider, for every $r \in [0, \beta R^2 T]$, the function

$$v(x, t) = \int_{S_R(\xi,\tau,r)} G(x, t, y, r)u(y, r)dy\,, \quad \forall (x, t) \in H_R(\xi, \tau, R^2 T)$$

(recall the definition (4.3) of $S_R(\xi,\tau,r)$). Since $u \geq 0$, by the comparison principle, we obtain $u(x,t) \geq v(x,t)$, for every $(x,t) \in H_R(\xi,\tau,R^2T)$, then

$$u(\bar{x},\bar{t}) \geq \int_{S_R(\xi,\tau,r)} G(\bar{x},\bar{t},y,r)u(y,r)dy. \tag{5.2}$$

Let $\delta' = \frac{\delta+\delta_0}{2}$ and consider, for any $\lambda > 0$, the set

$$\mathcal{S}(r,\lambda) = \{y \in S_{\delta' R}(\xi,\tau,r) : u(y,r) \geq \lambda\}.$$

Then inequality (5.2) and Proposition 4.3 imply that

$$1 = u(\bar{x},\bar{t}) \geq \int_{\mathcal{S}(r,\lambda)} G(\bar{x},\bar{t},y,r)u(y,r)dy \geq \frac{\lambda\kappa\,\mathrm{meas}(\mathcal{S}(r,\lambda))}{R^Q}. \tag{5.3}$$

We set

$$K = \frac{1}{2}\left(1+\frac{1}{\rho}\right) \qquad r(\lambda) = \frac{R}{\delta}\left(\frac{4}{\kappa\lambda(1-\rho)\mathrm{meas}(S)}\right)^{\frac{1}{Q}} \tag{5.4}$$

where ρ is the constant in Lemma 5.1. Note that, by (4.2), $r(\lambda)$ is such that

$$\mathrm{meas}\big(H^*_{\delta r(\lambda)}(\xi,\tau,(\delta r(\lambda))^2)\cap S_R(\xi,\tau,r)\big) = \frac{4\,R^Q}{\kappa\lambda(1-\rho)}, \tag{5.5}$$

for every $r \in [t-(\delta r(\lambda))^2, t]$.

We next prove the following statement. Let $\lambda > 0$ and $(x,t) \in H_{\delta' R}(\xi,\tau,R^2T)$ with $t \leq \tau + \beta R^2 T$ be such that $u(x,t) \geq \lambda$ and that $H^*_{r(\lambda)}(x,t,r(\lambda)^2T) \subset H_{\delta' R}(\xi,\tau,R^2T)$. Then there exists $(x',t') \in H^*_{r(\lambda)}(x,t,r(\lambda)^2T)$ such that $u(x',t') \geq K\lambda$. Indeed, from (5.3) it follows that

$$\mathrm{meas}\left(\mathcal{S}\left(t,\frac{\lambda}{2}(1-\rho)\right)\right) \leq \frac{2\,R^Q}{\lambda\,\kappa(1-\rho)}$$

so that, by (5.5), there is a $(\xi',\tau') \in H^*_{\delta r(\lambda)}(\xi,\tau,(\delta r(\lambda))^2T)$ such that $u(\xi',\tau') < \frac{\lambda}{2}(1-\rho)$. The claim then follows from Lemma 5.1.

We next show that there exists a positive constant M_0 such that $u(x,t) \leq M_0$ for every $(x,t) \in H^-$. The thesis then follows, since $u(\bar{x},\bar{t}) = \min_{\overline{H^+}} u = 1$. Fix a positive M and suppose that there is a $z_0 \in H^-$ such that $u(z_0) > M$. Then, by the preceding paragraph, there exists a (possibly infinite) sequence $\big(z_j\big)$ such that

$$u(z_j) \geq M\,K^j, \qquad z_{j+1} \in H^*_{r_j}(z_j, Tr_j^2), \qquad \text{where } r_j = r(M\,K^j),$$

provided that

$$H^*_{r_j}(z_j,Tr_j^2) \subset H_{\delta' R}(\xi,\tau,TR^2). \tag{5.6}$$

If we prove that (5.6) holds for every $j \in \mathbb{N}$, we find a sequence $u(z_j)$ which is unbounded and we get a contradiction with the continuity of u. In order to show that (5.6) holds for any $j \in \mathbb{N}$ we denote $z_j = (x_j,t_j)$ and we remark that

$$t_j > t_0 - \sum_{i=0}^{j-1} Tr_i^2 \geq t_0 - \frac{TR^2}{\delta^2}\left(\frac{4}{M\kappa(1-\rho)\mathrm{meas}(S)}\right)^{\frac{2}{Q}} \sum_{i=0}^{\infty} K^{-\frac{2i}{Q}},$$

for any $j \in \mathbb{N}$. Hence, if we set

$$T_0 = \frac{TR^2}{\delta^2}\left(\frac{4}{M\kappa(1-\rho)\mathrm{meas}(S)}\right)^{\frac{2}{Q}} \sum_{i=0}^{\infty} K^{-\frac{2i}{Q}},$$

we can choose a sufficiently big constant M such that $t_j > t_0 - T_0 \geq \tau$ for every $j \in \mathbb{N}$. We next note that

$$d(z_{j+1}, z_j) := \|z_j^{-1} \circ z_{j+1}\| \leq c_0 r_j, \quad \forall j \in \mathbb{N},$$

where $c_0 = \max_{z \in H_1^*(0,0,1), r \in [0,1]} \|z^{-1} \circ_r (0,0)\|$. Besides,

$$d(z_j, z_0) \leq \sum_{i=1}^{j} C_{T_0}^i d(z_i, z_{i-1}),$$

where C_{T_0} is the constant in (2.7). Hence

$$d(z_j, z_0) \leq c_0 \frac{R}{\delta} \left(\frac{4}{\kappa M (1-\rho) \text{meas}(S)} \right)^{\frac{1}{Q}} \sum_{i=1}^{\infty} C_{T_0}^i K^{-\frac{i}{Q}}.$$

Since the constant C_{T_0} in (2.7) goes to 1 as $T_0 \to 0$, it is possible to choose M so big that $C_{T_0} < K^{\frac{1}{Q}}$ and, then, the series $\sum_{i=1}^{\infty} C_{T_0}^i K^{-\frac{i}{Q}}$ is convergent. We finally remark that ζ_0 belongs to H^- and that $\overline{H^-}$ is a compact subset of $\text{int}(H_{\delta' R}(\xi, \tau, TR^2))$, we can then choose a positive M_0, that depends on α, δ, δ_0 but does not depend on R, such that (5.6) holds for any $j \in \mathbb{N}$. This accomplishes the proof. $\square$

In order to prove a non-local Harnack inequality we consider the cone $\mathcal{K}_\eta$ defined as

$$\mathcal{K}_\eta = \{\delta(\lambda)(x, -1) : \lambda > 0, |x| \leq \eta\}, \qquad \mathcal{K}_\eta(z_0) = z_0 \circ \mathcal{K}_\eta, \tag{5.7}$$

and we state the following

Proposition 5.2. *There exist three positive constants η, M and T_0, such that, if $u : \mathbb{R}^N \times]t_0, t_1[\to \mathbb{R}$ is a positive solution to $Lu = 0$, then*

$$u(\xi, \tau) \leq M\, u(x, t),$$

for every $(x, t) \in \mathbb{R}^N \times]t_0, t_1[$ and $(\xi, \tau) \in \mathcal{K}_\eta(x, t)$ such that $\tau \geq \max\{t - T_0, \frac{t+t_0}{2}\}$.

Proof. It is a simple corollary of Theorem 1.2, with $\alpha \leq 1/2 \leq \beta$ and $T_0 = T/2$. It is sufficient to consider a cylinder $H_R^*(x, t, R^2T) \subset \mathbb{R}^N \times]t_0, t_1[$ such that $(\xi, \tau) \in H_R^-(x, t, R^2T)$. $\square$

6 Non-local results

In this section we prove a non local Harnack inequality for positive solutions to $Lu = 0$ defined in a strip $\mathbb{R}^N \times J$, where J is an interval of $\mathbb{R}$. The method was introduced by Aronson [1] and used by Aronson and Serrin [2] in the study of uniformly parabolic operators, then it has been extended by Polidoro [28] to the non-Euclidean setting of homogeneous Kolmogorov operators. The idea is to use repeatedly the (local) Harnack inequality stated in Proposition 5.2. Here we set the problem in the theory of the optimal control for linear systems with quadratic cost, and we generalize the result in [28], since we drop the homogeneity property of the translation group.

Consider $(x, t), (y, s) \in \mathbb{R}^{N+1}$ with $t > s$, and let $\overline{\gamma} : [0, t-s] \to \mathbb{R}^{N+1}$ be a curve such that

$$\begin{cases} \dot{\overline{\gamma}}(\tau) = & \sum_{j=1}^{p_0} \lambda_j(\tau) X_j + Y(\overline{\gamma}(\tau)) \\ \overline{\gamma}(0) = & (x, t), \quad \overline{\gamma}(t-s) = (y, s), \end{cases} \tag{6.1}$$

where $X_1, \dots, X_{p_0}$ and Y are defined in (1.6) and the *controls* $\lambda_1, \dots \lambda_{p_0}$ belong to $L^2([0, t-s])$. If we denote $\overline{\gamma}(\tau) = (\gamma(\tau), t-\tau)$, with $\gamma : [0, t-s] \to \mathbb{R}^N$, then (6.1) can be state in the equivalent form

$$\begin{cases} \dot{\gamma}(\tau) = & B^T \gamma(\tau) + A^{\frac{1}{2}} \lambda(\tau) \\ \gamma(0) = & x, \quad \gamma(t-s) = y, \end{cases} \tag{6.2}$$

where $\lambda(\tau) = (\lambda_1(\tau), \dots, \lambda_{p_0}(\tau), 0, \dots, 0)^T \in \mathbb{R}^N$. It is known that the condition H_1 implies the existence of a path γ that solves (6.2) (see [19], Theorem 5, p. 81). Among the paths γ satisfying (6.2), we look for the one minimizing the *total cost*

$$C(\lambda) = \int_0^{t-s} |\lambda(s)|^2 ds. \tag{6.3}$$

The general control theory provides the optimal control and gives the explicit expression of the optimal cost. Our main result is the following

Theorem 6.1. *Let T_0, M be the constants in Proposition 5.2 and let $u : \mathbb{R}^N \times]s - T_0, s + T_0[\to \mathbb{R}$ be a positive solution to $Lu = 0$. Then*

$$u(y, s) \le M^{1 + \frac{1}{h} \langle \mathcal{C}^{-1}(t-s)(x - E(t-s)y), x - E(t-s)y \rangle} u(x, t),$$

for every $(x, t) \in \mathbb{R}^N \times]s, s + T_0[$ (h is a positive constant depending only on T_0, A and B).

The first step in the proof of Theorem 6.1 is the following

Lemma 6.2. *There exist two positive constants h, η such that, if $\gamma : [0, \sigma] \to \mathbb{R}^N$ is a solution to* (6.2), *and*

$$\int_0^\sigma |\lambda(\tau)|^2 d\tau \le h$$

then $(\gamma(\tau), t - \tau) \in \mathcal{K}_\eta(x, t)$ for every $\tau \in [0, \sigma]$.

Proof. The explicit solution to (6.2) is

$$\gamma(\tau) = E(-\tau)x + \int_0^\tau E(\rho - \tau) A^{\frac{1}{2}} \lambda(\rho) d\rho. \tag{6.4}$$

If we decompose

$$\gamma(\tau) = \left(\gamma^{(0)}(\tau), \gamma^{(1)}(\tau), \dots, \gamma^{(r)}(\tau)\right)^T,$$

with $\gamma^{(j)}(s) \in \mathbb{R}^{p_j}$, for $j = 0, \dots, r$, then a direct computation shows that $(\gamma(\tau), t - \tau) \in \mathcal{K}_\eta(x, t)$ if, and only if,

$$|(\gamma(\tau) - E(-\tau)x)^{(j)}| \le \eta\, \tau^{j + \frac{1}{2}}, \tag{6.5}$$

for $j = 0, \dots, r$ (here $|v|$ denotes the Euclidean norm of v). By Lemma 2.1, we find

$$|(\gamma(\tau) - E(-\tau)x)^{(j)}| \le c_j \int_0^\tau (\rho - \tau)^j |\lambda(\rho)| d\rho \le c'_j\, \tau^{j + \frac{1}{2}} \left(\int_0^\tau |\lambda(\rho)|^2 d\rho \right)^{\frac{1}{2}}$$

for $j = 0, \dots, r$, for some positive constants $c'_0, \dots, c'_r$ only depending on σ, A and B. Hence

$$|(\gamma(\tau) - E(-\tau)x)^{(j)}| \le c'_j \sqrt{h}\, \tau^{j + \frac{1}{2}} \qquad j = 0, \dots, r$$

for every $\tau \in [0, \sigma]$, and (6.5) follows by choosing h is sufficiently small. This accomplishes the proof. □

Lemma 6.3. *Let T_0, M and η be the constants in Proposition 5.2, let h be the constant in Lemma 6.2 and let $(x,t),(y,s) \in \mathbb{R}^{N+1}$ be such that $s < t < s + T_0$. Suppose that $u : \mathbb{R}^N \times]s - T_0, s + T_0[\to \mathbb{R}$ is a positive solution to $Lu = 0$, and that $\gamma : [0, t-s] \to \mathbb{R}^N$ is a solution to (6.2). Then*

$$u(y,s) \le M^{1+\frac{C(\lambda)}{h}} u(x,t),$$

where $C(\lambda)$ is the total cost of the control corresponding to γ.

Proof. If $\int_0^{t-s} |\lambda(\tau)|^2 d\tau \le h$, then the result is an immediate consequence of Lemma 6.2. If the above inequality is not satisfied, we set

$$k = \max \left\{ j \in \mathbb{N} : j\, h < \int_0^{t-s} |\lambda(\tau)|^2 d\tau \right\}. \tag{6.6}$$

and define

$$\sigma_j = \inf_{\sigma > 0} \int_0^{\sigma} |\lambda(\tau)|^2 d\tau > j\, h, \qquad j = 1, \dots, k.$$

Note that $0 < \sigma_1 < \dots < \sigma_k < t - s$, so that

$$t - \sigma_j > s > \max \left\{ t - T_0, \frac{t + (s - T_0)}{2} \right\}, \qquad \text{for } j = 1, \dots, k. \tag{6.7}$$

By Lemma 6.2 $(\gamma(\sigma_1), t - \sigma_1) \in \mathcal{K}_\eta(x,t)$, then $u(\gamma(\sigma_1), t - \sigma_1) \le M\, u(x,t)$.

We next repeat the above argument: Lemma 6.2 ensures that $(\gamma(\sigma_2), t - \sigma_2) \in \mathcal{K}_\eta(\gamma(\sigma_1), t - \sigma_1)$. Moreover (6.7) holds, then Proposition 5.2 gives $u(\gamma(\sigma_2), t - \sigma_2) \le M u(\gamma(\sigma_1), t - \sigma_1) \le M^2\, u(x,t)$. We repeat the above argument until, at the $(k+1)$-th step, we find

$$u(y,s) \le M u(\gamma(\sigma_k), t - \sigma_k) \le M^{k+1}\, u(x,t).$$

The thesis then follows from (6.6). $\square$

Proof of Theorem 6.1. Consider the Hamiltonian function $\mathcal{H}$ related to the control problem (6.2) in the interval $[0, t-s]$:

$$\mathcal{H}(x, q, \lambda) = q\, B^T x + q\, A^{\frac{1}{2}} \lambda + q_0 |\lambda|^2, \qquad q = (q_1, \dots, q_N),\ x = (x_1, \dots, x_N)^T. \tag{6.8}$$

The classical control theory for autonomous linear systems states that the optimal control is given by

$$q_0 = -\frac{1}{2}, \qquad \lambda(\tau) = \left(A^{\frac{1}{2}} \right)^T q(\tau)^T \text{ for some solution to } \dot{q} = -q\, B^T \tag{6.9}$$

(see [19], Theorem 3, p. 180). We use the above identity in (6.2), and we compute the explicit solution:

$$\gamma(\tau) = E(-\tau) \left(x + \mathcal{C}(\tau) q(0)^T \right)$$

for some constant vector $q(0)$ that is determined by the condition $\gamma(t-s) = y$. We find

$$q(0)^T = \mathcal{C}^{-1}(t-s)(E(t-s)y - x),$$

hence the optimal control is

$$\lambda(\tau) = \left(A^{\frac{1}{2}} \right)^T E(\tau)^T \mathcal{C}^{-1}(t-s)(E(t-s)y - x)$$

and the optimal cost is

$$C(\lambda) = \int_0^{t-s} |\lambda(\tau)|^2 d\tau = \langle \mathcal{C}^{-1}(t-s)(x - E(t-s)y), x - E(t-s)y \rangle.$$

The conclusion follows from Lemma 6.3. $\square$

Proof of Theorem 1.5. By Theorem 6.1 we have

$$\Gamma(x,t) \geq \Gamma\left(0, \frac{t}{2}\right) M^{-1-\frac{1}{h}\langle \mathcal{C}^{-1}\left(\frac{t}{2}\right)x,x\rangle}, \qquad \text{for every } (x,t) \in \mathbb{R}^N \times \mathbb{R}^+. \tag{6.10}$$

We next claim that there exist two positive constants T_1 and c_1 such that

$$\Gamma\left(0, \frac{t}{2}\right) \geq \frac{c_1}{\sqrt{\det \mathcal{C}(t)}}, \qquad \forall\, 0 < t < T_1. \tag{6.11}$$

Indeed, from Remark 2.3 and from the explicit expression (1.10) of Γ_Λ^- and Γ^+ we get

$$\Gamma\left(0, \frac{t}{2}\right) \geq \Lambda^{-N}\Gamma_\Lambda^-\left(0, \frac{t}{2}\right) - C\left(\frac{t}{2}\right)^{\frac{\alpha}{2}}\Gamma^+\left(0, \frac{t}{2}\right) \geq \frac{c_2 - c_3\left(\frac{t}{2}\right)^{\frac{\alpha}{2}}}{\sqrt{\det \mathcal{C}\left(\frac{t}{2}\right)}}.$$

The claim (6.11) then follows form the fact that $\det \mathcal{C}(t)$ is an increasing function, by (1.8). As a consequence of (6.10) and (6.11) we then find

$$\Gamma(x,t) \geq \frac{c_1}{\sqrt{\det \mathcal{C}(t)}} M^{-1-\frac{1}{h}\langle \mathcal{C}^{-1}\left(\frac{t}{2}\right)x,x\rangle}, \qquad \text{for every } (x,t) \in \mathbb{R}^N \times]0, T_1[. \tag{6.12}$$

We next show that there exist tree positive constants T_2, c' and c'' such that

$$c'\left\langle \mathcal{C}^{-1}(t)x,x\right\rangle \leq \left\langle \mathcal{C}^{-1}\left(t/2\right)x,x\right\rangle \leq c''\left\langle \mathcal{C}^{-1}(t)x,x\right\rangle \tag{6.13}$$

for every $x \in \mathbb{R}^N$, $0 < t < T_2$. By the above inequalities and (6.12) we obtain

$$\Gamma(x,t) \geq \frac{c_1}{\sqrt{\det \mathcal{C}(t)}} M^{-1-\frac{1}{h}\langle \mathcal{C}^{-1}(t)x,x\rangle} \tag{6.14}$$

for every $(x,t) \in \mathbb{R}^N \times]0, \min\{T_1, T_2\}[$. We next use (2.12) to prove (6.13). We have

$$\frac{1 - c_T\frac{t}{2}}{1 + c_T t} \cdot \frac{\langle \mathcal{C}_0^{-1}\left(\frac{t}{2}\right)x,x\rangle}{\langle \mathcal{C}_0^{-1}(t)x,x\rangle} \leq \frac{\langle \mathcal{C}^{-1}\left(\frac{t}{2}\right)x,x\rangle}{\langle \mathcal{C}^{-1}(t)x,x\rangle} \leq \frac{1 + c_T\frac{t}{2}}{1 - c_T t} \cdot \frac{\langle \mathcal{C}_0^{-1}\left(\frac{t}{2}\right)x,x\rangle}{\langle \mathcal{C}_0^{-1}(t)x,x\rangle},$$

for every $x \in \mathbb{R}^N$ and $t \in]0, T[$ such that $t < \frac{1}{c_T}$. We next use the second identity in (2.10)

$$\frac{\langle \mathcal{C}_0^{-1}\left(\frac{t}{2}\right)x,x\rangle}{\langle \mathcal{C}_0^{-1}(t)x,x\rangle} = \frac{\langle \mathcal{C}_0^{-1}\left(\frac{1}{2}\right)D\left(\frac{1}{\sqrt{t}}\right)x, D\left(\frac{1}{\sqrt{t}}\right)x\rangle}{\langle \mathcal{C}_0^{-1}(1)D\left(\frac{1}{\sqrt{t}}\right)x, D\left(\frac{1}{\sqrt{t}}\right)x\rangle},$$

and the claim (6.14) follows from the fact that the constant matrices $\mathcal{C}_0^{-1}\left(\frac{t}{2}\right)$ and $\mathcal{C}_0^{-1}(1)$ are strictly positive. From (1.10) and (6.14) we easily find that

$$\Gamma(x,t) \geq c^-\,\Gamma^-(x,t), \qquad \forall\, x \in \mathbb{R}^N,\ 0 < t < \min\{T_1, T_2\},$$

for some two positive constants μ and c^-, thus the proof is accomplished if $T \leq \min\{T_1, T_2\}$. If not, we apply repeatedly the reproduction property of the fundamental solution (2.20) and we conclude the proof after a finite number of steps. □

We now use Theorem 1.5 and the uniqueness result by result by Di Francesco and Pascucci ([11], Theorem 1.6) to prove Theorem 1.6.

Proof of Theorem 1.6. We first show that, if u be a non-negative solution of $Lu = 0$ on the strip $\mathbb{R}^N \times]0, T[$, then

$$u(x,t) \geq \int_{\mathbb{R}^N} \Gamma(x,t,\xi,\tau) u(\xi,\tau) d\xi \tag{6.15}$$

for every $(x,t) \in \mathbb{R}^N \times]0,T[$ and $0 < \tau < t$. For every $n \in \mathbb{N}$, we consider the function $\varphi_n(x,t) = 1 - \eta_n(x,t)$, where η_n is defined in (2.48). We recall that, for every $n \in \mathbb{N}$, $\varphi_n \in C_0^\infty(\mathbb{R}^N)$, $0 \leq \varphi_n \leq 1$, $\varphi_n(x,t) = 1$ if $\|(x,t)\| \leq \frac{n}{2}$ and $\varphi_n(x,t) = 0$ if $\|(x,t)\| \geq n$. For every $(x,t) \in \mathbb{R}^N \times]0,T[$ and $0 < \tau < t$, we set

$$u_n(x,t,\tau) = \int_{\mathbb{R}^N} \Gamma(x,t,\xi,\tau) \varphi_n(\xi) u(\xi,\tau) d\xi.$$

Clearly, we have that $Lu_n(\cdot,\tau) = 0$ in $\mathbb{R}^N \times]\tau, T[$ and

$$\lim_{(x,t)\to(y,\tau)} u_n(x,t,\tau) = \varphi_n(y) u(y,\tau) \leq u(y,\tau)$$

for every $y \in \mathbb{R}^N$. We then recall that $\varphi_n u$ is a compactly supported continuous function and use the upper bound (1.21) of Γ. We obtain

$$\lim_{|x|\to+\infty} \left(\sup_{\tau \leq t \leq T} u_n(x,t,\tau) \right) = 0$$

hence, by the maximum principle, we get

$$0 \leq u_n(x,t,\tau) \leq u(x,t) + \varepsilon \tag{6.16}$$

for every $(x,t) \in \mathbb{R}^N \times]\tau, T[$ and for any positive ε. Thus, being Γ and u positive, we find

$$\lim_{n\to+\infty} u_n(x,t,\tau) = \int_{\mathbb{R}^N} \Gamma(x,t,\xi,\tau) u(\xi,\tau) d\xi$$

and (6.15) plainly follows from (6.16).

In order to conclude the proof, we note that, for every $s \in]0,t[$ there exist two positive constants k_1, k_2, only depending on s, t and on the constant μ in (1.22), such that

$$\Gamma^-(0,t,\xi,\tau) \geq k_2 e^{-k_1|\xi|^2}$$

for every $\tau \in [0,s]$ and $\xi \in \mathbb{R}^N$. We then have

$$c_T^- k_2 \int_{\mathbb{R}^N} e^{-k_1|\xi|^2} u(\xi,\tau) d\xi \leq c_T^- \int_{\mathbb{R}^N} \Gamma^-(0,t,\xi,\tau) u(\xi,\tau) d\xi$$
$$\text{(by Theorem 1.5)} \leq \int_{\mathbb{R}^N} \Gamma(0,t,\xi,\tau) u(\xi,\tau) d\xi \leq \text{(by (6.15))} \leq u(0,t)$$

We next conclude the proof of the Theorem. Let u, v be two non negative solutions to the solution of the Cauchy problem (1.19). By integrating the above inequality with respect to $\tau \in]0,s[$, we obtain

$$\frac{c_T^- k_2}{s} \int_0^s \int_{\mathbb{R}^N} e^{-k_1|\xi|^2} |u(\xi,\tau) - v(\xi,\tau)| d\xi d\tau \leq u(0,t) + v(0,t),$$

then Theorem 1.6 in [11] implies that $u \equiv v$ in $\mathbb{R}^N \times [0,s[$. This accomplishes the proof. □

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