# Moser type estimates for a class of uniformly subelliptic ultraparabolic operators *


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**Abstract**

We consider a class of second order ultraparabolic differential equations in the form

$$\partial_t u = \sum_{j=1}^{m} X_j (AXu)_j + X_0 u,$$

where $A$ is a bounded, symmetric and uniformly positive matrix with measurable coefficients, under the assumption that the operator $\sum_{j=1}^{m} X_j^2 + X_0 - \partial_t$ is hypoelliptic and the vector fields $X_1, \ldots, X_m$ and $X_0 - \partial_t$ are invariant with respect to a suitable homogeneous Lie group. We adapt the Moser's iterative methods to the non-Euclidean geometry of the Lie groups and we prove an $L^\infty_{\rm loc}$ bound of the solution $u$ in terms of its $L^p_{\rm loc}$ norm.



*Keywords:* hypoelliptic equations, measurable coefficients, Moser's iterative method.


## 1 Introduction

Consider second order partial differential equations in the form

$$\mathcal{L}_A u := \sum_{i,j=1}^{m} X_j \left(a_{ij}(x,t) X_i u\right) + X_0 u - \partial_t u = 0, \tag{1.1}$$

where $(x,t) = (x_1, \ldots, x_N, t)$ denotes the point in $\mathbb{R}^{N+1}$, and $1 \le m \le N$. The $X_j$'s in (1.1) are smooth vector fields on $\mathbb{R}^N$, i.e.

$$X_j(x) = \sum_{k=1}^{N} b_k^j(x) \partial_{x_k}, \qquad k = 0, \ldots, m,$$

and any $b_k^j$ is a $C^\infty$ function. In the sequel we always denote by $z = (x,t)$ the point in $\mathbb{R}^{N+1}$, and by $A$ the $m \times m$ matrix $A = (a_{i,j})_{i,j=1,\ldots,m}$. Moreover we will use the following notations:

$$X = (X_1, \ldots, X_m), \qquad Y = X_0 - \partial_t, \qquad \mathrm{div}_X F = \sum_{j=1}^{m} X_j F_j, \tag{1.2}$$

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for every vector field $F = (F_1, \dots, F_m)$, so that the expression $\mathcal{L}_A u$ reads

$$\mathcal{L}_A u = \mathrm{div}_X(AXu) + Yu.$$

Finally, when $A$ is the $m \times m$ identity matrix, we will use the notation

$$\mathcal{L} := \sum_{k=1}^{m} X_k^2 + Y \tag{1.3}$$

We say that a curve $\gamma : [0, T] \to \mathbb{R}^{N+1}$ is $\mathcal{L}$-*admissible* if it is absolutely continuous and satisfies

$$\gamma'(s) = \sum_{k=1}^{m} \lambda_k(s) X_k(\gamma(s)) + \mu(s) Y(\gamma(s)), \qquad \text{a.e. in } [0, T],$$

for suitable piecewise constant real functions $\lambda_1, \dots, \lambda_m, \mu$, with $\mu \geq 0$. We suppose that:

[H.1] there exists a homogeneous Lie group $\mathbb{G} = \left(\mathbb{R}^{N+1}, \circ, \delta_\lambda\right)$ such that

(i) $X_1, \dots, X_m, Y$ are left translation invariant on $\mathbb{G}$;

(ii) $X_1, \dots, X_m$ are $\delta_\lambda$-homogeneous of degree one and $Y$ is $\delta_\lambda$-homogeneous of degree two;

[H.2] for every $(x, t), (\xi, \tau) \in \mathbb{R}^{N+1}$ with $t > \tau$, there exists an $\mathcal{L}$-admissible path $\gamma : [0, T] \to \mathbb{R}^{N+1}$ such that $\gamma(0) = (x, t)$, $\gamma(T) = (\xi, \tau)$.

We also assume the following *uniform ellipticity* condition:

[H.3] the coefficients $a_{ij}$, $1 \leq i, j \leq m$, are real valued, measurable functions of $z$. Moreover $a_{ij}(z) = a_{ji}(z)$, $1 \leq i, j \leq m$, and there exists a positive constant $\mu$ such that

$$\mu^{-1}|\xi|^2 \leq \sum_{i,j=1}^{m} a_{ij}(z)\xi_i\xi_j \leq \mu|\xi|^2,$$

for every $z \in \mathbb{R}^{N+1}$ and $\xi \in \mathbb{R}^m$.

We next give some comments about our hypotheses. We first note that, under the assumptions [H.1]-[H.2], $\mathcal{L}$ belongs to the class introduced by Kogoj and Lanconelli in [20] (of course, [H.3] is trivially satisfied by the identity matrix). We recall that [H.1]-[H.2] yield the well known Hörmander condition [17]:

$$\text{rank Lie}\{X_1, \dots, X_m, Y\}(z) = N + 1, \quad \text{for every } z \in \mathbb{R}^{N+1}, \tag{1.4}$$

then $\mathcal{L}$ is hypoelliptic (i.e. every distributional solution to $\mathcal{L}u = 0$ is a smooth, classic solution; see, for instance, Proposition 10.1 in [20]). Moreover, the fundamental solution $\Gamma(\cdot, \zeta)$ of $\mathcal{L}$, shares the main properties of the fundamental solution of the heat equation (see [20]). Due to this last fact, the operator $\mathcal{L}$ will be called *principal part* of $\mathcal{L}_A$.

Let us point out that several meaningful examples of operators of the form (1.1) belong to the class considered in this paper.

**Example 1.1** PARABOLIC OPERATORS ON CARNOT GROUPS. *Operators in the form* $\mathcal{L}_A = \mathrm{div}_X(AX\cdot) - \partial_t$ *satisfy assumptions [H.1]-[H.3] when the vector fields* $X_1, \dots, X_m$ *are the generators of a homogeneous Carnot group* $\mathbb{C} = \big(\mathbb{R}^N, \cdot, \widetilde{\delta}_\lambda\big)$. *In that case* $X_0 = 0$, $\Delta_{\mathbb{C}} = \sum_{k=1}^m X_k^2$ *is the sub-Laplacian on* $\mathbb{C}$ *and* $\mathcal{L}$ *is its heat operator*

$$\mathcal{L} = \Delta_{\mathbb{C}} - \partial_t. \tag{1.5}$$

*The operations of* $\mathbb{G} = \big(\mathbb{R}^{N+1}, \circ, \delta_\lambda\big)$ *are* $(x,t)\circ(\xi,\tau) = (x\cdot\xi, t+\tau)$ *and* $\delta_\lambda(x,t) = \big(\widetilde{\delta}_\lambda x, \lambda^2 t\big)$.

The theory developed by De Giorgi-Nash-Moser [28, 29], [30], in the study of uniformly parabolic equations has been applied in [37] to divergence form equations $\partial_t u = \mathrm{div}_X(AXu)$ by Saloff-Coste and Stroock.

Related results have been given by Bonfiglioli, Lanconelli and Uguzzoni in [4], where the analogous non-divergence form operator $\sum a_{ij}X_iX_j - \partial_t$ is considered. They assume that the coefficients $a_{ij}$ are Hölder continuous, and prove the existence and some pointwise estimates of the fundamental solution and of its derivatives. Under the same assumptions, Bonfiglioli and Uguzzoni prove in [5] a Harnack inequality for the positive solutions to $\sum a_{ij}X_iX_ju = \partial_t u$ and for the relevant elliptic equation $\sum a_{ij}X_iX_ju = 0$. We finally recall that, in [7, 8], Bramanti, Brandolini, Lanconelli and Uguzzoni extend the results proved in [4, 5] to Hörmander operators in the form $\sum a_{ij}X_iX_j - \partial_t$, without making the hypothesis that $\mathrm{Lie}\big\{X_1, \dots, X_m\big\}$ is stratified.

**Example 1.2** KOLMOGOROV-FOKKER-PLANCK OPERATORS. *Consider the equation*

$$\sum_{i,j=1}^m \partial_{x_i}\big(a_{ij}\partial_{x_j}u\big) + \langle Bx, \nabla u\rangle = \partial_t u, \tag{1.6}$$

*where* $B$ *is a constant* $N \times N$ *real matrix. In this case we have* $X_j = \partial_{x_j}, j = 1, \dots, m$ *and* $X_0 = \langle Bx, \nabla\rangle$. *We recall that assumptions [H.1]-[H.2] are equivalent to the following algebraic condition on matrix* $B$: *There exists a basis of* $\mathbb{R}^N$ *such that* $B$ *has the form*

$$B = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ B_1 & 0 & \dots & 0 & 0 \\ 0 & B_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & B_{\widetilde{k}} & 0 \end{pmatrix} \tag{1.7}$$

*where* $B_j$ *is a* $m_j \times m_{j+1}$ *matrix of rank* $m_{j+1}$, $j = 1, 2, \dots, \widetilde{k}$ *with*

$$m =: m_1 \ge m_2 \ge \dots \ge m_{\widetilde{k}+1} \ge 1, \qquad \text{and} \quad m_1 + \dots + m_{\widetilde{k}+1} = N.$$

*The Lie group product related to Kolmogorov equations is*

$$(x,t)\circ(\xi,\tau) = \big(\xi + e^{-\tau B}x, t+\tau\big), \qquad (x,t), (\xi,\tau) \in \mathbb{R}^{N+1},$$

*and the dilations are*

$$\delta_\lambda = \mathrm{diag}\big(\lambda I_{m_1}, \lambda^3 I_{m_2}, \dots, \lambda^{2\widetilde{k}+1} I_{m_{\widetilde{k}+1}}, \lambda^2\big), \qquad \lambda > 0, \tag{1.8}$$

*where* $I_{m_j}$ *denotes the* $m_j \times m_j$ *identity matrix (see Propositions 2.1 and 2.2 in [23]).*

We explicitly remark that Kolmogorov operators do not belong to the class considered in the Example 1.1, since the vector fields $\partial_{x_1}, \dots, \partial_{x_m}$ do not satisfy the Hörmander condition (1.4). On the other hand, this kind of operators arise in many applications. Indeed, the following equation

$$\partial_t f - \langle v, \nabla_x f \rangle = \sum_{i,j=1}^{n} \partial_{v_i} \left( a_{ij}(\cdot, f) \partial_{v_j} f \right), \qquad t \geq 0, \ x \in \mathbb{R}^m, \ v \in \mathbb{R}^m$$

where the *collision operator* at the right hand side of the equation can take either a linear or a non linear form, arises in kinetic theory (see, for instance, [11], [35], [9] and [24]). Equations of the form (1.6) occur in mathematical finance as well. More specifically, the following linear equation

$$S^2 \partial_{SS} V + f(S) \partial_M V - \partial_t V = 0, \qquad S, t > 0, \ M \in \mathbb{R}$$

with either $f(S) = \log S$ or $f(S) = S$, arises in the Black & Scholes theory when considering the problem of the pricing of Asian options (see [2]), and in the stochastic volatility model by Hobson & Rogers (see [16] and [12]).

Kolmogorov-Fokker-Planck operators have been studied by many authors. A systematic study of their principal parts has been carried out in [22], and [23]. We also quote the papers [39], [18], [13] [34], [33], [38], [25], [26], for the study of Kolmogorov operators with Hölder continuous coefficients $a_{ij}$, satisfying the uniformly ellipticity assumption [H.3]. The classical iterative method introduced by Moser [28, 29] has been used in the paper [32] concerning the equation (1.6) with measurable coefficients, and a pointwise upper bound for the solutions is proved. We also recall that the methods and results of [32] have been extended to Kolmogorov type operators on non-homogeneous Lie groups by Pascucci and the authors in [10].

**Example 1.3** Link of groups. *The notion of link of homogeneous groups has been introduced by Kogoj and Lanconelli in [20, 21]. If $\Delta_{\mathbb{G}}$ is a sub-Laplacian on a Carnot group $\mathbb{G}$ and $Y$ is a first order partial differential operator which is transverse to $\mathbb{G}$ (in the sense of Definition 4.5 of [21]), then*

$$\mathcal{L} = \Delta_{\mathbb{G}} + Y$$

*is left translation invariant, and homogeneous of degree two on the new homogeneous group obtained by linking $\mathbb{G}$ with the Kolmogorov group related to $Y$.*

The simplest example of that operator is

$$\mathcal{L} = (\partial_x + y\partial_s)^2 + (\partial_y - x\partial_s)^2 + x\partial_w - \partial_t, \tag{1.9}$$

defined for $(x, y, s, w) \in \mathbb{R}^4$. The operator acts on the variables $(x, y, s, t)$ as the heat equation on the Heisenberg group, and on the variables $(x, y, w, t)$ as a Kolmogorov operator. In this case $\mathcal{L} = X_1^2 + X_2^2 + Y$, where $Y = x\partial_w - \partial_t$, $X_1 = \partial_x + y\partial_s$, $X_2 = \partial_y - x\partial_s$. We recall that, as an application of the procedure of linking of groups, sequences of homogeneous groups of dimension and step arbitrarily large have been constructed in [21].

Few results concerning the regularity of the solutions of $\mathcal{L}_A u = 0$ have been proved for operators of this kind. Bramanti and Brandolini consider in [6] operators in the form

$\mathcal{L}_A = \sum_{i,j=1}^m a_{ij}(x) X_i X_j + a_0 Y$ where $a_{ij}$ and $a_0$ belong to the class $V.M.O.$ of the *Vanishing Mean Oscillation* functions. They extend the general theory of function spaces developed by Folland [14], Rothschild and Stein [36] for Hörmander operators.

In this paper we use the iterative method introduced by Moser to prove an $L^\infty_{\rm loc}$ bound of the solution $u$ of (1.1) in terms of its $L^p_{\rm loc}$ norm. It is well known that the Moser iteration is based on a combination of a Caccioppoli type estimate with the classical embedding Sobolev inequality. The method has been adapted to the non-Euclidean geometry of the operators considered in the Example 1.1 by Saloff-Coste and Stroock [37]. The Caccioppoli type inequalities plainly extend to this kind of operators and give some $L^2_{\rm loc}$ bounds of the first order derivatives $X_1 u, \dots, X_m u$ of the solution $u$ of (1.1). The Moser procedure can be accomplished by using the so called Sobolev-Stein inequalities.

In our more general case that argument fails since, even if the Caccioppoli type inequalities still give an $L^2_{\rm loc}$ bound of $X_1 u, \dots, X_m u$, we cannot rely on the Sobolev-Stein inequalities, due to the fact that some information on the norm of $X_0 u$ is needed to conclude the procedure. This problem has been previously encountered in the study of Kolmogorov operators [32] and has been solved by using a suitable Sobolev-type inequality which only holds for the solutions to (1.1). We extend here the technique used in [32] to the general class of operators satisfying assumptions [H.1]-[H.3]. The main idea is to prove a Sobolev type inequality by using a representation formula for the solution $u$ in terms of the fundamental solution of the principal part $\mathcal{L}$ of $\mathcal{L}_A$. More specifically, if $u$ is a solution to (1.1), then

$$\mathcal{L} u = (\mathcal{L}_A - \mathcal{L}) u = \mathrm{div}_X F, \tag{1.10}$$

where

$$F_i = \sum_{j=1}^m (\delta_{ij} - a_{ij}) X_j u, \qquad i = 1, \dots, m.$$

Since the $F_i$'s depend only on the first order derivatives $X_j u$, $j = 1, ..., m$, the Caccioppoli inequality yields an $H^{-1}_{\rm loc}$-estimate of the right hand side of (1.10). Thus, by using some potential estimate for the fundamental solution of $\mathcal{L}$, we prove the needed bound for the $L^p_{\rm loc}$ norm of $u$.

We conclude this introduction with a couple of remarks. We recall that Saloff-Coste and Stroock accomplish the Moser method in [37] by proving an invariant Harnack inequality. They rely on the Poincaré type inequality due to Jerison [19], but a similar inequality which is suitable for our operators $\mathcal{L}_A$ has not yet established. On the other hand, the method introduced by Aronson in [1], in the study of uniformly parabolic equations, only relies on the local upper bound proved by Moser and provides an pointwise bound of the fundamental solution in its whole domain. That method has previously adapted by Pascucci and Polidoro [31] to homogeneous Kolmogorov operators, and we plan in a future study to further extend the same method to the operators $\mathcal{L}_A$ considered here.

The plan of the paper is the following. In Section 2 we introduce some notations and our main result (see Theorem 2.2), in Section 3 we prove the Caccioppoli and Sobolev type inequalities and in Section 4 we prove the pointwise bounds of the positive solutions. Section

4 also contains related results concerning positive super and sub-solutions (see Propositions 4.2 and 4.4), changing sign solutions (see Proposition 4.3).

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## 2 Notations and main results

We first introduce some notations, then we state our main results. We refer to the monograph [3] for a detailed treatment of the subject. A Lie group $\mathbb{G} = \left(\mathbb{R}^{N+1}, \circ\right)$ is said *homogeneous* if a family of dilations $(\delta_\lambda)_{\lambda>0}$ exists on $\mathbb{G}$ and is an automorphism of the group:

$$\delta_\lambda(z \circ \zeta) = (\delta_\lambda z) \circ (\delta_\lambda \zeta), \quad \text{for all } z, \zeta \in \mathbb{R}^{N+1} \text{ and } \lambda > 0.$$

As we stated in the Introduction, hypotheses [H.1]-[H.2] imply the Hörmander condition (1.4). Moreover the dilation $\delta_\lambda$ induces a direct sum decomposition on $\mathbb{R}^N$

$$\mathbb{R}^N = V_1 \oplus \cdots \oplus V_k, \tag{2.1}$$

as follows. If we denote $x = x^{(1)} + x^{(2)} + \cdots + x^{(k)}$ with $x^{(j)} \in V_j$, then

$$\delta_\lambda \left(x^{(1)} + x^{(2)} + \cdots + x^{(k)}, t\right) = \left(\lambda x^{(1)} + \lambda^2 x^{(2)} + \cdots + \lambda^k x^{(k)}, \lambda^2 t\right). \tag{2.2}$$

The decomposition (2.1) is well known when considering Carnot groups $\mathbb{C} = \left(\mathbb{R}^N, \cdot, \widetilde{\delta}_\lambda\right)$. In that case $V_1 = \text{span}\left\{X_1(0), \ldots, X_m(0)\right\}$, $V_j = [V_{j-1}, V_1]$ for $j = 2, \ldots, k$, and $[V_k, V_1] = \left\{0\right\}$.

Note that some of the $V_j$'s may be the trivial space $\left\{0\right\}$, as in the case of Kolmogorov operators occurs. Indeed, according to (1.8) and (2.1), we have $\mathbb{R}^N = V_1 \oplus V_3 \oplus \cdots \oplus V_{2\widetilde{k}+1}$, with $\text{dim}V_{2j-1} = m_j$, for $j = 1, \ldots, \widetilde{k}+1$. We explicitly note that $V_j = \left\{0\right\}$ for every even $j$.

The natural number

$$Q := \text{dim}V_1 + 2\,\text{dim}V_2 + \cdots + k\,\text{dim}V_k + 2 \tag{2.3}$$

is usually called the *homogeneous dimension* of $\mathbb{G}$ with respect to $(\delta_\lambda)$.

Since $\mathbb{G} = \left(\mathbb{R}^{N+1}, \circ\right)$ is a homogeneous Lie group, we have $X_j^* = -X_j$ for $j = 1, \ldots, m$ and $Y^* = -Y$. Thus we can give the following

**Definition 2.1** *A weak solution of* (1.1) *in a subset* $\Omega$ *of* $\mathbb{R}^{N+1}$ *is a function* $u$ *such that* $u, X_1 u, \ldots, X_m u, Y u \in L^2_{\text{loc}}(\Omega)$ *and*

$$\int_\Omega -\langle AXu, X\varphi\rangle + \varphi Y u = 0, \qquad \forall \varphi \in C_0^\infty(\Omega). \tag{2.4}$$

Here $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^m$. In the sequel we will also consider *weak sub-solutions* of (1.1), namely functions $u$ such that $u, X_1 u, \ldots, X_m u, Y u \in L^2_{\text{loc}}(\Omega)$ and

$$\int_\Omega -\langle AXu, X\varphi\rangle + \varphi Y u \geq 0, \qquad \forall \varphi \in C_0^\infty(\Omega),\ \varphi \geq 0. \tag{2.5}$$

Moreover $u$ is a *weak super-solution* of (1.1) if $-u$ is a sub-solution. Clearly, if $u$ is a sub and super-solution of (1.1), then it is a solution.

We define the *unit cylinder*

$$R_1 = \left\{(x,t) \in \mathbb{R}^{N+1} \;\mid\; \sum_{j=1}^{N} x_j^2 < 1,\ |t| < 1\right\},$$

and, for every $z_0 \in \mathbb{R}^{N+1}$ and $r > 0$, we set

$$R_r(z_0) := z_0 \circ \delta_r\left(R_1\right) = \{z \in \mathbb{R}^{N+1} \;\mid\; z = z_0 \circ \delta_r(\zeta),\ \zeta \in R_1\}. \tag{2.6}$$

Our main result is the following

**Theorem 2.2** *Let $u$ be a non-negative weak solution of* (1.1) *in $\Omega$. Let $z_0 \in \Omega$ and $r, \varrho$, $0 < \frac{r}{2} \leq \varrho < r$, be such that $\overline{R_r(z_0)} \subseteq \Omega$. Then there exists a positive constant $c$ which only depends on the operator $\mathcal{L}_A$ such that, for every $p > 0$, it holds*

$$\sup_{R_\varrho(z_0)} u^p \leq \frac{c}{(r-\varrho)^Q} \int_{R_r(z_0)} u^p \tag{2.7}$$

*($Q$ is defined in* (2.3)*). Estimate* (2.7) *also holds for every $p < 0$ such that $u^p \in L^1(R_r(z_0))$.*

We next recall some useful facts about homogeneous Lie groups. We first explicitly note that [H.1] and (1.4) imply that $\text{span}\big\{X_1(0), \dots, X_m(0)\big\} = V_1$; then we may assume $m = \text{dim}V_1$ and $X_j(0) = \mathbf{e}_j$ for $j = 1, \dots, m$ where $\{\mathbf{e}_i\}_{1 \leq i \leq N}$ denotes the canonical basis of $\mathbb{R}^N$. We set

$$|x|_{\mathbb{G}} = \left(\sum_{j=1}^{k} \sum_{i=1}^{m_j} \left(x_i^{(j)}\right)^{\frac{2k!}{j}}\right)^{\frac{1}{2k!}}, \qquad \|(x,t)\|_{\mathbb{G}} = \left(|x|_{\mathbb{G}}^{2k!} + |t|^{k!}\right)^{\frac{1}{2k!}},$$

and we observe that the above functions are homogeneous of degree 1, on $\mathbb{R}^N$ and $\mathbb{R}^{N+1}$, respectively, in the sense that

$$\left|\left(\lambda x^{(1)} + \cdots + \lambda^k x^{(k)}\right)\right|_{\mathbb{G}} = \lambda |x|_{\mathbb{G}}, \qquad \|\delta_\lambda(x,t)\|_{\mathbb{G}} = \lambda \|(x,t)\|_{\mathbb{G}},$$

for every $(x,t) \in \mathbb{R}^{N+1}$ and for any $\lambda > 0$. We define the quasi-distance in $\mathbb{G}$ as

$$d(z,\zeta) := \|\zeta^{-1} \circ z\|_{\mathbb{G}}, \qquad \text{for all } z, \zeta \in \mathbb{R}^{N+1}. \tag{2.8}$$

We finally recall that, for every compact set $K \subset \mathbb{R}^{N+1}$ there exist two positive constants $c_K^-$ and $c_K^+$, such that

$$c_K^- |z - \zeta| \leq d(z,\zeta) \leq c_K^+ |z - \zeta|^{\frac{1}{k}}, \qquad \text{for all } z, \zeta \in K, \tag{2.9}$$

(here $|\cdot|$ denotes the usual Euclidean modulus, see for instance, Proposition 11.2 in [15]).

We finally recall some useful facts on the fundamental solution of the hypoelliptic operators defined in (1.3). If $\Gamma(\cdot, \zeta)$ is the fundamental solution of $\mathcal{L}$ with pole at $\zeta \in \mathbb{R}^{N+1}$, then $\Gamma$ is smooth out of the diagonal of $\mathbb{R}^{N+1} \times \mathbb{R}^{N+1}$ and has the following properties:

*i)* for any $z \in \mathbb{R}^{N+1}, \Gamma(\cdot, z)$ and $\Gamma(z, \cdot)$ belong to $L^1_{\text{loc}}(\mathbb{R}^{N+1})$;

*ii)* for every $\varphi \in C_0^\infty(\mathbb{R}^{N+1})$ and $z \in \mathbb{R}^{N+1}$ we have

$$\mathcal{L} \int_{\mathbb{R}^{N+1}} \Gamma(z, \zeta)\varphi(\zeta) d\zeta = \int_{\mathbb{R}^{N+1}} \Gamma(z, \zeta)\mathcal{L}\varphi(\zeta) d\zeta = -\varphi(z);$$

*iii)* $\mathcal{L}\Gamma(\cdot, \zeta) = -\delta_\zeta$ (Dirac measure supported at $\zeta$);

*iv)* $\Gamma(z, \zeta) \geq 0$, and $\Gamma(x, t, \xi, \tau) > 0$ if, and only if, $t > \tau$;

*v)* if we define $\Gamma^*(z, \zeta) := \Gamma(\zeta, z)$, then $\Gamma^*$ is the fundamental solution of the adjoint $\mathcal{L}^*$ of $\mathcal{L}$;

*vi)* there exists a constant $C > 0$ such that

$$\Gamma(z, \zeta) \leq C \|\zeta^{-1} \circ z\|_{\mathbb{G}}^{2-Q}, \qquad \forall\, z, \zeta \in \mathbb{R}^{N+1}.$$

*vii)* $\Gamma(z, \zeta) = \Gamma(\zeta^{-1} \circ z, 0) =: \Gamma(\zeta^{-1} \circ z)$, for every $z, \zeta \in \mathbb{R}^{N+1}, z \neq \zeta$;

*viii)* $\Gamma(\delta_\lambda(z)) = \lambda^{-Q+2}\Gamma(z)$, for every $z \in \mathbb{R}^{N+1} \setminus \{0\},\ \lambda > 0$.

(see Theorem 2.7, Proposition 2.8 and Corollary 2.9 in [20]). Moreover the following upper bound for $\Gamma$ holds

$$\Gamma(x, t) \leq C t^{-\frac{Q-2}{2}} \exp\left(-\frac{|x|_{\mathbb{G}}^2}{C\,t}\right), \qquad \forall\ (x, t) \in \mathbb{R}^N \times \mathbb{R}^+, \tag{2.10}$$

for some positive constant $C$ (see [20], section 5.1).

We define the $\mathcal{L}$-*potential* of the function $f \in L^1(\mathbb{R}^{N+1})$ as follows

$$\Gamma(f)(z) = \int_{\mathbb{R}^{N+1}} \Gamma(z, \zeta) f(\zeta) d\zeta, \qquad z \in \mathbb{R}^{N+1}. \tag{2.11}$$

Let us explicitly write the potential $\Gamma(X_j f)$ of any $f \in C_0^\infty(\mathbb{R}^{N+1})$, for $j = 1, \ldots, m$. It holds

$$\Gamma(X_j f)(z) = \int_{\mathbb{R}^{N+1}} X_j^R \Gamma(\eta) f(z \circ \eta^{-1}) d\eta$$

where $X_j^R$ denotes the *right* invariant vector fields that agrees with $X_j$ at the origin. Also note that $X_j^R\Gamma$ is a $\delta_\lambda$-homogeneous function of degree $1 - Q$. We finally remark that

$$\int_{\mathbb{R}^{N+1}} X_j^R \Gamma(\eta) f(z \circ \eta^{-1}) d\eta = \int_{\mathbb{R}^{N+1}} X_j^{(\zeta)} \Gamma(\zeta^{-1} \circ z) f(\zeta) d\zeta, \tag{2.12}$$

where the superscript in $X_j^{(\zeta)}$ indicates that we are differentiating w.r.t. the $\zeta$ variable.

We next recall a result that extends the classical potential estimates to homogeneous Lie groups (see, for instance, Proposition (1.11) in [14]). As a plain consequence, we see that the above definition is well posed.

**Theorem 2.3** *Let $\alpha \in ]0, Q[$, and let $G \in C(\mathbb{R}^{N+1} \setminus \{0\})$ be a $\delta_\lambda$-homogeneous function of degree $\alpha - Q$. Consider $f \in L^p(\mathbb{R}^{N+1})$ for some $p \in ]1, +\infty[$. Then the function*

$$G_f(z) := \int_{\mathbb{R}^{N+1}} G(\zeta^{-1} \circ z) f(\zeta) d\zeta$$

*is defined almost everywhere and there exists a constant $c = c(Q, p)$ such that*

$$\|G_f\|_{L^q(\mathbb{R}^{N+1})} \leq c \, \|f\|_{L^p(\mathbb{R}^{N+1})},$$

*where $q$ is defined by*

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}.$$

**Corollary 2.4** *Let $f \in L^2(\mathbb{R}^{N+1})$. There exists a positive constant $c = c(Q)$ such that*

$$\|\Gamma(f)\|_{L^{2\widetilde{\kappa}}(\mathbb{R}^{N+1})} \leq c\|f\|_{L^2(\mathbb{R}^{N+1})},$$
$$\|(\Gamma X_1 f, \dots, \Gamma X_m f)\|_{L^{2\kappa}(\mathbb{R}^{N+1})} \leq c\|f\|_{L^2(\mathbb{R}^{N+1})},$$

*where $\widetilde{\kappa} = 1 + \frac{4}{Q-4}$ and $\kappa = 1 + \frac{2}{Q-2}$.*

PROOF. It is an immediate consequence of Theorem 2.3 and of the homogeneity of $\Gamma$ and of $\left(X_1^R \Gamma, \dots, X_m^R \Gamma\right)$. $\square$

As in [32], [10], we can use the fundamental solution $\Gamma$ as a test function in the definition of sub and super-solution.

**Lemma 2.5** *Let $v$ be a weak sub-solution of $\mathcal{L}_A u = 0$ in $\Omega$. For every $\varphi \in C_0^\infty(\Omega)$, $\varphi \geq 0$, and for almost every $z \in \mathbb{R}^{N+1}$, we have*

$$-\int_\Omega \langle AXv, X(\Gamma(z, \cdot)\varphi)\rangle + \int_\Omega \Gamma(z, \cdot)\, \varphi Y v \geq 0.$$

*An analogous result holds for weak super-solutions.*

*Proof.* For every $\varepsilon > 0$, we set

$$\chi_\varepsilon(z, \zeta) = \chi\left(\frac{\|\zeta^{-1} \circ z\|_{\mathbb{G}}}{\varepsilon}\right), \qquad z, \zeta \in \mathbb{R}^{N+1},$$

where $\chi \in C^1([0, +\infty[, [0, 1])$ is such that $\chi(s) = 0$ for $s \in [0, 1]$, $\chi(s) = 1$ for $s \geq 2$ and $0 \leq \chi' \leq 2$. By (2.5), for every $\varepsilon > 0$ and $z \in \mathbb{R}^{N+1}$, we have

$$0 \leq -\int_\Omega \langle AXv, X(\Gamma(z, \cdot)\chi_\varepsilon(z, \cdot)\varphi)\rangle + \int_\Omega \Gamma(z, \cdot)\chi_\varepsilon(z, \cdot)\varphi Y v = -I_{1,\varepsilon}(z) + I_{2,\varepsilon}(z) - I_{3,\varepsilon}(z),$$

where

$$I_{1,\varepsilon}(z) = \int_\Omega \langle AXv, X(\Gamma(z,\cdot))\rangle \chi_\varepsilon(z,\cdot)\varphi,$$

$$I_{2,\varepsilon}(z) = \int_\Omega \Gamma(z,\cdot)\chi_\varepsilon(z,\cdot)\left(-\langle AXv, X\varphi\rangle + \varphi Yv\right),$$

$$I_{3,\varepsilon}(z) = \int_\Omega \langle AXv, X\chi_\varepsilon(z,\cdot)\rangle \Gamma(z,\cdot)\varphi.$$

Consider the first integral. Since $\varphi\chi_\varepsilon(z,\cdot)AXv \to \varphi AXv$ in $L^2(\mathbb{R}^{N+1})$, as $\varepsilon \to 0$, Corollary 2.4 and (2.12) give

$$I_{1,\varepsilon}(z) \to \int_\Omega \langle AXv, X(\Gamma(z,\cdot))\rangle\varphi,$$

as $\varepsilon \to 0$, for almost every $z \in \mathbb{R}^{N+1}$. The same argument applies to the second and third integrals. In particular, we find $I_{3,\varepsilon}(z) \to 0$ as $\varepsilon \to 0$, then the proof is accomplished. $\square$

# 3 Caccioppoli and Sobolev inequalities

We recall the notations (1.2), (2.6) and, by simplicity, we shall write $R_r$ instead of $R_r(0)$.

**Theorem 3.1 [Caccioppoli type inequalities]** *Let $u$ be a non-negative weak solution of (1.1) in $R_1$. Let $p \in \mathbb{R}, p \neq 0, p \neq 1/2$ and let $\varrho, r$ be such that $\frac{1}{2} \le \varrho < r \le 1$. If $u^p \in L^2(R_r)$ then $Xu^p \in L^2(R_\varrho)$, and there exists a constant $c$, only dependent on the operator $\mathcal{L}_A$, such that*

$$\|\,(X_1u^p, \dots, X_mu^p)\,\|_{L^2(R_\varrho)} \le \frac{c\sqrt{\mu(\mu+\varepsilon)}}{\varepsilon(r-\varrho)}\;\|u^p\|_{L^2(R_r)}, \quad \textit{where} \quad \varepsilon = \frac{|2p-1|}{4p}. \tag{3.1}$$

*Proof.* We first consider the case $p < 1, p \neq 0, p \neq 1/2$. We preliminarily assume that $u \ge u_0$ for some positive constant $u_0$. This assumption will be removed in the sequel. We let $v = u^p$ and we note that, since $u$ is a weak solution to $\mathcal{L}_A u = 0$ and $u \ge u_0$, then $v, Xv, Yv \in L^2(R_r)$. For every $\psi \in C_0^\infty(R_1)$ we consider the function $\varphi = u^{2p-1}\psi^2$. Note that $\varphi$ and $X\varphi \in L^2(R_1)$, then we can use $\varphi$ as a test function in (2.4). We find

$$\begin{aligned} 0 &= \frac{p}{2}\int_{R_1} \langle AXu, X\varphi\rangle - \varphi Yu \\ &= \frac{p}{2}\int_{R_1} (2p-1)u^{2p-2}\psi^2\langle AXu, Xu\rangle + 2\psi u^{2p-1}\langle AXu, X\psi\rangle - u^{2p-1}\psi^2 Yu \\ &= \int_{R_1} \left(1 - \frac{1}{2p}\right)\psi^2\langle AXv, Xv\rangle + v\psi\langle AXv, X\psi\rangle - \frac{\psi^2}{4}Y(v^2) = \end{aligned}$$

(using the identity

$$\psi^2 Y(v^2) = Y(\psi^2 v^2) - 2v^2\psi Y\psi$$

and applying the divergence theorem)

$$= \int_{R_1} \left(1 - \frac{1}{2p}\right) \psi^2 \langle AXv, Xv\rangle + v\psi\langle AXv, X\psi\rangle + \frac{v^2\psi}{2} Y\psi.$$

Setting $\varepsilon = \frac{|2p-1|}{4p}$ and using the estimate

$$v\psi \, |\langle AXv, X\psi\rangle| \leq \varepsilon\psi^2\langle AXv, Xv\rangle + \frac{v^2}{4\varepsilon}\langle AX\psi, X\psi\rangle,$$

we finally obtain

$$\varepsilon \int_{R_1} \psi^2\langle AXv, Xv\rangle \leq \frac{1}{4}\int_{R_1} v^2 \left(\frac{1}{\varepsilon}\langle AX\psi, X\psi\rangle + 2\,|\psi Y\psi|\right). \tag{3.2}$$

The thesis follows by making a suitable choice of the function $\psi$ in (3.2). More precisely, we set

$$\psi(x,t) = \chi\left(\|(x,0)\|_{\mathbb{G}}\right)\chi\left(|t|^{\frac{1}{2}}\right) \tag{3.3}$$

where $\chi \in C^\infty(\mathbb{R}, [0,1])$ is such that

$$\chi(s) = 1 \text{ if } s \leq \varrho, \qquad \chi(s) = 0 \text{ if } s \geq r, \qquad |\chi'| \leq \frac{2}{r-\varrho}.$$

We observe that

$$|\partial_t\psi|\,, |X_j\psi| \leq \frac{c_1}{r-\varrho}, \qquad j = 0, 1, \dots, m \tag{3.4}$$

where $c_1$ is a positive constant only depending on the operator. Then, accordingly to (3.2), we obtain

$$\begin{aligned} \frac{\varepsilon}{\mu}\int_{R_\varrho} |Xu^p|^2 &\leq \varepsilon\int_{R_r} \psi^2\langle AXu^p, Xu^p\rangle \\ &\leq \frac{1}{4}\int_{R_r} u^{2p}\left(\frac{mc_1^2\mu}{\varepsilon(r-\varrho)^2} + \frac{4c_1}{r-\varrho}\right) \leq \frac{c_2}{(r-\varrho)^2}\left(1+\frac{\mu}{\varepsilon}\right)\int_{R_r} u^{2p}, \end{aligned} \tag{3.5}$$

and this proves (3.1). In order to remove the assumption that $\inf u > 0$ it is sufficient to apply estimate (3.5) to the solution $u + \frac{1}{n}$, $n \in \mathbb{N}$ and to rely on the monotone convergence theorem.

We next consider the case $p \geq 1$. We proceed as in the proof of Lemma 1 in the paper [27] by Moser. For any $n \in \mathbb{N}$, we define the function $g_{n,p}$ on $]0, +\infty[$ as follows

$$g_{n,p}(s) = \begin{cases} s^p, & \text{if } 0 < s \leq n, \\ n^p + pn^{p-1}(s-n), & \text{if } s > n, \end{cases}$$

then we apply the same argument used above to the function $v_{n,p} = g_{n,p}(u)$. By using

$$\varphi = g_{n,p}(u)g'_{n,p}(u)\psi^2, \qquad \psi \in C_0^\infty(R_1)$$

as a test function in (2.4), we find

$$\varepsilon \int_{R_1} \psi^2 \langle AXv_{n,p}, Xv_{n,p}\rangle \le \frac{1}{4}\int_{R_1} v_{n,p}^2 \left(\frac{1}{\varepsilon}\langle AX\psi, X\psi\rangle + \frac{1}{2}\,|\psi Y\psi|\right),$$

where $\varepsilon = \frac{|2p-1|}{4p}$. The claim then follows by letting $n \to \infty$. For more details we refer to [27] or [32]. □

Next Proposition extends Theorem 3.1 to super and sub-solutions. We omit the proof, since it follows the same lines of Theorem 3.1.

**Proposition 3.2** *Let $u$ be a non-negative weak sub-solution of* (1.1) *in $R_1$. Let $\varrho, r$ be such that $\frac{1}{2} \le \varrho < r \le 1$, and $p \ge 1$ or $p < 0$. If $u^p \in L^2(R_r)$ then $Xu^p \in L^2(R_\varrho)$ and there exists a constant $c$, only dependent on the operator $\mathcal{L}_A$, such that*

$$\|\,(X_1u^p, \dots, X_mu^p)\,\|_{L^2(R_\varrho)} \le \frac{c\sqrt{\mu(\mu+\varepsilon)}}{\varepsilon(r-\varrho)}\,\|u^p\|_{L^2(R_r)}, \quad \textit{where} \quad \varepsilon = \frac{|2p-1|}{4p}.$$

*The same statement holds when $u$ is a non-negative weak super-solution of* (1.1) *and $p \in ]0, 1/2[$.*

**Theorem 3.3 [Sobolev type inequalities for super and sub-solutions].** *Let $v$ be a non-negative weak sub-solution of* (1.1) *in $R_1$. Then $v \in L^{2\kappa}_{\mathrm{loc}}(R_1)$, $\kappa = 1 + \frac{2}{Q-2}$, and there exists a constant $c$, only dependent on the operator $\mathcal{L}_A$, such that*

$$\|v\|_{L^{2\kappa}(R_\varrho)} \le \frac{c}{r-\varrho}\left(\|v\|_{L^2(R_r)} + \|\,(X_1v, \dots, X_mv)\,\|_{L^2(R_r)}\right), \tag{3.6}$$

*for every $\varrho, r$ with $\frac{1}{2} \le \varrho < r \le 1$.*

*The same statement holds for non-negative super-solutions.*

*Proof.* Let $v$ be a non-negative sub-solution of $\mathcal{L}_Au = 0$. We represent $v$ in terms of the fundamental solution $\Gamma$. To this end, we consider the cut-off function $\psi$ introduced in (3.3). For every $z \in R_\varrho$, we have

$$v(z) = v\psi(z) = \int_{R_r} \left[\langle X(v\psi), X\Gamma(z,\cdot)\rangle - \Gamma(z,\cdot)Y(v\psi)\right](\zeta)d\zeta = I_1(z) + I_2(z) + I_3(z), \tag{3.7}$$

where

$$I_1(z) = \int_{R_r} \left[\langle X\psi, X\Gamma(z,\cdot)\rangle v\right](\zeta)d\zeta - \int_{R_r} \left[\Gamma(z,\cdot)vY\psi\right](\zeta)d\zeta,$$
$$I_2(z) = \int_{R_r} \left[\langle (I_m - A)\,Xv, X\Gamma(z,\cdot)\rangle\psi\right](\zeta)d\zeta - \int_{R_r} \left[\Gamma(z,\cdot)\langle AXv, X\psi\rangle\right](\zeta)d\zeta,$$
$$I_3(z) = \int_{R_r} \left[\langle AXv, X\left(\Gamma(z,\cdot)\psi\right)\rangle - \Gamma(z,\cdot)\psi Yv\right](\zeta)d\zeta$$

(here and in the sequel $I_m$ denotes the $m \times m$ identity matrix). Since the function $v$ is a weak sub-solution of (1.1), it follows from Lemma 2.5 that $I_3 \le 0$, then

$$0 \le v(z) \le I_1(z) + I_2(z) \quad \text{for a.e. } z \in R_\varrho.$$

To prove our claim it is sufficient to estimate $v$ by a sum of $\mathcal{L}$-potentials.

We start by estimating $I_1$. Denote by $I_1'$ and $I_1''$ the first and the second integral in $I_1$, respectively. Then $I_1'$ can be estimate by Corollary 2.4 (and using (2.12)) as follows

$$\|I_1'\|_{L^{2\kappa}(R_\varrho)} \le c\|vX\psi\|_{L^2(\mathbb{R}^{N+1})} \le \frac{c}{r-\varrho}\|v\|_{L^2(R_r)},$$

where the last inequality follows from (3.4). Here and in the sequel we use the notation $\|F\|_{L^2(\Omega)} = \|\,(F_1,\dots,F_m)\,\|_{L^2(\Omega)}$, for every $F \in L^2(\Omega,\mathbb{R}^m)$. To estimate $I_1''$ we use the first statement of Corollary 2.4:

$$\|I_1''\|_{L^{2\kappa}(R_\varrho)} \le \text{meas}(R_\varrho)^{1/Q}\|I_1''\|_{L^{2\tilde{\kappa}}(R_\varrho)} \le c\|vY\psi\|_{L^2(\mathbb{R}^{N+1})} \le \frac{c}{r-\varrho}\|v\|_{L^2(R_r)}.$$

We can use the same technique to prove that

$$\|I_2\|_{L^{2\kappa}(R_\varrho)} \le \frac{c}{r-\varrho}\|Xv\|_{L^2(R_r)},$$

for some constant $c = c(Q,\mu)$, thus our first claim is proved.

A similar argument proves the thesis when $v$ is a super-solution. In this case, we introduce the following auxiliary operator

$$\widetilde{\mathcal{L}} := \text{div}_X X + \widetilde{Y}, \qquad \widetilde{Y} = -X_0 - \partial_t.$$

Since $R_r$ is a domain which is symmetric with respect to the time variable $t$, we can use the change of variable $(x,t) \mapsto (x,-t)$ and find that

$$\iint_{R_r} \Big(-\langle A(x,-t)Xv(x,-t), X\varphi(x,t)\rangle - \varphi(x,t)\widetilde{Y}v(x,-t)\Big)\,dx\,dt \le 0,$$

for every $\varphi \in C_0^\infty(R_r)$, $\varphi \ge 0$. Note that the fundamental solution $\widetilde{\Gamma}$ of $\widetilde{\mathcal{L}}$ satisfies the assumptions of Corollary 2.4, then we deduce from the above inequality and Lemma 2.5 that

$$\iint_{R_r}\Big(-\langle A(\xi,-\tau)Xv(\xi,-\tau), X(\widetilde{\Gamma}(x,t,\xi,\tau)\psi(\xi,\tau))\rangle - \widetilde{\Gamma}(x,t,\xi,\tau)\psi(\xi,\tau)\widetilde{Y}v(\xi,-\tau)\Big)\,d\xi\,d\tau \le 0,$$

for almost any $(x,t) \in R_r$, with $\psi$ as in (3.3). Then the claim follows from the same argument used above, by a representation formula analogous to (3.7), written in terms of $\widetilde{\Gamma}$ instead of $\Gamma$. For more details we refer to [32] or [10]. $\square$

# 4 The Moser method

We start this section with some preliminary remarks. We first note that a transformation of the form

$$\zeta \longmapsto z_0 \circ \delta_r(\zeta), \qquad r > 0,\ z_0 \in \mathbb{R}^{N+1}, \tag{4.1}$$

preserves the class of differential equations considered. More specifically, if $u$ is a weak solution of (1.1) in the cylinder $R_r(z_0)$ then the function $v(\zeta) = u(z_0 \circ \delta_r(\zeta))$ is a solution to the equation $\mathrm{div}_X(\widetilde{A}Xv) + Yv = 0$ in $R_1$ where $\widetilde{A}(\zeta) = A(z_0 \circ \delta_r(\zeta))$ satisfies hypothesis [H.3] with the same constant $\mu$ as $A$.

Since $\mathbb{G} = \left(\mathbb{R}^{N+1}, \circ\right)$ is a homogeneous Lie group, we have $\det(\mathcal{J}_{\tau_z}) = 1$, and $\det(\delta_\lambda) = \lambda^Q$, so that

$$\int_\Omega f(z \circ w) dw = \int_{\tau_z(\Omega)} f(\zeta) d\zeta, \qquad \int_\Omega f(\delta_\lambda w) dw = \lambda^{-Q} \int_{\delta_\lambda(\Omega)} f(\zeta) d\zeta, \tag{4.2}$$

for every $f \in L^1(\Omega)$ (here $\tau_z(w) = z \circ w$).

**Lemma 4.1** *There exists a positive constant $\bar{c}$ such that, for every positive $\varrho, r$ with $\frac{r}{2} \leq \varrho < r$ and $z_0 \in \mathbb{R}^{N+1}$, it holds*

$$R_{\bar{c}(r-\varrho)}(z) \subseteq R_r(z_0), \qquad \forall z \in R_\varrho(z_0). \tag{4.3}$$

*Proof.* By the change of variables $z = z_0 \circ \delta_r(\zeta)$, it suffices to prove (4.3) for $z_0 = 0$ and $r = 1$. We next recall the expression (2.2) of the dilations $(\delta_\lambda)$, and note that only positive integer power of $\lambda$ there occur, as a consequence we find

$$R_\varrho \subseteq \left\{ (x,t) \in \mathbb{R}^{N+1} \ \mid \ \sum_{j=1}^N x_j^2 \leq \varrho^2,\ |t| \leq \varrho \right\}, \qquad \forall \varrho \in ]0,1[.$$

Hence,

$$\min\left\{ |w - z| : w \in R_\varrho, z \in \partial R_1 \right\} \geq 1 - \varrho,$$

and, if we apply the first inequality in (2.9) (with $K = \overline{R_1}$), we find

$$\min\left\{ d(w,z) : w \in R_\varrho, z \in \partial R_1 \right\} \geq c_K^-(1 - \varrho).$$

In other terms,

$$\mathcal{B}(z, c_K^-(1-\varrho)) := \left\{ w \in \mathbb{R}^{N+1} : d(w,z) \leq c_K^-(1-\varrho) \right\} \subset \overline{R_1}, \qquad \text{for every } z \in R_\varrho.$$

Then the thesis is a consequence of the following inclusion

$$R_s(z) \subset \mathcal{B}(z, ks), \quad \text{where} \quad k = \max\left\{ d(w,0) : w \in \overline{R_1} \right\}$$

which is a direct consequence of (2.6) and of the fact that $\mathcal{B}(z, ks) = z \circ \delta_{ks}\left(\mathcal{B}(0,1)\right)$. $\square$

*Proof of Theorem 2.2.* As said before, we first prove the claim in the unit cylinder $R_1$. Namely, we show that there exists a positive constant $c_1$, only depending on the Lie group $\mathbb{G} = \left(\mathbb{R}^{N+1}, \circ, \delta_\lambda\right)$ and on the constant $\mu$ in hypothesis [H.3], such that

$$\sup_{R_{1/2}} u^p \le c_1 \int_{R_1} u^p \tag{4.4}$$

for every positive solution $u$ of $\mathcal{L}_A u = 0$. As a consequence we have that

$$\sup_{R_{\theta/2}(z)} v^p \le \frac{c_1}{\theta^Q} \int_{R_\theta(z)} v^p$$

for every positive solution $v$ of $\mathcal{L}_A v = 0$ in $R_\theta(z)$, where $z \in \mathbb{R}^{N+1}$ and $\theta > 0$. Indeed, the change of variable $u(\zeta) = v\left(z \circ \delta_\theta\left(\zeta\right)\right)$ is in the form (4.1), and (4.2) holds. As a consequence, if $v$ is a positive solution of (1.1) in $R_r(z_0)$, and we set $\theta = \bar{c}(r - \varrho)$, where $\bar{c}$ is the constant in Lemma 4.1, we find

$$\sup_{R_\varrho(z_0)} v^p \le \frac{c_1}{\bar{c}^Q (r-\varrho)^Q} \int_{R_r(z_0)} v^p,$$

and the Theorem is proved.

We are left with the proof of (4.4). We first consider the case $p > 0$ which is technically more complicated. Combining Theorems 3.1 and 3.3, we obtain the following estimate: if $q, \sigma$ are two positive constants verifying the condition

$$|q - 1/2| \ge \sigma,$$

then there exists a positive constant $c_\sigma = c(\sigma, Q, \mu)$, such that

$$\|u^q\|_{L^{2\kappa}(R_\varrho)} \le \frac{c_\sigma}{(r-\varrho)^2} \|u^q\|_{L^2(R_r)}, \tag{4.5}$$

for every $\varrho, r$, $\frac{1}{2} \le \varrho < r \le 1$, where $\kappa = 1 + \frac{2}{Q-2}$.

Fixed a suitable $\sigma > 0$ as we shall specify later and $p > 0$, we iterate inequality (4.5) by choosing

$$\varrho_n = \frac{1}{2}\left(1 + \frac{1}{2^n}\right), \quad p_n = \frac{p\kappa^n}{2}, \qquad n \in \mathbb{N} \cup \{0\}.$$

We set $v = u^{\frac{p}{2}}$. If $p > 0$ is such that

$$|p\kappa^n - 1| \ge 2\sigma, \qquad \forall n \in \mathbb{N} \cup \{0\}, \tag{4.6}$$

we obtain from (4.5)

$$\|v^{\kappa^n}\|_{L^{2\kappa}(R_{\varrho_{n+1}})} \le \frac{c_\sigma}{(\varrho_n - \varrho_{n+1})^2} \|v^{\kappa^n}\|_{L^2(R_{\varrho_n})}, \qquad \forall n \in \mathbb{N} \cup \{0\},$$

that can be written in the equivalent form

$$\|v\|_{L^{2\kappa^{n+1}}(R_{\varrho_{n+1}})} \le \left(\frac{c_\sigma}{(\varrho_n - \varrho_{n+1})^2}\right)^{\frac{1}{\kappa^n}} \|v\|_{L^{2\kappa^n}(R_{\varrho_n})}, \qquad \forall n \in \mathbb{N} \cup \{0\}.$$

Iterating this inequality, we obtain

$$\|v\|_{L^{2\kappa^{n+1}}(R_{\varrho_{n+1}})} \leq \prod_{j=0}^{n} \left( \frac{c_\sigma}{(\varrho_j - \varrho_{j+1})^2} \right)^{\frac{1}{\kappa^j}} \|v\|_{L^2(R_1)},$$

and letting $n$ go to infinity, we get

$$\sup_{R_{1/2}} v \leq \bar{c}_\sigma \ \|v\|_{L^2(R_1)}, \quad \text{where} \quad \bar{c}_\sigma := \prod_{j=0}^{\infty} \left( \frac{c_\sigma}{(\varrho_j - \varrho_{j+1})^2} \right)^{\frac{1}{\kappa^j}}$$

is a finite constant, dependent on $\sigma$. Thus, we have proved (4.4) with $c_1 = \bar{c}_\sigma^2$, for every $p > 0$ which verifies condition (4.6).

We now make a suitable choice of $\sigma > 0$, only dependent on the homogeneous dimension $Q$, in order to show that (4.4) holds for every positive $p$. We remark that, if $p$ is a number of the form

$$p_j = \frac{\kappa^j}{\kappa + 1}, \qquad j \in \mathbb{Z},$$

then (4.6) is satisfied with $\sigma = (2Q - 2)^{-1}$, for every $j \in \mathbb{Z}$. Therefore (4.4) holds for such a choice of $p$, with $c_1$ only dependent on $Q, \mu$. On the other hand, if $p$ is an arbitrary positive number, we consider $j \in \mathbb{Z}$ such that

$$p_j = \frac{\kappa^j}{\kappa + 1} \leq p < p_{j+1}. \tag{4.7}$$

Hence, by (4.4), we have

$$\sup_{R_{1/2}} u \leq \left( c_1 \int_{R_1} u^{p_j} \right)^{\frac{1}{p_j}} \leq c_1^{\frac{1}{p_j}} \operatorname{meas}(R_1)^{\frac{1}{p_j} - \frac{1}{p}} \left( \int_{R_1} u^p \right)^{\frac{1}{p}}$$

so that, by (4.7), we obtain

$$\sup_{R_{1/2}} u^p \leq c_1^{\frac{p}{p_j}} \operatorname{meas}(R_1)^{\frac{p}{p_j} - 1} \int_{R_1} u^p \leq c_1^\kappa \operatorname{meas}(R_1)^{\kappa - 1} \int_{R_1} u^p.$$

This concludes the proof of (4.4) for $p > 0$.

We next consider $p < 0$. In this case, assuming that $u \geq u_0$ for some positive constant $u_0$, estimate (2.7) can be proved as in the case $p > 0$ or even more easily since condition (4.5) is satisfied for every $p < 0$. On the other hand, if $u$ is a non-negative solution, it suffices to apply (2.7) to $u + \frac{1}{n}$, $n \in \mathbb{N}$, and to let $n$ go to infinity, by the monotone convergence theorem. $\square$

We end this section with some further statements in the spirit of Theorem 2.2.

**Proposition 4.2** *Let $z_0 \in \Omega$ and $r, \varrho$, $0 < \frac{r}{2} \leq \varrho < r$, be such that $\overline{R_r(z_0)} \subseteq \Omega$.*

*(i) If $u$ is a non-negative weak sub-solution of* (1.1) *such that $u^p \in L^1(R_r(z_0))$, for $p \geq 1$ or $p < 0$, then* (2.7) *holds;*

*(ii) if $u$ is a non-negative weak super-solution of* (1.1), *with $p \in ]0, \frac{1}{2}[$, then* (2.7) *holds. In this case, the constant $c$ in* (2.7) *also depends on $p$.*

*Proof.* Proceeding as in the proof of Theorem 2.2, we obtain that

$$\sup_{R_\varrho(z_0)} u \le \left( \frac{c}{(r-\varrho)^Q} \int_{R_r(z_0)} u^p \right)^{\frac{1}{p}}, \qquad \forall p \ge 1, \tag{4.8}$$

$$\inf_{R_\varrho(z_0)} u \ge \left( \frac{c}{(r-\varrho)^Q} \int_{R_r(z_0)} u^p \right)^{\frac{1}{p}}, \qquad \forall p < 0, \tag{4.9}$$

where $c = c(Q, \mu)$. Estimate (4.9) is meaningful only when $u^p \in L^1(R_r(z_0))$. $\square$

**Proposition 4.3** *Let $u$ be a weak solution of* (1.1) *in* $\Omega$. *Let $z_0, \varrho, r$ as in Theorem 2.2. Then, we have*

$$\sup_{R_\varrho(z_0)} |u| \le \left( \frac{c}{(r-\varrho)^Q} \int_{R_r(z_0)} |u|^p \right)^{\frac{1}{p}}, \qquad \forall p \ge 1, \tag{4.10}$$

*where $c = c(Q, \mu)$.*

*Proof.* We consider a sequence $(g_n)_{n\in\mathbb{N}}$ in $C^\infty(\mathbb{R}, [0, +\infty[)$ with the following properties:

$$g_n(s) \downarrow \max(0, s), \qquad s \in \mathbb{R}, \quad \text{as } n \to \infty,$$

and, for every $n \in \mathbb{N}$, $g_n$ is a monotone increasing, convex function which is linear out of a fixed compact set. Then, $(g_n(u))$ and $(g_n(-u))$ are sequences of non-negative sub-solutions of (1.1), which converge to $u^+ = \max(0, u)$ and $u^- = \max(0, -u)$ respectively (see Lemma 1 in [27] for a detailed proof of the above statement). Thus, the thesis follows applying (4.8) of to $g_n(u)$, $g_n(-u)$ and passing at limit as $n$ goes to infinity. $\square$

The following result restores the analogy with the classical result by Moser. Denote $R_r^-(x_0, t_0) = R_r(x_0, t_0) \cap \{t < t_0\}$, then

**Proposition 4.4** *Let $u$ be a non-negative weak sub-solution of* (1.1) *in* $\Omega$. *Let $z_0 \in \Omega$ and $r, \varrho$, $0 < \frac{r}{2} \le \varrho < r$, be such that $\overline{R_r^-(z_0)} \subseteq \Omega$. Suppose that $u^p \in L^1(R_r^-(z_0))$, for $p < 0$ or $p \ge 1$. Then there exists a positive constant $c$, which only depends on the operator $\mathcal{L}_A$, such that*

$$\sup_{R_\varrho^-(z_0)} u^p \le \frac{c}{(r-\varrho)^Q} \int_{R_r^-(z_0)} u^p.$$

*Proof.* As in [32], we follow the lines of the proof of Theorem 2.2, by using the following two estimates:

$$\|Xu^p\|_{L^2(R_\varrho^-)} \le \frac{c\sqrt{\mu(\mu+\varepsilon)}}{\varepsilon(r-\varrho)} \|u^p\|_{L^2(R_r^-)}, \quad \text{where} \quad \varepsilon = \frac{|2p-1|}{4p}, \tag{4.11}$$

and

$$\|u^p\|_{L^{2\kappa}(R_\varrho^-)} \le \frac{c}{r-\varrho} \left( \|u^p\|_{L^2(R_r^-)} + \|Xu^p\|_{L^2(R_r^-)} \right), \tag{4.12}$$

for every real $p \notin [0,1[$ and for any $\varrho, r$ such that $\frac{r}{2} \leq \varrho < r$.

The Sobolev type inequality (4.12) can be proved exactly as Theorem 3.3, since the fundamental solution $\Gamma(x,t,\xi,\tau)$ vanishes in the set $\{\tau > t\}$.

In order to prove the Caccioppoli type inequality (4.11) we follow the method used in the proof of Theorem 3.1, by setting $v = u^p$ and using $\varphi = \chi_n(t) u^{2p-1}\psi^2$ as a test function in (2.5), where $\psi \in C_0^\infty(R_1)$ and $\chi_n(t)$ is defined as

$$\chi_n(s) = \begin{cases} 1, & \text{if } \; s \leq 0, \\ 1 - ns, & \text{if } \; 0 \leq s \leq 1/n, \\ 0, & \text{if } \; s \geq 1/n, \end{cases}$$

for every $n \in \mathbb{N}$. Then, by letting $n \to \infty$, we find

$$\int_{R_1^-} \left(1 - \frac{1}{2p}\right) \psi^2 \langle AXv, Xv\rangle + \psi \langle AXv, X\psi\rangle + \frac{v^2\psi}{2} Y\psi \leq 0.$$

After that, we follow the same line used in the proof of Theorem 3.1 and we obtain (4.11). □

# References


[1] D. G. Aronson, *Bounds for the fundamental solution of a parabolic equation*, Bull. Amer. Math. Soc., 73 (1967), pp. 890–896.

[2] E. Barucci, S. Polidoro, and V. Vespri, *Some results on partial differential equations and Asian options*, Math. Models Methods Appl. Sci., 11 (2001), pp. 475–497.

[3] A. Bonfiglioli, E. Lanconelli, and F. Uguzzoni, *Stratified Lie Groups and PotentialTheory for their sub–Laplacians*, (to appear).

[4] A. Bonfiglioli, E. Lanconelli, and F. Uguzzoni, *Fundamental solutions for non-divergence form operators on stratified groups*, Trans. Amer. Math. Soc., 356 (2004), pp. 2709–2737.

[5] A. Bonfiglioli and F. Uguzzoni, *Harnack inequality for non-divergence form operators on stratified groups*, (preprint), (2005).

[6] M. Bramanti and L. Brandolini, *$L^p$ estimates for uniformly hypoelliptic operators with discontinuous coefficients on homogeneous groups*, Rend. Sem. Mat. Univ. Politec. Torino, 58 (2000), pp. 389–433 (2003).

[7] M. Bramanti, L. Brandolini, E. Lanconelli, and F. Uguzzoni, *Heat kernels for non-divergence operators of Hörmander type*, (preprint).

[8] ——, *Non-divergence equations structured on Hörmander vector fields: heat kernels and Harnack inequalities*, (in preparation).

[9] C. Cercignani, *The Boltzmann equation and its applications*, Springer-Verlag, New York, 1988.

[10] C. Cinti, A. Pascucci, and S. Polidoro, *Pointwise estimates for solutions to a class of non-homogeneous Kolmogorov equations,*, preprint, (2005).

[11] L. Desvillettes and C. Villani, *On the trend to global equilibrium in spatially inhomogeneous entropy-dissipating systems: the linear Fokker-Planck equation*, Comm. Pure Appl. Math., 54 (2001), pp. 1–42.

[12] M. Di Francesco and A. Pascucci, *On the complete model with stochastic volatility by Hobson and Rogers*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., 460 (2004), pp. 3327–3338.

[13] S. D. Eidelman, S. D. Ivasyshen, and H. P. Malytska, *A modified Levi method: development and application*, Dopov. Nats. Akad. Nauk Ukr. Mat. Prirodozn. Tekh. Nauki, 5 (1998), pp. 14–19.

[14] G. B. Folland, *Subelliptic estimates and function spaces on nilpotent Lie groups*, Ark. Mat., 13 (1975), pp. 161–207.

[15] P. Hajłasz and P. Koskela, *Sobolev met Poincaré*, Mem. Amer. Math. Soc., 145 (2000), pp. x+101.

[16] D. G. Hobson and L. C. G. Rogers, *Complete models with stochastic volatility*, Math. Finance, 8 (1998), pp. 27–48.

[17] L. Hörmander, *Hypoelliptic second order differential equations*, Acta Math., 119 (1967), pp. 147–171.

[18] A. M. Il′in, *On a class of ultraparabolic equations*, Dokl. Akad. Nauk SSSR, 159 (1964), pp. 1214–1217.

[19] D. Jerison, *The Poincaré inequality for vector fields satisfying Hörmander's condition*, Duke Math. J., 53 (1986), pp. 503–523.

[20] A. E. Kogoj and E. Lanconelli, *An invariant Harnack inequality for a class of hypoelliptic ultraparabolic equations*, Mediterr. J. Math., 1 (2004), pp. 51–80.

[21] ——, *Link of groups and homogeneous Hörmander operators*, to appear on Proc. of Amer. Math. Soc., 1 (2006).

[22] L. P. Kupcov, *The fundamental solutions of a certain class of elliptic-parabolic second order equations*, Differencial′nye Uravnenija, 8 (1972), pp. 1649–1660, 1716.

[23] E. Lanconelli and S. Polidoro, *On a class of hypoelliptic evolution operators*, Rend. Sem. Mat. Univ. Politec. Torino, 52 (1994), pp. 29–63. Partial differential equations, II (Turin, 1993).

[24] P.-L. Lions, *On Boltzmann and Landau equations*, Philos. Trans. Roy. Soc. London Ser. A, 346 (1994), pp. 191–204.

[25] A. Lunardi, *Schauder estimates for a class of degenerate elliptic and parabolic operators with unbounded coefficients in* $\mathbb{R}^N$, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 24 (1997), pp. 133–164.

[26] M. Manfredini, *The Dirichlet problem for a class of ultraparabolic equations*, Adv. Differential Equations, 2 (1997), pp. 831–866.

[27] J. Moser, *A new proof of De Giorgi's theorem concerning the regularity problem for elliptic differential equations*, Comm. Pure Appl. Math., 13 (1960), pp. 457–468.

[28] ——, *A Harnack inequality for parabolic differential equations*, Comm. Pure Appl. Math., 17 (1964), pp. 101–134.

[29] ——, *On a pointwise estimate for parabolic differential equations*, Comm. Pure Appl. Math., 24 (1971), pp. 727–740.

[30] J. Nash, *Continuity of solutions of parabolic and elliptic equations*, Amer. J. Math., 80 (1958), pp. 931–954.

[31] A. Pascucci and S. Polidoro, *A Gaussian upper bound for the fundamental solutions of a class of ultraparabolic equations*, J. Math. Anal. Appl., 282 (2003), pp. 396–409.

[32] ——, *The Moser's iterative method for a class of ultraparabolic equations*, Commun. Contemp. Math., 6 (2004), pp. 395–417.

[33] S. Polidoro, *On a class of ultraparabolic operators of Kolmogorov-Fokker-Planck type*, Matematiche (Catania), 49 (1994), pp. 53–105.

[34] ——, *A global lower bound for the fundamental solution of Kolmogorov-Fokker-Planck equations*, Arch. Rational Mech. Anal., 137 (1997), pp. 321–340.

[35] H. Risken, *The Fokker-Planck equation: Methods of solution and applications*, Springer-Verlag, Berlin, second ed., 1989.

[36] L. P. Rothschild and E. M. Stein, *Hypoelliptic differential operators and nilpotent groups*, Acta Math., 137 (1976), pp. 247–320.

[37] L. Saloff-Coste and D. W. Stroock, *Opérateurs uniformément sous-elliptiques sur les groupes de Lie*, J. Funct. Anal., 98 (1991), pp. 97–121.

[38] J. I. Šatyro, *The smoothness of the solutions of certain degenerate second order equations*, Mat. Zametki, 10 (1971), pp. 101–111.

[39] M. Weber, *The fundamental solution of a degenerate partial differential equation of parabolic type*, Trans. Amer. Math. Soc., 71 (1951), pp. 24–37.