Parametrix approximations for option prices

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Abstract

We propose the use of a classical tool in PDE theory, the parametrix method, to build approximate solutions to generic parabolic models for pricing and hedging contingent claims. We obtain an expansion for the price of an option using as starting point the classical Black&Scholes formula. The approximation can be truncated to any number of terms and easily computable error measures are available.

1 Introduction and motivation

Under the standard dynamically complete market hypotheses, the forward price $O_t = f(S_t, T-t)$ computed at time t of an European option expiring at time T with payoff $H(S_T)$, where the underlying asset S evolves according to the stochastic differential equation

$$dS_t = \mu(t, S_t)dt + \sigma(t, S_t)dW_t, \tag{1.1}$$

is given by

$$\widetilde{E}_t(H(S_T)) \tag{1.2}$$

where $\widetilde{E}$ represents the expected value with respect to the martingale measure under which the dynamics of the forward price of the underlying is driftless. Accordingly, in the case of null riskless interest rate, f is also the solution of the Cauchy problem

$$\begin{cases} \partial_t f = \frac{\sigma^2}{2}\partial_{SS} f, \\ f(x,0) = H(x). \end{cases} \tag{1.3}$$

From both (1.2) or (1.3) we have the representation

$$f(S_t, t) = \int_{\mathbb{R}} H(x)\Gamma(S_t, t; x, 0)dx, \tag{1.4}$$

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where $\Gamma(S, t; x, 0)$ is the transition density of S_t from $(x, 0)$ which corresponds to the so called "fundamental solution" of the PDE in (1.3).

As a matter of fact the fundamental solution Γ is explicitly known only for a rather small set of models. Among these the most relevant cases are the arithmetic and geometric Brownian motions (Gaussian and log-normal densities), the general linear case (affine models studied e.g. in [7]), the square root process (e.g. [3]) and classes of models derived via transforms from these models (see e.g. [1]). In view of the paramount advantages, both in terms of understanding and computation time, given by the existence of an analytical solution for (1.3), actual modeling has largely been restricted to this rather small set of diffusions. On the other hand the analytical tractability of these models is not accompanied by good statistical properties in the sense that the distributions implied by these models give poor fit to actual market data.

This motivates a growing interest for models whose solution can be computed only by numerical methods (deterministic or Montecarlo based). A major problem which severely limits the use of these models is that, while their practical relevance has been found in the valuation of exotic or very far from the money vanilla options, the numerical burden implied by their use for such payoffs is still by far too big to allow widespread application. It is to be noticed that such a burden can be excessive even in the case of standard model when applied to the computation of hedging parameters for some exotic payoff.

Even if we do not consider the numerical problem, a second relevant obstacle to the implementation of more statistically satisfactory but less tractable models is that the lack of analytical solution severely restricts the ability of the practitioner to understand, pending the reaction times allowed by the market, the implications of a given model and its possible weak points. This is relevant, in particular, when a real time position risk management is required.

A third and connected problem with analytically untractable models is that they do not allow for an easy valuation of the consequences of model misspecification. In an applied milieu where model risk management is becoming a central portion of the financial decision making process, such weakness is rapidly becoming an heavy burden for elastic but not tractable models.

The standard practitioner's way out of these problems has been a clever and often very creative use of inconsistent behaviours. In practice solvable models are implemented with clear understanding of their inadequacies, ad hoc fixtures are used in order to (and with the hope of) avoiding the consequences of these inadequacies.

Among the most frequently applied fixtures we quote:

1. Parameter recalibration: model parameters which should be constant in the model are periodically recalibrated so that the model replicates observed prices.

2. Use of different models for pricing and hedging derivatives which should share the same risk neutral distribution: As an example, even with recalibration simple models are not capable of correct pricing for options expiring the same day but with different strikes. As a consequence different values for the same parameter are used when hedging different options with the same expiry date but different strikes.

3. Computation of "greeks" for parameters which should be constants of the model (e.g. the *vega*).

4. Initial conditions recalibration. This is quite frequent in interest rates and credit risk models. As an instance: in HJM inspired models the term structure of interest rates is an initial condition of the model. This condition is an input of the model only at inception date, afterward the model specifies all possible shapes for the future term structure and the probabilities of these. It is often the case that among these shapes we cannot find the observed one. In this case the model is restarted from the observed term structure.

From these attitudes, and similar ones, it sprung a lot of practitioner's lore which is now standard in the market. Just to quote some examples which shall be discussed in what follows we cite smile fitting, that is, the attitude of using different volatilities for different strikes on the same day and the connected use of "skew correction" for hedging parameters like the *delta*.

This correction, which can take various shapes (see e.g. [4]), tries to account for the change of implied volatility which may accompany the change in moneyness of a given option. In a particular specification, often termed "sticky delta", if Δ_{BS} is the Black&Scholes' *delta*, *vega* is the standard Black&Scholes' *vega* and $\sigma(K/S)_S$ is the derivative, w.r.t. the price, of the volatility used for evaluating the option with strike K with moneyness K/S, we have a skew corrected *delta* computed as

$$\Delta = \Delta_{\mathrm{BS}} + vega * \sigma(K/S)_S.$$

While inconsistent, these behaviours are often sensible and hold up to market strains, at least up to the unpredictable moment when they break down: the current unsatisfactory state of correlation modeling in applied credit risk modeling is an example of this. It should then be a purpose of research either to offer tools for avoiding such ad hoc behaviours or to offer tools for reinterpreting them in a consistent way which could point out their effettive scope of validity and possible extensions thereof.

Two possible ways out of this problem can be suggested. The first one is to extend the class of analytically solvable models; the second one is to develop tools capable of calibrating analytically computable approximations to non analytically computable models and to evaluate the error of this approximation.

This paper is concerned with the second alternative. We suggest the use of a classical tool in PDE theory: the parametrix expansion. A parametrix expansion can be used to build an approximate fundamental solution to a generic parabolic PDE using as starting point the explicit solution of a simpler parabolic PDE. The approximation can be truncated to any number of terms and easily computable error measures are available. A comprehensive presentation of the parametrix method for uniformly parabolic PDEs can be found, for instance, in [10]: in [5] a more recent presentation of this technique applied to a wider class of (possibly degenerate) PDEs can be found.

While well-known in the classical theory of parabolic PDEs, the parametrix series is, as far as we know, unknown in the field of mathematical finance (with the exception of [2]). Apart from being a tool for the approximate solution to pricing problems, the parametrix series can be of use as a tool for model risk management and as a way to unify a number of sparse results about approximate option valuation already available in the financial literature under a common principle. A welcome bonus of the parametrix is that, far from being an abstract mathematical tool, it yields to an interesting financial interpretation.

The paper is organized as follows. In Section 2, we give a heuristic derivation of the parametrix series, connected to the evaluation of pricing and hedging errors implied by the use of a "wrong" model. We also give a financial interpretation to the derivation of the parametrix. In Section 3, we formally introduce the parametrix series, derive it under conditions suitably general but easy to assess in the case of financial application and present explicit valuations of error terms: in this section our main results, the forward and backward parametrix expansion Theorems 3.6 and 3.14, are stated. In Section 4, we use the parametrix series to approximate the solution for already solvable models and compare the resulting approximations with exact results.

2 A heuristic derivation of the parametrix series

The purpose of this section is to stress the financial intuition underlying the meaning and the derivation of the parametrix expansion. In so doing it is sufficient for us to work in the one dimensional (space variable) case. In the following section the parametrix will be derived in its full generality for the case of any finite number of space variables.

We begin by noticing that a parametrix expansion can be computed both for the standard parabolic PDE implied in a valuation problem and for its adjoint PDE. While the fundamental solutions of the two equations are strictly related, the two parametrix approximations are distinct and offer two slightly different and very interesting financial interpretations so we will discuss both.

Let us start by sketching the derivation of the parametrix for the standard parabolic PDE. We are interested in the fundamental solution $\Gamma = \Gamma(z;\zeta)$ where $z = (x,t)$ and $\zeta = (\xi,\tau) \in \mathbb{R}\times\mathbb{R}$ of L

$$Lu = a(z)\partial_{xx}u - \partial_t u = 0, \tag{2.1}$$

that is: $L\Gamma(\cdot;\zeta) = 0$ in $\mathbb{R}^2 \setminus \{\zeta\}$ and for every suitable function $H(x)$ a classical solution of the Cauchy problem

$$\begin{cases} Lu = 0, & x \in \mathbb{R},\ t > 0, \\ u(x,0) = H(x), & x \in \mathbb{R}, \end{cases} \tag{2.2}$$

is given by

$$u(x,t) = \int_{\mathbb{R}} H(\xi)\Gamma(x,t;\xi,0)d\xi. \tag{2.3}$$

In financial terms, formula (2.3) gives the (forward) price at time to maturity t for an European option expiring at time to maturity 0 with payoff H.

Suppose now that a is such that equation (2.2) cannot be solved explicitly. It is then inviting to find an approximation formula for (2.3) whose first term is given by, or at least similar to, the Black&Scholes formula. In the sequel we assume $\zeta = (\xi,0)$ and use the notation $z_j = (x_j,t_j)$ for $j \in \mathbb{N}$.

The parametrix approximation is based on two ideas. The first one is to locally approximate $\Gamma(z;\zeta)$ by the so-called parametrix $Z(z;\zeta) = \Gamma_\zeta(z;\zeta)$ where Γ_w is the fundamental solution to the "frozen" operator[1]

$$L_w = a(w)\partial_{xx} - \partial_t. \tag{2.4}$$

[1] The context of this paper suggests us to use, as a parametrix, the fundamental solution of the heat equation which is strictly connected with the Black&Scholes formula. However it is to be noticed that, in principle, the fundamental solution of any solvable parabolic PDE could be used as a starting point. Ideally that parametrix should be used which solves the parabolic problem "most similar" to the problem under analysis.

The second idea is that of supposing that the fundamental solution Γ of L is in the form (recall that $\zeta = (\xi, 0)$)

$$\Gamma(z;\zeta) = Z(z;\zeta) + \int_0^t \int_{\mathbb{R}} Z(z;z_0)\Phi(z_0;\zeta)dz_0. \tag{2.5}$$

In order to identify Φ we notice that from $L\Gamma = 0$ in $\mathbb{R}\times]0,t[$, we get

$$0 = LZ(z;\zeta) + L\int_0^t \int_{\mathbb{R}} Z(z;z_0)\Phi(z_0;\zeta)dz_0. \tag{2.6}$$

But formally it holds

$$L\int_0^t \int_{\mathbb{R}} Z(z;z_0)\Phi(z_0;\zeta)dz_0 = -\Phi(z;\zeta) + \int_0^t \int_{\mathbb{R}} LZ(z;z_0)\Phi(z_0;\zeta)dz_0 \tag{2.7}$$

so that

$$\Phi(z;\zeta) = LZ(z;\zeta) + \int_0^t \int_{\mathbb{R}} LZ(z;z_0)\Phi(z_0;\zeta)dz_0. \tag{2.8}$$

Formula (2.8) yields an iteration on Φ so that:

$$\begin{aligned}\Phi(z;\zeta) &= LZ(z;\zeta) + \int_0^t \int_{\mathbb{R}} LZ(z;z_0)LZ(z_0;\zeta)dz_0 + \int_0^t \int_{\mathbb{R}} LZ(z;z_1)\int_0^{t_1} \int_{\mathbb{R}} LZ(z_1;z_0)\Phi(z_0;\zeta)dz_0dz_1\\ &= LZ(z;\zeta) + \int_0^t \int_{\mathbb{R}} LZ(z;z_0)LZ(z_0;\zeta)dz_0\\ &+\sum_{n=0}^{\infty} \int_0^t \int_{\mathbb{R}} LZ(z;z_{n+1})\int_0^{t_{n+1}} \int_{\mathbb{R}} LZ(z_{n+1};z_n)\cdots\int_0^{t_1} \int_{\mathbb{R}} LZ(z_1;z_0)LZ(z_0;\zeta)dz_0\cdots dz_n dz_{n+1}.\end{aligned} \tag{2.9}$$

In terms of formula (2.3) this implies an expansion of the option price given by:

$$u(z) = \sum_{n=0}^{\infty} u_n(z) \tag{2.10}$$

where

$$\begin{aligned}u_0(z) &= \int_{\mathbb{R}} H(\xi)Z(z;\xi,0)d\xi,\\ u_1(z) &= \int_0^t \int_{\mathbb{R}} Z(z;\zeta)Lu_0(\zeta)d\zeta,\end{aligned} \tag{2.11}$$

and in general, for $n \in \mathbb{N}$,

$$u_n(z) = \int_0^t \int_{\mathbb{R}} Z(z;\zeta) L U_{n-1}(\zeta) d\zeta, \qquad U_{n-1}(z) := \sum_{k=0}^{n-1} u_k(z). \tag{2.12}$$

Indeed, by (2.5) and (2.9), we have

$$u_1(z) = \int_{\mathbb{R}} H(\xi) \int_0^t \int_{\mathbb{R}} Z(z;z_0) L Z(z_0;\xi,0) dz_0 d\xi = \int_0^t \int_{\mathbb{R}} Z(z;z_0) L \underbrace{\int_{\mathbb{R}} H(\xi) Z(z_0;\xi,0) d\xi}_{=u_0(z_0)} dz_0,$$

that proves (2.11). Moreover

$$\begin{aligned} u_2(z) &= \int_{\mathbb{R}} H(\xi) \int_0^t \int_{\mathbb{R}} Z(z;z_1) \int_0^{t_1} \int_{\mathbb{R}} L Z(z_1;z_0) L Z(z_0;\xi,0) dz_0 dz_1 d\xi = \\ &= \int_0^t \int_{\mathbb{R}} Z(z;z_1) \int_0^{t_1} \int_{\mathbb{R}} L Z(z_1;z_0) L \underbrace{\int_{\mathbb{R}} H(\xi) Z(z_0;\xi,0) d\xi}_{=u_0(z_0)} dz_0 dz_1 \\ &= \int_0^t \int_{\mathbb{R}} Z(z;z_1) \Bigg(L \underbrace{\int_0^{t_1} \int_{\mathbb{R}} Z(z_1;z_0) L u_0(z_0) dz_0}_{=u_1(z_1)} + L u_0(z_1) \Bigg) dz_1, \end{aligned}$$

and this proves (2.12) for $n = 2$. The general case can be proved by induction.

This is the expansion which is usually used in the classical PDEs' theory to prove the *existence* of a fundamental solution to L. Before interpreting the result let us examine the equivalent expansion derived from the adjoint PDE. The use of the adjoint parametrix seems to be convenient by several points of view: first of all, we are able to derive an approximating expansion whose first term is given exactly by the Black&Scholes formula, while the subsequent terms can be expressed as solutions to suitable Cauchy problems related to constant coefficients operators. Secondly, the approximating terms generated in this way are convolutions of a Gaussian function $\Gamma_z(z;\cdot)$ for fixed z: this seems to be convenient from a numerical point of view since we may rely upon several known efficient numerical techniques. We define the *backward parametrix*

$$P(z;\zeta) = \Gamma_z(z;\zeta). \tag{2.13}$$

In the next section we prove that the solution to the option pricing problem has an expansion of the form (2.10) where now

$$u_0(z) = \int_{\mathbb{R}} H(\xi) P(z;\xi,0) d\xi, \tag{2.14}$$

and

$$u_n(z) = \int\limits_0^t \int\limits_{\mathbb{R}} P(z;\zeta) L U_{n-1}(\zeta) d\zeta, \qquad U_{n-1}(z) := \sum_{k=0}^{n-1} u_k(z), \qquad n \in \mathbb{N}. \tag{2.15}$$

The main differences between the two parametrix expansions hinted before are now clear. Each term in the expansion is an "expected value" with respect to the distributions with density $Z(z;\zeta)$ or $P(z;\zeta)$. But, while $P(z;\zeta)$ is the same Gaussian density for each value of the integration variable (z is frozen and the integration is performed varying ζ) and so is a true PDF, $Z(z;\zeta)$ is a different Gaussian (different variance) for each value of the integrating variable ζ.

Let us examine the first term of the expansion for the parametrix Z:

$$u_0(z) = \int_{\mathbb{R}} H(\xi) Z(z;\xi,0) d\xi. \tag{2.16}$$

Since the explicit expression of $Z(z;\xi,0) = \Gamma_{(\xi,0)}(z;\xi,0)$ is known,

$$\Gamma_{(\xi,0)}(x,t;\xi,0) = \frac{1}{\sqrt{4\pi t\, a(\xi,0)}} \exp\left(-\frac{(x-\xi)^2}{4ta(\xi,0)}\right), \qquad t>0, \tag{2.17}$$

we see that u_0 in (2.16) is very similar to the solution of a Cauchy problem for a constant coefficients operator. On the other hand, the integration in (2.16) is performed with respect to the variable ξ which also appears in $L_{(\xi,0)}$ as the point where the operator L is frozen. Hence the first term of the expansion is an "expected value" of the terminal payoff which uses as density a Gaussian with a different volatility (corresponding to the "true" diffusion coefficient) for each point in the integration range. Due to this reason this "state dependent" Gaussian is not a density as it is nonnegative but does not integrate, in general, to one (it is obviously possible to normalize it).

This seems quite sensible a starting point and can obviously compared with standard "implied volatility" approximations. With implied volatility we use a different Gaussian distribution (for $\log S$) for each strike. Here the suggestion is to use the same distribution but with a different volatility for each terminal value of the stock. As we will see in the empirical section of the paper, this rough, zero order, approximation can give good results for interesting payoffs.

Let us now pass to the adjoint parametrix expansion zero order term:

$$u_0(z) = \int_{\mathbb{R}^N} H(\xi) P(z;\xi,0) d\xi.$$

Here the interpretation is straightforward: the zero order term is simply the Black&Scholes option value. Indeed since

$$P(z;\zeta) = \Gamma_{(x,t)}(x,t;\xi,0),$$

then the parametrix $P(z;\zeta)$ is the same density for the full range of the integrating variable ξ and is the terminal log-price density corresponding to the heat operator frozen at (x,t).

Notice, however, that a different "volatility" value is used for each initial pair (x,t). Accordingly, if we compute the derivative of the option w.r.t. the price $S = e^x$ we have that the Delta for the zero order approximation is given, with the obvious notation, by

$$\Delta = \Delta_{\text{BS}} + vega * \frac{\partial \sigma}{\partial S}$$

It is clear how this way of computing the Delta is a direct reinterpretation of the skew corrected delta introduced above.

Next we consider the following terms in the expansions. Both expansions are similar in that each new term is an expected value. The difference that makes the adjoint parametrix more readable is that in that case the term is a true expected value (with respect to the same "frozen" Gaussian $P(z;\zeta)$) while in the case of the standard parametrix, $Z(z;\xi,0)$ does not correspond to an exact density.

Since each new term can be read as the value (exact or approximate) of a new option in a Black&Scholes world, it is interesting to understand the meaning of such options. The solution to this problem comes from the understanding how the L operator (the original operator for both expansions) acts.

In both expansions L acts on the first argument of both P and Z that is, on the argument not involved in the expectation integral. Moreover (cf. (2.12)) the operator L in the term of order n acts on the "option approximation" derived up to order $n-1$.

Each action of the L operator can be interpreted as a check of the fact that the approximation of order $n-1$ satisfies $LU_{n-1}=0$. In other words LU_{n-1} is a measure of the error implied in supposing that U_{n-1} satisfies $LU_{n-1}=0$. This represents a "transaction cost" for the new option. This error is a function of the variable on which L acts and the term u_n is then computed as the expected value of the error using the P density or the Z "density".

We see how the parametrix expansion partitions the value of a given option computed in a non Black&Scholes world (the governing PDE is not the heat equation) into a series of option values each computed in the Black&Scholes world. This is exact in the case of the adjoint parametrix and approximately exact, if we recall that Z is not a density, in the standard parametrix case. The transaction cost for each option is a valuation of the error made by valuing the option implied in the previous term in the Black&Scholes world and not in the world described by L.

In the following section we will see how it is possible to bound the overall error derived by truncating the series at the n^{th} term with explicit and easily computable bounds uniformly decreasing in n. Moreover in the applications section we will see how the iterative nature of the parametrix series definition allows a fast implementation of the valuation algorithm.

Even at this intuitive level we see how the parametrix series can become a useful tool in model risk management. Suppose a risk manager is willing to use a price model based on a given operator L which is believed to faithfully represent the statistical properties of observed underlying and options prices. It is likely that this operator will not yield to explicit computation. The risk manager can then compute a number of terms in the parametrix series each of which will be the value of a (Black&Scholes) option and will be hedgeable as such. The risk manager will also be able to compute a measure of error, which, as we will see in the next section, will be interpretable as the value of another option computed in the Black&Scholes world. As such it will be easy for the risk manager to interpret this option/error value as the price of the approximate model valuation and hedge it, if necessary.

3 Forward and backward parametrix expansion

In this section we present the parametrix expansion in its full generality. Consider a parabolic differential equation in the form

$$Lu := \sum_{i,j=1}^{N} a_{ij}(z)\partial_{x_i x_j} u + \sum_{i=1}^{N} b_i(z)\partial_{x_i} u + c(z)u - \partial_t u = 0, \tag{3.1}$$

where $A(z) = (a_{ij}(z))$ is a symmetric and positive definite matrix. Throughout the section we systematically denote by $z = (x,t)$ and $\zeta = (\xi, \tau)$ the points in $\mathbb{R}^{N+1}$. We also denote by $\lambda_1(z), \dots, \lambda_N(z)$ the eigenvalues of $A(z)$ and set

$$m := \inf_{\substack{i=1,\dots,N \\ z\in\mathbb{R}^{N+1}}} \lambda_i(z), \qquad M := \sup_{\substack{i=1,\dots,N \\ z\in\mathbb{R}^{N+1}}} \lambda_i(z)\mu(z).$$

Our main hypotheses are the following:

[H1] M, m are positive and finite[2];

[H2] the coefficients of L are bounded functions: moreover, $a_{ij} \in C^{1,\frac{1}{2}}(\mathbb{R}^{N+1})$ that is

$$|a_{ij}(x,t) - a_{ij}(x',t')| \le \alpha \left(|x - x'| + |t - t'|^{\frac{1}{2}} \right), \qquad (x,t),(x',t') \in \mathbb{R}^{N+1},\ i,j = 1,\dots,N, \tag{3.2}$$

for some positive constant α.

As a consequence of **[H1]** we have the usual uniformly parabolicity condition:

$$m|\eta|^2 \le \sum_{i,j=1}^{N} a_{ij}(z)\xi_i\xi_j \le M|\eta|^2, \qquad \forall \xi \in \mathbb{R}^N,\ z \in \mathbb{R}^{N+1}. \tag{3.3}$$

It is known that, under the above hypotheses, the operator L has a fundamental solution $\Gamma(z;\zeta)$. Given $w \in \mathbb{R}^{N+1}$, we denote by $\Gamma_w(z;\zeta)$ the fundamental solution to the *frozen operator* L_w defined by

$$L_w = \sum_{i,j=1}^{N} a_{ij}(w)\partial_{x_i x_j} - \partial_t; \tag{3.4}$$

then we have $\Gamma_w(z;\zeta) = \Gamma_w(z - \zeta)$ where

$$\Gamma_w(x,t) := \Gamma_w(x,t;0) = \frac{(4\pi t)^{-\frac{N}{2}}}{\sqrt{\det A(w)}} \exp\left(-\frac{\langle A^{-1}(w)x, x\rangle}{4t} \right), \qquad x \in \mathbb{R}^N, t > 0. \tag{3.5}$$

Given a constant $\mu > 0$, we also denote by Γ^μ the fundamental solution to the heat operator

$$\mu \sum_{i=1}^{N} \partial_{x_i x_i} - \partial_t.$$

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[2] Equivalently we may use by $M := \sup_{z\in\mathbb{R}^{N+1}} \mu(z)$ and $m := \inf_{z\in\mathbb{R}^{N+1}} \mu(z)$ where $\mu(z)$ is the Euclidean norm of $A(z)$ in $\mathbb{R}^{N+1} \times \mathbb{R}^{N+1}$ (also equal to the Euclidean norm of $(\lambda_1(z), \dots, \lambda_N(z))$, the vector of the eigenvalues of $A(z)$). This gives less precise, but more easily computable, estimates.

Lemma 3.1. *For every $z, \zeta, w \in \mathbb{R}^{N+1}$ with $z \neq \zeta$, it holds*

$$\left(\frac{m}{M}\right)^{\frac{N}{2}} \Gamma^m(z;\zeta) \leq \Gamma_w(z;\zeta) \leq \left(\frac{M}{m}\right)^{\frac{N}{2}} \Gamma^M(z;\zeta).$$

Proof. We only prove the second inequality in the case $\zeta = 0$. The thesis follow directly from condition (3.3) keeping in mind formula (3.5): indeed we have

$$\Gamma_w(z) \leq \frac{1}{(4\pi t m)^{\frac{N}{2}}} \exp\left(-\frac{|x|^2}{4tM}\right) = \left(\frac{M}{m}\right)^{\frac{N}{2}} \Gamma^M(z).$$

□

Lemma 3.2. *For every $\varepsilon, \mu > 0$ and $n \in \mathbb{N} \cup \{0\}$ it holds*

$$\left(\frac{|x|}{\sqrt{t}}\right)^n \Gamma^\mu(x,t) \leq \left(\frac{n}{\varepsilon}\right)^{\frac{n}{2}} (\mu+\varepsilon)^n \left(\frac{\mu+\varepsilon}{\mu}\right)^{\frac{N}{2}} \Gamma^{\mu+\varepsilon}(x,t),$$

for any $x \in \mathbb{R}^N$ and $t > 0$.

Proof. Setting $a = \frac{|x|}{\sqrt{t}}$, we have

$$\left(\frac{|x|}{\sqrt{t}}\right)^n \Gamma^\mu(z,0) = a^n (4\pi\mu t)^{-\frac{N}{2}} \exp\left(-\frac{a^2}{4\mu}\right) \leq (4\pi\mu t)^{-\frac{N}{2}} \exp\left(-\frac{a^2}{4(\mu+\varepsilon)}\right) \sup_{\mathbb{R}_+} \Phi,$$

where

$$\Phi(a) = a^n \exp\left(-\left(\frac{1}{4\mu} - \frac{1}{4(\mu+\varepsilon)}\right) a^2\right). \tag{3.6}$$

The thesis follows since a straightforward computation shows that Φ attains a global maximum at $\bar{a} = \sqrt{\frac{2n\mu(\mu+\varepsilon)}{\varepsilon}}$ and

$$\Phi(\bar{a}) = \left(\frac{2n\mu(\mu+\varepsilon)}{e\varepsilon}\right)^{\frac{n}{2}} \leq \left(\frac{n}{\varepsilon}\right)^{\frac{n}{2}} (\mu+\varepsilon)^n.$$

□

3.1 Forward parametrix expansion

For $z \neq \zeta$, we define the *forward parametrix*

$$Z(z;\zeta) = \Gamma_\zeta(z;\zeta). \tag{3.7}$$

Notation 3.3. *In order to avoid confusion, when necessary, we write $L^{(z)}$ in order to indicate that the operator L is acting in the variable z.*

We remark explicitly that

$$L_\zeta^{(z)} Z(z;\zeta) = 0, \qquad \text{for } z \neq \zeta. \tag{3.8}$$

We first prove some preliminary result will be crucial in the development of the parametrix expansion.

Lemma 3.4. *For every $\varepsilon > 0$ and $i, j = 1, \ldots, N$ it holds*

$$|\partial_{x_i}\Gamma_w(z;\zeta)| \leq \frac{1}{2\sqrt{\varepsilon(t-\tau)}} \left(\frac{M+\varepsilon}{m}\right)^{\frac{N}{2}+1} \Gamma^{M+\varepsilon}(z;\zeta), \tag{3.9}$$

$$\left|\partial_{x_ix_j}\Gamma_w(z;\zeta)\right| \leq \frac{1}{\varepsilon(t-\tau)} \left(\frac{M+\varepsilon}{m}\right)^{\frac{N}{2}+2} \Gamma^{M+\varepsilon}(z;\zeta), \tag{3.10}$$

for any $z, \zeta, w \in \mathbb{R}^{N+1}$ with $t > \tau$.

Proof. For sake of simplicity, we prove the above estimates in the case $\zeta = 0$. We have

$$|\partial_{x_i}\Gamma_w(z)| = \frac{1}{2}\frac{\left|\left(A^{-1}(w)x\right)_i\right|}{t}\Gamma_w(z) \leq$$

(by Lemma 3.1)

$$\leq \frac{1}{2m\sqrt{t}}\left(\frac{M}{m}\right)^{\frac{N}{2}} \frac{|x|}{\sqrt{t}}\Gamma^M(z)$$

and (3.10) follows applying Lemma 3.2 with $\mu = M$ and $n = 1$.

Moreover

$$\left|\partial_{x_ix_j}\Gamma_w(z)\right| = \frac{1}{2t}\left|A^{-1}(w)_{ij} + \frac{1}{2t}\left(A^{-1}(w)x\right)_i\left(A^{-1}(w)x\right)_j\right|\Gamma_w(z) \leq \frac{1}{2t}\left(\frac{1}{m} + \frac{|x|^2}{2m^2t}\right)\Gamma_w(z),$$

and (3.9) easily follows by Lemmas 3.1 and 3.2 with $\mu = M$. □

Lemma 3.5. *For every positive ε, we have*

$$\left|L^{(z)}Z(z;\zeta)\right| \leq \frac{\eta_\varepsilon}{\sqrt{t-\tau}}\Gamma^{M+\varepsilon}(z;\zeta), \qquad \forall z, \zeta \in \mathbb{R}^{N+1},\ t > \tau, \tag{3.11}$$

where

$$\eta_\varepsilon := \alpha N^2 \left(\frac{2}{\varepsilon}\right)^{\frac{3}{2}} \left(\frac{M+\varepsilon}{m}\right)^{\frac{N}{2}+2} \left(M + \varepsilon + \sqrt{\frac{\varepsilon}{2}}\right) + \beta\frac{N}{2\sqrt{\varepsilon}}\left(\frac{M+\varepsilon}{m}\right)^{\frac{N}{2}+1} + \gamma\left(\frac{M+\varepsilon}{m}\right)^{\frac{N}{2}}\sqrt{t-\tau} \tag{3.12}$$

and

$$\beta := \sup_{\substack{i=1,\ldots,N \\ z\in\mathbb{R}^{N+1}}} |b_i(z)|, \qquad \gamma := \sup_{z\in\mathbb{R}^{N+1}} |c(z)|$$

and α is the constant in (3.2).

Proof. For $t > \tau$, we have

$$|LZ(z;\zeta)| = |(L - L_\zeta)Z(z;\zeta)| \leq I_1 + I_2 + I_3$$

where

$$I_1 = \sum_{i,j=1}^{N} |a_{ij}(z) - a_{ij}(\zeta)|\left|\partial_{x_ix_j}Z(z;\zeta)\right| \leq$$

(by (3.2))

$$\leq \alpha N^2 \left(|x-\xi| + \sqrt{t-\tau}\right) \max_{i,j} \left|\partial_{x_i x_j} Z(z;\zeta)\right| \leq$$

(by Lemma 3.4)

$$\leq \frac{\alpha N^2}{\varepsilon} \left(\frac{M+\varepsilon}{m}\right)^{\frac{N}{2}+2} \left(1 + \frac{|x-\xi|}{\sqrt{t-\tau}}\right) \Gamma^{M+\varepsilon}(z;\zeta) \leq$$

(by Lemma 3.2)

$$\begin{aligned}
&\leq \frac{\alpha N^2}{\varepsilon} \left(\frac{M+\varepsilon}{m}\right)^{\frac{N}{2}+2} \left(\frac{M+2\varepsilon}{M+\varepsilon}\right)^{\frac{N}{2}} \left(1 + \frac{M+2\varepsilon}{\sqrt{\varepsilon}}\right) \Gamma^{M+2\varepsilon}(z;\zeta) \\
&\leq \frac{\alpha N^2}{\varepsilon^{\frac{3}{2}}} \left(\frac{M+2\varepsilon}{m}\right)^{\frac{N}{2}+2} \left(M + 2\varepsilon + \sqrt{\varepsilon}\right) \Gamma^{M+2\varepsilon}(z;\zeta).
\end{aligned}$$

Moreover, by Lemma 3.4, we have

$$I_2 = \sum_{i=1}^{N} |b_i(z)| \left|\partial_{x_i} Z(z;\zeta)\right| \leq \beta \frac{N}{2\sqrt{\varepsilon(t-\tau)}} \left(\frac{M+\varepsilon}{m}\right)^{\frac{N}{2}+1} \Gamma^{M+\varepsilon}(z;\zeta);$$

finally, by Lemma 3.1, we have

$$I_3 = |c(z)| Z(z;\zeta) \leq \gamma \left(\frac{M+\varepsilon}{m}\right)^{\frac{N}{2}} \Gamma^{M+\varepsilon}(z;\zeta).$$

□

We can now state the *forward parametrix expansion theorem.*

Theorem 3.6. *Assume hypotheses* **[H1]** *and* **[H2]**. *Then for every* $\zeta \in \mathbb{R}^{N+1}$, *the following expansion of the fundamental solution* Γ *holds*

$$\Gamma(z;\zeta) = Z(z;\zeta) + \int_{\tau}^{t} \int_{\mathbb{R}^N} Z(z;w)\Phi(w;\zeta)dw, \qquad t > \tau, \tag{3.13}$$

where

$$\Phi(z;\zeta) = \sum_{k=1}^{+\infty} (LZ)_k(z;\zeta), \tag{3.14}$$

with

$$\begin{aligned}
(LZ)_1(z;\zeta) &= L^{(z)} Z(z;\zeta), \\
(LZ)_{k+1}(z;\zeta) &= \int_{\tau}^{t} \int_{\mathbb{R}^N} L^{(z)} Z(z;w)(LZ)_k(w;\zeta)dw,
\end{aligned}$$

and, for every $T>0$, the series in (3.14) *converges uniformly in the strip $\mathbb{R}^N \times]\tau, \tau+T[$. Moreover, for every positive ε, we have the following estimate for the approximation truncated at the n-th term:*

$$\begin{aligned}\left|\Gamma(z;\zeta) - Z(z;\zeta) - \sum_{k=1}^{n-1} \int_\tau^t \int_{\mathbb{R}^N} Z(z;w)(LZ)_k(w;\zeta)dw\right| \\ \leq \sqrt{\frac{2}{\pi}} \left(\frac{M+\varepsilon}{m}\right)^{\frac{N}{2}} f_n\left(\eta_\varepsilon \sqrt{2\pi(t-\tau)}\right) \Gamma^{M+\varepsilon}(z;\zeta)\end{aligned} \tag{3.15}$$

for $t>\tau$, where η_ε is defined in (3.12),

$$f_n(\eta) = e^{\frac{\eta^2}{2}}(\eta+1) \frac{\left(\frac{\eta^2}{2}\right)^{\left[\frac{n+1}{2}\right]}}{\left[\frac{n+1}{2}\right]!}, \tag{3.16}$$

and $[a]$ denotes the integer part of $a \in \mathbb{R}$.

Remark 3.7. We remark explicitly that, when $\eta = \eta_\varepsilon \sqrt{2\pi(t-\tau)} \ll 1$ in (3.16), then the rate of convergence of the parametrix approximation is very fast. This is the case, for instance, when $t-\tau \ll 1$, i.e. for short time to maturity. Also note that (3.15) is a global estimate w.r.t. the spatial variables.

As a consequence of Theorem 3.6, we have the following forward parametrix expansion for solutions to the Cauchy problem for L.

Corollary 3.8. *The solution to the Cauchy problem*

$$\begin{cases} Lu(x,t) = 0, & x \in \mathbb{R}^N, \ t>0, \\ u(x,0) = H(x), & x \in \mathbb{R}^N, \end{cases} \tag{3.17}$$

has an expansion of the form (2.10)-(2.11)-(2.12).

The proof of Theorem 3.6 is based on the following preliminary result.

Lemma 3.9. *For every $\varepsilon>0$ and $k \geq 1$ the following estimate for the term $(LZ)_k$ in* (3.14) *holds:*

$$|(LZ)_k(z;\zeta)| \leq \frac{\Gamma_E\left(\frac{1}{2}\right)^k}{\Gamma_E\left(\frac{k}{2}\right)} \frac{\eta_\varepsilon^k}{(t-\tau)^{1-\frac{k}{2}}} \Gamma^{M+\varepsilon}(z;\zeta), \qquad \forall z,\zeta \in \mathbb{R}^{N+1}, \ t>\tau, \tag{3.18}$$

where η_ε is defined in (3.12) *and Γ_E denotes the Euler's Gamma function.*

Proof. We prove (3.18) by induction on k. The case $k=1$ was proved in Lemma 3.5. Let us now assume that (3.18) holds for k and prove it for $k+1$. We have

$$|(LZ)_{k+1}(z;\zeta)| = \left| \int_\tau^t \int_{\mathbb{R}^N} L^{(z)} Z(z;w)(LZ)_k(w;\zeta)dw \right| \leq$$

(by Lemma 3.5, the inductive hypothesis and denoting $(y, s) = w$)

$$\leq \eta_\varepsilon^{k+1} \frac{\Gamma_E\left(\frac{1}{2}\right)^k}{\Gamma_E\left(\frac{k}{2}\right)} \int_\tau^t \frac{1}{\sqrt{t-s}(s-\tau)^{1-\frac{k}{2}}} \int_{\mathbb{R}^N} \Gamma^{M+\varepsilon}(x,t;y,s)\Gamma^{M+\varepsilon}(y,s;\xi,\tau)dyds =$$

(by the reproduction property[3] for $\Gamma^{M+\varepsilon}$ and by the change of variable $s = (1-r)\tau + rt$)

$$= \frac{\eta_\varepsilon^{k+1}}{(t-\tau)^{1-\frac{k+1}{2}}} \frac{\Gamma_E\left(\frac{1}{2}\right)^k}{\Gamma_E\left(\frac{k}{2}\right)} \int_0^1 \frac{1}{r^{1-\frac{k}{2}}\sqrt{1-r}} dr\, \Gamma^{M+\varepsilon}(z;\zeta),$$

and the thesis follows by the known properties[4] of the Euler's Gamma function. □

Proof. (of Theorem 3.6)
Estimate (3.18) directly implies the convergence of the series (3.14) uniformly in $S_{\tau,\tau+T}$ for every fixed $\zeta \in \mathbb{R}^{N+1}$ and $T > 0$. This also implies that Φ solves the integral equation

$$\Phi(z;\zeta) = L^{(z)}Z(z;\zeta) + \int_\tau^t \int_{\mathbb{R}^N} L^{(z)}Z(z;w)\Phi(w;\zeta)dw.$$

The hard part of the proof consists in showing that

$$G(z;\zeta) := Z(z;\zeta) + \int_\tau^t \int_{\mathbb{R}^N} Z(z;w)\Phi(w;\zeta)dw, \qquad t > \tau,$$

is a fundamental solution of L: this is based on the study of some singular integral and can be performed following the classical theory (see, for instance, [10] or the more recent exposition in [6]).

Next we prove (3.15):

$$\left|\Gamma(z;\zeta) - Z(z;\zeta) - \sum_{k=1}^{n-1} \int_\tau^t \int_{\mathbb{R}^N} Z(z;w)(LZ)_k(w;\zeta)dw\right| \leq \sum_{k=n}^{\infty} \int_\tau^t \int_{\mathbb{R}^N} Z(z;w)\left|(LZ)_k(w;\zeta)\right| dw$$

(by Lemma 3.1, estimate (3.18) and the reproduction property)

$$\leq \left(\frac{M+\varepsilon}{m}\right)^{\frac{N}{2}} \Gamma^{M+\varepsilon}(z;\zeta) \sum_{k=n}^{\infty} \int_\tau^t \frac{\Gamma_E\left(\frac{1}{2}\right)^k}{\Gamma_E\left(\frac{k}{2}\right)} \frac{\eta_\varepsilon^k}{(s-\tau)^{1-\frac{k}{2}}} ds$$

[3]For every $x, \xi \in \mathbb{R}^N$ and $\tau < s < t$, it holds

$$\int_{\mathbb{R}^N} \Gamma^{M+\varepsilon}(z;y,s)\Gamma^{M+\varepsilon}(y,s;\zeta)dy = \Gamma^{M+\varepsilon}(z;\zeta).$$

[4]It holds

$$\int_0^1 \frac{1}{r^{1-\frac{k}{2}}\sqrt{1-r}} dr = \frac{\Gamma_E\left(\frac{1}{2}\right)\Gamma_E\left(\frac{k}{2}\right)}{\Gamma_E\left(\frac{k+1}{2}\right)}.$$

(using the properties of the Gamma function[5])

$$= \left(\frac{M+\varepsilon}{m}\right)^{\frac{N}{2}} \Gamma^{M+\varepsilon}(z;\zeta)\sqrt{\frac{2}{\pi}}\sum_{k=n}^{\infty}\frac{\left(\eta_\varepsilon\sqrt{2\pi(t-\tau)}\right)^k}{k!!}. \tag{3.19}$$

Then estimate (3.15) follows from some elementary computation. Indeed, if n is even then $\left[\frac{n+1}{2}\right] = \frac{n}{2}$ and we have

$$\sum_{k=n}^{\infty}\frac{\eta^k}{k!!} = \sum_{k=\frac{n}{2}}^{\infty}\frac{\eta^{2k}}{(2k)!!} + \sum_{k=\frac{n}{2}+1}^{\infty}\frac{\eta^{2k-1}}{(2k-1)!!} \le \sum_{k=\frac{n}{2}}^{\infty}\frac{\eta^{2k}}{(2k)!!} + \sum_{k=\frac{n}{2}+1}^{\infty}\frac{\eta^{2k-1}}{(2k-2)!!} =$$

(since $(2k)!! = 2^k k!$)

$$= \sum_{k=\frac{n}{2}}^{\infty}\frac{1}{k!}\left(\frac{\eta^2}{2}\right)^k + \sum_{k=\frac{n}{2}}^{\infty}\frac{\eta^{2k+1}}{2^k k!} = f_n(\eta),$$

with f_n as in (3.16) and using the fact that

$$\sum_{k=n}^{\infty}\frac{\eta^k}{k!} = \frac{e^\eta \eta^n}{n!}.$$

The case of n odd can be treated analogously and is omitted. □

As a byproduct of the parametrix method, we obtain the following upper Gaussian estimate of the fundamental solution.

Theorem 3.10. *For every $\varepsilon > 0$, we have*

$$\Gamma(z;\zeta) \le \left(\frac{M+\varepsilon}{m}\right)^{\frac{N}{2}} \left(1+\eta_\varepsilon\sqrt{2\pi(t-\tau)}\right) e^{\pi(t-\tau)\eta_\varepsilon^2}\,\Gamma^{M+\varepsilon}(z;\zeta), \qquad z,\zeta\in\mathbb{R}^{N+1},\ t>\tau,$$

with η_ε as in (3.12)*.*

Proof. By Theorem 3.6 we have

$$\Gamma(z;\zeta) = Z(z;\zeta) + \sum_{k=1}^{\infty}\int_\tau^t\int_{\mathbb{R}^N} Z(z;w)(LZ)_k(w;\zeta)dw;$$

therefore, as in (3.19), we get

$$\Gamma(z;\zeta) \le \left(\frac{M+\varepsilon}{m}\right)^{\frac{N}{2}} \Gamma^{M+\varepsilon}(z;\zeta)\sum_{k=0}^{\infty}\frac{\left(\eta_\varepsilon\sqrt{2\pi(t-\tau)}\right)^k}{k!!},$$

[5] Recall that

$$\frac{\Gamma_E\left(\frac{1}{2}\right)^k}{\Gamma_E\left(\frac{k}{2}\right)} = \frac{(2\pi)^{\frac{k-1}{2}}}{(k-2)!!}$$

and the thesis follows since

$$\sum_{k=0}^{\infty} \frac{\eta^k}{k!!} \le (1+\eta) e^{\frac{\eta^2}{2}},$$

for $\eta > 0$. □

3.2 Backward parametrix expansion

We begin by stating the dual version of Lemma 3.4.

Lemma 3.11. *For every $\varepsilon > 0$ and $i, j = 1, \dots, N$ it holds*

$$|\partial_{\xi_i} \Gamma_w(z;\zeta)| \le \frac{1}{2\sqrt{\varepsilon(t-\tau)}} \left(\frac{M+\varepsilon}{m}\right)^{\frac{N}{2}+1} \Gamma^{M+\varepsilon}(z;\zeta),$$

$$\left|\partial_{\xi_i \xi_j} \Gamma_w(z;\zeta)\right| \le \frac{1}{\varepsilon(t-\tau)} \left(\frac{M+\varepsilon}{m}\right)^{\frac{N}{2}+2} \Gamma^{M+\varepsilon}(z;\zeta),$$

for any $z, \zeta, w \in \mathbb{R}^{N+1}$ with $t > \tau$.

The proof is analogous to that of Lemma 3.4. In the sequel we assume the following additional hypothesis which allows to introduce the adjoint operator of L:

[H3] the derivatives $\partial_{x_i} a_{ij}, \partial_{x_i x_j} a_{ij}, \partial_{x_i} b_i$ are bounded functions.

We define as usual the adjoint operator $\widetilde{L}$ of L:

$$\widetilde{L}u = \sum_{i,j=1}^{N} a_{ij} \partial_{x_i x_j} u + \sum_{i=1}^{N} \widetilde{b}_i \partial_{x_i} u + \widetilde{c} u + \partial_t u \tag{3.20}$$

where

$$\widetilde{b}_i = -b_i + 2\sum_{j=1}^{N} \partial_{x_i} a_{ij}, \qquad \widetilde{c} = c + \sum_{i,j=1}^{N} \partial_{x_i x_j} a_{ij} - \sum_{i=1}^{N} \partial_{x_i} b_i. \tag{3.21}$$

Then we have

$$\int_{\mathbb{R}^{N+1}} \varphi L \psi = \int_{\mathbb{R}^{N+1}} \psi \widetilde{L} \varphi, \qquad \forall \varphi, \psi \in C_0^{\infty}(\mathbb{R}^{N+1}),$$

and the following classical result holds (cf. for instance [10] Cap. 1 Theor. 15):

Theorem 3.12. *There exists a fundamental solution $\widetilde{\Gamma}$ of $\widetilde{L}$ and it holds*

$$\Gamma(z;\zeta) = \widetilde{\Gamma}(\zeta; z), \qquad z, \zeta \in \mathbb{R}^{N+1}, \ z \ne \zeta. \tag{3.22}$$

For $z \ne \zeta$, we define the *backward parametrix*

$$P(z;\zeta) = \Gamma_z(z;\zeta). \tag{3.23}$$

By Theorem 3.12, the backward parametrix satisfies

$$P(z;\zeta) = \Gamma_z(z;\zeta) = \widetilde{\Gamma}_z(\zeta;z), \tag{3.24}$$

and, analogously to (3.8), we have

$$\widetilde{L}_z^{(\zeta)} P(z;\zeta) = 0, \qquad \text{for } z \neq \zeta.$$

Next we recall Notation 3.3 and state the dual version of Lemma 3.5.

Lemma 3.13. *Under hypothesis* **[H3]***, for every positive* ε*, we have*

$$\left|\widetilde{L}^{(\zeta)} P(z;\zeta)\right| \leq \frac{\widetilde{\eta}_\varepsilon}{\sqrt{t-\tau}}\, \Gamma^{M+\varepsilon}(z;\zeta), \qquad \forall z,\zeta \in \mathbb{R}^{N+1},\ t > \tau, \tag{3.25}$$

where

$$\widetilde{\eta}_\varepsilon := \alpha N^2 \left(\frac{2}{\varepsilon}\right)^{\frac{3}{2}} \left(\frac{M+\varepsilon}{m}\right)^{\frac{N}{2}+2} \left(M+\varepsilon+\sqrt{\frac{\varepsilon}{2}}\right) + \widetilde{\beta}\frac{N}{2\sqrt{\varepsilon}} \left(\frac{M+\varepsilon}{m}\right)^{\frac{N}{2}+1} + \widetilde{\gamma}\left(\frac{M+\varepsilon}{m}\right)^{\frac{N}{2}} \sqrt{t-\tau} \tag{3.26}$$

where

$$\widetilde{\beta} := \sup_{\substack{i=1,\dots,N \\ z\in\mathbb{R}^{N+1}}} |\widetilde{b}_i(z)|, \qquad \widetilde{\gamma} := \sup_{z\in\mathbb{R}^{N+1}} |\widetilde{c}(z)|.$$

The proof, analogous to that of Lemma 3.5, is based on Lemma 3.11 and is omitted.

Theorem 3.14. *Assume hypotheses* **[H1]**, **[H2]** *and* **[H3]***. Then for every* $\zeta \in \mathbb{R}^{N+1}$*, the following expansion of the fundamental solution* Γ *holds*

$$\Gamma(z;\zeta) = P(z;\zeta) + \int_\tau^t \int_{\mathbb{R}^N} P(z;w)\Psi(w;\zeta)dw, \qquad t > \tau, \tag{3.27}$$

where

$$\Psi(z;\zeta) = \sum_{k=1}^{+\infty} (LP)_k(z;\zeta), \tag{3.28}$$

with

$$\begin{aligned} (LP)_1(z;\zeta) &= L^{(z)} P(z,\zeta), \\ (LP)_{k+1}(z;\zeta) &= \int_\tau^t \int_{\mathbb{R}^N} L^{(z)} Z(z;w)(LP)_k(w;\zeta)dw, \end{aligned}$$

and, for every $T > 0$*, the series in* (3.14) *converges uniformly in the strip* $\mathbb{R}^N \times]\tau, \tau+T[$*. Moreover, for every positive* ε*, we have the following estimate for the approximation truncated at the* n*-th term:*

$$\left|\Gamma(z;\zeta) - P(z;\zeta) - \sum_{k=1}^{n-1} \int_\tau^t \int_{\mathbb{R}^N} P(z;w)(LP)_k(w;\zeta)dw\right| \leq$$
$$\sqrt{\frac{2}{\pi}} \left(\frac{M+\varepsilon}{m}\right)^{\frac{N}{2}} f_n\left(\widetilde{\eta}_\varepsilon \sqrt{2\pi(t-\tau)}\right) \Gamma^{M+\varepsilon}(z;\zeta)$$

for $t > \tau$*, where* $\widetilde{\eta}_\varepsilon$ *is defined in* (3.26) *and* f_n *in* (3.16)*. As a consequence, the solution to the Cauchy problem* (3.17) *has an expansion of the form* (2.14)-(2.15).

Proof. Proceeding as in the "forward case", one can prove that

$$\widetilde{\Gamma}(\zeta; z) = \widetilde{\Gamma}_z(\zeta; z) + \int_\tau^t \int_{\mathbb{R}^N} \widetilde{\Gamma}_w(\zeta; w)\widetilde{\Phi}(w; z)dw, \qquad t > \tau, \tag{3.29}$$

where

$$\widetilde{\Phi}(\zeta; z) = \sum_{k=1}^{+\infty} I_k(\zeta; z), \tag{3.30}$$

with

$$I_1(\zeta; z) = \widetilde{L}^{(\zeta)}\widetilde{\Gamma}_z(\zeta; z),$$
$$I_{k+1}(\zeta; z) = \int_\tau^t \int_{\mathbb{R}^N} \widetilde{L}^{(\zeta)}\widetilde{\Gamma}_w(\zeta; w)I_k(w; z)dw,$$

and the series converges uniformly on the strips. Moreover an error estimate analogous to (3.15) holds. In order to conclude the proof, it suffices to invoke Theorem 3.12 and prove that the terms of the expansions (3.27)-(3.28) and (3.29)-(3.30) coincide, that is

$$\int_\tau^t \int_{\mathbb{R}^N} P(z; w)(LP)_k(w; \zeta)dw = \int_\tau^t \int_{\mathbb{R}^N} \widetilde{\Gamma}_w(\zeta; w)I_k(w; z)dw \tag{3.31}$$

for every $k \in \mathbb{N}$.

For $k = 1$, recalling (3.24), we have

$$\int_\tau^t \int_{\mathbb{R}^N} \widetilde{\Gamma}_w(\zeta; w)I_1(w; z)dw = \int_\tau^t \int_{\mathbb{R}^N} P(w; \zeta)\widetilde{L}^{(w)}P(z; w)dw,$$

so that the thesis follows immediately integrating by parts since we have no contribution at borders. Indeed, denoting $w = (y, s)$, *formally* we have

$$\int_\tau^t \int_{\mathbb{R}^N} \Gamma_w(w; \zeta)\partial_s\Gamma_z(z; w)dw = \bar{I} - \int_\tau^t \int_{\mathbb{R}^N} \partial_s\Gamma_w(w; \zeta)\Gamma_z(z; w)dw,$$

where

$$\bar{I} = \int_{\mathbb{R}^N} \Gamma_{(y,t)}(y, t; \xi, \tau)\Gamma_{(x,t)}(x, t; y, t)dy - \int_{\mathbb{R}^N} \Gamma_{(y,\tau)}(y, \tau; \xi, \tau)\Gamma_{(x,t)}(x, t; y, \tau)dy = 0$$

since $\Gamma_{(x,t)}(x, t; y, t) = \delta_x(y)$ and $\Gamma_{(y,\tau)}(y, \tau; \xi, \tau) = \delta_\xi(y)$. On the other hand the above argument can be made rigorous by performing the integration by parts on a thinner strip $S_{\tau+\delta,t-\delta}$ and then applying the dominated convergence theorem as $\delta \to 0^+$ combined with the summability estimate (3.25).

For $k = 2$, we have

$$\begin{aligned}
&\int_\tau^t \int_{\mathbb{R}^N} \Gamma_{z_0}(z_0; \zeta) \int_{t_0}^t \int_{\mathbb{R}^N} \widetilde{L}^{(z_0)}\Gamma_{z_1}(z_1; z_0)\widetilde{L}^{(z_1)}\Gamma_z(z; z_1)dz_1dz_0 \\
&= \int_\tau^t \int_{\mathbb{R}^N} \Gamma_{z_0}(z_0; \zeta)\Big(\widetilde{L}^{(z_0)} \int_{t_0}^t \int_{\mathbb{R}^N} \Gamma_{z_1}(z_1; z_0)\widetilde{L}^{(z_1)}\Gamma_z(z; z_1)dz_1 \\
&+ \int_{\mathbb{R}^N} \Gamma_{(y,t_0)}(y, t_0; z_0)\widetilde{L}^{(y,t_0)}\Gamma_z(z; y, t_0)dy\Big)dz_0 \equiv J_1 + J_2,
\end{aligned}$$

where, using again that $\Gamma_{(y,t_0)}(y,t_0;z_0)=\delta_{x_0}(y)$, we get

$$J_2 = \int_\tau^t \int_{\mathbb{R}^N} \Gamma_{z_0}(z_0;\zeta)\widetilde{L}^{(z_0)}\Gamma_z(z;z_0)dz_0 =$$

(proceeding as in the case $k=1$)

$$= \int_\tau^t \int_{\mathbb{R}^N} L^{(z_0)}\Gamma_{z_0}(z_0;\zeta)\Gamma_z(z;z_0)dz_0;$$

on the other hand

$$J_1 = \int_\tau^t \int_{\mathbb{R}^N} \Gamma_{z_0}(z_0;\zeta)\widetilde{L}^{(z_0)} \int_{t_0}^t \int_{\mathbb{R}^N} \Gamma_{z_1}(z_1;z_0)\widetilde{L}^{(z_1)}\Gamma_z(z;z_1)dz_1dz_0 =$$

(by parts as before)

$$\begin{aligned} &= \int_\tau^t \int_{\mathbb{R}^N} L^{(z_0)}\Gamma_{z_0}(z_0;\zeta) \int_{t_0}^t \int_{\mathbb{R}^N} L^{(z_1)}\Gamma_{z_1}(z_1;z_0)\Gamma_z(z;z_1)dz_1dz_0 \\ &- \int_{\mathbb{R}^N} \Gamma_{(y,\tau)}(y,\tau;\xi,\tau) \int_\tau^t \int_{\mathbb{R}^N} L^{(z_1)}\Gamma_{z_1}(z_1;y,\tau)\Gamma_z(z;z_1)dz_1dy = \end{aligned}$$

(since $\Gamma_{(y,\tau)}(y,\tau;\xi,\tau)=\delta_\xi(y)$)

$$= \int_\tau^t \int_{\mathbb{R}^N} L^{(z_0)}\Gamma_{z_0}(z_0;\zeta) \int_{t_0}^t \int_{\mathbb{R}^N} L^{(z_1)}\Gamma_{z_1}(z_1;z_0)\Gamma_z(z;z_1)dz_1dz_0 - \int_\tau^t \int_{\mathbb{R}^N} L^{(z_1)}\Gamma_{z_1}(z_1;\zeta)\Gamma_z(z;z_1)dz_1.$$

Combining the expressions of J_1 and J_2, eventually we obtain

$$\begin{aligned} &\int_\tau^t \int_{\mathbb{R}^N} \Gamma_{z_0}(z_0;\zeta) \int_{t_0}^t \int_{\mathbb{R}^N} \widetilde{L}^{(z_0)}\Gamma_{z_1}(z_1;z_0)\widetilde{L}^{(z_1)}\Gamma_z(z;z_1)dz_1dz_0 \\ &= \int_\tau^t \int_{\mathbb{R}^N} L^{(z_0)}P(z_0;\zeta) \int_{t_0}^t \int_{\mathbb{R}^N} L^{(z_1)}P(z_1;z_0)P(z;z_1)dz_1dz_0, \end{aligned}$$

which concludes the proof. As before the previous argument should be made rigorous by some approximating procedure. The general case can be straightforwardly achieved by induction. □

4 Numerical tests

In this section we validate the parametrix expansion by testing its computational performance on some well-known non-constant volatility model. The following examples are some of the simplest non-trivial cases commonly used in practice. While more complicated models could have been considered, here we only aim to present some preliminary test and refer to a forthcoming paper for a more detailed and extensive analysis of the numerical efficiency of the parametrix method for computing option prices and the related greeks. Moreover we emphasize that in the particular examples examined below, we merely consider parametrix approximations of order zero or one, which involve only up to two terms in the expansion; it is clear that better results can be obtained by using higher order approximations.

The expression of the first term of the expansion, as given in (2.16), is

$$u_0(z) = \int_{\mathbb{R}} H(\xi)P(z;\xi,0)d\xi$$

where H is the payoff function and $P(z;\xi,0) = \Gamma_z(z;\xi,0)$ is the backward parametrix whose expression is explicitly given in (3.5): specifically it is a Gaussian function corresponding to the fundamental solution of a constant coefficients parabolic equation. The expression of the second term u_1 of the expansion is given in (2.15) with $n = 1$.

As a first example, we consider the CEV model that assumes the following dynamic for the underlying asset:

$$\frac{dS_t}{S_t} = \mu dt + \sigma S_t^{-\alpha} dW_t$$

with $\alpha \in [0, 1[$. The key feature of the model is the inverse relationship between volatility level and stock price; for $\alpha = 0$ we recover the standard Black&Scholes dynamic. We recall that closed form formulas for option prices in the CEV model are available (cf., for instance, Epps [8]) so that we can obtain exact error estimates for the parametrix approximation. Figure 1 reports the differences $u_k - u$ between the approximated prices u_k for $k = 0, 1$ and the exact CEV price corresponding to $\sigma = 30\%$, $r = 0.05$ and $\alpha = 0.25$ of a Call option with strike $K = 1$ and maturity $T = 0.5$. In this case the parametrix expansion gives very good results with an absolute error of the order of 10^{-3}: since the at-the-money price $u_1(1) = 0.0964$, this corresponds to a relative error of the order of 0.1%.

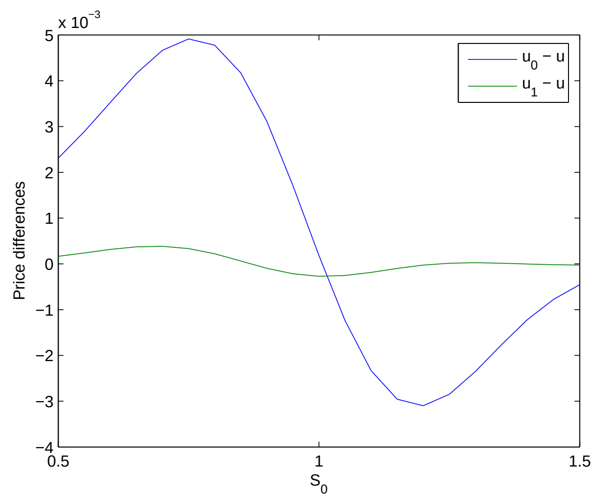

Figure 1: Price differences of Call options with strike $K = 1$ in the CEV model for different values of the underlying asset, ranging from 0.5 to 1.5, and for $\alpha = 0.25$, $r = 0.05$, $\sigma = 30\%$, $K = 1$, $T = 0.5$.

For a more refined testing in Figure 2 we also report the the implied volatilities in the CEV model and in the first order backward parametrix approximation.

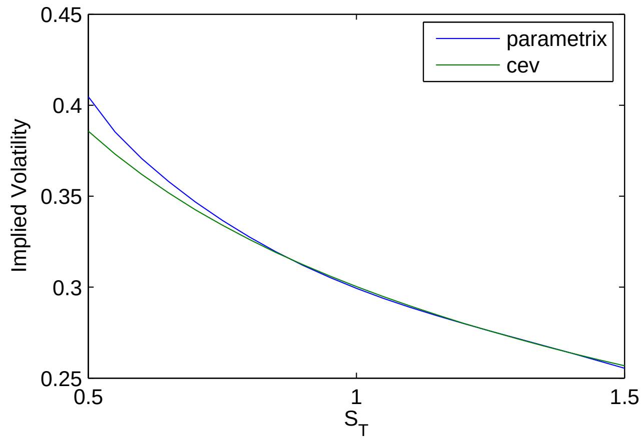

Figure 2: Implied volatilities in the CEV model and in the first order backward parametrix approximation.

As a second experiment we consider a local volatility model with the following specification of the volatility function

$$\sigma(x,t) = 0.2\left(1 + \frac{x^2}{1+t}\right). \tag{4.1}$$

As in the first example we have a two dimensional problem, however in this case there is no available closed formula for option prices. Therefore we compare the parametrix performance with a Monte Carlo approximation with a very large number of simulations and standard variance reduction techniques. Figure 3 shows the surface of Call prices, with strike $K = 1$, as a function of time to maturity and stock price in the first order backward parametrix and Monte Carlo approximations. Figure 4 plots the corresponding implied volatilities: this figure enlightens the fact that, according to formulas (3.15)-(3.16), the parametrix method gives best results for short times to maturity. The more relevant errors correspond to deep in/out of the money values for which the Monte Carlo approximation seems to be not completely reliable.

As a final test, we examine the slightly more demanding problem of the valuation of a Call option in a path-dependent volatility model. Specifically we consider an extension of the local volatility model in which the volatility is defined as a function of the whole trajectory of the underlying asset (and not only of the spot price). Path dependent volatility was first introduced by Hobson&Rogers [11] and recently generalized by Foschi and one of the authors [9]. A key feature is that it generally leads to a complete market model; moreover there are evidences about the effectiveness of the model and superior in the hedging performance with respect to standard stochastic volatility.

In order to briefly introduce the path-dependent volatility model, we consider an average weight ψ that is a non-negative, piecewise continuous and integrable function on $]-\infty, T]$. We assume that ψ is strictly positive in $[0,T]$ and we set

$$\Psi(t) = \int_{-\infty}^{t} \psi(s)ds.$$

Then we define the average process as

$$Y_t = \frac{1}{\Psi(t)} \int_{-\infty}^{t} \psi(s) Z_s ds, \qquad t \in]0, T],$$

where $Z_t = \log(e^{-rt} S_t)$ denotes the log-discounted price process: the Hobson&Rogers model corresponds to the specification $\psi(t) = e^{\lambda t}$ for some positive parameter λ. Then by Itô formula we have

$$dY_t = \frac{\varphi(t)}{\Phi(t)} (Z_t - Y_t)\, dt,$$

and assuming the following dynamic for the log-price

$$dZ_t = \mu(Z_t - Y_t)dt + \sigma(Z_t - Y_t)dW_t,$$

we obtain the pricing PDE

$$\frac{\sigma^2(z-y)}{2} (\partial_{zz} f - \partial_z f) + \frac{\varphi(t)}{\Phi(t)} (z-y)\partial_y f + \partial_t f = 0, \qquad (t, z, y) \in [0, T[\times \mathbb{R}^2. \tag{4.2}$$

The idea is that, in case of large movements of the underlying asset far from its "normal trend", a path-dependent volatility model is designed to automatically increase the level of volatility in order to undertake market dynamics in a more natural way.

Note that (4.2) is not a uniformly parabolic PDE, since the quadratic form associated with the second order part is represented by the singular matrix

$$\begin{pmatrix} \frac{\sigma^2}{2} & 0 \\ 0 & 0 \end{pmatrix},$$

so that our results (in particular, Theorem 3.14) do not directly apply. However we recall that the parametrix method has been adapted in [6] to a class of PDEs including (4.2). Therefore the results of the present paper are likely to be potentially extended to the more general setting of (4.2). The empirical findings confirm this belief as it is apparent by Figures 5 and 6: we consider options with strike $K = 1$, in the Hobson&Rogers model with $\lambda = 1$ and the following specification of the volatility function

$$\sigma(z - y) = \max\{\eta\sqrt{1 + 5(z - y)^2}, 3\},$$

where $\eta = 0.25$. Figures 5 and 6 respectively show the surface of Call prices as a function of time to maturity and stock price in the first order backward parametrix and Monte Carlo approximations for different values of y, namely $y = 0.1$ and $y = -0.2$. Since prices of at-the-money options are of the order of 0.1, also in this case we find that the relative errors are of the order of 0.1%.

Acknowledgement. We thank Paolo Foschi and Valeria Volpe for having helped us in performing the numerical tests of this section.

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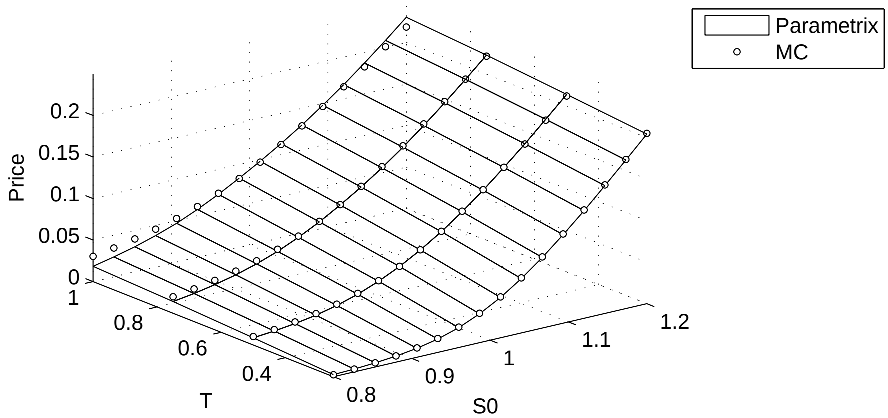

Figure 3: Call prices in the local volatility model with σ specified in (4.1) and $K = 1$: Monte Carlo and first order backward parametrix approximations.

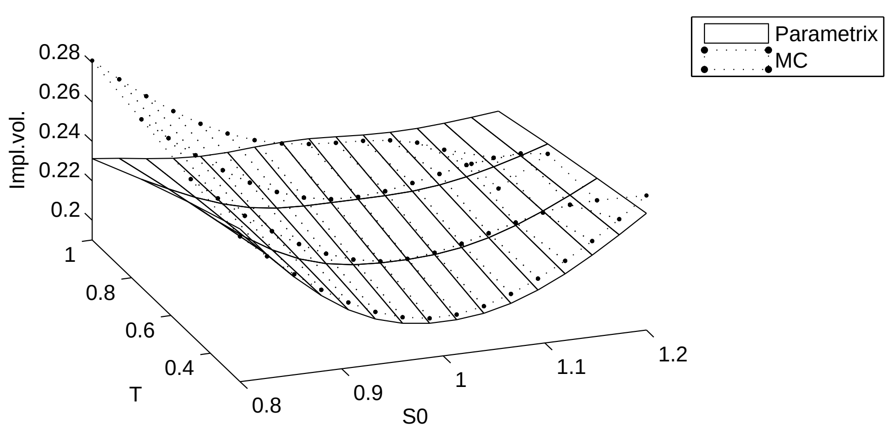

Figure 4: Implied volatilities in the local volatility model with σ specified in (4.1): Monte Carlo and first order backward parametrix approximations.

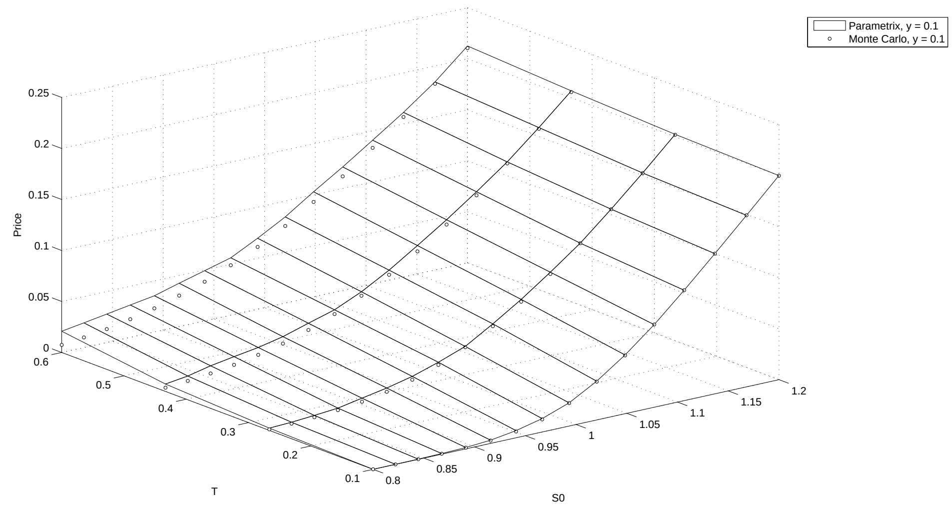

Figure 5: Price differences of Call options, with strike $K = 1$, in the path-dependent volatility model: Monte Carlo and first order backward parametrix approximations for $y = 0.1$.

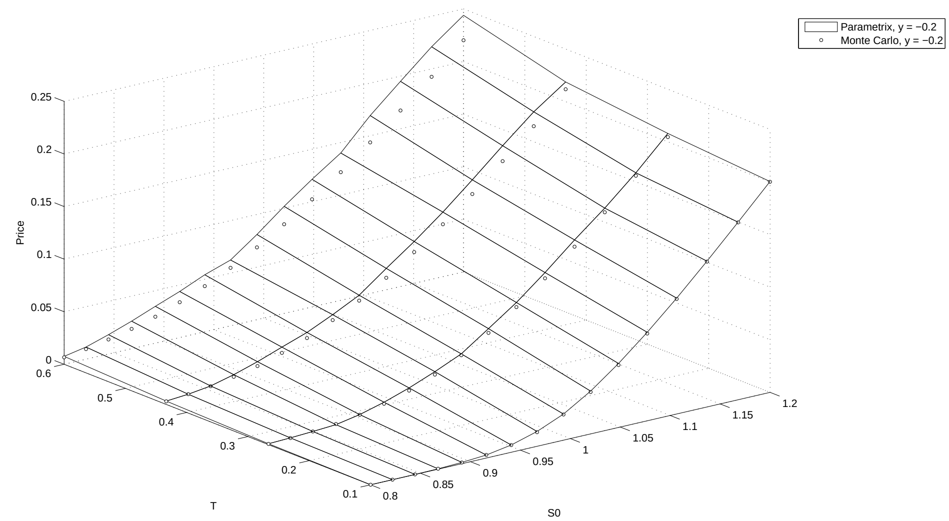

Figure 6: Price differences of Call options, with strike $K = 1$, in the path-dependent volatility model: Monte Carlo and first order backward parametrix approximations for $y = -0.2$.