CURVATURE ESTIMATES FOR MINIMAL SURFACES WITH TOTAL BOUNDARY CURVATURE LESS THAN 4π.

GIUSEPPE TINAGLIA

Abstract. We establish a curvature estimate for classical minimal surfaces with total boundary curvature less than 4π. The main application is a bound on the genus of these surfaces depending solely on the geometry of the boundary curve. We also prove that the set of simple closed curves which do not bound an embedded minimal surface of a given genus g is open in the $C^{2,\alpha}$ topology.

Introduction

In this paper we discuss the geometry and topology of compact minimal surfaces whose boundary Γ has total curvature $T(\Gamma)$ less than 4π. Recall that $T(\Gamma) = \int_\Gamma |k|$, where k is the curvature of Γ. If Γ is a connected simple closed curve, we denote by $M(\Gamma)$ the family of compact minimal surfaces whose boundary is Γ. It is well known that if Γ is a connected piecewise C^1 simple closed curve, then $M(\Gamma)$ contains at least an immersed minimal disk [2, 11, 13]. We stress that Ekholm, White, and Wienholtz proved in [3] that a classical minimal surface with boundary Γ such that $T(\Gamma) \leq 4\pi$ must in fact be embedded, regardless of the topological type.

The main theorem of this paper is the following curvature estimate.

Theorem 0.1. *Let $\Gamma \subset \mathbb{R}^3$ be a $C^{2,\alpha}$ connected simple closed curve such that $T(\Gamma) < 4\pi$. Then there exists a constant $C = C(\Gamma)$ such that*

$$\sup_{\Sigma \in M(\Gamma)} |K_\Sigma| \leq C,$$

where K_Σ is the Gaussian curvature of Σ.

The constant C depends on the size of the largest embedded tubular neighborhood around Γ as well as on upperbounds for the length of Γ and $\|\Gamma\|_{C^{2,\alpha}}$.

A finer quantative version of Theorem 0.1 is to be found in Section 2.

The proof of Theorem 0.1 is based on a compactness argument. We now give a brief sketch of the proof. Suppose that there exists a sequence of embedded minimal surfaces and points on these surfaces which are away from the boundary and where the curvature blows up. This being the case, we use a rescaling argument to obtain a new sequence that converges to a complete nonplanar embedded minimal surface. A key point in the argument is a delicate application of the density estimate [3] which is used to prove that this surface must in fact be a plane, thus yelding a contradiction. A similar, yet more refined argument, is needed when the sequence of points converges to a boundary point.

Using the Gauss-Bonnet formula together with an area estimate for minimal surfaces, our curvature estimate establishes a bound on the genus of each Σ in $M(\Gamma)$

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that depends only on the geometry of Γ. Consequently, the topology of the elements in $M(\Gamma)$ cannot be arbitrary. One of the outstanding conjectures in the subject—formulated by Ekholm, White, and Wienholtz in [3]—claims the following.

Conjecture 0.2. *Let Γ be a smooth simple closed curve in $\mathbb{R}^3$ with $T(\Gamma) \leq 4\pi$. Then in addition to a unique minimal disk, Γ bounds either:*

(1) *no other minimal surfaces, or*
(2) *exactly one minimal Möbius strip and no other minimal surfaces, or*
(3) *exactly two minimal Möbius strips and no other minimal surfaces.*

This paper advances our understanding of this conjecture. In addition to Theorem 0.1, we prove that the bound on the curvature varies continuously with Γ, relative to the $C^{2,\alpha}$ topology. This is used to establish that *the set of simple closed curves whose total curvature is less than 4π and which do not bound an embedded minimal surface of a given genus g is open in the $C^{2,\alpha}$ topology,* (Theorem 3.9). There are many special curves Γ with total curvature less than 4π for which it has been proven that $M(\Gamma)$ consists only of disks (e.g. if Γ lies on the boundary of a convex set [7]). Theorem 3.9 reveals that if Γ' is a new curve which is obtained by a slight modification of such a Γ, then $M(\Gamma')$ must still consist only of disks.

This paper is organized as follow. In the first section we prove Theorem 0.1. Next we examine the constant in Theorem 0.1 to prove that it depends only on a few geometric quantities associated with the curve Γ. Finally, we discuss some interesting applications of our curvature estimate. For completeness, in the appendix we state the density estimates which were proven in [3] and are used in this paper.

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1. Proof of Theorem 0.1

Notice first that the fact that Γ is $C^{2,\alpha}$ does imply that for any $\Sigma \in M(\Gamma)$ there exists a constant $C(\Sigma)$ such that $\sup_\Sigma |K_\Sigma| \leq C(\Sigma)$ [9]. In this paper, we are showing that the bound does not depend on the surface but on the geometry of the boundary. In order to prove our theorem we are going to use the density estimates given in [3] which, for completness can be found in the appendix (see (A-4) and (A-5)).

Recall that in [3], Ekholm, White, and Wienholtz proved that any $\Sigma \in M(\Gamma)$ is embedded and recall also that, using the Gauss equation, one can show that for a minimal surface $-2K_\Sigma = |A|^2$, where $|A| = \sqrt{k_1^2 + k_2^2}$ is the norm of the second fundamental form, k_1 and k_2 are the principal curvatures.

To prove the bound on the Gaussian curvature we use a compactness argument. The proof is by contradiction. Assuming that the statement is false, we can find a sequence of minimal surfaces Σ_n such that

$$\max_{\Sigma_n} |A_n| > n.$$

Let $p_n \in \Sigma_n$ such that

$$|A_n(p_n)| = \max_{\Sigma_n} |A_n| > n.$$

There are two possible cases: Either $|A_n(p_n)|\text{dist}_{\Sigma_n}(p_n, \Gamma)^*$ is going infinity or not.

First Case, $|A_n(p_n)|dist(p_n, \Gamma)$ is going to infinity. Assuming that $|A_n(p_n)|\text{dist}(p_n, \Gamma)$ is going to infinity, consider the sequence of minimal surfaces, which we will still call Σ_n, obtained by rescaling Σ_n by a factor $|A(p_n)|$. After a translation that takes p_n to the origin, Σ_n is a sequence of embedded minimal surfaces with $|A_n|$ uniformly bounded, $|A_n(p_n)| = 1$. Up to a subsequence, it converges to a complete embedded oriented minimal surfaces $\overline{\Sigma}$ with bounded curvature and curvature one at the origin. Notice that thanks to the results in [3] there exists a $\delta > 0$ such that

$$\frac{Area(\Sigma_n \cap B_r(q))}{\pi r^2} < 2 - \delta < 2 \quad \text{for any } q \in \Sigma_n \text{ and any } r > 0. \tag{1.1}$$

In particular, (1.1) gives the following.

$$\frac{Area(\overline{\Sigma} \cap B_r(q))}{\pi r^2} < 2 - \delta < 2 \quad \text{for any } q \in \overline{\Sigma} \text{ and any } r > 0. \tag{1.2}$$

Our goal is to show that $\overline{\Sigma}$ is a plane. Since $\overline{\Sigma}$ has quadratic area growth, it is possible to take the cone at infinity. In other words, there exists a sequence $t_n > 0$ approaching zero such that $t_n\overline{\Sigma}$ converges to a stationary cone C and its density at the origin must be less than $2-\delta$. It is known that the intersection of C with the unit sphere consists of a collection of geodesic arcs. We claim that it consists of a single great circle with multiplicity one. In order to prove that we rule out the possibility that there is a point which is the end point of more than two arcs. Clearly, there can not be a point where there meet more than 3 arcs, otherwise the density at that point would be 2 and that would contradict (1.2). Assume there is a point where there meet 3 arcs. That implies that we can find a large circle which is transverse to $\overline{\Sigma}$ and intersects $\overline{\Sigma}$ in exactly 3 points. However, this contradicts the fact that the intersection number must be zero mod 2 [6].

Second Case, $|A_n(p_n)|dist_{\Sigma}(p_n, \Gamma)$ does not go to infinity. After reasoning similarly to the previous case, we obtain a sequence of minimal surfaces Σ_n that converges to a complete orientable embedded minimal surface $\overline{\Sigma}$ bounded by a straight line L such that the curvature is bounded and one at the origin. Moreover, there exists a $\delta > 0$ such that the following is true.

$$\frac{Area(\overline{\Sigma} \cap B_r(q))}{\pi r^2} < 2 - \delta < 2 \quad \text{for any } q \in M\backslash L \text{ and any } r > 0 \tag{1.3}$$

and also

$$\frac{Area(\overline{\Sigma} \cap B_r(q))}{\pi r^2} < \frac{3}{2} - \delta < \frac{3}{2} \quad \text{for any } q \in L \text{ and any } r > 0. \tag{1.4}$$

Our goal is to show that $\overline{\Sigma}$ is a half-plane. Let C be the cone at infinity. The intersection of C with the unit sphere consists of a collection of geodesic arcs and it

*dist$(p_n, \Gamma) = \min_{q \in \Gamma} |p_n - q|$

must contain two antipodal points P, Q. We claim that it consists of a single half great circle which implies that $\overline{\Sigma}$ is a plane. We can always assume that $P = (0, 1, 0)$ and $Q = (0, -1, 0)$. Reasoning similarly to the previous case gives that it consists of half great circles where P, Q are the endpoints. Moreover, there can be at most two of them otherwise the density at the origin would be $\frac{3}{2}$ and that contradicts (1.4). Assume there are two half great circles. Using Schwarz reflection gives a complete embedded minimal surface with no boundary M' and let C' be its cone at infinity. We have already shown that it consists of at most two great circles. Let π be a plane through the origin which does not contain either great and let $\omega(R) = \{(0,0,t) : -R \leq t \leq R\} \cup \{(t, 0, R) : 0 \leq t \leq R\} \cup \{(R, 0, t) : -R \leq t \leq R\} \cup \{(t, 0, -R) : 0 \leq t \leq R\}$. If there are two great circle, because of the simmetry of M', for R large $\omega(R)$ intersects M' in an odd number of points. However, this contradicts the fact that the intersection number must be zero mod 2 [6]. □

Remark 1.3. *Certainly there are continuous but not $C^{2,\alpha}$ simple closed curve Γ and minimal surfaces $\Sigma \in M(\Gamma)$ such that* $\sup_\Sigma |K_\Sigma| = \infty$ [9, 10].

Remark 1.4. *The hypothesis that Γ is connected is redundant. In fact by the Fenchel-Borsuk theorem the total curvature of a connected simple closed curve Γ is always greater than or equal to 2π and equality holds if and only if it is a convex planar curve* [1, 4, 5, 8].

2. The constant C

With little modification of the argument in the proof of Theorem 0.1, one can prove the following theorem.

Theorem 2.5. *Let $\Gamma \subset \mathbb{R}^3$ be a $C^{2,\alpha}$ connected simple closed curve such that $T(\Gamma) < 4\pi$. There exists a $\rho > 0$ and $C(\Gamma, \rho)$ such that if Γ' is a simple closed curve and $\|\Gamma - \Gamma'\|_{C^{2,\alpha}} \leq \rho$, then*

$$\sup_{\Sigma \in M(\gamma)} |K_\Sigma| \leq C(\Gamma, \rho).$$

Let us denote by $E(\Gamma)$ the size of the largest tubular neighborhood around Γ. Using Theorem 2.5 we can now prove a finer quantative version of the main theorem. For simplicity we state the theorem assuming that the length of Γ is less than one. After rescaling, one can restate the theorem without that assumption.

Theorem 2.6. *Given $\varepsilon > 0, \Delta > 0$ and $\theta < 4\pi$ there exists a constant $C(\varepsilon, \Delta, \theta)$ such that the following holds.*

Let $\Gamma \subset \mathbb{R}^3$ be a $C^{2,\alpha}$ connected simple closed curve whose lenght is less than one and such that $E(\Gamma) \geq \varepsilon$, $\|\Gamma\|_{C^{2,\alpha}} \leq \Delta$, and $T(\Gamma) \leq \theta$, then

$$\sup_{\Sigma \in M(\Gamma)} |K_\Sigma| \leq C(\varepsilon, \Delta, \theta).$$

Proof. The proof is by contradiction. Suppose that there exists a sequence of Γ_n and $\Sigma_n \in M(\Gamma_n)$ for which the curvature goes to infinity. The conditions on Γ_n guarantee that there exists a connected simple closed curve Γ whose total curvature is less than or equal to θ and a subsequence Γ_{n_k} such that $\|\Gamma - \Gamma_{n_k}\|_{C^{2,\alpha}}$ is going to zero. We can apply Theorem 2.5 to reach a contradiction. □

3. Applications

Theorem 0.1 clearly implies the following compactness theorem.

Theorem 3.7. *Let $\Gamma \subset \mathbb{R}^3$ be a $C^{2,\alpha}$ connected simple closed curve such that $T(\Gamma) < 4\pi$. The space of minimal surfaces bounded by Γ is compact.*

Also, using Gauss-Bonnet Theorem plus an isoperimetric inequality [12], the curvature estimate presented in this paper clearly determines a bound on the genus of any $\Sigma \in M(\Gamma)$ that depends only on Γ.

Corollary 3.8. *Let $\Gamma \subset \mathbb{R}^3$ be a $C^{2,\alpha}$ connected simple closed curve such that $T(\Gamma) < 4\pi$. There exists a constant $N(\Gamma)$ such that the genus of any $\Sigma \in M(\Gamma)$ is less than or equal to $N(\Gamma)$.*

Proof. The Gauss-Bonnet Theorem gives that

$$\int_\Gamma \vec{k} \cdot \vec{n} ds + \int_\Sigma K_\Sigma = 2\pi\chi(\Sigma)$$

where $\vec{k}$ is the curvature vector of the curve Γ, $\vec{n}$ is the exterior normal of Σ, and $\chi(\Sigma)$ is the Euler characteristic of Σ. The first integral is bounded in absolute value by the total curvature of Γ while the second integral is bounded by the area of Σ times the bound on the curvature given by Theorem 0.1. □

Moreover, the next corollary shows that the set of simple closed curves whose total curvature is less than 4π and which do not bound an embedded minimal surface of a given genus g is open in the $C^{2,\alpha}$ topology.

Corollary 3.9. *Let Γ and $N(\Gamma)$ be as above. There exists an $\varepsilon > 0$ such that if Γ' is a connected simple closed curve such that $\|\Gamma - \Gamma'\|_{C^{2,\alpha}} < \varepsilon$, then the genus of any $\Sigma \in M(\Gamma')$ is bounded by $N(\Gamma)$.*

Proof. Assuming that the statement is false, for any $n > 0$ there exist Γ_n and $\Sigma_n \in M(\Gamma_n)$ such that $\|\Gamma - \Gamma_n\|_{C^{2,\alpha}} < \frac{1}{n}$ and the genus of Σ_n is greater than $N(\Gamma)$. However, the second fundamental forms of the Σ_n's are uniformly bounded therefore, after going to a subsequence, Σ_n converges to a surface $\Sigma \in M(\Gamma)$ with genus greater than $N(\Gamma)$. That is a contradiction. □

There are many special curves Γ with total curvature less than 4π for which it has been proven that a minimal surface whose boundary is Γ has to be a disk [7]. This result reveals that a minimal surface whose boundary is a new curve obtained by a slight modification of such a Γ must still be a disk.

Using a compactness argument like the one in the proof of Theorem 0.1 one can prove that the bound on the genus depends on the constants described in Theorem 0.1.

Theorem 3.10. *Given $\varepsilon > 0, \Delta > 0$ and $\theta < 4\pi$ there exists a constant $N(\varepsilon, \Delta, \theta)$ such that the following holds.*

Let $\Gamma \subset \mathbb{R}^3$ be a $C^{2,\alpha}$ connected simple closed curve whose lenght is less than one and such that $E(\Gamma) \geq \varepsilon$, $\|\Gamma\|_{C^{2,\alpha}} \leq \Delta$, and $T(\Gamma) \leq \theta$, then the genus of any $\Sigma \in M(\Gamma)$ is bounded by $N(\Gamma)$.

APPENDIX-A

In this appendix we provide the density estimates which are proven in [3] by Ekholm, White, and Wienholtz.

Theorem 3.11. *Let M be a minimal surface in $\mathbb{R}^N$ with rectifiable boundary Γ. Let p be any point in $\mathbb{R}^N$. Then*

$$\Theta(M,p) \leq \Theta(Cone_p\Gamma, p) \tag{A-1}$$

where $\Theta(M,p)$ is the density of M at p. Furthermore, the inequality is strict unless $M = Cone_p\Gamma$.

In particular, this implies the following corollary.

Corollary 3.12. *Let Γ be a simple closed curve with $TC(\Gamma) \leq 4\pi$. For any $p \in M(\Gamma)$ the following density estimates are true:*

$$\Theta(M,p) \leq 2 \qquad \text{if} \quad p \in M(\Gamma)\backslash\Gamma \tag{A-2}$$

$$\Theta(M,p) \leq \frac{3}{2} \qquad \text{if} \quad p \in \Gamma \tag{A-3}$$

with strict inequality unless $M = Cone_p\Gamma$.

Notice that, if M is contained in a cone, then both the mean curvature and the scalar curvature vanish, so M is contained in a plane.

Furthermore, if the total curvature is strictly less than 4π then these density estimates are sharp. In other words the following holds.

Corollary 3.13. *Given $\theta > 0$ there exists $\delta = \delta(\theta) > 0$ such that the following holds.*

Let Γ be a simple closed curve with $TC(\Gamma) \leq \theta < 4\pi$. For any $p \in M(\Gamma)$ the following density estimates are true:

$$\Theta(M,p) \leq 2 - \delta \qquad \text{if} \quad p \in M(\Gamma)\backslash\Gamma \tag{A-4}$$

$$\Theta(M,p) \leq \frac{3}{2} - \delta \qquad \text{if} \quad p \in \Gamma. \tag{A-5}$$

Giuseppe Tinaglia gtinagli@nd.edu
Mathematics Department, University of Notre Dame, Notre Dame, IN, 46556-4618

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