STRUCTURE THEOREMS FOR EMBEDDED DISKS WITH MEAN CURVATURE BOUNDED IN L^P

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Abstract. After appropriate normalizations an embedded disk whose second fundamental form has large norm contains a multi-valued graph, provided the L^p norm of the mean curvature is sufficiently small. This generalizes to non-minimal surfaces a well known result of Colding and Minicozzi.

Introduction

In [8] Colding and Minicozzi proved that a minimal disk embedded in $\mathbb{R}^3$ whose Gaussian curvature is large at a point contains a multi-valued graph around that point. This means that, locally, the disk looks like a piece of a suitably scaled helicoid (see Figure 1). This was later generalized in [20] to the constant mean curvature case. The structure theorem in [8] has been used as a key ingredient in their series of papers [3, 7, 8, 9, 10] dealing with the geometry of embedded minimal surfaces of fixed genus. Moreover, the new ideas provided by their recent work have been applied by the authors to solve several longstanding problems in the field; see for instance [2, 5, 13, 14].

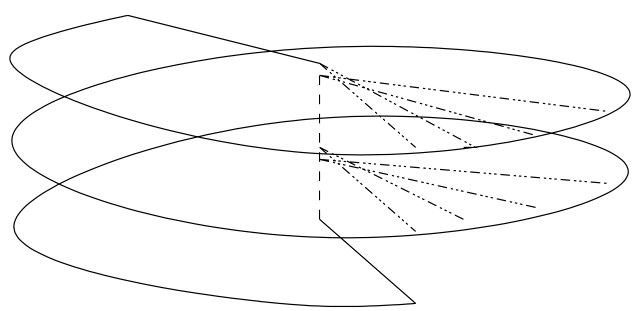

Figure 1. Half of the the helicoid

In this paper we discuss the geometry of disks embedded in $\mathbb{R}^3$ for which the L^p norm of the mean curvature, $\|H\|_{L^p}$, is suitably bounded. We point out that $\|H\|_{L^p}$ is a natural quantity to consider as it appears, for instance, in such classical results as the monotonicity formulae [19] and related applications [11, 12]. Loosely speaking, the principle established in our main result is that an embedded disk whose second fundamental form is bounded but large at the origin and whose L^q-norm of

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the gradient of the mean curvature is bounded, $q > 2$, must contain a multi-valued graph if the L^p-norm of the mean curvature is suitably small. Here is a simplified version of the main theorem.

Theorem 0.1. *Given $N \in \mathbb{Z}_+$, $T \geq 0$, $q > 2$ and $p \geq 1$ there exist $C_1 = C_1(N) > 0$, $C_2 = C_2(N, T, p, q) > 0$, and $l = l(N, p) > 1$ such that the following holds.*
If $\Sigma \subset \mathbb{R}^3$ is an embedded disk with $0 \in \Sigma \subset B_l(0)$, $\partial\Sigma \subset \partial B_l(0)$, $\|H\|_{L^p} \leq C_2$, $\|\nabla H\|_{L^q} \leq T$, and

$$\sup_{\Sigma \cap B_l(0)} |A_\Sigma| \leq 2C_1 = 2|A_\Sigma|(0),$$

then $\Sigma \cap B_1(0)$ contains an N-valued graph that forms around the origin.

Here $B_l(0)$ is the euclidean ball of radius l centered at the origin. We recall that if Σ is a surface and k_1 and k_2 are its principal curvatures, then the mean curvature is $H = \frac{k_1+k_2}{2}$. The norm of the second fundamental form is $|A_\Sigma| = \sqrt{k_1^2 + k_2^2}$ and the Gaussian curvature is $K_\Sigma = k_1 k_2$. A precise definition of an N-valued graph as well as a finer quantative version of Theorem 0.1 is to be found in Section 4.

Our new generalization of the Colding-Minicozzi structure theorem is intended as a first step towards classifying the singularities of the limit of a sequence of embedded disks with $\|H\|_{L^p}$ bounded. While this problem has been successfully studied for minimal disks [3, 6, 10, 16], it remains unsolved in this more general setting. In fact, if the norm of the second fundamental form of these disks is uniformly bounded, they do converge to a well defined surface, although not necessarily an embedded one. The main objective is therefore understanding what happens as the norm of the second fundamental becomes large. As with minimal surfaces, the answer to this question will surely provide new tools for the study of the global properties of surfaces with $\|H\|_{L^p}$ bounded.

The proof of Theorem 0.1 uses a new compactness argument which, somewhat unexpectedly, does not require a bound on the area.

1. Minimal Surfaces

1. **Definition.** Let $\Sigma \subset \mathbb{R}^3$ be a 2-dimensional smooth orientable surface (possibly with boundary) with unit normal N_Σ. Given a function ϕ in the space $C_0^\infty(\Sigma)$ of infinitely differentiable (i.e., smooth), compactly supported functions on Σ, consider the one-parameter variation

$$\Sigma_{t,\phi} = \{x + t\phi(x)N_\Sigma(x)/x \in \Sigma\}$$

and let $A(t)$ be the area functional,

$$A(t) = Area(\Sigma_{t,\phi}).$$

The so-called first variation formula of area is the equation (integration is with respect to $d$area)

$$A'(0) = \int_\Sigma \phi H, \tag{1.1}$$

where H is the mean curvature of Σ. When H is identically zero the surface Σ is a critical point for the area functional and it is called a *minimal* surface [17, 4]; concrete examples of minimal surfaces are planes, the helicoid and the catenoid.

In general, if Σ is given as graph of a function u then

$$H = \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right). \tag{1.2}$$

2. **Limits of minimal surfaces.** In this section we discuss limits of minimal surfaces. Some of this material is covered in great detail (including proofs) in [18, Section 4].

Let Σ be a surface in $\mathbb{R}^3$ and let $\mathrm{T}_p\Sigma$ denote its tangent plane at p. Given $p \in \Sigma$ and $r > 0$ we label by

$$D(p,r) = \{p + v/v \in \mathrm{T}_\Sigma, |v| < r\} \tag{1.3}$$

the tangent disk of radius r. $W(p,r)$ stands for the infinite solid cylinder of radius r around the affine normal line at p,

$$W(p,r) = \{q + tN(p)/q \in D(p,r), t \in \mathbb{R}\}. \tag{1.4}$$

Inside $W(p,r)$ and for $\varepsilon > 0$, we have the compact slice

$$W(p,r,\varepsilon) = \{q + tN(p)/q \in D(p,r), |t| < \varepsilon\}. \tag{1.5}$$

Definition 1.2. *Let Σ_n be a sequence of surfaces embedded in an open set O. We say that Σ_n converges C^k with finite multiplicity to a surface Σ_∞ on compact sets if for any $K \subset O$ compact there exist $r, \varepsilon > 0$ such that*

(1) *for any $p \in \Sigma_\infty \cap K$, $\Sigma_\infty \cap W(p,r,\varepsilon)$ can be represented as the graph of a function $u\colon D(p,r) \to \mathbb{R}$;*
(2) *for all n large enough, $\Sigma_n \cap K \cap W(p,r,\varepsilon)$ consists of a finite number of graphs (independent of n) over $D(p,r)$ which converge to u in the usual C^k topology.*

Given a sequence of subsets $\{F_n\}_n$ in an open domain O, its *accumulation set* is defined by $\{p \in O/\exists p_n \in F_n \text{ with } p_n \longrightarrow p\}$

The next two theorems are compactness theorems for sequences of embedded minimal surfaces. The first assumes a uniform bound on the area and on the norm of the second fundamental form while the second is slightly more general and does not assume a bound on the area. Their proofs are similar in nature. We will only sketch the proof of Theorem 1.4, pointing out where to use the uniform bound on the area to prove Theorem 1.3.

Theorem 1.3. *Let Σ_n be a sequence of minimal surfaces embedded in an open set O. Suppose that Σ_n has an accumulation point and that there exist C_1 and C_2 such that*

$$Area(\Sigma_n) < C_1 \text{ and } \sup_{\Sigma_n} |A_n| < C_2$$

uniformly. Then, there exists a subsequence Σ_{n_k} and a minimal surface Σ embedded in O such that Σ_{n_k} converges smoothly with finite multiplicity to Σ on compact subsets of O.

Theorem 1.4. *Let Σ_n be a sequence of minimal surfaces embedded in an open set O. Suppose that there exists a sequence $p_n \in \Sigma_n$ converging to a point $p \in O$ and that*

$$\sup_{\Sigma_n} |A_n| < C$$

uniformly. Then, there exists a subsequence Σ_{n_k} and a connected minimal surface Σ in O satisfying

(1) *Σ is contained in the accumulation set of Σ_n;*
(2) *$p \in \Sigma$ and $|A|(p_n) = \lim |A_n|(p_n)$;*
(3) *Σ is embedded in O;*
(4) *Any divergent path in Σ either diverges in O or has infinite length.*

Sketch of the proof of 1, 2 and 3. As p_n accumulates at $p \in O$, the uniform bound on the second fundamental form implies that there exists $r > 0$ such that for ε small, $W(p_n, r, \varepsilon) \cap \Sigma_n$ consists of a collection of graphs, u_n^k, over $D(p_n, r)$ (a uniform bound on the area would give a bound for the number of graphs which is independent of n). After going to a subsequence we can assume that $T_{p_n}\Sigma_n$ converges to a plane π and that the graphs u_n^k are graphs over π. Moreover, $|u_n^k|$, $|\nabla u_n^k|$ and $|\nabla^2 u_n^k|$ are uniformly bounded. Since u_n is a minimal graph, thanks to the minimal graph equation we have uniform $C^{2,\alpha}$ estimates for u_n^k. In this situation, Arzela-Ascoli's Theorem implies that a subsequence of u_n^k converges C^2 to a graph u. Due to the C^2 convergence, u is also a minimal graph. An analytic prolongation argument allows us to construct a subsequence Σ_k and a maximal sheet Σ in the accumulation set of Σ_k which extends the graph u. By construction, the minimal surface Σ satisfies items 1 and 2. Σ must be embedded because transversal self-intersections of it would give rise to transversal self intersection of Σ_n for n large and tangential self-intersections would contradict the maximum principle, thus we have 3. $\square$

2. Compactness Theorem

In this section we prove more general compactness theorems which will be used in the proof of the structure theorem. We start generalizing Theorem 1.3 and 1.4 to more general surfaces. Before doing that, we need to establish some notation. Let f be a function defined over Σ and let $p \geq 1$, then $\|f\|_{L^p(\Sigma)}$ is the L^p norm of f, while $[f]_\alpha := \sup_\Sigma \frac{|f(x)-f(y)|}{dist_\Sigma(x,y)}$.

Theorem 2.5. *Let Σ_n be a sequence of surfaces embedded in an open set O. Suppose that Σ_n has an accumulation point and that there exist C_1, C_2, and T such that*

$$Area(\Sigma_n) < C_1, \sup_{\Sigma_n} |A_n| < C_2 \text{ and } [H_n]_\alpha < T \text{ uniformly.}$$

Suppose also that $\|H_n\|_{L^p(\Sigma_n)}$ is going to zero. Then, there exists a subsequence Σ_{n_k} and a minimal surface Σ embedded in O such that Σ_{n_k} converges C^2 with finite multiplicity to Σ on compact sets of O.

Theorem 2.6. *Let Σ_n be a sequence of surfaces properly embedded in an open set O. Suppose that there exists a sequence $p_n \in \Sigma_n$ converging to a point $p \in O$ and that there exist C and T such that*

$$\sup_{\Sigma_n} |A_n| < C \text{ and } [H_n]_\alpha < T.$$

Suppose also that $\|H_n\|_{L^p(\Sigma_n)}$ is going to zero. Then, there exists a subsequence Σ_{n_k} and a connected minimal surface Σ in O satisfying

(1) *Σ is contained in the accumulation set of Σ_n;*
(2) *$p \in \Sigma$ and $|A|(p_n) = \lim |A_n|(p_n)$;*
(3) *Σ is embedded in O;*
(4) *Any divergent path in Σ either diverges in O or has infinite length.*

Their proofs are a slight modification of the proofs of Theorem 1.3 and 1.4. As before, the uniform bound on the second fundamental form implies that $W(p_n, r, \varepsilon) \cap \Sigma_n$ consists of a collection of graphs. In the proofs of Theorem 1.3 and 1.4 we needed the surfaces to be minimal in order to obtain uniform $C^{2,\alpha}$ estimates for these graphs. In fact, it can be shown that in order to obtain $C^{2,\alpha}$ estimates, it suffices to know that $\|H_n\|_{L^p(\Sigma_n)}$ and $[H]_\alpha$ are bounded. Once uniform $C^{2,\alpha}$ estimates are obtained we can apply Arzela-Ascoli Theorem to extract a subsequence of graphs which converges C^2 to a graph. The fact that $\|H_n\|_{L^p(\Sigma_n)}$ is going to zero is ultimately used to show that the limit graph is minimal.

Recall that by the Sobolev embedding theorem, a bound on $\||\nabla H_n|\|_{L^q(\Sigma_n)}$, $q > 2$, gives a bound on $[H]_\alpha$. We could therefore restate Theorem 1.3 and 1.4 replacing the uniform bound on $[H]_\alpha$ with a uniform bound on $\|\nabla H_n\|_{L^q(\Sigma_n)}$, $q > 2$. Furthermore, if $[H]_\alpha$ is not bounded uniformily, although we are not able to exctract a subsequence of graphs which converges C^2 to a minimal graph, it is still possible to exctract a subsequence of graphs which converges C^1 to a minimal graph. Moreover, an upper bound on the norm of the second fundamental form of the limit is still valid. In other words, the following theorem follows.

Theorem 2.7. *Let Σ_n be a sequence of surfaces embedded in an open set O. Suppose that there exists a sequence $p_n \in \Sigma_n$ converging to a point $p \in O$ and that there exist C such that*

$$\sup_{\Sigma_n} |A_n| < C.$$

Suppose also that $\|H_n\|_{L^p(\Sigma_n)}$ is going to zero. Then, there exists a subsequence Σ_{n_k} and a connected minimal surface Σ in O satisfying

(1) *Σ is contained in the accumulation set of Σ_n;*
(2) $\sup_\Sigma |A| < C$*;*
(3) *Σ is embedded in O;*
(4) *Any divergent path in Σ either diverges in O or has infinite length.*

In the next theorem we use Theorem 2.7 to describe more accurately the accumulation set of a suquence of surface whose $\|H_n\|_{L^p}$ is going to zero.

Theorem 2.8. *Let Σ_n be a sequence of surfaces embedded in $B_n(0)$ such that $0 \in \Sigma \subset B_n(0)$, $\partial\Sigma$(if non-empty) $\subset \partial B_n(0)$. Suppose that there exists a constant C such that $\sup_{\Sigma_n} |A_n| < C$ and that $\|H_n\|_{L^p}$ is going to zero as n goes to infinity then, up to a subsequence, the accumulation set of Σ_n is non-empty and it consists either of a connected complete properly embedded minimal surface Σ or of a collection of parallel planes.*

Moreover, Σ_n^1 converges C^1 with multeplicity one to Σ^1, where Σ_n^1 is the connected component of $\Sigma_n \cap B_1(0)$ which contains the origin and Σ^1 is, depending on the accumulation set, the connected component of $\Sigma \cap B_1(0)$ which contains the origin or a unit disk centered at the origin.

Proof. Supposing that it is false, let Σ_n be a sequence of surfaces embedded in $B_n(0)$ such that $0 \in \Sigma_n \subset B_n(0)$, $\partial\Sigma_n \subset \partial B_n(0)$ and $\|H_n\|_{L^p} < \frac{1}{n}$. Theorem 2.7 implies that there exists a complete connected embedded minimal surface, Σ, which contains the origin and it is contained in the accumulation set of Σ_n. Furthermore, Σ has bounded second fundamental form and therefore it is properly embedded, see [15]. If Σ_n has another accumulation point which is not in Σ then the same argument shows that there exists another complete connected properly embedded minimal surface, Σ', which is contained in the accumulation set of Σ_n and it is disjoint from Σ. The results in [1, 21] imply that they must be parallel planes.

Let $\varepsilon > 0$ and let $TN(\varepsilon)$ be an embedded tubular neighborhood of Σ^1 of size ε. Choose r and ε, $r > 2\varepsilon > 0$ such that for any $p \in \Sigma$, $W(p, r, \varepsilon) \cap \Sigma_n^1$ consist of a collection of graphs. Let u_0 be the minimal graph over $D(0, r)$ which locally rapresents Σ^1, and let u_0^n be the graph in $W(p, r, \varepsilon) \cap \Sigma_1^n$ containing the origin. From the way Σ has been obtained, u_0^n converges C^1 to u_0. For any $q \in \partial W(0, r, \varepsilon)$ if we let u_q represents $W(q, r, \varepsilon) \cap \Sigma^1$ and u_q^n represents the connected component of $W(q, r, \varepsilon) \cap \Sigma_n^1$ which intersects u_0^n, then we can assume that u_q^n converges C^1 to u_q. Since Σ is properly embedded, Σ^1 is compact. After finitely many steps it is possible to continue u_0^n to get a one sheeted cover of Σ_1. $\square$

Loosely speaking, in the next theorems we describe the geometry away form the boundary of a surface whose L^p norm of the mean curvature is small.

Theorem 2.9. *Given $C > 0$ there exist $R = R(C) > 2$, $\varepsilon = \varepsilon(C) > 0$ such that the following holds.*

Let Σ be a surface embedded in $B_R(0)$ such that $0 \in \Sigma \subset B_R(0)$, $\partial\Sigma$(if non-empty) $\subset \partial B_R(0)$ and $\|H\|_{L^p} < \frac{1}{R}$ then Σ^1 is properly embedded and has an embedded tubular neighborhood of size ε.

Proof. Let us first show that Σ^1 is properly embedded. Assuming false, let Σ_n be a sequence of surfaces embedded in $B_n(0)$ such that $0 \in \Sigma_n \subset B_n(0)$, $\partial\Sigma \subset \partial B_n(0)$ and $\|H_n\|_{L^p} < \frac{1}{n}$. Theorem 2.8 implies that, after going to a subsequence, Σ_n^1 converges C^1 with multeplicity one to a compact properly embedded minimal surface. This implies that Σ_n^1 must be properly embedded for n large. This contradicts our assumption and proves that Σ^1 is properly embedded.

The fact that Σ^1 is properly embedded implies that it admits an embedded tubular neighborhood. However, its size might depend on Σ^1. Arguing by contradiction and using a compacteness argument as above one can prove that the size of the tubular neighborhood does not depend on Σ^1. In fact, it becoming smaller would contradict the multiplicity one convergence. □

Note that embeddedness is the only topological assumption. For instance, we are not assuming that the surface separates the ball or any restriction on the genus.

Another easy consequence of Theorem 2.9 is some bound on the area of an embedded surface with bounded L^p norm of the mean curvature and bounded second fundamental form. Although the area of such a surface is not necessarily bounded, we show that it is possible to bound the area of connected pieces which are sufficiently away from the boundary.

Corollary 2.10. *Given $C > 0$ there exist $K = K(C) > 0$ and $R = R(C) > 2$ such that the following holds.*
Let $0 \in \Sigma$ be an embedded surface such that $\Sigma \subset B_R(0)$, $\partial\Sigma \subset \partial B_R(0)$, $\|H\|_{L^p} < \frac{1}{R}$ and $\sup_\Sigma |A| < C$ then the area of Σ^1 is bounded by K.

In the following compactness theorem we prove that if the elements of the sequence in Theorem 2.9 are embedded disks, so is the limit.

Theorem 2.11. *Given $C > 0$ there exists $R = R(C) > 2$ such that the following holds. Let Σ_n be a surface embedded in $B_R(0)$ such that $0 \in \Sigma \subset B_R(0)$, $\partial\Sigma$(if non-empty) $\subset \partial B_R(0)$ and $\sup_{\Sigma_n} |A_n| < C$. Suppose also that $\|H\|_{L^p}$ is going to zero as n goes to infinity and then Σ_n is simply connected. Then, up to a subsequence, Σ_n^1 converges C^1 to a properly embedded minimal disk Σ^1.*

Proof. In light of Theorem 2.9, all that needs to be showed is that Σ_n^1 is a disk. This will be discussed in the next section. □

3. Weak Convex Hull Properties

In this section we prove a weak convex hull property for surfaces with bounded L^p norm of the mean curvature.

In the next lemma we prove that if Σ is surface contained in a compact set, its second fundamental form is bounded and its boundary is contained in a certain ball then, if the L^p norm of the mean curvature is small enough, the surface cannot live too far outside the ball. The proof is by contradiction and uses a compactness argument. The idea is that after taking a convergent subsequence, since the limit minimal surface satisfies a convex hull property, an analogous property has to be satisfied by the elements in the sequence. Notice that we need the elements of the sequence to be contained in a compact set otherwise one could take a sequence of spheres with radii going to infinity. The L^p norm of the mean curvature of these spheres is going to zero, $p > 2$, but they do not satisfy any weak convex hull property.

Lemma 3.12. *Given $l > 1$, $\varepsilon > 0$ and $1 \leq p < \infty$ there exists an $n = n(l, \varepsilon, p) > 0$ such that the following holds.*

Suppose Σ is a compact surface such that $\Sigma \subset B_l(0)$, $\|H\|_{L^p} < \frac{1}{n}$ and $\sup_\Sigma |A| < C$ and let $\Sigma^1 \subset \Sigma$ be a surface such that $\partial\Sigma^1 \subset B_1(0)$. Then $\Sigma^1 \subset B_{1+\varepsilon}(0)$.

Proof. The proof is a proof by contradiction. Assume that there exists a sequence of Σ_n and $\Sigma_n^1 \subset \Sigma_n$ such that $\Sigma_n \subset B_l(0)$, $\|H_n\|_{L^p} < \frac{1}{n}$, $\sup_{\Sigma_n} |A_n| < C$, $\partial\Sigma_n^1 \subset B_1(0)$ and $\Sigma_n^1 \not\subset B_{1+\varepsilon}(0)$. Let $p_n \in \Sigma_n^1$ such that

$$l \geq |p_n| = \max_{q \in \Sigma_n^1 \cap \mathbb{R}^3 \setminus B_1(0)} |q| > 1 + \varepsilon. \tag{3.6}$$

After going to a subsequence we can assume that p_n converges to a point $p \in B_l(0) \setminus B_{1+\varepsilon}(0)$. Consider $\delta \leq \frac{\varepsilon}{2}$ such that the connected component of $W(p_n, \delta, \delta)$ that contains p_n consists of a graph over $D(p_n, \delta)$. In particular, after going to a subsequence, the graph containing p_n would converge C^1 to a minimal graph which is tangent to $B_{|p|}(0)$ and contained inside its convex side. This contradicts the maximum principle and proves the theorem. $\square$

In the case when $p = \infty$ a stronger weak convex hull property that does not require a bound on the second fundamental form and can be proved without using a compactness argument.

Lemma 3.13. *Fix $l > 1$ and let Σ be an embedded surface such that $\Sigma \subset B_l(0)$, $\sup_\Sigma |H| < \frac{1}{2l}$. Let $\Sigma' \subset \Sigma$ be a compact surface such that $\partial\Sigma' \subset B_r(p)$, $r > 0$, then $\Sigma' \subset B_r(p)$.*

Proof. If Σ' is not contained in $B_r(p)$ then there exists an R, $r < R < 2l$, such that Σ' is contained inside $B_R(p)$ and it is tangent to its boundary. Let $k_1(q)$ and $k_2(q)$ be the principal curvatures at q. Clearly, $k_1(q)$ and $k_2(q)$ have the same sign, and $|k_i(q)| \geq \frac{1}{R} > \frac{1}{2l}$. Consequently, $|H(q)| \geq \frac{1}{2l}$. This contradicts the assumption and proves the lemma. $\square$

4. Structure Theorem

In this section we prove the structure theorem for embedded disks with bounded L^p norm of the mean curvature, $p \geq 1$, and bounded L^q norm of the gradient of the mean curvature, $q > 2$. For simplicity we are going to state the theorems when $p = \infty$ and assuming a bound on $[H]_\alpha$.

This is the definition of multi-valued graph:

Definition 4.14 (Multi-valued graph)**.** *Let D_r be the disk in the plane centered at the origin and of radius r and let $\mathcal{P}$ be the universal cover of the punctured plane $\mathbb{C}\backslash 0$ with global coordinates (ρ, θ) so $\rho > 0$ and $\theta \in \mathbb{R}$. An N-valued graph of a function u on the annulus $D_s \backslash D_r$ is a single valued graph over $\{(\rho, \theta) | r \leq \rho \leq s, |\theta| \leq N\pi\}$.*

When dealing with multi-valued graphs, the surface to keep in mind is the helicoid, Fig. 1. A parametrization of the helicoid that illustrates the existence of such an N-valued graph is the following

$$(s \sin t, s \cos t, t) \qquad \text{where } (s,t) \in \mathbb{R}^2.$$

It is easy to see that it contains the N-valued graph ϕ defined by

$$\phi(\rho, \theta) = \theta \qquad \text{where } (\rho, \theta) \in \mathbb{R}^+ \backslash 0 \times [-N\pi, N\pi].$$

This is what Colding and Minicozzi proved:

Theorem 4.15. [8, **Theorem 0.4.**] *Given $N \in \mathbb{Z}_+$, $\omega > 1$ and $\varepsilon > 0$, there exist $C = C(N, \omega, \varepsilon) > 0$ such that the following holds.*

Let $0 \in \Sigma \subset B_R \subset \mathbb{R}^3$ be an embedded minimal disk such that $\partial \Sigma \subset \partial B_R$. If

$$\sup_{\Sigma \cap B_{r_0}} |A|^2 \leq 4C^2 r_0^{-2} \text{ and } |A|^2(0) = C^2 r_0^{-2}$$

for some $0 < r_0 < R$, then there exists $\overline{R} < \frac{r_0}{\omega}$ and (after a rotation) an N-valued graph $\Sigma_g \subset \Sigma$ over $D_{\omega \overline{R}} \backslash D_{\overline{R}}$ with gradient $\leq \varepsilon$ and $\mathrm{dist}_\Sigma(0, \Sigma_g) \leq 4\overline{R}$.

This is our main result:

Theorem 4.16. *Given $N \in \mathbb{Z}_+$, $\omega > 1$, $\varepsilon > 0$, $p \geq 1$, and $T > 0$ there exist $C_1 = C_1(N, \omega, \varepsilon) > 0$, $C_2 = C_2(N, \omega, \varepsilon, T, p) > 0$, and $l = l(N, \omega, \varepsilon, p) > 1$ such that the following holds.*

If $\Sigma \subset \mathbb{R}^3$ is an embedded disk with $0 \in \Sigma \subset B_{r_0 l}(0)$, $\partial \Sigma \subset \partial B_{r_0 l}(0)$,

$$\sup_{\Sigma \cap B_{r_0 l}(0)} |A|^2 \leq 4C_1^2 r_0^{-2} \text{ and } |A|^2(0) = C_1^2 r_0^{-2},$$

$$\|H\|_{L^p} \leq r_0^{\frac{2-p}{p}} C_2 \text{ and } r_0^{1+\alpha} [H]_\alpha \leq T \tag{4.7}$$

for some $r_0 > 0$, then there exists $\overline{R} < \frac{r_0}{\omega}$ and (after a rotation) an N-valued graph $\Sigma_g \subset \Sigma$ over $D_{\omega \overline{R}} \backslash D_{\overline{R}}$ with gradient $\leq \varepsilon$ and $\mathrm{dist}_\Sigma(0, \Sigma_g) \leq 4\overline{R}$.

Proof. Theorem 4.16 will follow by rescaling after we prove it for $r_0 = 1$. Assuming $r_0 = 1$ the hypotheses become $0 \in \Sigma \subset B_l(0)$, $\partial\Sigma \subset \partial B_l(0)$,

$$\sup_{\Sigma \cap B_{\overline{l}}(0)} |A|^2 \leq 4C_1^2 \text{ and } |A|^2(0) = C_1^2,$$

$$|H| \leq C_2 \text{ and } [H]_\alpha \leq T.$$

We have to prove that fixed $T > 0$ there exists C_2 such that if the above is true, then Σ contains a multi-valued graph in the ball of radius 1. The proof is by contradiction and uses a compactness argument.

Assuming that the theorem is false, let C_1 be as big as given by Theorem 4.15, let l be as given by Theorem 2.11 and let Σ_n be a sequence of embedded disks satisfying the hypotheses of the statement that does not contain a multi-valued graph and with $|H|$ less than $\frac{1}{n}$. As n goes to infinity, Theorem 2.11 gives that, up to a subsequence, Σ_n^1 (the connected component of $\Sigma_n \cap B_1(0)$ containing the origin) converges C^2 with multiplicity one to a minimal disk Σ^1. The minimal disk containing the origin satisfies the hypotheses of Theorem 4.15 and therefore it contains a multi-valued graph. Since the limit contains a multi-valued graph, Σ_n^1 must also contain a multi-valued graph for n large. This gives a contradiction and proves the theorem. □

Notice that the C^2 convergence guarantees that not only does Σ_n contain an N-valued graph, but the properties of this graph, such as the upper bound on the gradient, are preserved.

When the mean curvature is bounded in L^∞ we can prove the next two corollaries. This is due to the fact that in such a case we have a stronger weak convex hull property. For simplicity we will not state them in full generality as we did for Theorem 4.16. The general versions can be easily obtained using a rescaling argument.

In the next corollary we prove that if the second fundamental form of an embedded disk at a point is bigger than what it is necessary to prove existence of an N-valued graph and it is almost its maximum, and the disk satisfies (4.7) in Theorem 4.16 then the disk contains a multi-valued graph, possibly on a smaller scale.

Corollary 4.17. *Given $N \in \mathbb{Z}_+$, $\omega > 1$, $\varepsilon > 0$, and $T > 0$ there exist $C_1 = C_1(N, \omega, \varepsilon) > 0$, $C_2 = C_2(N, \omega, \varepsilon) > 0$, and $l_1 = l_1(N, \omega, \varepsilon) > 1$ such that the following holds.*

If $\Sigma \subset \mathbb{R}^3$ is an embedded disk with $0 \in \Sigma \subset B_l(0)$, $\partial\Sigma \subset \partial B_l(0)$,

$$\sup_{\Sigma \cap B_l(0)} |A|^2 \leq 4(C+\beta)^2 \textit{ and } |A|^2(0) = (C+\beta)^2,$$

$$|H| \leq \min(C_2, \frac{1}{2l}) \textit{ and } [H]_\alpha \leq T$$

for some $\alpha > 0$, $l > l_1$ then there exists $\overline{R} < \frac{C}{\omega(C+\beta)}$ and (after a rotation) an N-valued graph $\Sigma_g \subset \Sigma$ over $D_{\omega\overline{R}} \backslash D_{\overline{R}}$ with gradient $\leq \varepsilon$.

Proof. Consider the rescaled surface $\Sigma' = \frac{C+\beta}{C}\Sigma$ and let Σ'' be the connected component of $\Sigma' \cap B_l(0)$ that contains the origin. Thanks to the weak convex hull property Σ'' is still a disk and since $\frac{C+\beta}{C} > 1$ it satisfies the hypothesis of Theorem 4.16. It follows that there exists $\overline{R} < \frac{1}{\omega}$ and (after a rotation) an N-valued graph $\Sigma_g \subset \Sigma''$ over $D_{\omega\overline{R}} \backslash D_{\overline{R}}$ with gradient $\leq \varepsilon$. Thus, rescaling back proves the corollary. $\square$

In the next corollary we prove that if the second fundamental form of an embedded disk is big at a point but not necessarily almost its maximum, and the disk satisfies (4.7) in Theorem 4.16 then the disk contains a multi-valued graph, possibly around another point.

Corollary 4.18. *Given $N \in \mathbb{Z}_+$, $\omega > 1$, $\varepsilon > 0$, and $T > 0$ there exist $C_1 = C_1(N, \omega, \varepsilon) > 0$, $C_2 = C_2(N, \omega, \varepsilon) > 0$, and $l_1 = l_1(N, \omega, \varepsilon) > 1$ such that the following holds.*

If $\Sigma \subset \mathbb{R}^3$ is an embedded disk with $0 \in \Sigma \subset B_l(0)$, $\partial\Sigma \subset \partial B_l(0)$, $|A|(0) = C$,

$$|H| \leq \min(C_2, \frac{1}{2l}) \text{ and } [H]_\alpha \leq T$$

for some $l > l_1$ then there exist $p \in \Sigma$, $\overline{R} < \frac{1}{\omega}$, and $0 < \delta < 1$ such that, after a translation that takes p to the origin and possibly after a rotation, Σ contains an N-valued graph Σ_g over $D_{\delta\omega\overline{R}} \backslash D_{\delta\overline{R}}$ with gradient $\leq \varepsilon$.

Proof. If

$$\sup_{\Sigma} |A|^2 \leq 4C^2$$

then we are done, because of Theorem 4.16.

If instead $\sup_\Sigma |A|^2 > 4C^2$ then consider the non negative function $F(x) = (l - |x|)^2 |A(x)|^2$. F is zero on the boundary of Σ therefore it obtains its maximum at a point in the interior. Let p be that point, i.e.

$$F(p) = \max_{\Sigma} F(x) = (|p| - l)^2 |A|^2(p) \geq F(0) > 4l^2C^2.$$

Let $2\sigma < l - |p|$ such that

$$4\sigma^2 |A|^2(p) = 4l^2C^2.$$

Since F achieves its maximum at p,

$$\sup_{B_\sigma(p)\cap\Sigma} \sigma^2 |A|^2 \leq \sup_{B_\sigma(p)\cap\Sigma} \sigma^2 \frac{F(x)}{(|x| - l)^2} \leq \tag{4.8}$$

$$\leq \frac{4\sigma^2}{(|p| - l)^2} \sup_{B_\sigma(p)\cap\Sigma} F(x) = \frac{4\sigma^2}{(|p| - l)^2} F(p) = 4\sigma^2 |A|^2(p).$$

From the weak convex hull property we know that $B_\sigma(p) \cap \Sigma$ consists of a collection of disks. Rescale $B_\sigma(p) \cap \Sigma$ by a factor of $\frac{l}{\sigma} \geq 1$ and translate p to the origin. Let

$\Sigma' \subset B_l(0)$ be the rescaled connected component that contains the origin. Σ' is an embedded disk such that the following holds: $\Sigma' \subset \mathcal{B}_l(0), \partial\Sigma' \subset \partial\mathcal{B}_l(0)$,

$$|H'| \leq \frac{\sigma}{l}\min(C_2, \frac{1}{2l}) \leq \min(C_2, \frac{1}{2l}), [H']_\alpha \leq (\frac{\sigma}{l})^{1+\alpha} T \leq T \text{ and } \sup_{\Sigma'} \leq 4C^2 = 4|A|^2(0).$$

Theorem 4.16 gives that there exists $\overline{R} < \frac{1}{\omega}$ and (after a rotation) an N-valued graph $\Sigma_g \subset \Sigma'$ over $D_{\omega\overline{R}} \backslash D_{\overline{R}}$ with gradient $\leq \varepsilon$. Thus, rescaling back proves the corollary. □

5. Counterexamples

In this section we are going to show that in order to have a multi-valued graph form in a smooth surface, it is not enough to assume that the mean curvature is small relative to the second fundamental form, which is big. How small the mean curvature has to be must also depend on the Holder norm of the mean curvature.

Let us consider the graph of the function

$$u(x, y) = xy \log \sqrt{x^2 + y^2}.$$

Let $r = \sqrt{x^2 + y^2}$ then the following holds.

(1) $\nabla u = (y \log r + \frac{x^2 y}{r^2}, x \log r + \frac{y^2 x}{r^2})$
(2) $\Delta u = 4\frac{xy}{r^2}$
(3) $u_{xy} = \log r + 1 - 2\frac{x^2 y^2}{r^4}$

Let us take a sequence of mollification $u_\sigma(x, y)$ that approximates u. As σ goes to zero $\max Hess(u_\sigma)$ goes to infinity. For any $C > 0$ consider the new sequence of graphs $\frac{C}{\max Hess(u_\sigma)} u_\sigma$. This is a sequence of graphs whose mean curvature is going to zero and that is converging C^1 to a plane, but whose maximum of the second fundamental form is C. Clearly though, a graph does not contain a multi-valued graph around any point.

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