The structure of embedded constant mean curvature disks: extending the multi-valued graph

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Abstract

It is known that a constant mean curvature embedded disk whose Gaussian curvature is large contains a multi-valued graph, provided that the mean curvature is sufficiently small. This multi-valued graph forms on the scale of the Gaussian curvature. In this paper we show that in fact, this multi-valued graph extends horizontally to a larger scale. This generalizes to non-minimal surfaces a well known result of Colding and Minicozzi.

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1 Introduction.

This paper is a preliminary version. In a series of papers [2, 5, 6, 7, 8], Colding and Minicozzi discussed the geometry of embedded minimal surfaces of fixed genus. The new ideas provided by their recent work have been applied to solve several longstanding problems in the field; see for instance [1, 4, 10, 11]. An important ingredient has been describing the structure of minimal disks embedded in $\mathbb{R}^3$.

In [6] they proved that a minimal disk embedded in $\mathbb{R}^3$ whose Gaussian curvature is large at a point contains a highly sheeted multi-valued graph around that point. This means that, locally, the disk looks like a piece of a suitably scaled helicoid (see Figure **??**). This result was later generalized in [15] to the constant mean curvature case. The multi-valued graph described in [6] and [15] forms on the scale of the Gaussian curvature which, loosely speaking, means that as the Gaussian curvature becomes larger, the multivalued graph becomes well-defined closer to the point of large curvature. In [5] Colding and

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Minicozzi proved that this multi-valued graph which locally forms on the scale of the Gaussian curvature, in fact it extends horizontally, to form a multi-valued graph on a larger scale.

The goal of this paper is to continue studying the structure of embedded constant mean curvature disks. In particular we generalize the results in [5] to the non-zero constant mean curvature case. The main theorem is the following.

Theorem 1.1 *Given $\Delta_1 < 1$, $\Delta_2 > 1$ there exist $\Omega_2(\Delta_1, \Delta_2) = \Omega_2$, $C_3 = C_3(\Delta_1, \Delta_2) > 0$, and $C_4 = C_4(\Delta_1, \Delta_2) > 0$ such that the following holds.*

Let $\Sigma \subset \mathbb{R}^3$ be an embedded and simply connected constant mean curvature equal to $H > 0$ surface with $0 \in \Sigma \subset B_l$, $\partial\Sigma \subset \partial B_l$, $l > \Omega_1$ and $\alpha > 1$. If $H < \min\{C_4, \frac{1}{2l}\}$ and

$$\sup_{\Sigma \cap B_1(0)} |A| \leq 2\alpha C_3 = 2|A|(0),$$

then (after a rotation) there exists a 2-component multi-graph $\Sigma_g \subset \Sigma$ over $D_{\Delta_2} \backslash D_{(1-\Delta_1)(1+\frac{1}{\alpha})}$.

2 Background.

Let $\Sigma \subset \mathbb{R}^3$ be a 2-dimensional smooth orientable surface (possibly with boundary) with unit normal N_Σ. Given a function ϕ in the space $C_0^\infty(\Sigma)$ of infinitely differentiable (i.e., smooth), compactly supported functions on Σ, consider the one-parameter variation

$$\Sigma_{t,\phi} = \{x + t\phi(x)N_\Sigma(x)/x \in \Sigma\}$$

and let $A(t)$ be the area functional,

$$A(t) = Area(\Sigma_{t,\phi}).$$

The so-called first variation formula of area is the equation (integration is with respect to $d$area)

$$A'(0) = \int_\Sigma \phi H, \tag{1}$$

where H is the mean curvature of Σ. When H is identically zero the surface Σ is a critical point for the area functional and it is called a *minimal* surface [13, 3]; concrete examples of minimal surfaces are planes, the helicoid and the catenoid.

When H is constant the surface Σ is a critical point for the area functional restricted to those variations which preserved the enclosed volume, in other words ϕ must satisfy the condition,

$$\int_\Sigma \phi = 0. \tag{2}$$

These surfaces are called *constant mean curvature* (CMC) surfaces and examples are spheres, cylinders, and Delaunay surfaces.

In general, if Σ is given as graph of a function u then

$$H = \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right). \tag{3}$$

In order to prove the main theorem we are going to need a few results about minimal surfaces which are contained in [5, 15, 14].

The first two results are contained are contained in [5].

Theorem 2.1 (Theorem II.0.21. in [5]) *Given $\tau > 0$, there exist $N_1, \varepsilon_1, \Omega_1 > 0$ such that the following holds.*

Let $\delta \le 1$, $\Sigma \subset B_{R_0}$ be a stable embedded minimal disk with $\partial\Sigma \subset B_{\delta r_0} \cup \partial B_{R_0} \cup \{x_1 = 0\}$ where $\partial\Sigma\backslash\partial B_{R_0}$ is connected. If $\Omega_1 r_0 < 1 < R_0/\Omega_1$ and Σ contains an N_1-valued graph Σ_g over $D_\delta\backslash D_{\delta r_0}$ with gradient $\le \varepsilon_1$,

$$\Pi^{-1}(D_{\delta r_0}) \cap \Sigma^M \subset \{|x_3| \le \varepsilon_1 \delta r_0\},$$

and a curve η connects Σ_g to

$$\partial\Sigma\backslash\partial B_{R_0},$$

then Σ contains a 2-valued graph Σ_d over $D_{R_0/\Omega_1}\backslash D_{\delta r_0}$ with gradient $\le \tau$.

Where Σ^M is the middle sheet, see [5].

Theorem 2.2 (Lemma I.0.9. in [5]) *Let $\Gamma \subset \{|x_3| \le \beta h\}$ be a stable embedded minimal surface. There exists $C_g, \beta_s > 0$ so that if $\beta \le \beta_s$ and E is a component of*

$$\mathbb{R}^2\backslash T_h(\Pi(\partial\Gamma)),$$

then each component of $\Pi^{-1}(E) \cap \Gamma$ is a graph over E of a function u with

$$|\nabla_{\mathbb{R}^2} u| \le C_g \beta.$$

The following result can be easily proved combing the results in [15] and [14].

Theorem 2.3 *Given $N \in \mathbb{Z}_+, \omega > 1$ and $\varepsilon > 0$, there exist $C_1 = C_1(N, \omega, \varepsilon) > 0$, $C_2 = C_2(N, \omega, \varepsilon) > 0$ and $\bar{l} > 1$ such that the following holds.*

Let $\Sigma \subset \mathbb{R}^3$ be an embedded and simply connected constant mean curvature equal to $H > 0$ surface with $0 \in \Sigma \subset B_l$, $\partial\Sigma \subset \partial B_l$ and $\alpha > 1$. If $l > \bar{l}$, $H < \min\{C_2, \frac{1}{2l}\}$ and $|A|(0) = \alpha C_1$, then there exists $\delta < 1$ and $p \in \Sigma \cap B_{\frac{1}{\alpha}}(0)$ such that after a translation that takes p to the origin and possibly a rotation, Σ contains an N-valued graph, Σ_g over $D_{\frac{\omega \bar{R}}{\alpha}}\backslash D_{\frac{\bar{R}}{\alpha}}$ where $\bar{R} < \frac{1}{\omega}$ (with gradient $\le \epsilon$).

In particular, if $\sup_{\Sigma\cap B_1(0)} |A| \le 2\alpha C_1$ then p is the origin.

3 Multi-valued minimal graph.

The first step in the proof is to use Theorem 2.1, 2.2 and 2.3 to show that Σ is contained between the sheets of a large, minimal 2-valued graph Σ_{min}. In order to construct Σ_{min} we are going to use Theorem 2.3 to find a simple closed curve γ which is the boundary of a disk in Σ and such that the following holds. After using the results in [12] by Meeks and Yau to find a stable embedded minimal disk Σ' bounded by γ, using Theorem 2.2 and 2.3, we can show that Σ' satisfies the hypothesis of Theorem 2.1.

Theorem 3.1 *Given $\tau > 0$ there exist $\omega = \omega(\tau) > 1$, $\Omega_1(\tau) = \Omega_1$, $C_1 = C_1(\tau) > 0$, and $C_2 = C_2(\tau) > 0$ such that the following holds.*

Let $\Sigma \subset \mathbb{R}^3$ be an embedded and simply connected constant mean curvature equal to $H > 0$ surface with $0 \in \Sigma \subset B_l$, $\partial\Sigma \subset \partial B_l$, $l > \Omega_1$ and $\alpha > 1$. If $H < \min\{C_2, \frac{1}{2l}\}$ and

$$\sup_{\Sigma\cap B_1(0)} |A| \leq 2\alpha C_1 = 2|A|(0),$$

then there exist $\overline{R} < \frac{1}{\omega}$ and a stable minimal disk Σ' such that (after a rotation) $\Pi^{-1}(\partial D_{\frac{\overline{R}}{\alpha}}) \cap \Sigma \subset \partial\Sigma'$ which contains a 2-valued minimal graph Σ_{min} over $D_{\frac{l}{\Omega_1}} \backslash D_{\frac{(5\omega-1)\overline{R}}{4\alpha\omega}}$ with gradient less than τ.

Proof. Fix $\tau > 0$ and let N_1, ε_1 and Ω_1 be given by Theorem 2.1. Let $N = N_1$, $\varepsilon < \min\{\frac{\varepsilon_1}{2\pi N}, \frac{\varepsilon_1(1-\frac{1}{\omega})}{8\pi\omega C_g}\}$ and $\omega > 1$ such that $\frac{5\omega-1}{2\omega^2-2}\Omega_1 < 1$. Let $\bar{l}$, C_1 and C_2 be the ones given in Theorem 2.3 and let $l_1 = \max\{\bar{l}, \frac{1}{\Omega_1}\}$.

Using Theorem 2.3 gives that Σ (after a rotation) contains an N-valued graph, Σ_g, over $D_{\frac{\omega\bar{R}}{\alpha}} \backslash D_{\frac{\bar{R}}{\alpha}}$ where $\bar{R} < \frac{1}{\omega}$, with gradient $\leq \varepsilon$. Let $\Delta = \frac{\omega\overline{R}-\overline{R}}{\alpha}$, $Q = (\frac{1}{H} + \frac{\overline{R}}{\alpha} + \frac{\Delta}{2}, 0, 0)$ and consider $\Sigma \cap B_{\frac{1}{H}}(Q)$. $\Sigma \cap B_{\frac{1}{H}}(Q) \cap \Sigma_g$ consists of N segments with alternating orientation. Consider S_t, S_b be the first segments from the top, respectively from the bottom, such that the normal to Σ is pointing down, respectively up. We want to show that components γ_t and γ_b of $\Sigma \cap B_{\frac{1}{H}}(Q)$ which contain S_t and respectively S_b are simple curves such that $\partial\gamma_t$ and $\partial\gamma_b$ are contained in $\partial\Sigma$. In other words, we want to show that we can extend Σ_t and Σ_b all the way to the boundary of Σ. What we need to show is that S_t and S_b are not closed curves. Suppose that S_t is closed then, since Σ is a disk, S_t is the boundary of a disk Λ_t. Clearly, Λ_t must be contained inside $B_{\frac{1}{H}}(Q)$. However, the normal projection of the normal vector has to point inside the region on the sphere enclosed by S_t otherwise it woud contradict the maximum princple. However, S_t cannot contain any of the other segments. Therefore, the only possibility is that all the other segments are contained in the region enclosed by S_t. However, an analogous argument can be done using S_b instead of S_t, which gives a contradiction and proves the claim. Let now S_t and S_b denote a path on $B_{\frac{1}{H}}(\frac{1}{H}+\frac{\overline{R}}{\alpha}+\frac{\Delta}{2},0,0)$ which connects γ_t and γ_b to the boundary of Σ and let $\Gamma_2 \subset \partial\Sigma$ a path that connects S_t and S_b. Let $\Gamma = \Gamma_1 \cup \Gamma_t \cup S_t \cup \Gamma_2 \cup S_b \cup \Gamma_b$ where $\Gamma_1 = \Pi^{-1}(\partial D_{\frac{\overline{R}}{\alpha}}) \cap \Sigma_g$ and Γ_t and Γ_b are segments to connect Γ_1 to S_t respectively S_b. Γ bounds a disk on Σ and we can use the result of Meeks and Yau in [12] to produce an embedded stable minimal disk Σ' contained in one of the two connected components of $B_R(0)\setminus\Sigma$ such that $\partial\Sigma' = \Gamma$.

Our goal is now to show that Σ' satisfies the hypothesis of Theorem 2.1. Using a linking argument one can clearly show that Σ' contains points in between the sheets of Σ_g. What we are going to show is that Σ' contains an N-valued graph over $D_{\frac{\omega\overline{R}}{\alpha}-\frac{\Delta}{2}-\frac{\Delta}{4\omega}} \setminus D_{\frac{\overline{R}}{\alpha}+\frac{\Delta}{4\omega}}$. Let $p \in \Pi^{-1}(\partial D_{\frac{\overline{R}}{\alpha}+\frac{\Delta}{4}})$ and let Θ be the connected component containing p of the intersection of Σ' with the vertical cylinder centered at p of radius $\frac{\Delta}{4}$ and let Θ' be the connected component containing p of the intersection of Σ' with the vertical cylinder centered at p of radius $\frac{\Delta}{4}-\frac{\Delta}{4\omega}$. We want to apply Theorem 2.2 to show that Θ' is a graph with gradient less than ε_1. Θ is certainly containd in a slab of width $2\pi\varepsilon\omega\overline{R}$ because it is sandwiched between to sheets of Σ_g. Moreover, it satisfies the hypotheses of Theorem 2.2 with $h = \frac{\Delta}{4\omega}$. Let $\beta \leq \frac{\varepsilon_1}{C_g}$ then, in order to apply Theorem 2.2 we need that $\frac{2\pi\varepsilon\omega\overline{R}}{\alpha} < \frac{\varepsilon_1\Delta}{4\omega C_g}$ which follows from our choices of constants. This shows that Σ' contains an N-valued graph over $D_{\frac{\omega\overline{R}}{\alpha}-\frac{\Delta}{2}-\frac{\Delta}{4\omega}} \setminus D_{\frac{\overline{R}}{\alpha}+\frac{\Delta}{4\omega}}$ with greadient less than ε_1.

Notice that in our case $r_0\delta = \frac{\overline{R}}{\alpha}+\frac{\Delta}{4\omega}$ and $\delta = \frac{\omega\overline{R}}{\alpha}-\frac{\Delta}{2}+\frac{\Delta}{4\omega}$. Hence, $r_0 = \frac{5\omega-1}{2\omega^2-2}$ and $r_0\Omega_1 < 1$. The last thing we need to check is that $\Pi^{-1}(D_{\delta r_0}) \cap \Sigma^M \subset \{|x_3| \leq \varepsilon_1\delta r_0\}$ which is true since $\Pi^{-1}(D_{\delta r_0}) \cap \Sigma' \subset \{|x_3| \leq 2\pi N\varepsilon\delta r_0\}$ □

Theorem 3.1 can be easily generalized to the following result which will be used in the next section to extend the multi-valued graph.

Remark 3.2 *Given $\tau > 0$ there exist $\omega = \omega(\tau) > 1$, $\varepsilon = \varepsilon(\tau) > 0$, $N_1 = N_1(\tau) > 0$, $\Omega_1(\tau) = \Omega_1$, $C_1 = C_1(\tau) > 0$, and $C_2 = C_2(\tau) > 0$ such that the following holds:*

Let $\Sigma \subset \mathbb{R}^3$ be an embedded and simply connected constant mean curvature equal to $H > 0$ surface with $0 \in \Sigma \subset B_l$, $\partial\Sigma \subset \partial B_l$, $l > \Omega_1$ and $\alpha > 1$. If $H < \min\{C_2, \frac{1}{2l}\}$ and

$$\sup_{\Sigma \cap B_1(0)} |A| \leq 2\alpha C_1 = 2|A|(0),$$

then Σ (after a rotation) contains an N-valued graph, Σ_g, over $D_{\frac{\omega\overline{R}}{\alpha}} \setminus D_{\frac{\overline{R}}{\alpha}}$ where $\bar{R} < \frac{1}{\omega}$, with gradient $\leq \varepsilon$. If there exist two simple curves $\mu_t, \mu_b \subset \{x_1 \geq frac\omega_1\overline{R}\alpha\}$ which respectively connect the top sheet and the bottom sheet of the N_2-valued graph to $\partial\Sigma$ then there exist $\overline{R} < \frac{1}{\omega}$ and a stable minimal disk Σ' such that (after a rotation) $\Pi^{-1}(\partial D_{\frac{\overline{R}}{\alpha}}) \cap \Sigma \subset \partial\Sigma'$ which contains a 2-valued minimal graph Σ_{min} over $D_{\frac{l}{\Omega_1}} \setminus D_{\frac{(5\omega-1)\overline{R}}{4\alpha\omega}}$ with gradient less than τ.

4 Extending the multi-valued CMC graph.

In order to prove the main Theorem we are going to need the following lemma. Loosely speaking Lemma 4.1 says the following: If Σ is contained in a thin, horizontal slab and there is a point where it is not graphical (i.e. the normal vector is horizontal) then the second fundamental form near that point must be large.

Lemma 4.1 *Given $\delta > 0$ there exists $C = C(\delta) > 0$, $\varepsilon = \varepsilon(\delta) > 2\delta$ such that the following holds.*

Let $\Sigma \subset \{|x_3| \leq \delta\}$ and $p \in \Sigma$ such that $\sup_{B_\varepsilon(p)\cap\Sigma} |A| < C$ and $\partial\Sigma \subset \mathbb{R}^3 \backslash B_\varepsilon(p)$ Then, $N(p)$ is not horizontal.

In order to extend the multi-valued graph the idea is the following. Theorem 3.1 implies that a subset of Σ is contained in between a minimal 2-valued graph. Our goal is to show that the part which is contained between the sheets is graphical. Suppose there exists a point where the surface is not graphical then, using Lemma 4.1 the curvature must be large nearby. We can then apply Theorem 2.3 to show that a multi-valued graph forms. Using the nodoid foliation discussed in Section 5 we can connect a point on the top sheet of the multi-valued graph and a point on the bottom sheet to points on the boundary of Σ. In this way, similarly to the argument described in the proof of Theorem 3.1 and Remark 3.2 we obtain a simple closed curve γ which bounds a disk in Σ and the following holds. After using the results in [12] by Meeks and Yau we find a stable embedded minimal disk, Σ', bounded by γ and disjoint from Σ. However, using Theorem 2.1, Σ' contains a 2-valued graph over a certain annulus and this annulus is sufficently large so that Σ' is forced to intersect Σ thus giving a contradiction.

Theorem 4.2 *Given $\Delta_1 < 1$,$\Delta_2 > 1$ and there exist $\Omega_2(\Delta_1, \Delta_2) = \Omega_2$, $C_3 = C_3(\Delta_1, \Delta_2) > 0$, and $C_4 = C_4(\Delta_1, \Delta_2) > 0$ such that the following holds.*

Let $\Sigma \subset \mathbb{R}^3$ be an embedded and simply connected constant mean curvature equal to $H > 0$ surface with $0 \in \Sigma \subset B_l$, $\partial\Sigma \subset \partial B_l$, $l > \Omega_1$ and $\alpha > 1$. If $H < \min\{C_4, \frac{1}{2l}\}$ and

$$\sup_{\Sigma\cap B_1(0)} |A| \leq 2\alpha C_1 = 2|A|(0),$$

then (after a rotation) there exists a 2-valued graph $\Sigma_g \subset \Sigma$ over $D_{\Delta_2} \backslash D_{(1-\Delta_1)(1+\frac{1}{\alpha})}$.

Proof. As described in Remark 3.2, fix $\tau_1 = \frac{1}{2}$ and let $\overline{\varepsilon} = \varepsilon(\tau) > 0$, $\overline{N_1} = N_1(\tau) > 0$, $\overline{\omega} = \omega(\frac{1}{2}) > 1$, $\overline{\Omega_1} = \Omega_1(\frac{1}{2})$, $\overline{C_1} = C_1(\frac{1}{2}) > 0$, and $\overline{C_2} = C_2(\frac{1}{2}) > 0$. Fix β such that $\Delta_1 = 1 - \frac{1}{\beta}$ and using Lemma 4.1 fix $\tau > 0$ such that if $p \in \Sigma \subset \{|x_3| \leq 10\pi\tau\Delta_2\}$ then $|A|(p) = 2\beta\overline{C_1}$. Let $\Omega_2 = \Delta_2\Omega_1(\tau)^2$, $C_3 = \beta C_1(\tau) > 0$, and $C_4 = \min\{\frac{\rho_1}{20\pi\tau\Delta_2}, C_2(\tau)\}$, where $\Omega_1(\tau)$, $C_1(\tau)$, and $C_2(\tau)$ are given by Theorem 3.1.

In sum, if Σ satisfies the hypothesis of the theorem then $\Sigma \subset \mathbb{R}^3$ is an embedded and simply connected constant mean curvature equal to $H > 0$ surface with $\partial\Sigma \subset \partial B_l(0)$ for $l > \Omega_1$ and such that $H < \min\{C_4, \frac{1}{2l}\}$ and

$$\sup_{\Sigma \cap B_1(0)} |A| \leq 2\alpha C_1 = 2|A|(0), \text{ for } \alpha > 1.$$

Using Theorem 3.1 gives that there exist a stable minimal disk Σ' such that (after a rotation) $\Pi^{-1}(\partial D_{\frac{\overline{R}}{\alpha}}) \cap \Sigma \subset \partial\Sigma'$ which contains a 2-valued minimal graph Σ_{min} over $D_{\frac{l}{\Omega_1}} \backslash D_{\frac{(5\omega-1)\overline{R}}{4\alpha\omega}}$ with gradient less than τ.

We want to show that the constant mean curvature disk is graphical between the sheets of this 2-valued minimal graph. Let p be a point in Σ which is between the sheets (using a linking argument it can be showed that p exists) and such that $\frac{(5\omega-1)\overline{R}}{4\alpha\beta\omega} + \frac{1}{\beta} < |\Pi(p)| < \Delta_2$. Since the gradient of the 2-valued minimal graph is less than τ, Σ is contained a slab of height $\{10\pi\tau\Delta_2\}$. Suppose that Σ is not graphical aroun p then, due to the way we have chosen τ, we have that $|A|(p) = 2\beta\overline{C_1}$. Using Theorem 2.3 and Remark 3.2 gives that an N_1-valued with gradient less than $\overline{\varepsilon}$ forms around a point $q \in B_{\frac{1}{\beta}}(p)$ over $D_{\frac{\overline{\omega}R'}{\beta}} \backslash D_{\frac{R'}{\beta}}$, $R' < \frac{1}{\overline{\omega}}$. This multi-valued is contained in the slab. Using the catenoid foliation we can connect the the top sheet and the bottom sheet of the multi-valued graph to $\Sigma \cap \partial B_{\frac{l}{\Omega_1}}$ in such a way that we can apply Remark 3.2. This gives a large 2-valued minimal graph Σ' around q over a disk of radius $\frac{l}{\Omega_1^2} > \Delta_2$. This being the case Σ' must intersec Σ_{min}. However, form the way it is obtained, Σ' must be disjoint from Σ_{min} thus giving a contradiction. □

5 Nodoid foliation.

In this section we generalize some results about minimal surfaces given by Hoffman and Meeks in [9] and Colding and Minicozzi in [5] to the context of constant mean curvature surfaces.

Before we discuss the nodoid foliation we need to establish some notation. If p, q are points in $\mathbb{R}^3$ then $\gamma_{p,q}$ is the segment between p and q. If S is a subset of $\mathbb{R}^3$ then $T_h(S) = \bigcup_{x \in S} B_h(x)$. Let Nod' be the vertical nodoid whose mean curvature is equal to $\frac{1}{2}$ and such that the shortest closed geodesic consists of the circle $\{x_1^2 + x_2^2 = 1, x_3 = 0\}$. Let C be the cone centered at the origin which intersects Nod' tangentially in two circles, C_t and C_b. With a slight abuse of notation, denote by Nod' the connected component of $Nod \backslash C_t \cup C_b$ and denote by Nod the solid neck such that $\partial Nod = Nod' \cup D_t \cup D_b$ where D_t and D_b are the disks bounded by C_t and C_b. Dilation of Nod' are all disjoint and, consequently, give a CMC foliation of the solid open cone and the mean curvature of the leaves inside Nod is less than $\frac{1}{2}$. Let Σ be a CMC surface with mean curvature $H < \frac{1}{2}$ and let p be a point in the interior of Σ. If Σ is tangent at p to the interior of δNod, $\delta \leq 1$, then, locally, Σ cannot be in the mean convex side of δNod (i.e. outside the neck). In other words, the leaves of the nodoid foliation are level sets of the function f given by

$$\frac{x}{f(x)} \in Nod'$$

and the following holds.

Lemma 5.1 *Let $\Sigma \subset C$ be a CMC with mean curvature $H < \frac{1}{2}$ then f_Σ has no nontrivial interior local extrema inside Nod.*

Let $\rho_1 \in \mathbb{R}_+$ small enough such that

$$\{x||x_3| \leq 2\rho_1\} \backslash B_{\frac{1}{8}}(0) \subset C$$

and

$$\{x|f(x) = \frac{3}{16}\} \cap \{x||x_3| \leq 2\rho_1\} \subset B_{\frac{7}{32}}(0).$$

Lemma 5.2 *Let $\Sigma \subset \{|x_3| \leq 2\delta\rho_1\}$ be a CMC surface such that $H < \frac{1}{2\delta}$, $\partial\Sigma \subset \partial B_\delta(0)$ and $B_{\frac{3\delta}{4}}(0) \cap \Sigma \neq \emptyset$ then*

$$B_{\frac{\delta}{4}}(0) \cap \Sigma.$$

Iterating Lemma 5.2 along a chain of balls and a rescaling argument gives the following corollary.

Corollary 5.3 *Let $\Sigma \subset \{|x_3| \leq 2\delta\rho_1\}$ be a CMC surface such that $H < \frac{1}{2\delta}$ and $\partial\Sigma \subset \partial B_R(0)$, $R > 10\delta$. Let $p, q \in \{x_3 = 0\}$ such that $T_\delta(\gamma_{p,q}) \cap \partial\Sigma = \emptyset$, and $y_p \in B_{\frac{\delta}{4}}(p) \cap \Sigma$. There exist a curve ν such that $\nu \subset T_\delta(\gamma_{p,q}) \cap \Sigma$ and $\partial\nu = \{y_p, y_q\}$ where $y_q \in B_\delta(q)$.*

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