Combinatorial properties of the numbers of tableaux of bounded height

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Abstract

We introduce an infinite family of lower triangular matrices $\Gamma^{(s)}$, where $\gamma_{n,i}^s$ counts the standard Young tableaux on n cells and with at most s columns on a suitable subset of shapes. We show that the entries of these matrices satisfy a three-term row recurrence and we deduce recursive and asymptotic properties for the total number $\tau_s(n)$ of tableaux on n cells and with at most s columns.

1   Introduction

The first simple expressions of the number of standard Young tableaux of given shape were given by Frobenius and Young (see [6] and [15]) and by Frame-Robinson-Thrall [5]. More recently, the number of standard Young tableaux has been studied according to the height of their shape. Regev [12] gave asymptotic values for the numbers $\tau_s(n)$ of standard Young tableaux whose shape consists of n cells and at most s columns and Stanley (see [14]) discussed the algebraic or differentiably finite nature of the corresponding generating functions. More recently, many authors entered this vein. For example, D.Gouyou-Beauchamps gave both exact forms in [13] and recurrence formulas (see [9]) for the numbers $\tau_s(n)$ when $s \leq 5$ by combinatorial tools, while F.Bergeron and F.Gascon [4] found some recurrence formulas for the numbers $\tau_s(n)$ by analytic argumentations.

In this paper we present a different approach to the subject. Our starting point is the remark that the elements of the Ballot matrix introduced

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by M.Aigner in [1] correspond bijectively to the integers counting standard Young tableaux of a given shape with at most 2 columns. Firstly, we arrange the entries of the Ballot Matrix in a new lower triangular matrix A in such a way that the entries of the n-th row count standard Young tableaux with precisely n cells. The integer $\tau_2(n)$ can be therefore recovered by summing the elements of the n-th row of A.

In the following sections, we extend the results to the general case of standard Young tableaux with at most s columns. For every fixed $s \in \mathbb{N}$ we define an infinite matrix $\Gamma^{(s)}$ whose (n, i)-th entry is the total number of standard Young tableaux with at most s columns and such that the difference between the length of the second and of the third column is i. The total number $\tau_s(n)$ of tableaux on n cells with at most s columns is the n-th row sum of the matrix $\Gamma^{(s)}$. The matrix $\Gamma^{(s)}$ presents a three-term row-recurrence property, which yields a recurrence law satisfied by the integers $\tau_s(n)$. This recurrence allows to get a combinatorial interpretation of the asymptotic behaviour of the ratio

$$\frac{\tau_s(n)}{\tau_s(n-1)}.$$

In Section 3 we treat more extensively the case $s = 3$, in order to exhibit recursive and asymptotic properties for the integers $\tau_3(n) = M_n$, where M_n denotes the n-th Motzkin number.

2 The two-column case

In the following, we will denote the shape of a standard Young tableau T as a list containing the length of the first, second, ..., last column of T.

We first consider the standard Young tableaux whose shape consists of at most two columns. We define an infinite, lower triangular matrix $A = (\alpha_{n,i})$, where $\alpha_{n,i}$ is the number of standard Young tableaux whose shape corresponds to the list $(n - i, i)$, with $i \in \mathbb{N}$.

The matrix A can be recursively constructed as follows:

Proposition 1 *The matrix A is defined by the following initial conditions:*

  i) $\alpha_{i,0} = 1$ *for every* $i \geq 0$;

  ii) $\alpha_{1,i} = 0$ *for every* $i > 0$;

iii) $\alpha_{n,i} = 0$ *for every* $i > \lfloor \frac{n}{2} \rfloor$

and by the recurrence:

$$\alpha_{n,i} = \alpha_{n-1,i} + \alpha_{n-1,i-1} \quad for \quad n \geq 1, 1 \leq i \leq \lfloor \frac{n}{2} \rfloor. \tag{1}$$

Proof The initial conditions follow immediately by the definition of the matrix A. In order to prove the recurrence, remark that a Young tableau with n cells can be obtained from a tableau with $n-1$ cells by adding a new box containing the symbol n in a corner position. In particular, in a Young tableau of shape $(n-i,i)$, $i > 0$, the symbol n can be placed in at most two corner cells. Hence, the entry $\alpha_{n,i}$ is the sum of the two integers $\alpha_{n-1,i-1}$ and $\alpha_{n-1,i}$, that count the Young tableaux of appropriate shape. $\square$

The recurrence formula (1) has as an immediate fallout the following result:

Corollary 2 *The entries of the matrix A satisfy the following columnwise recurrence:*

$$\alpha_{n,i} = \sum_{h=2i-1}^{n-1} \alpha_{h,i-1}$$

for every $n, i > 0$.

We remark that the elements of the matrix A correspond bijectively to the entries of the Ballot Matrix defined by Aigner in [1]. More precisely, the Ballot Matrix $\widetilde{A} = (\tilde{\alpha}_{i,j})$ can be obtained rearranging the entries of the matrix A as follows:

$$\tilde{\alpha}_{j,k} = \alpha_{2j-k,j-k}.$$

As a consequence, for every n, the number of standard Young tableaux with two columns of the same length n is the n-th Catalan number C_n:

$$\alpha_{2n,n} = C_n \tag{2}$$

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Denote by $\tau_2(n)$ the number of Young tableaux with n cells and at most two columns. Obviously, we have:

$$\tau_2(n) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} a_{n,i}.$$

In the next theorem we deduce a recurrence formula for the sequence $\tau_2(n)_{n\geq 1}$:

Theorem 3 *The integers $\tau_2(n)$, $n \geq 1$, satisfy the recurrence*

$$\tau_2(n) = 2\tau_2(n-1) - E(n-1) \cdot C_{\frac{n-1}{2}}, \tag{3}$$

where C_i denotes the i-th Catalan number and

$$E(n) = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

Proof   By Proposition 1, we have:

$$\tau_2(n) = \sum_{i=0}^{\lfloor\frac{n}{2}\rfloor} a_{n,i} = \sum_{i=0}^{\lfloor\frac{n}{2}\rfloor} a_{n-1,i-1} + \sum_{i=0}^{\lfloor\frac{n}{2}\rfloor} a_{n-1,i} =$$

$$= 2\tau_2(n) - E(n-1) \cdot \alpha_{n-1,\frac{n-1}{2}} = 2\tau_2(n) - E(n-1) \cdot C_{\frac{n-1}{2}},$$

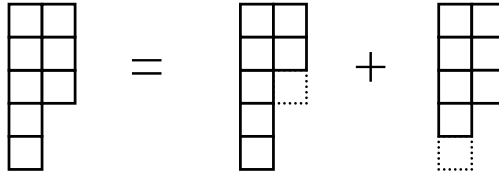as desired.   □

Figure 1: The recurrence formula for the entry $\alpha_{8,3}$.

The preceding result yields an alternative proof of the following theorem, originally proved by Regev in [12]:

Theorem 4 *The number of standard Young tableaux with exactly n cells and at most two columns is*

$$\tau_2(n) = \binom{n}{\lfloor\frac{n}{2}\rfloor}.$$

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Proof   It is well known that the central binomial coefficients $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ satisfy the recurrence (3). Remarking that $\tau_2(0) = 1 = \binom{0}{0}$, we have the assertion.
□

Moreover, identity (3) yields the following asymptotic property of the integers $\tau_2(n)$:

Proposition 5 *We have:*

$$\lim_{n \to \infty} \frac{\tau_2(n)}{\tau_2(n-1)} = 2$$

Proof   By (3) we get:

$$\frac{\tau_2(n)}{\tau_2(n-1)} = \frac{2\tau_2(n-1) - E(n+1) \cdot C_{\frac{n-1}{2}}}{\tau_2(n-1)}.$$

This implies that the assertion is proved as soon as we show that

$$\lim_{n \to \infty} \frac{E(n+1) \cdot C_{\frac{n-1}{2}}}{\tau_2(n-1)} = 0.$$

If n is even, this identity trivially holds. If n is odd, the described correction term is negligeable with respect to $\tau_2(n-1)$ as n goes to infinity. In fact, we get:

$$\lim_{n \to \infty} \frac{\frac{1}{n}\binom{n-1}{\frac{n-1}{2}}}{\binom{n-1}{\frac{n-1}{2}}} = \lim_{n \to \infty} \frac{1}{n} = 0,$$

as required.
□

3   The three-column case

We now extend the results of Section 2 to the case of standard Young tableaux with at most three columns. Define a matrix $B = (\beta_{n,i})$ such that the (n,i)-th entry of B is the total number of standard Young tableaux of shape $(n-i-2k, i+k, k)$, for every possible value of k. In other terms, the integer $\beta_{n,i}$ is the cardinality of the set $Y_{n,i}^3$ containing all standard tableaux on n

cells with at most 3 columns such that the difference between the second and third column is i.

The entries of the matrix B satisfy a three-term recurrence:

Proposition 6 *The integers $\beta_{n,0}$ satisfy the identity:*

$$\beta_{n,0} = \beta_{n-1,0} + \beta_{n-1,1}. \tag{4}$$

Moreover, for every $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$, we have:

$$\beta_{n,i} = \beta_{n-1,i-1} + \beta_{n-1,i} + \beta_{n-1,i+1} - r_{n,i}, \tag{5}$$

where

$$r_{n,i} = \begin{cases} \text{number of tableaux of shape } (\frac{n+i-2}{3}, \frac{n+i-2}{3}, \frac{n-2i+1}{3}) & \text{if } n - 2i \equiv_3 2 \\ 0 & \text{otherwise.} \end{cases}$$

Proof Let T be a standard Young tableau T with 3 columns of shape $(n - i - 2k, i + k, k)$, for some k, and let T' be the tableau obtained from T by removing the cell containing the symbol n. Obviously, the tableau T' has either shape $(n - i - 2k - 1, i + k, k)$ or $(n - i - 2k, i + k - 1, k)$ or $(n - i - 2k, i + k, k - 1)$. This implies that the correspondence $T \mapsto T'$ maps the set $Y^3_{n,i}$ into the union $Y^3_{n-1,i-1} \cup Y^3_{n-1,i} \cup Y^3_{n-1,i+1}$. However, such map is not in general surjective. In fact, it is easy to check that the difference between the set $Y^3_{n-1,i-1} \cup Y^3_{n-1,i} \cup Y^3_{n-1,i+1}$ and the image of the map consists exactly of the tableaux of shape $(\frac{n+i-2}{3}, \frac{n+i-2}{3}, \frac{n-2i+1}{3})$. This proves (5). The first recurrence can be proved by the same arguments, observing that, in this case, the map defined above is surjective. $\square$

The preceding result yields a recurrence formula satisfied by the total number $\tau_3(n)$ of standard Young tableaux with n cells and at most 3 columns:

Theorem 7 *The numbers $\tau_3(n)$, $n \geq 3$, satisfy the recurrence*

$$\tau_3(n) = 3\tau_3(n - 1) - R(n) - E(n - 1) \cdot C_{\frac{n-1}{2}} - \beta_{n-1,0}, \tag{6}$$

where

$$R(n) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} r_{n,i}.$$

Proof By definition of $\beta_{n,i}$ we have:

$$\tau_3(n) = \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \beta_{n,i}.$$

Hence,

$$\tau_3(n) = \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \beta_{n-1,i-1} + \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \beta_{n-1,i} + \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \beta_{n-1,i+1} - \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} r_{n-1,i-1}. \qquad (7)$$

Remark that

$$\tau_3(n-1) = \sum_{i=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \beta_{n-1,i}.$$

The proof is an immediate consequence of the following considerations:

- the first summand in the right hand side of Identity (7) is equal to $\tau_3(n-1)$ if n is even. In case of n odd, instead, this term exceeds $\tau_3(n-1)$ by the Catalan number $C_{\frac{n-1}{2}}$,

- the second summand in the right hand side of Identity (7) is exactly $\tau_3(n-1)$,

- the third summand in the right hand side of Identity (7) exceeds the integer $\tau_3(n-1)$ exactly by $\beta_{n-1,0}$,

- the fourth summand in the right hand side of Identity (7) is obviously equal to $R(n-1)$.

$\square$

Remark that an explicit evaluation of the integers $Q(n)$ and $R(n)$ can be obtained via the well known Hook Length Formula.

Previous considerations lead to the following result concerning the asymptotic behaviour of the sequence $\tau_3(n)$:

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Proposition 8 *For every integer $n \geq 1$, we have:*

$$\frac{\tau_3(n)}{\tau_3(n-1)} < 3. \tag{8}$$

Moreover,

$$\lim_{n \to \infty} \frac{\tau_3(n)}{\tau_3(n-1)} = 3. \tag{9}$$

Proof Inequality (8) follows directly by previous considerations. In order to prove (9) we remark that, by Theorem (7), the ratio $\frac{\tau(n)}{\tau(n-1)}$ can be written as:

$$\frac{\tau_3(n)}{\tau_3(n-1)} = 3 - (U_1(n-1) + U_2(n-1) + U_3(n-1)),$$

where

$$U_1(n-1) = \frac{E(n+1) \cdot C_{\frac{n-1}{2}}}{\tau_3(n-1)}$$

$$U_2(n-1) = \frac{\beta_{n-1,0}}{\tau_3(n-1)}$$

$$U_3(n-1) = \frac{R(n-1)}{\tau_3(n-1)}.$$

The statement will be proved as soon as we show that each summand $U_i(n-1)$ goes to 0 as n goes to infinity. The numerator of each $U_i(n-1)$ is a sum of a suitable finite number f_λ. For every f_λ appearing in one of these expression, we can single out an appropriate subset S_λ of summands whose cardinality grows as n goes to infinity, such that:

$$\lim_{n \to \infty} \frac{f_\lambda}{\tau_3(n-1)} \leq \lim_{n \to \infty} \frac{f_\lambda}{\sum_{\mu \in S_\lambda} f_\mu} = 0.$$

For instance, consider the rectangular shape ρ over $n-1$ blocks, namely, the shape corresponding to the values $i = 0$, $k = \frac{n-1}{3}$. Obviously, such a shape exists only if $n - 1 \equiv 0 \pmod 3$. In this case, the family S_ρ consists of the shapes over $n-1$ cells and corresponding to the values $i = h$, $k = k = \frac{n-1}{3} - h$, where h ranges from 1 to $k = \frac{n-1}{6}$. By Hook Length Formula, we get:

$$\lim_{n \to \infty} \frac{f_\rho}{\tau_3(n-1)} \leq \lim_{n \to \infty} \frac{f_\rho}{\sum_{\mu \in S_\rho} f_\mu} = \lim_{n \to \infty} \frac{f_\rho}{\sum_{h=1}^{\lfloor \frac{n-1}{6} \rfloor} (h+1)^3 f_\rho} \leq \lim_{n \to \infty} \frac{1}{8 \lfloor \frac{n-1}{6} \rfloor} = 0.$$

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The sets S_ν corresponding to the other summand of each of the numerators of the $U_i(n-1)$ can be described analogously. ☐

As proved in [12], $\tau_3(n) = M_n$, where M_n is the n-th Motzkin number. Hence, these last arguments yield a combinatorial proof of Propositions 4 and 5 in [2].

4   The general case

In this section we extend the previous argumentations to the general case of standard Young tableaux with at most s columns. Define a matrix $\Gamma^{(s)} = (\gamma_{n,i}^{(s)})$ such that the (n,i)-th entry is the total number of standard Young tableaux with at most s columns and such that the difference between the length of the second and of the third column is i.

Also in this case, when $s \geq 4$ and $n \geq s$, the entries of this matrix satisfy a three-term recurrence. In fact, the following proposition can be proved by the same arguments used to prove Proposition 6:

Proposition 9 *The integers $\gamma_{n,0}^{(s)}$ satisfy the identity:*

$$\gamma_{n,0}^{(s)} = (s-2)\gamma_{n-1,0}^{(s)} + \gamma_{n-1,1}^{(s)} - \sum_{j=3}^{s-1} r_j^{(s)}(n-1,0). \qquad (10)$$

Moreover, for every $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$, we have:

$$\gamma_{n,i}^{(s)} = \gamma_{n-1,i-1}^{(s)} + (s-2)\gamma_{n-1,i}^{(s)} + \gamma_{n-1,i+1}^{(s)} - r_1^{(s)}(n-1,i-1)$$

$$- \sum_{j=3}^{s-1} r_j^{(s)}(n-1,i), \qquad (11)$$

where $r_j^{(s)}(n-1,i)$ is the number of standard Young tableaux with at most s columns such that the difference between the length of the second and of the third column is i and the j-th and the $(j+1)$-th columns have the same length.

☐

Similarly, exploiting the same arguments as in Proposition 8, we get:

Proposition 10 *The numbers $\tau_s(n)$ satisfy the recurrence:*

$$\tau_s(n) = s\tau_s(n-1) - E(n-1)C_{\frac{n-1}{2}} - \gamma_{n-1,0}^{(s)} - R^{(s)}(n-1), \qquad (12)$$

where $R^{(s)}(n-1)$ is the sum of all the correction terms appearing in Formula (11).

□

Now we can easily deduce from these identities some information concerning the asymptotic behaviour of the sequence $\tau_s(n)$:

Proposition 11 *The sequence $\tau_s(n)$ satisfies the following properties:*

$$\frac{\tau_s(n)}{\tau_s(n-1)} < s$$

$$\lim_{n \to \infty} \frac{\tau_s(n)}{\tau_s(n-1)} = s.$$

□

Acknowledgments

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