The signed Eulerian numbers on involutions

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Abstract. We define an analogue of signed Eulerian numbers $f_{n,k}$ for involutions of the symmetric group and derive some combinatorial properties of this sequence. In particular, we exhibit both an explicit formula and a recurrence for $f_{n,k}$ arising from the properties of its generating function.

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1 Introduction

Let σ be a permutation in S_n. We say that σ has a *descent* at position i whenever $\sigma(i) > \sigma(i+1)$. Analogously, we say that σ has a *rise* at position i whenever $\sigma(i) < \sigma(i+1)$. The number of descents (respectively rises) of a permutation σ is denoted by $des(\sigma)$ (resp. $ris(\sigma)$). The polynomial

$$A_n(t) = \sum_{\sigma \in S_n} t^{des(\sigma)} = \sum_{k=0}^{n-1} a_{n,k} t^k,$$

is known as the *Eulerian polynomial*, and the integers $a_{n,k}$, i.e., the number of permutations $\sigma \in S_n$ with $des(\sigma) = k$, are called the *Eulerian numbers*. We recall that the Eulerian numbers satisfy the property $a_{n,k} = a_{n,n-1-k}$, that implies

$$\sum_{\sigma \in S_n} t^{des(\sigma)} = \sum_{\sigma \in S_n} t^{ris(\sigma)}. \tag{1}$$

The study of the distribution of the descent statistic has been carried out both in the case of the symmetric group and of some particular subsets of

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permutations. For example, Eulerian distribution on the set of involutions $\mathscr{I}_n \subseteq S_n$ has been deeply investigated by several authors ([5], [4], [6], and [1]).

Loday [8], in his study of the cyclic homology of commutative algebras, introduced the sequence $(b_{n,k})$ of *signed Eulerian numbers*, namely, the coefficients of the polynomial

$$B_n(t) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) t^{ris(\sigma)} = \sum_{k=0}^{n-1} b_{n,k} t^k.$$

In [3], Désarménien and Foata showed several combinatorial properties satisfied by the integers $b_{n,k}$ by exploiting their relations with the Eulerian numbers. More recently, Tanimoto (see, e.g., [11]) developedthe study of signed Eulerian numbers from a different point of view.

In this paper we study the signed Eulerian numbers on involutions, i.e., the coefficients of the *signed Eulerian polynomial on involutions*

$$F_n(t) = \sum_{\sigma \in \mathscr{I}_n} \mathrm{sgn}\sigma t^{ris(\sigma)} = \sum_{k=0}^{n-1} f_{n,k} t^k.$$

To this aim, we exploit a map introduced in [2] that associates an involution with a family of generalized involutions, which share the shape of the corresponding semistandard Young tableau. This map yields an explicit formula for the number of involutions in $\mathscr{I}_n$ with a given number of rises. The above map yields also an analogue of Worpitzky Identity for the sequence $f_{n,k}$:

$$\sum_{j=0}^{s-1} \binom{n+j}{j} f_{n,s-j-1} = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \binom{\binom{s+1}{2}+j-1}{j} \binom{s}{n-2j}$$

and, hence, an explicit formula for the signed Eulerian numbers on involutions.
This formula gives the following expression for the generating function of the signed Eulerian polynomials on involutions:

$$\sum_{n \geq 0} F_n(t) \frac{u^n}{(1-t)^{n+1}} = \sum_{r \geq 0} t^r \frac{(1+u)^{r+1}}{(1+u^2)^{\binom{r+2}{2}}}.$$

Applying the Maple package $ZeilbergerRecurrence(T, n, k, s, 0..n)$ to this last identity, we find a recurrence satisfied by the integers $f_{n,k}$.
Moreover, we exhibit an explicit formula, a recurrence, and a generating function for the sequence $F_n(1)$, with $n \in \mathbb{N}$, namely, the difference between the number of even and odd involutions in $\mathscr{I}_n$.

2 The signed Eulerian numbers

In order to investigate the combinatorial properties of the signed Eulerian polynomial on involutions, we exploit the relations between involutions and particular biwords, called generalized involutions, introduced in [1].

A *generalized involution* of length n is defined to be a biword:

$$\alpha = \binom{x}{y} = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{pmatrix},$$

such that:

- for every $1 \leq i \leq n$, there exists an index j with $x_i = y_j$ and $y_i = x_j$,
- $x_i \leq x_{i+1}$,
- $x_i = x_{i+1} \Longrightarrow y_i \geq y_{i+1}$.

We say that an integer a is a *repetition* of multiplicity r for the generalized involution α if

$$x_i = y_i = x_{i+1} = y_{i+1} = \cdots = x_{i+r-1} = y_{i+r-1} = a.$$

In [2], the authors introduced a map Π from the set of generalized involutions of length n to the set of involutions $\mathscr{I}_n$ defined as follows: if

$$\alpha = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{pmatrix},$$

then $\Pi(\alpha)$ is the involution σ

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ y'_1 & y'_2 & \cdots & y'_n \end{pmatrix},$$

where $y'_i = 1$ if y_i is the least symbol occurring in the word y, $y'_j = 2$ if y_j is the second least symbol in y and so on. In the case $y_i = y_j$, with $i > j$, we consider y_i to be less then y_j. We call the involution $\sigma = \Pi(\alpha)$ the *polarization* of α.

We denote by $\mathrm{Gen}_m(\sigma)$ the set of generalized involutions of length n, with symbols taken from $[m]$, whose polarization is σ. Then, we have the following result proved in [1]:

Proposition 1 *Let $\sigma \in \mathscr{I}_n$ be an involution with t rises. Then,*

$$|Gen_m(\sigma)| = \binom{n+m-t-1}{n}. \tag{2}$$

$\diamond$

We recall that the sign of an involution $\sigma \in \mathscr{I}_n$ is determined by the number $fix(\sigma)$ of fixed points of σ. More precisely:

$$\mathrm{sgn}(\sigma) = (-1)^{\frac{n-fix(\sigma)}{2}}.$$

It is well known [9] that the integer $fix(\sigma)$ depends only on the shape of the tableau T_σ. On the other hand, it is immediately seen that the semistandard tableau associated with any generalized involution in $\mathrm{Gen}_m(\sigma)$ by the Robinson-Schensted-Knuth algorithm has the same shape as the standard tableau T_σ associated with σ. We are interested in counting separately the number of even and odd involutions with a given number of rises. Consequently, we can define an *even generalized involution* to be a generalized involution whose polarization is an even involution.
Remark that the sign of a generalized involution depends only on the multiplicity of its repetitions. In fact, given a generalized involution α, we define $gfix(\alpha)$ to be the number of repetitions of odd multiplicity of α (note that, if $\alpha \in \mathscr{I}_n$, then $gfix(\alpha) = fix(\alpha)$). It is easy to verify that a generalized involution α of length n is even if and only if

$$\frac{n - gfix(\sigma)}{2}$$

is even. This remark allows to compute explicitly the number $a^+_{n,m}$ of generalized involutions of length n over $[m]$:

Proposition 2 *We have:*

$$a^+_{n,m} = \sum_{h=0}^{\lfloor \frac{n}{4} \rfloor} \binom{m}{n-4h} \binom{\binom{m+1}{2} + 2h - 1}{2h}.$$

Proof Let α be a generalized involution. We remarked that α is even whenever $\frac{n-gfix(\sigma)}{2}$ is even. We want to count generalized involutions with $\frac{n-gfix(\sigma)}{2} = 2h$ with fixed h. We can choose the $n-4h$ fixed points of α in $\binom{m}{n-4h}$ ways. Then, we choose $2h$ biletters (i,j) in the set $B = \{(i,j)|1 \le i \le j \le m\}$, whose cardinality is $\binom{m+1}{2}$. We complete the generalized involution by adding a biletter (j,i) for every biletter (i,j) previously chosen, hence getting an even generalized involution.

$\diamond$

On the other hand, recalling that the total number of generalized involutions of length n over $[m]$ is (see, e.g., [1])

$$a_{n,m} = \sum_{h=0}^{\lfloor \frac{n}{2} \rfloor} \binom{m+n-2h-1}{n-2h} \binom{\binom{m}{2} + h - 1}{h},$$

we get an explicit formula for the number $a^-_{n,m}$ of odd generalized involutions of length n over $[m]$:

$$a^-_{n,m} = \sum_{h=0}^{\lfloor \frac{n}{2} \rfloor} \binom{m+n-2h-1}{n-2h} \binom{\binom{m}{2} + h - 1}{h} - \sum_{h=0}^{\lfloor \frac{n}{4} \rfloor} \binom{m}{n-4h} \binom{\binom{m+1}{2} + 2h - 1}{2h}.$$

We remark that, given $\sigma \in \mathscr{I}_n$, all generalized involutions in $\mathrm{Gen}_m(\sigma)$ have the same sign as σ. Hence, Proposition 1 implies that

$$a^+_{n,m} = \sum_{k=0}^{m-1} \binom{n+k}{k} f^+_{n,m-k-1} \tag{3}$$

$$a^-_{n,m} = \sum_{k=0}^{m-1} \binom{n+k}{k} f^-_{n,m-k-1}, \tag{4}$$

where $f^+_{n,k}$ and $f^-_{n,k}$ denote the number of of positive and negative involutions in $\mathscr{I}_n$ with k rises, respectively. These identities allow to state the following:

Theorem 3 *We have:*

$$f_{n,k} = \sum_{m=0}^{k+1} (-1)^{k-m+1} \binom{n+1}{k-m+1} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \binom{\binom{m+1}{2}+j-1}{j} \binom{m}{n-2j}. \tag{5}$$

Proof Set $\hat{a}_{n,m} = a^+_{n,m} - a^-_{n,m}$. Then, Identities (3) and (4) imply

$$\hat{a}_{m,n} = \sum_{k=0}^{m-1} \binom{n+k}{k} f_{n,m-k-1}. \tag{6}$$

By inversion, we have:

$$f_{n,k} = \sum_{m=0}^{k+1} (-1)^{k-m+1} \binom{n+1}{k-m+1} \hat{a}_{n,m} =$$

$$= \sum_{m=0}^{k+1} (-1)^{k-m+1} \binom{n+1}{k-m+1} \left(2 \sum_{j=0}^{\lfloor \frac{n}{4} \rfloor} \binom{m}{n-4j} \binom{\binom{m+1}{2}+2j-1}{2j} \right.$$

$$\left. - \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{m+n-2j-1}{n-2j} \binom{\binom{m}{2}+j-1}{j} \right)$$

that is equivalent to (5).

$\diamond$

Remark that Identity (6) gives the following analogue of the Worpitzky Identity:

$$\sum_{j=0}^{s-1} \binom{n+j}{j} f_{n,s-j-1} = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \binom{\binom{s+1}{2}+j-1}{j} \binom{s}{n-2j}.$$

3 The signed Eulerian polynomial on involutions

We define the n-th *signed Eulerian polynomial for involutions* to be the polynomial:

$$F_n(t) = \sum_{\sigma \in \mathscr{I}_n} \mathrm{sgn}(\sigma) t^{ris(\sigma)} = \sum_{k=0}^{n-1} f_{n,k} t^k.$$

As shown in the previous section, the relation between involutions and generalized involutions is crucial in our analysis. We denote by

$$R_n(t) = \sum_{m \geq 0} \hat{a}_{n,m} t^m$$

$$C_m(t) = \sum_{n \geq 0} \hat{a}_{n,m} t^n$$

the row and column generating functions of the array $\hat{a}_{n,m}$. As seen in the proof of Theorem 3, we have

$$\hat{a}_{n,m} = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \binom{\binom{m+1}{2} + j - 1}{j} \binom{m}{n-2j}.$$

and hence

$$C_m(t) = \frac{(1+u)^m}{(1+u^2)^{\binom{m+1}{2}}}.$$

Theorem 4 *The polynomial $F_n(t)$ satisfies the identity*

$$\sum_{n \geq 0} F_n(t) \frac{u^n}{(1-t)^{n+1}} = \sum_{m \geq 0} t^m \frac{(1+u)^{m+1}}{(1+u^2)^{\binom{m+2}{2}}}. \tag{7}$$

Proof The binomial relation between the sequences $f_{n,k}$ and $\hat{a}_{n,m}$ yields the following identity:

$$\frac{tF_n(t)}{(1-t)^{n+1}} = R_n(t).$$

Hence:

$$\sum_{n \geq 0} F_n(t) \frac{u^n}{(1-t)^{n+1}} = \sum_{n \geq 0} \sum_{m \geq 0} \hat{a}_{n,m+1} t^m u^n =$$

$$= \sum_{m\geq 0} C_{m+1} t^m = \sum_{m\geq 0} t^m \frac{(1+u)^{m+1}}{(1+u^2)^{\binom{m+2}{2}}}$$

as desired.

$\diamond$

We apply the Maple package $ZeilbergerRecurrence(T, n, k, s, 0..n)$ to the sequence $f_{n,k}$, following along the lines of [6], and we deduce by Identity (7) the following recurrence formula:

$$nf_{n,k} = (2+k-n)f_{n-1,k} + (2n-k-1)f_{n-1,k-1} - (n+3k+k^2)f_{n-2,k}+$$
$$+(-2+4k+2k^2-2kn)f_{n-2,k-1} + (2-k-k^2+2kn-n^2)f_{n-2,k-2}+$$
$$+(-n-k^2-2k+2)f_{n-3,k} + (-7+4k+3k^2+2n-2kn)f_{n-3,k-1}+$$
$$+(8-2k-3k^2-2n+4kn-n^2)f_{n-3,k-2} + (-3+k^2+n-2kn+n^2)f_{n-3,k-3}.$$

In conclusion, we study the combinatorial properties of the sequence $F_n(1)$, with $n \in \mathbb{N}$. Obviously, $F_n(1)$ is the difference between the number i_n^+ of even involutions on n objects and the number i_n^- of odd involutions. First of all, we have:

Proposition 5 *The evaluation of the polynomial $F_n(t)$ at 1 is*

$$F_n(1) = 2\sum_{h=0}^{\lfloor \frac{n}{4} \rfloor} \frac{n!}{(2h)!(n-4h)!2^{2h}} - \sum_{h=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!}{(n-2h)!2^h}.$$

Proof Fix an integer $h \leq \lfloor \frac{n}{4} \rfloor$. We count the number of involutions whose cycle decomposition consists of $2h$ transpositions. Choose a word $w = w_1 \cdots w_n$ consisting of distinct letters taken from $[n]$. We have $n!$ choices for w. This word correspond to a unique even involution τ with $n-4h$ fixed points defined by the following conditions:

$$\tau(w_1) = w_2 \quad \ldots \quad \tau(w_{4h-1}) = w_{4h};$$

$$\tau(w_{4h+j}) = w_{4h+j}$$

with $0 < j \leq n - 4h$. It is easily checked that the involution τ arises from $(n-4h)!(2h)!2^{2h}$ different words w. These considerations imply that

$$i_n^+ = \sum_{h=0}^{\lfloor \frac{n}{4} \rfloor} \frac{n!}{(2h)!(n-4h)!2^{2h}}.$$

On the other hand, it is well known that

$$|\mathscr{I}_n| = \sum_{h=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!}{(n-2h)!2^h}.$$

Hence, since $F_n(1) = i_n^+ - i_n^- = 2i_n^+ - |\mathscr{I}_n|$, we get the assertion.

$\diamond$

Proposition 6 *The sequence $F_n(1)$ satisfies the recurrence*

$$F_n(1) = F_{n-1}(1) - (n-1)F_{n-2}(1).$$

Proof Consider an even involution $\tau \in \mathscr{I}_n$. If $\tau(1) = 1$, the restriction of τ on the set $\{2, \ldots, n\}$ is an even involution on $n-1$ objects. If $\tau(1) = j \neq 1$, the restriction of τ on $\{2, \ldots, n\} \setminus \{j\}$ is an odd involution on $n-2$ objects. This implies that:

$$i_n^+ = i_{n-1}^+ + (n-1)i_{n-2}^-$$

and, analogously,

$$i_n^- = i_{n-1}^- + (n-1)i_{n-2}^+.$$

These identities gives immediately the assertion.

$\diamond$

These results allow to deduce the following expression for the exponential generating function of the sequence $F_n(1)$:

$$\sum_{n \geq 0} \frac{F_n(1)}{n!} t^n = e^{t - \frac{t^2}{2}}.$$

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