BILINEAR FORMS ON THE DIRICHLET SPACE

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1. Introduction and Summary

Hankel operators on the Hardy space of the disk, $H^2(\mathbb{D})$, can be studied as linear operators from $H^2(\mathbb{D})$ to its dual space, as conjugate linear operators from $H^2(\mathbb{D})$ to itself, or, in the viewpoint we will take here, as bilinear functionals on $H^2(\mathbb{D}) \times H^2(\mathbb{D})$. In that formulation, given a holomorphic *symbol function* b we consider the bilinear Hankel form, defined initially for f, g in $\mathcal{P}(\mathbb{D})$, the space of polynomials, by

$$S_b(f,g) := \langle fg, b\rangle_{H^2}.$$

The norm of S_b is

$$\|S_b\|_{H^2\times H^2} = \sup\{|S_b(f,g)| : \|f\|_{H^2} = \|g\|_{H^2} = 1\}.$$

Nehari's classical criterion for the boundedness of S_b can be cast in modern language using Fefferman's duality theorem. We say a positive measure μ on the disk is a Carleson measure for H^2 if

$$\|\mu\|_{CM(H^2)} := \sup\left\{\int_{\mathbb{D}} |f|^2\, d\mu : \|f\|_{H^2} = 1\right\} < \infty$$

and that b is in the space BMO if

$$\|b\|_{BMO} := |b(0)| + \left\| |b'(z)|^2 (1-|z|^2) dA \right\|_{CM(H^2)} < \infty.$$

Nehari's theorem [15] is the equivalence $\|S_b\|_{H^2\times H^2} \approx \|b\|_{BMO}$.

Our main result is an analogous statement for a similar class of bilinear forms on the Dirichlet space $\mathcal{D}(\mathbb{D}) = \mathcal{D}$. We will give the definitions and statement now with further background discussion in the next section. Recall that $\mathcal{D}$ is the Hilbert space of holomorphic functions on the disk with inner product

$$\langle f, g\rangle_{\mathcal{D}} = f(0)\overline{g(0)} + \int_{\mathbb{D}} f'(z)\overline{g'(z)}\, dA,$$

and normed by $\|f\|_{\mathcal{D}}^2 = \langle f, f\rangle_{\mathcal{D}}$. We consider a holomorphic *symbol function* b and define the associated bilinear form, initially for $f, g \in \mathcal{P}(\mathbb{D})$, by

$$T_b(f,g) := \langle fg, b\rangle_{\mathcal{D}}.$$

The norm of T_b is

$$\|T_b\|_{\mathcal{D}\times\mathcal{D}} = \sup\{|T_b(f,g)| : \|f\|_{\mathcal{D}} = \|g\|_{\mathcal{D}} = 1\}.$$

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We say a positive measure μ on the disk is a Carleson measure for $\mathcal{D}$ if

$$\|\mu\|_{CM(\mathcal{D})} := \sup\left\{\int_{\mathbb{D}} |f|^2 \, d\mu : \|f\|_{\mathcal{D}} = 1\right\} < \infty,$$

and that the function b is in the space $\mathcal{X}$ if

$$\|b\|_{\mathcal{X}} := |b(0)| + \left\| |b'(z)|^2 \, dA \right\|_{CM(\mathcal{D})} < \infty.$$

Our main result is

Theorem 1.

$$\|T_b\|_{\mathcal{D}\times\mathcal{D}} \approx \|b\|_{\mathcal{X}}$$

In the next section we give some background and show how Theorem 1 can be reformulated as a duality result for a space which is presented by its *weak factorization.* In notation introduced there we show that $(\mathcal{D} \odot \mathcal{D})^* = \mathcal{X}$. We also combine our result with earlier work to conclude that $I\left(D\mathcal{D} \odot \mathcal{D}\right) = \mathcal{D} \odot \mathcal{D}$.

The third section contains the proof of the main theorem. It is easy to see that $\|T_b\|_{\mathcal{D}\times\mathcal{D}} \leq C \|b\|_{\mathcal{X}}$. To obtain the other inequality we must use the boundedness of T_b to show $|b'|^2 \, dA$ is a Carleson measure. Analysis of the capacity theoretic characterization of Carleson measures due to Stegenga allows us to focus attention on a certain set V in $\mathbb{D}$ and the relative sizes of $\int_V |b'|^2$ and the capacity of the set $\bar{V} \cap \partial\bar{\mathbb{D}}$. To compare these quantities we construct $V_{\exp}$, an expanded version of the set V which satisfies two conflicting conditions. First, $V_{\exp}$ is not much larger than V, either when measured by $\int_{V_{\exp}} |b'|^2$ or by the capacity of the $\overline{V_{\exp}} \cap \partial\bar{\mathbb{D}}$. Second, $\mathbb{D}\backslash V_{\exp}$ is well separated from V in a way that allows the interaction of quantities supported on the two sets to be controlled. Once this is done we can construct a function $\Phi_V \in \mathcal{D}$ which is approximately one on V and which has Φ_V' approximately supported on $\mathbb{D}\backslash V_{\exp}$. Using Φ_V we build functions f and g with the property that

$$|T_b(f,g)| = \int_V |b'|^2 + \text{ error.}$$

The technical estimates on Φ_V allow us to show that the error term is small and the boundedness of T_b then gives the required control of $\int_V |b'|^2$.

2. Background on Hankel Forms

2.1. **Bilinear Hankel Forms.** Similar bilinear forms can be defined for many spaces of holomorphic functions. Suppose $\mathcal{H}$ is a Hilbert space of holomorphic functions on a domain Ω and suppose for convenience that $\mathcal{P}(\Omega) \subset \mathcal{H}$. Given a function b holomorphic in Ω and $f, g \in \mathcal{P}(\Omega)$ we can define a Hankel form, K_b, by

$$K_b(f,g) = \langle fg, b\rangle_{\mathcal{H}}.$$

One can then ask what conditions on b are necessary and sufficient for K_b to be bounded. These questions have been studied in very many contexts; examples include [15], [9], [12], [6] [11], [13], [20]. Also, in a context not involving function theory, the boundedness result of Maz'ya and Verbitsky, [14], also appears to be part of this pattern. It is fascinating that although there is a great deal of variety in the techniques used in the proofs, there is a surprising similarity in the answers obtained. The answer, quite generally, is that for some differential operator $\mathfrak{D}$, $|\mathfrak{D}b|^2$ can be used to define a Carleson measure for $\mathcal{H}$. And, although this paper provides another instance of this, the authors do not have a heuristic explanation for the pattern.

There is a close connection between boundedness results for this type of bilinear form, duality theorems for function spaces, and weak factorization results for function spaces; see for instance [1] and [8]. Here is how those ideas play out in this context. Define the weakly factored space $\mathcal{D}\odot\mathcal{D}$ to be the completion of finite sums $h=\sum f_j g_j$ under the norm

$$\|h\|_{\mathcal{D}\odot\mathcal{D}}=\inf\left\{\sum\|f_j\|_{\mathcal{D}}\|g_j\|_{\mathcal{D}}:h=\sum f_j g_j\right\}.$$

A corollary of Theorem 1 is

Corollary 1. *With the pairing $(h,b)=\langle h,b\rangle_{\mathcal{D}}=T_b(h,1)$ we have that $(\mathcal{D}\odot\mathcal{D})^*=\mathcal{X}$. That is, if $\Lambda\in(\mathcal{D}\odot\mathcal{D})^*$, there is a unique $b\in\mathcal{X}$ with $\Lambda h=T_b(h,1)$ for $h\in\mathcal{P}(\mathbb{D})$, and $\|\Lambda\|=\|T_b\|\approx\|b\|_{\mathcal{X}}$.*

Proof. If $b\in\mathcal{X}$ and $h\in\mathcal{D}\odot\mathcal{D}$, say $h=\sum f_i g_i$ with $\sum\|f_i\|_{\mathcal{D}}\|g_i\|_{\mathcal{D}}\leq\|h\|_{\mathcal{D}\odot\mathcal{D}}+\varepsilon$, then

$$\begin{aligned}|\langle h,b\rangle_{\mathcal{D}}| &= \left|\sum_{i=1}^{\infty}\langle f_i g_i,b\rangle_{\mathcal{D}}\right|=\left|\sum_{i=1}^{\infty}T_b\langle f_i,g_i\rangle\right|\\ &\leq \|T_b\|\left|\sum_{i=1}^{\infty}\|f_i\|_{\mathcal{D}}\|g_i\|_{\mathcal{D}}\right|\leq\|T_b\|\left(\|h\|_{\mathcal{D}\odot\mathcal{D}}+\varepsilon\right).\end{aligned}$$

It follows that $\Lambda_b h=\langle h,b\rangle_{\mathcal{D}}$ defines a continuous linear functional on $\mathcal{D}\odot\mathcal{D}$ with $\|\Lambda_b\|\leq\|T_b\|$.

Conversely, if Λ is a continuous linear functional on $\mathcal{D}\odot\mathcal{D}$ with norm $\|\Lambda\|$, then

$$|\Lambda h|=|\Lambda(h\cdot 1)|\leq\|\Lambda\|\,\|h\|_{\mathcal{D}}\|1\|_{\mathcal{D}}=\|\Lambda\|\,\|h\|_{\mathcal{D}},$$

for $h\in\mathcal{D}$, and so there is a unique $b\in\mathcal{D}$ such that $\Lambda h=\Lambda_b h$ for $h\in\mathcal{D}$. Finally, if $h=fg$ with $f,g\in\mathcal{D}$ we have

$$\begin{aligned}|T_b(f,g)| &= |\langle fg,b\rangle_{\mathcal{D}}|=|\Lambda_b h|=|\Lambda h|\\ &\leq \|\Lambda\|\,\|h\|_{\mathcal{D}\odot\mathcal{D}}\leq\|\Lambda\|\,\|f\|_{\mathcal{D}}\|g\|_{\mathcal{D}},\end{aligned}$$

which shows that T_b extends to a continuous bilinear form on $\mathcal{D}\odot\mathcal{D}$ with $\|T_b\|\leq\|\Lambda\|$. By the theorem we conclude $b\in\mathcal{X}$ and also, with the other estimates, that $\|\Lambda\|=\|T_b\|\approx\|b\|_{\mathcal{X}}$. ∎

The corresponding statement in the Hardy space is Fefferman's duality theorem, $\left(H^2\odot H^2\right)^*=BMO$. However our result is only a partial analog. In the Hardy space context one also has internal characterizations of the spaces involved. That is, not only is $H^2\odot H^2=H^1$ but H^1 can be defined intrinsically using integral means; similarly BMO can be defined using oscillation and without reference to Carleson measures..In contrast we do not have a satisfactory intrinsic characterization of either the functions in $\mathcal{D}\odot\mathcal{D}$ of those in $\mathcal{X}$.

2.2. **Other Hankel Forms.** On the Hardy space, letting P denote the projection from $L^2(\mathbb{T})$ to H^2, we have, for any $f,g\in H^2$,

$$S_b(f,g)=\langle fg,b\rangle_{H^2}=\langle f,\bar{g}b\rangle_{H^2}=\langle f,P(\bar{g}b)\rangle_{H^2}$$

Hence the study of S_b is roughly equivalent to the study of the conjugate linear map $\widetilde{S_b}:g\to P(\bar{g}b)$. (The use of *conjugate* linear operators is an inessential convenience.)

Similar comments apply for Hankel forms on, for example, the Bergman space. However this type of rewriting does not work in $\mathcal{D}$ because it is not true that for three holomorphic functions f, g, and b we have $\langle fg, b\rangle_{\mathcal{D}} = \langle f, \bar{g}b\rangle_{\mathcal{D}}$.

Focusing for the moment on functions which vanish at the origin, we have

$$\begin{aligned} T_b(f,g) &= \int (fg)'\,\overline{b'}\,dA \\ &= \int (f'g + fg')\overline{b'}\,dA. \\ &= \int f'g\overline{b'}dA + \int fg'\overline{b'}\,dA. \end{aligned}$$

Let P_B be the orthogonal projection of $L^2(\mathbb{D}, dA)$ onto the Bergman space. We can analyze the first summand as

$$\begin{aligned} \int f'g\overline{b'}dA &= \int g\overline{\overline{f'}b'}dA \\ &= \int g\overline{P_B(\overline{f'}b')}dA \\ &= \left\langle g, I^2 P_B(\overline{f'}b')\right\rangle_{\mathcal{D}}. \end{aligned}$$

Here I is the operator of indefinite integration. This operator is similar in spirt to $\widetilde{S_b}$ but some symmetry has been lost; it is not true that $\left\langle f, \widetilde{T_b}g\right\rangle$ equals $\left\langle g, \widetilde{T_b}f\right\rangle$ and in fact both terms are needed to reconstruct $T_b(f,g)$. Operators such as $\widetilde{S_b}$. and $\widetilde{T_b}$ are also sometimes called Hankel operators. It is shown in [17] that $\widetilde{T_b}$ is bounded exactly if $\|b\|_{\mathcal{X}} < \infty$. An analogous result for real variable operators is in [10].

Define the space $I(D\mathcal{D} \odot \mathcal{D})$ to be the completion of the space of functions h such that h' can be written as a finite sum, $h' = \sum f_j' g_j$,with the norm

$$\|h\|_{I(D\mathcal{D}\odot\mathcal{D})} = \inf\left\{\sum \|f_j\|_{\mathcal{D}}\|g_j\|_{\mathcal{D}} : h' = \sum f_j' g_j\right\}.$$

Define $\mathcal{X}_0$ to be the norm closure in $\mathcal{X}$ of the polynomials.

Corollary 2. $\mathcal{X}_0^* = I(D\mathcal{D} \odot \mathcal{D}) = \mathcal{D} \odot \mathcal{D}$.

Proof. It is shown by Wu in [18] that the first two spaces are the same. It is immediate that $\mathcal{D} \odot \mathcal{D} \subset I(D\mathcal{D} \odot \mathcal{D})$. It is shown in [17] that $I(D\mathcal{D} \odot \mathcal{D})^* = \mathcal{X}$. Combining that with the previous corollary we see that the second and third spaces have the same dual. Hence by the Hahn-Banach theorem the two spaces agree. ∎

2.3. **Preliminary Steps of the Proof.**

2.3.1. *The Easy Direction.* Suppose that μ_b is a $\mathcal{D}$-Carleson measure. For $f,g\in\mathcal{P}(\mathbb{D})$ we have

$$\begin{aligned}
|T_b(f,g)| &= \left| f(0)\,g(0)\,\overline{b(0)} + \int\limits_{\mathbb{D}} \left(f'(z)\,g(z)+f(z)\,g'(z)\right)\overline{b'(z)}dA \right| \\
&\leq |f(0)\,g(0)\,b(0)| + \int\limits_{\mathbb{D}} |f'(z)\,g(z)\,b'(z)|\,dA + \int\limits_{\mathbb{D}} |f(z)\,g'(z)\,b'(z)|\,dA \\
&\leq |(fgb)(0)| + \|f\|_{\mathcal{D}} \left(\int\limits_{\mathbb{D}} |g|^2\,d\mu_b\right)^{\frac{1}{2}} + \|g\|_{\mathcal{D}} \left(\int\limits_{\mathbb{D}} |f|^2\,d\mu_b\right)^{\frac{1}{2}} \\
&\leq C\left(|b(0)| + \|\mu_b\|_{\mathcal{D}-Carleson}\right)\|f\|_{\mathcal{D}}\,\|g\|_{\mathcal{D}}\,. \\
&= C\,\|b\|_{\mathcal{X}}\,\|f\|_{\mathcal{D}}\,\|g\|_{\mathcal{D}}\,.
\end{aligned}$$

Thus T_b has a bounded extension to $\mathcal{D}\times\mathcal{D}$ with $\|T_b\|\leq C\,\|b\|_{\mathcal{X}}$

We also note for later use that if T_b extends to a bounded bilinear form on $\mathcal{D}$ then $b\in\mathcal{D}$. Setting $g=1$ we obtain

$$|\langle f,b\rangle_{\mathcal{D}}| = |T_b(f,1)| \leq \|T_b\|\,\|f\|_{\mathcal{D}}\,\|1\|_{\mathcal{D}}$$

for all polynomials $f\in\mathcal{P}(\mathbb{D})$, which shows that $b\in\mathcal{D}$ and

$$\|b\|_{\mathcal{D}} \leq C\,\|T_b\|\,. \tag{2.1}$$

2.3.2. *Capacity and Tree Extremal Functions.* For an interval I in the circle we let I_m be its midpoint and $z(I)=\left(1-\frac{|I|}{2\pi}\right)z$ be the associated index point in the disk. In the other direction we set $I(z)$ to be the interval such that $z(I(z))=z$. We set $T(I)$, the tent over I to be the convex hull of I and $z(I)$ and let $T(z)=T(z(I)):=T(I)$. More generally, for any open subset H of the circle $\mathbb{T}$, we set

$$T(H)=\cup_{I\subset H}T(I)$$

To complete the proof we will show that μ_b given by

$$\mu_b=|b'|^2\,dA$$

is a $\mathcal{D}$-Carleson measure by verifying a condition due to Stegenga [19]. He works with a capacity defined by, for G in the circle $\mathbb{T}$,

$$\operatorname{Cap}_{\mathbb{D}} G = \inf\left\{\|\psi\|_{\mathcal{D}}^2 : \psi(0)=0, \operatorname{Re}\psi(z)\geq 1 \text{ for } z\in G\right\}. \tag{2.2}$$

and shows that μ is a Carleson measure exactly if for any finite collection of disjoint arcs $\{I_j\}_{j=1}^N$ in the circle $\mathbb{T}$ we have

$$\mu\left(\dot{\cup}_{j=1}^N T(I_j)\right) \leq C\ Cap_{\mathbb{D}}\left(\dot{\cup}_{j=1}^N I_j\right). \tag{2.3}$$

In our proof we use functions which are approximate extremals for measuring capacity, that is functions for which the equality in (2.2) is approximately attained.

We will also need to understand how the capacity of a set changes if we expand it in certain ways. More precisely for I an open arc and $0<\rho\leq 1$, let I^ρ be the arc concentric with I having length $|I|^\rho$. For G open in $\mathbb{T}$ let

$$G^\rho=\cup_{I\subset G}T(I^\rho)\,.$$

In both our study of approximate extremals and of capacity of expanded sets we find it convenient to transfer our arguments to and from rotations of the Bergman tree $\mathcal{T}_1$ associated with $\mathbb{D}$ and to work with tree capacities instead. A detailed description and properties of these trees are presented in [4] and we follow that treatment.

Consider a dyadic tree T together with the following notation. If x is an element of the tree T, x^{-1} denotes its immediate predecessor in T. If z is an element of the sequence $Z \subset T$, Pz denotes its predecessor in Z: $Pz \in Z$ is the maximum element of $Z \cap [o, z)$ (we assume $o \in Z$ for convenience). Let $\mathcal{T}_1(\theta)$ be the rotation of the tree $\mathcal{T}_1$ by the angle θ, and let Cap_θ be the tree capacity associated with $\mathcal{T}_1(\theta)$:

$$Cap_\theta(G) = \inf \left\{ \sum_{\kappa \in \mathcal{T}_1(\theta)} f(\kappa)^2 : f(o) = 0,\ f(\beta) \geq 1 \text{ for } \beta \in \mathcal{T}_1(\theta),\, I(\beta) \subset G \right\}.$$

We say that $S \subset \mathcal{T}_1$ is a stopping region if every pair of distinct points in S are incomparable in $\mathcal{T}_1$.

Let $\Omega \subseteq T$. A point $x \in T$ is in the interior of Ω if $x, x^{-1}, x_+, x_- \in \Omega$. A function H is *harmonic* in Ω if

$$H(x) = \frac{1}{3}[H(x^{-1}) + H(x_+) + H(x_-)] \tag{2.4}$$

for every point x which is interior in Ω. Let $Ih(x) = \sum_{y \in [o,x]} h(y)$. If $H = Ih$ is harmonic in Ω, then we have that

$$h(x) = h(x_+) + h(x_-) \tag{2.5}$$

whenever x is in the interior of Ω.

The following proposition and remark are essentially proved in [4].

Proposition 1. *Let T be a dyadic tree and suppose that E and F are subsets as above.*

1. *There is an extremal function $H = Ih$ such that $Cap(E, F) = \|h\|^2_{\ell^2}$.*
2. *The function H is harmonic on $T \setminus (E \cup F)$.*
3. *If S is a stopping region in $\mathcal{T}_1$, then $\sum_{\kappa \in S} |h(\kappa)| \leq 2Cap(E, F)$.*
4. *The function h is positive on geodesics joining a point of E to a point of F, and zero everywhere else.*

Remark 1. *There is the following useful formula for computing capacities: given $z < \zeta$ and $U_\pm \subset S(\zeta_\pm) \subset S(z_+)$,*

$$Cap(z, U_+ \cup U_-) = \frac{Cap(\zeta, U_+) + Cap(\zeta, U_-)}{1 + d(z,\zeta)[Cap(\zeta, U_+) + Cap(\zeta, U_-)]}. \tag{2.6}$$

2.3.3. *The Capacity of Tree Blowups.* We also need the tree analogue $G^\rho(\theta)$ of the blowup G^ρ of an open set G:

$$G^\rho(\theta) = \cup \{T(A^\rho \kappa) : \kappa \in \mathcal{T}_1(\theta) \text{ with } I(\kappa) \subset G\},$$

where $A^\rho\kappa$ denotes the unique element in the tree $\mathcal{T}_1(\theta)$ satisfying

$$\begin{aligned} o &\leq A^\rho\kappa \leq \kappa, \\ \rho d(\kappa) &< d(A^\rho\kappa) \leq \rho d(\kappa) + 1. \end{aligned}$$

Consider a subtree $\{o, \kappa, \beta, \gamma\}$ of distinct elements in the dyadic tree $\mathcal{T}_1$ where $o < \kappa = \beta \wedge \gamma$ and β, γ are incomparable. Given $\varepsilon > 0$, suppose that $\beta < \beta^*$ and $\gamma < \gamma^*$ where

$$\frac{d(\beta^*)}{d(\beta)} \leq 1 + \varepsilon \text{ and } \frac{d(\gamma^*)}{d(\gamma)} \leq 1 + \varepsilon.$$

We claim that

$$\frac{Cap_o\{\beta, \gamma\}}{Cap_o\{\beta^*, \gamma^*\}} \leq 1 + \varepsilon. \tag{2.7}$$

Indeed, with $a = d(\kappa)$, $b = d(\kappa, \beta)$, $c = d(\kappa, \gamma)$ and b^*, c^* defined by

$$a + b^* = (1 + \varepsilon)(a + b) \text{ and } a + c^* = (1 + \varepsilon)(a + c),$$

we have

$$\begin{aligned} \frac{1}{Cap_o\{\beta, \gamma\}} &= d(\kappa) + \frac{1}{\frac{1}{d(\kappa,\beta)} + \frac{1}{d(\kappa,\gamma)}} \\ &= a + \frac{1}{\frac{1}{b} + \frac{1}{c}} = a + \frac{bc}{b + c}, \end{aligned}$$

and

$$\begin{aligned} \frac{1}{Cap_o\{\beta^*, \gamma^*\}} &= d(\kappa) + \frac{1}{\frac{1}{d(\kappa,\beta^*)} + \frac{1}{d(\kappa,\gamma^*)}} \\ &\leq a + \frac{1}{\frac{1}{b^*} + \frac{1}{c^*}} = a + \frac{b^* c^*}{b^* + c^*} \\ &= a + \frac{[b + \varepsilon(a + b)][c + \varepsilon(a + c)]}{[b + \varepsilon(a + b)] + [c + \varepsilon(a + c)]} \\ &= \frac{1}{Cap_o\{\beta, \gamma\}} \\ &\quad + \frac{[b + \varepsilon(a + b)][c + \varepsilon(a + c)]}{[b + \varepsilon(a + b)] + [c + \varepsilon(a + c)]} - \frac{bc}{b + c}. \end{aligned}$$

Thus we need to prove that

$$\frac{[b + \varepsilon(a + b)][c + \varepsilon(a + c)]}{[b + \varepsilon(a + b)] + [c + \varepsilon(a + c)]} - \frac{bc}{b + c} \leq \frac{\varepsilon}{Cap_o\{\beta, \gamma\}} = \varepsilon\left(a + \frac{bc}{b + c}\right).$$

A calculation reveals that the left side is

$$\begin{aligned} &\frac{(b + c)[b + \varepsilon(a + b)][c + \varepsilon(a + c)] - bc\{b + c + \varepsilon(2a + b + c)\}}{\{b + c + \varepsilon(2a + b + c)\}(b + c)} \\ = \ &\varepsilon \frac{(b + c)(ab + ac + 2bc) - bc(2a + b + c)}{\{b + c + \varepsilon(2a + b + c)\}(b + c)} \\ &+ \varepsilon \frac{\varepsilon(a + b)(a + c)}{\{b + c + \varepsilon(2a + b + c)\}}, \end{aligned}$$

so that after dividing by ε and multiplying by $b + c$, we are left with showing that

$$\frac{(b + c)(ab + ac + 2bc) - bc(2a + b + c) + \varepsilon(a + b)(a + c)(b + c)}{\{b + c + \varepsilon(2a + b + c)\}} \leq ab + ac + bc.$$

The left hand side is of the form $f(\varepsilon) = (\alpha + \beta\varepsilon) / (\gamma + \delta\varepsilon)$ with all the quantities positive. Hence $f'(\varepsilon)$ is continuous and of constant sign and it suffices to verify the inequality at $\varepsilon = 0$ and in the limit as $\varepsilon \to \infty$. At $\varepsilon = 0$ we have

$$\begin{aligned} \frac{(b+c)(ab+ac+2bc) - bc(2_A + b + c)}{(b+c)} &= ab + ac + bc - \frac{2abc}{b+c} \\ &\leq ab + ac + bc. \end{aligned}$$

For the limit as $\varepsilon \to \infty$ we need to check

$$\frac{(a+b)(a+c)(b+c)}{2a+b+c} \leq ab + ac + bc.$$

or

$$(a+b)(a+c)(b+c) \leq (2a+b+c)(ab+ac+bc).$$

When we do the multiplication the same monomials appear on both sides and the coefficients on the left are never larger.

The analogue of (2.7) holds by the same proof for the "virtual edges" created in the algorithm for computing capacities in [4]. Applying induction we obtain the following result.

Proposition 2. *Let $Z = \{z_j\}_{j=1}^N$ be a stopping time in $\mathcal{T}_1$. Choose elements $z_j^* \geq z_j$ such that $d(z_j^*) \leq (1+\varepsilon) d(z_j)$ for $1 \leq j \leq N$. Then with $Z^* = \{z_j^*\}_{j=1}^N$ we have*

$$\frac{Cap_o Z}{Cap_o Z^*} \leq 1 + \varepsilon.$$

We will use the following corollary. Given a point $w \in \mathcal{T}_1$ and $0 < \rho < 1$, define w^ρ to be the unique point in $\mathcal{T}_1$ satisfying $o \leq w^\rho \leq w$ and

$$\rho d(w) < d(w^\rho) \leq \rho d(w) + 1.$$

Corollary 3. *Let $W = \{w_\ell\}_{\ell=1}^M$ be a stopping time in $\mathcal{T}_1$. Suppose $0 < \rho < 1$ and let $Z = \{z_j\}_{j=1}^N$ consist of the minimal tree elements in the set $\{w_\ell^\rho\}_{\ell=1}^M$. Then*

$$Cap_o Z < \frac{1}{\rho} Cap_o W.$$

Proof. For each j, there is w_ℓ such that $z_j = w_\ell^\rho$. Fix such a choice and let $z_j^* = w_\ell$. Then

$$d(z_j^*) = d(w_\ell) < \frac{1}{\rho} d(w_\ell^\rho) = \frac{1}{\rho} d(z_j)$$

for $1 \leq j \leq N$. The corollary now follows from Proposition 2 with $\frac{1}{\rho} = 1 + \varepsilon$. ∎

The conclusion of the corollary can be stated

$$Cap(\{o\}, Z) < \frac{1}{\rho} Cap(\{o\}, W).$$

Unfortunately we cannot extend these arguments in the opposite direction to obtain a condenser estimate,

$$Cap(Z, W) < \frac{1}{1-\rho} Cap(\{o\}, W).$$

Indeed, the left side can even be finite while the right side is finite as shown by an example below. Thus the geometric blowup construction does not lead to a useful capacity estimate for the condenser $Cap(W^\rho, W)$. On the other hand we do have a useful capacity estimate for W^ρ in terms of $CapW$ while achieving a good geometric separation between W^ρ and W, a fact that plays a crucial role in using Schur's test to estimate an integral below, as well as in estimating an error term near the end of the paper.

To construct an appropriate condenser, we will instead use a method based on a *capacitary* extemal and a comparison principle. Let W be a stopping time in $\mathcal{T}_1$. By Proposition 1, there is a unique extremal function $H = Ih$ such that

$$\begin{aligned} H(o) &= 0, \\ H(x) &= 1 \text{ for } x \in W, \\ CapW &= \|h\|_{\ell^2}^2, \end{aligned} \tag{2.8}$$

where Cap denotes tree capacity in T_1. Given stopping times $E, F \subset \mathcal{T}_1$ we say that $E \succ F$ if for every $x \in E$ there is $y \in F$ with $y < x$. For stopping times $E \succ F$ denote by $\mathcal{G}(E, F)$ the union of all those geodesics connecting a point of $x \in E$ to the point $y \in E$ lying above it, i.e. $y < x$.

Given a stopping time W, the corresponding extremal H satisfying (2.8), and $0 < \rho < 1$, define the *capacitary* blowup $\widetilde{W^\rho}$ of W (as opposed to the *geometric* blowup W^ρ of W) by

$$\widetilde{W^\rho} = \{t \in \mathcal{G}(\{o\}, W) : H(t) \geq \rho \text{ and } H(x) \leq \rho \text{ for } x < t\}.$$

Lemma 1. *$Cap\widetilde{W^\rho} \leq \frac{1}{\rho^2} CapW$.*

Proof: Let $H^\rho = \frac{1}{\rho} H$ and $h^\rho = \frac{1}{\rho} h$ where $h = \triangle H$ and H is the extremal for W in (2.8). Then H^ρ is a candidate for the infimum in the definition of capacity of $\widetilde{W^\rho}$, and hence by the "comparison principle",

$$Cap\widetilde{W^\rho} \leq \|h^\rho\|_{\ell^2}^2 = \left(\frac{1}{\rho}\right)^2 \|h\|_{\ell^2}^2 = \frac{1}{\rho^2} CapW.$$

This capacitary blowup $\widetilde{W^\rho}$, unlike the geometric blowup W^ρ, does indeed satisfy a condenser inequality. Note that by (2.22) below, it suffices to obtain a condenser inequality only for W with small capacity.

Lemma 2. *$Cap\left(W, \widetilde{W^\rho}\right) \leq \frac{4}{(1-\rho)^2} CapW$ provided $CapW \leq \frac{1}{4}(1-\rho)^2$.*

Proof: Let H be the extremal for W in (2.8). For $t \in \widetilde{W^\rho}$ we have by our assumption,

$$h(t) \leq \|h\|_{\ell^2} \leq \sqrt{CapW} \leq \frac{1}{2}(1-\rho),$$

and so

$$H(t) = H(At) + h(t) \leq \rho + \frac{1}{2}(1-\rho) = \frac{1+\rho}{2}.$$

If we define $\widetilde{H}(t) = \frac{2}{1-\rho}\left\{H(t) - \frac{1+\rho}{2}\right\}$, then $\widetilde{H} \leq 0$ on $\widetilde{W^\rho}$ and $\widetilde{H} = 1$ on W. Thus $\widetilde{H}$ is a candidate for the capacity of the condenser and so by the "comparison principle",

$$\begin{aligned} Cap\left(W, \widetilde{W^\rho}\right) &\leq \left\|\triangle\widetilde{H}\right\|_{\ell^2\left(\mathcal{G}\left(\widetilde{W^\rho}, W\right)\right)}^2 \leq \left\|\triangle\widetilde{H}\right\|_{\ell^2(\mathcal{T}_1)}^2 \\ &= \left(\frac{2}{1-\rho}\right)^2 \|h\|_{\ell^2(\mathcal{T}_1)}^2 = \frac{4}{(1-\rho)^2} CapW. \end{aligned}$$

Corollary 4. *Let Z be a stopping time in $\mathcal{T}_1$ and suppose that $0<\gamma<\alpha<1$. Set $E=Z^{\alpha}$ and $F=\widetilde{E^{\rho}}$ where $\rho=\sqrt{\frac{\gamma}{\alpha}}$. Then*

$$\begin{aligned} Cap F &\leq \frac{1}{\gamma} Cap Z, \\ Cap\left(E,F\right) &\leq \frac{4}{\alpha\left(1-\sqrt{\frac{\gamma}{\alpha}}\right)^2} Cap Z. \end{aligned}$$

Proof: We have from Lemma 1 and Corollary 3,

$$Cap F = Cap \widetilde{E^{\rho}} \leq \frac{1}{\rho^2} Cap E = \frac{\alpha}{\gamma} Cap Z^{\alpha} < \frac{\alpha}{\gamma}\frac{1}{\alpha} Cap Z = \frac{1}{\gamma} Cap Z,$$

and then from Lemma 2 and Corollary 3,

$$Cap\left(E,F\right) = Cap\left(E, \widetilde{E^{\rho}}\right) \leq \frac{4}{\left(1-\rho\right)^2} Cap E < \frac{4}{\alpha\left(1-\rho\right)^2} Cap Z.$$

Recall that $T\left(G\right)$ is the tent above G as defined in the tree $\mathcal{T}_1$, and similarly for $T\left(G^t\right)$, $0<t<1$. Given a stopping time Z we let $\mathcal{S}_Z$ be the shadow of Z on the circle $\mathbb{T}$, so that $T_0\left(\mathcal{S}_Z\right)=Z$. Consider the condenser $\left(E,F\right)$ where $E=T_0\left(G^{\alpha}\right)=\left\{w_k^{\alpha}\right\}_k$ and $F=\widetilde{E}^{\sqrt{\frac{\gamma}{\alpha}}}=\left\{w_k^{\gamma}\right\}_k$. Note that each point w_k^{α} in E is a descendent of a unique w_{ℓ}^{γ} in F. By Proposition 1, there is a unique extremal function $H=Ih$ such that

$$\begin{aligned} H &= 0 \text{ on } F, \\ H &= 1 \text{ on } E, \\ Cap\left(E,F\right) &= \left\|h\right\|_{\ell^2}^2, \end{aligned}$$

where Cap denotes tree capacity in $\mathcal{T}_1$. It follows from Corollary 4 that

$$Cap\ G \leq Cap\left(E,F\right) \leq \frac{4}{\alpha\left(1-\sqrt{\frac{\gamma}{\alpha}}\right)^2} Cap\ G. \tag{2.9}$$

Notation 1. *Let $0<\beta<\gamma<\alpha<1$. We will use the shorthand notations*

$$\begin{aligned} V_G^{\alpha} &= T\left(G^{\alpha}\right), \\ V_G^{\gamma} &= \widetilde{V_G^{\alpha}}^{\sqrt{\frac{\gamma}{\alpha}}}, \\ V_G^{\beta} &= T\left(\mathcal{S}_{V_G^{\gamma}}^{\frac{\beta}{\gamma}}\right). \end{aligned}$$

We also define maximal intervals $\left\{J_k^{\alpha}\right\}_k, \left\{J_k^{\gamma}\right\}_k, \left\{J_k^{\beta}\right\}_k$ such that

$$V_G^{\alpha} = \cup T\left(J_k^{\alpha}\right) \text{ and } V_G^{\gamma} = \cup T\left(J_k^{\gamma}\right) \text{ and } V_G^{\beta} = \cup T\left(J_k^{\beta}\right).$$

Thus V_G^α is a geometric blowup of $T(G)$, V_G^γ is a capacitary blowup of V_G^α, and V_G^β is again a geometric blowup of V_G^γ. We have from above the estimates

$$\begin{aligned} Cap\left(V_G^\alpha\right) &< \frac{1}{\alpha} CapG, \\ Cap\left(V_G^\gamma\right) &< \frac{1}{\gamma} CapG, \\ Cap\left(V_G^\beta\right) &< \frac{1}{\beta} CapG, \\ Cap\left(V_G^\alpha, V_G^\gamma\right) &< C_{\alpha,\gamma} CapG. \end{aligned}$$

Remark 2. *The geometric blowups have good geometric separation properties (useful when estimating Bergman type kernels) while the capacitary blowup has a good condenser estimate (useful in constructing holomorphic extremals). The geometric separation between V_G^α and $T(G)$ is used in (2.25) below in conjunction with the Schur test, the good condenser estimate of $\left(V_G^\alpha, V_G^\gamma\right)$ is used in the construction of a holomorhic extermal in Lemma 3 below, and finally the geometric separation between V_G^β and V_G^γ is used in estimating term (4_{ABA}) near the end of the paper.*

2.3.4. *Holomorphic Approximate Extremals and Capacity Estimates.* Now we define a holomorphic approximation Φ to the function $H = Ih$ on $\mathcal{T}_1$ constructed in Proposition 1. We will use a parameter s. In addition to specific assumptions made at various places we will always assume

$$s > -1.$$

Define $\varphi_\kappa(z) = \left(\frac{1-|\kappa|^2}{1-\overline{\kappa}z}\right)^{1+s}$ and

$$\Phi(z) = \sum_{\kappa\in\mathcal{T}_1} h(\kappa)\varphi_\kappa(z) = \sum_{\kappa\in\mathcal{T}_1} h(\kappa)\left(\frac{1-|\kappa|^2}{1-\overline{\kappa}z}\right)^{1+s}. \tag{2.10}$$

Note that

$$\sum_{\kappa\in\mathcal{T}_1} h(\kappa) I\delta_\kappa(z) = I\left(\sum_{\kappa\in\mathcal{T}_1} h(\kappa)\delta_\kappa\right)(z) = Ih(z) = H(z),$$

and so

$$\Phi(z) - H(z) = \sum_{\kappa\in\mathcal{T}_1} h(\kappa)\left\{\varphi_\kappa - I\delta_\kappa\right\}(z). \tag{2.11}$$

We will also need to write Φ in terms of the projection operator

$$\Gamma_s h(z) = \int_{\mathbb{D}} h(\zeta)\frac{\left(1-|\zeta|^2\right)^s}{\left(1-\overline{\zeta}z\right)^{1+s}} dA, \tag{2.12}$$

namely as $\Phi = \Gamma_s g$ where

$$g(\zeta) = \sum_{\kappa\in\mathcal{T}_1} h(\kappa)\frac{1}{|B_\kappa|}\frac{\left(1-\overline{\zeta}\kappa\right)^{1+s}}{\left(1-|\zeta|^2\right)^s}\chi_{B_\kappa}(\zeta), \tag{2.13}$$

and B_κ is the Euclidean ball centered at κ with radius $c(1-|\kappa|)$ for a sufficiently small positive constant c to be chosen later. The function Φ satisfies the following estimates.

Lemma 3. *Suppose $z \in \mathcal{T}_1$ and $s > -1$. Then we have*

$$\left\{ \begin{array}{lll} \left|\Phi(z) - \Phi(w_k^\alpha)\right| & \leq & C Cap(E,F), \quad z \in T(J_k^\alpha) \\ \operatorname{Re}\Phi(w_k^\alpha) & \geq & c > 0, \quad 1 \leq k \leq M_\alpha \\ \left|\Phi(w_k^\alpha)\right| & \leq & C, \quad 1 \leq k \leq M_\alpha \\ \left|\Phi(z)\right| & \leq & C Cap(E,F), \quad z \notin V_G^\gamma \end{array} \right. . \tag{2.14}$$

Lemma 4. *Furthermore, if $s > -\frac{1}{2}$ then $\Phi = \Gamma_s g$ where*

$$\left|g(\zeta)\right|^2 dA \leq C Cap(E,F). \tag{2.15}$$

Corollary 5. *For $s > \frac{1}{2}$, (2.15) and Lemma 2.4 of* [5] *show that*

$$\|\Phi\|_{\mathcal{D}}^2 \leq \int_{\mathbb{D}} \left|g(\zeta)\right|^2 dA \leq C Cap(E,F). \tag{2.16}$$

Remark 3. *In the case that $V_G^\alpha = T(J_1^\alpha)$ consists of a single arc, we may divide the function $\Phi(z)$ by $\Phi(w_k^\alpha)$ to obtain a holomorphic function that is close to* 1 *on V_G^α and small outside V_G^γ as in Lemma 4.1 in* [5].

Proof. From (2.11) we have

$$\begin{aligned} \left|\Phi(z) - H(z)\right| &\leq \sum_{\kappa \in [o,z]} \left|h(\kappa)\{\varphi_\kappa(z) - 1\}\right| + \sum_{\kappa \notin [o,z]} \left|h(\kappa)\varphi_\kappa(z)\right| \\ &= I(z) + II(z). \end{aligned}$$

We also have that h is nonnegative and supported in $V_G^\gamma \setminus V_G^\alpha$. We first show that

$$II(z) \leq \sum_{\kappa \notin [o,z]} h(\kappa) \left|\frac{1 - |\kappa|^2}{1 - \overline{\kappa} z}\right|^{1+s} \leq C Cap(E,F).$$

For $A > 1$ let

$$\Omega_k = \left\{\kappa \in \mathcal{T}_1 : A^{-k-1} < \left|\frac{1 - |\kappa|^2}{1 - \overline{\kappa} z}\right| \leq A^{-k}\right\}.$$

If we choose A sufficiently close to 1, then for every k the set Ω_k is a stopping time for $\mathcal{T}_1$, i.e. any two distinct elements in Ω_k are incomparable in $\mathcal{T}_1$. Now we use the stopping time property 3 in Proposition 1 to obtain

$$\sum_{\kappa \in \Omega_k} h(\kappa) \leq C Cap(E,F), \ k \geq 0.$$

Altogether we then have

$$II(z) \leq \sum_{k=0}^{\infty} \sum_{\kappa \in \Omega_k} h(\kappa) A^{-k(1+s)} \leq C_s Cap(E,F).$$

If $z \in \mathcal{T}_1 \setminus V_G^\gamma$, then $I(z) = 0$ and $H(z) = 0$ and we have

$$\left|\Phi(z)\right| = \left|\Phi(z) - H(z)\right| \leq II(z) \leq C_s Cap(E,F),$$

which is the fourth line in (2.14).

If $z \in V_G^\alpha \cap \mathcal{T}_1$, let k be such that $z \geq w_k^\alpha$. Then for $\kappa \notin [o,z]$ we have

$$\left|\varphi_\kappa(w_k^\alpha)\right| \leq C \left|\varphi_\kappa(z)\right|,$$

and for $\kappa \in [o,z]$ we have

$$\left|\varphi_\kappa(z) - \varphi_\kappa(w_k^\alpha)\right| = \left|\left(\frac{1-|\kappa|^2}{1-\overline{\kappa}z}\right)^{1+s} - \left(\frac{1-|\kappa|^2}{1-\overline{\kappa}w_k^\alpha}\right)^{1+s}\right| \leq C_s \frac{|z-w_k^\alpha|}{1-|\kappa|^2}.$$

Thus

$$\begin{aligned} |\Phi(z) - \Phi(w_k^\alpha)| &\leq \sum_{\kappa\in[o,w_k^\alpha]} h(\kappa)\left|\varphi_\kappa(z) - \varphi_\kappa(w_k^\alpha)\right| + C \sum_{\kappa\notin[o,z]} h(\kappa)\left|\varphi_\kappa(z)\right| \\ &\leq C_s \sum_{\kappa\in[o,w_k^\alpha]} h(\kappa)\frac{|z-w_k^\alpha|}{1-|\kappa|^2} + CII(z) \\ &\leq C_s Cap(E,F), \end{aligned}$$

since $h(\kappa) \leq CCap(E,F)$ and $\sum_{\kappa\in[o,w_k^\alpha]} \frac{1}{1-|\kappa|^2} \approx \frac{1}{1-|w_k^\alpha|^2}$. This proves the first line in (2.14).

Moreover, we note that for $s=0$ and $\kappa \in [o,w_k^\alpha]$,

$$\operatorname{Re}\varphi_\kappa(w_k^\alpha) = \operatorname{Re}\frac{1-|\kappa|^2}{1-\overline{\kappa}w_k^\alpha} = \operatorname{Re}\frac{1-|\kappa|^2}{|1-\overline{\kappa}w_k^\alpha|^2}\left(1-\kappa\overline{w_k^\alpha}\right) \geq c > 0.$$

A similar result holds for $s > -1$ provided the Bergman tree $\mathcal{T}_1$ is constructed sufficiently thin depending on s. It then follows from $\sum_{\kappa\in[o,w_k^\alpha]} h(\kappa) = 1$ that

$$\begin{aligned} \operatorname{Re}\Phi(w_k^\alpha) &= \sum_{\kappa\in[o,w_k^\alpha]} h(\kappa)\operatorname{Re}\varphi_\kappa(w_k^\alpha) + \sum_{\kappa\notin[o,z]} h(\kappa)\operatorname{Re}\varphi_\kappa(w_k^\alpha) \\ &\geq c\sum_{\kappa\in[o,w_k^\alpha]} h(\kappa) - CCap(E,F) \geq c' > 0. \end{aligned}$$

We trivially have

$$|\Phi(w_k^\alpha)| \leq I(z) + II(z) \leq C\sum_{\kappa\in[o,w_k^\alpha]} h(\kappa) + CCap(E,F) \leq C,$$

and this completes the proof of (2.14).

Finally we prove (2.15). From property 1 of Proposition 1 we obtain

$$\begin{aligned} \int_{\mathbb{D}} |g(\zeta)|^2 dA &= \int_{\mathbb{D}} \left|\sum_{\kappa\in\mathcal{T}_1} h(\kappa)\frac{1}{|B_\kappa|}\frac{\left(1-\overline{\zeta}\kappa\right)^{1+s}}{\left(1-|\zeta|^2\right)^s}\chi_{B_\kappa}(\zeta)\right|^2 dA \\ &= \sum_{\kappa\in\mathcal{T}_1} |h(\kappa)|^2 \frac{1}{|B_\kappa|^2}\int_{B_\kappa}\frac{\left|1-\overline{\zeta}\kappa\right|^{2+2s}}{\left(1-|\zeta|^2\right)^{2s}}dA \\ &\approx \sum_{\kappa\in\mathcal{T}_1} |h(\kappa)|^2 \approx Cap(E,F). \end{aligned}$$

∎

Corollary 6. *Let G be a finite union of arcs in the circle $\mathbb{T}$. Then*

$$Cap_{\mathcal{T}_1}(G) \approx Cap_{\mathbb{D}}(G), \tag{2.17}$$

where $Cap_{\mathbb{D}}$ denotes Stegenga's capacity on the circle $\mathbb{T}$.

Proof. The inequality $\lessapprox$ follows easily from Lemma 3 which provides a candidate for testing the Stegenga capacity of G. Indeed, let c, C be the constants in Lemma 3, and suppose that $Cap\,(E,F) \leq \frac{c}{3C}$. Set $\Psi(z) = \frac{3}{c}(\Phi(z) - \Phi(0))$. Then $\Psi(0) = 0$,

$$\begin{aligned}\operatorname{Re}\Psi(z) &= \frac{3}{c}\{\operatorname{Re}\Phi(z) - \operatorname{Re}\Phi(0)\} \\ &\geq \frac{3}{c}\{c - 2C\operatorname{Cap}(E,F)\} \geq 1,\ z \in V_G^{\alpha},\end{aligned}$$

and by (2.16) we have

$$\|\Psi\|_{\mathcal{D}}^2 = \left(\frac{3}{c}\right)^2 \|\Phi\|_{\mathcal{D}}^2 \leq \left(\frac{3}{c}\right)^2 C Cap\,(E,F).$$

Continuing by invoking Corollary **??** we obtain that for $G \subset \mathbb{T}$,

$$\|\Psi\|_{\mathcal{D}}^2 \leq \left(\frac{3}{c}\right)^2 C Cap\,(E,F) \leq C Cap_{\mathcal{T}_1} G.$$

Conversely, to obtain the inequality $\gtrapprox$, let $\psi \in \mathcal{D}$ be an extremal function for $Cap_{\mathbb{D}} G$. Define $h(o) = 0$ and

$$h(\kappa) = (1 - |\kappa|) \int_{Q(\kappa)} |\psi'(z)|\, d\lambda(z),\ \kappa \in \mathcal{T}_1 \setminus \{o\},$$

where $Q_h(\kappa)$ is the hyperbolic cube corresponding to κ in $\mathcal{T}_1$, and $d\lambda(z)$ is invariant measure on the disk $\mathbb{D}$. One easily verifies as in [2] that $Ih(o) = 0$,

$$\begin{aligned}\|Ih\|_{B_2(\mathcal{T}_1)}^2 &= \|h\|_{\ell^2(\mathcal{T}_1)}^2 = \sum_{\kappa \in \mathcal{T}_1} (1 - |\kappa|)^2 \left(\int_{Q(\kappa)} |\psi'(z)|\, d\lambda(z)\right)^2 \\ &\leq C \sum_{\kappa \in \mathcal{T}_1} \int_{Q(\kappa)} |\psi'(z)|\, dA = C\,\|\psi\|_{\mathcal{D}}^2,\end{aligned}$$

and

$$Ih(\beta) = \sum_{\kappa \in [o,\beta]} h(\kappa) \geq \operatorname{Re}\psi(\beta) \geq c > 0,\ \text{for } S(\beta) \subset G,$$

since if $B_h(\kappa, R)$ is the hyperbolic ball of radius R about κ, then for R large enough,

$$\begin{aligned}|\psi(\beta)| &\leq \sum_{\kappa \in [o,\beta]} \left|\psi(\kappa) - \psi(\kappa^{-1})\right| \\ &\leq \sum_{\kappa \in [o,\beta]} \left|\frac{1}{|B_h(\kappa,1)|} \int_{B_h(\kappa,1)} \psi(z)\, dA - \frac{1}{|B_h(\kappa^{-1},1)|} \int_{B_h(\kappa^{-1},1)} \psi(z)\, dA\right| \\ &\leq C \sum_{\kappa \in [o,\beta]} \frac{1 - |\kappa|^2}{|B_h(\kappa,1)|} \int_{B_h(\kappa,R)} |\psi'(z)|\, dA \\ &\leq C \sum_{\kappa \in [o,\beta]} \left(1 - |\kappa|^2\right) \int_{Q(\kappa)} |\psi'(z)|\, d\lambda(z) = C \sum_{\kappa \in [o,\beta]} h(\kappa),\end{aligned}$$

where the final inequality is the submean value property for $|\psi'(z)|$. It follows that

$$\begin{aligned} Cap_{\mathcal{T}_1} G &= \inf\left\{\|H\|^2_{B_2(\mathcal{T}_1)} : H(0) = 0, \operatorname{Re} H(\kappa) \geq 1 \text{ if } S(\kappa) \subset G\right\} \\ &\leq \left\|\frac{1}{c} Ih\right\|^2_{B_2(\mathcal{T}_1)} \leq \frac{C}{c^2}\|\psi\|^2_{\mathcal{D}} = \frac{C}{c^2} Cap_{\mathbb{D}} G. \end{aligned}$$

∎

Lemma 2.14 of Bishop [7] says that

$$Cap_{\mathbb{D}}\left(\cup_{j=1}^N I_j^\rho\right) \leq C_\rho Cap_{\mathbb{D}}\left(\cup_{j=1}^N I_j\right), \tag{2.18}$$

for a constant C_ρ depending only on $\rho < 1$. In the next Corollary we use the uniform versions of this, i.e $C_\rho \searrow 1$ as $\rho \nearrow 1$, given by Proposition 2 and its corollaries. Set $d\sigma = d\sigma/2\pi$ on $\mathbb{T}$.

Now let G be an open set in $\mathbb{T}$ such that

$$\frac{\int_{\mathbb{T}} \mu_b(T_\theta(G))\, d\sigma}{\int_{\mathbb{T}} Cap_\theta(G)\, d\sigma} = M := \sup_{E \text{ open } \subset \mathbb{T}} \frac{\int_{\mathbb{T}} \mu_b(T_\theta(E))\, d\sigma}{\int_{\mathbb{T}} Cap_\theta(E)\, d\sigma}. \tag{2.19}$$

Corollary 7. *With M as in (2.19) we have $\|\mu_b\|^2_{\mathcal{D}-Carleson} \approx M$.*

Proof. Using Corollary 6 and $T_\theta(E) \subset T(E)$, we have

$$\begin{aligned} M &\leq C \sup_{E \text{ open } \subset \mathbb{T}} \frac{\int_{\mathbb{T}} \mu_b(T(E))\, d\sigma}{\int_{\mathbb{T}} Cap_{\mathbb{D}}(E)\, d\sigma} \\ &= C \sup_{E \text{ open } \subset \mathbb{T}} \frac{\mu_b(T(E))}{Cap_{\mathbb{D}}(E)} \approx \|\mu_b\|^2_{\mathcal{D}-Carleson}, \end{aligned}$$

where the final comparison is Stegenga's theorem. Conversely, one can verify using the argument in (2.21) above that for $0 < \rho < 1$,

$$\begin{aligned} \mu_b(E) &\leq C \int_{\mathbb{T}} \mu_b(T_\theta(E^\rho))\, d\sigma \\ &\leq CM \int_{\mathbb{T}} Cap_\theta(E^\rho)\, d\sigma \\ &\approx CM Cap_{\mathbb{D}}(E^\rho) \\ &\leq CM Cap_{\mathbb{D}}(E), \end{aligned}$$

where the third line uses (2.17) with E^ρ and $\mathcal{T}_1(\theta)$ in place of G and $\mathcal{T}_1$, and the final inequality follows from (2.18). Thus from Stegenga's theorem we obtain

$$\|\mu_b\|^2_{\mathcal{D}-Carleson} \approx \sup_{E \text{ open } \subset \mathbb{T}} \frac{\mu_b(E)}{Cap_{\mathbb{D}}(E)} \leq CM.$$

∎

We need to know that $\mu_b(V_G^\beta \setminus V_G)$ is small.

Proposition 3. *Given $\varepsilon > 0$ we can find $\beta = \beta(\varepsilon) < 1$ to that*

$$\mu_b(V_G^\beta \setminus V_G) \leq \varepsilon \mu_b(V_G), \tag{2.20}$$

Proof. Corollary 3 shows that $Cap_\theta\left(G^\rho\left(\theta\right)\right)\leq\rho^{-1}Cap_\theta\left(G\right),\ 0\leq\theta<2\pi$,and if we integrate on $\mathbb{T}$ we obtain

$$\int_{\mathbb{T}}Cap_\theta\left(G^\rho\left(\theta\right)\right)d\sigma\leq\frac{1}{\rho}\int_{\mathbb{T}}Cap_\theta\left(G\right)d\sigma.$$

From (2.19) we thus have

$$\begin{aligned}\int_{\mathbb{T}}\mu_b\left(T_\theta\left(G^\rho\left(\theta\right)\right)\right)d\sigma &\leq M\int_{\mathbb{T}}Cap_\theta\left(G^\rho\left(\theta\right)\right)d\sigma\\ &\leq M\frac{1}{\rho}\int_{\mathbb{T}}Cap_\theta\left(G\right)d\sigma\\ &= \frac{1}{\rho}\int_{\mathbb{T}}\mu_b\left(T_\theta\left(G\right)\right)d\sigma.\end{aligned}$$

It follows that

$$\begin{aligned}&\int_{\mathbb{T}}\mu_b\left(T_\theta\left(G^\rho\left(\theta\right)\right)\setminus T_\theta\left(G\right)\right)d\sigma\\ =&\int_{\mathbb{T}}\mu_b\left(T_\theta\left(G^\rho\left(\theta\right)\right)\right)d\sigma-\int_{\mathbb{T}}\mu_b\left(T_\theta\left(G\right)\right)d\sigma\\ \leq&\left(\frac{1}{\rho}-1\right)\int_{\mathbb{T}}\mu_b\left(T_\theta\left(G\right)\right)d\sigma.\end{aligned}$$

Now with $\sigma=\frac{\rho+1}{2}$ halfway between ρ and 1,

$$\begin{aligned}&\int_{\mathbb{T}}\mu_b\left(T_\theta\left(G^\rho\left(\theta\right)\right)\setminus T_\theta\left(G\right)\right)d\sigma\\ =&\int_{\mathbb{T}}\int_{T_\theta\left(G^\rho\left(\theta\right)\right)\setminus T_\theta\left(G\right)}d\mu_b\left(z\right)d\sigma\\ \geq&\int_{\mathbb{T}}\int_{T_\theta\left(G^\rho\left(\theta\right)\right)\setminus T\left(G\right)}d\mu_b\left(z\right)d\sigma\\ =&\int_{\mathbb{D}}\left\{\frac{1}{2\pi}\int_{\left\{\theta:z\in T_\theta\left(G^\rho\left(\theta\right)\right)\setminus T\left(G\right)\right\}}d\sigma\right\}d\mu_b\left(z\right)\\ \geq&\frac{1}{2}\int_{T\left(G^\sigma\right)\setminus T\left(G\right)}d\mu_b\left(z\right),\end{aligned}\tag{2.21}$$

since every $z\in T\left(G^\sigma\right)$ lies in $T_\theta\left(G^\rho\left(\theta\right)\right)$ for at least half of the θ's in $[0,2\pi)$. Here we may assume that the components of G^ρ have small length since otherwise we trivially have $\int_{\mathbb{T}}Cap_{\mathcal{T}_1(\theta)}\left(G\right)d\sigma\geq c>0$ and so then

$$M\leq\frac{1}{c}\int d\mu_b\leq\frac{1}{c}\left\|b\right\|_{\mathcal{D}}^2\leq\frac{C}{c}\left\|T_b\right\|^2.\tag{2.22}$$

Combining the above inequalities and using $\sigma=\frac{\rho+1}{2}$, $\frac{1}{2}\leq\rho<1$, we obtain

$$\begin{aligned}\mu_b\left(T\left(G^\sigma\right)\setminus T\left(G\right)\right) &\leq 2\left(\frac{1}{\rho}-1\right)\int_{\mathbb{T}}\mu_b\left(T_\theta\left(G\right)\right)d\sigma\\ &= 2\left(\frac{1}{\rho}-1\right)\int_{\mathbb{T}}\mu_b\left(T_\theta\left(G\right)\right)d\sigma\\ &\leq \frac{8}{3}\left(1-\sigma\right)\int_{\mathbb{T}}\mu_b\left(T_\theta\left(G\right)\right)d\sigma,\end{aligned}$$

for $\frac{3}{4} \leq \sigma < 1$. Recalling $V_G^{\sigma} = T(G^{\sigma})$ and $V_G = T(G)$ this becomes

$$\mu_b\left(V_G^{\sigma} \setminus V_G\right) \leq \frac{8}{3}(1-\sigma)\int_{\mathbb{T}} \mu_b\left(T_\theta(G)\right) d\sigma \leq \frac{8}{3}(1-\sigma)\,\mu_b\left(V_G\right), \ \frac{3}{4} \leq \sigma < 1,$$

since $T_\theta(G) \subset T(G) = V_G$ for all θ. Thus given $\varepsilon > 0$ it is possible to select β to that (2.20) holds. ∎

2.3.5. *Schur Estimates and a Bilinear Operator on Trees.* We begin with a bilinear version of Schur's well known theorem.

Theorem 2. *Let (X,μ), (Y,ν) and (Z,ω) be measure spaces and $H(x,y,z)$ be a nonnegative measurable function on $X \times Y \times Z$. Define*

$$T(f,g)(x) = \int_{Y\times Z} H(x,y,z)\, f(y)\, d\nu(y)\, g(z)\, d\omega(z)\,, \ x \in X,$$

at least initially for nonnegative functions f,g. Then if $1 < p < \infty$, T is bounded from $L^p(\nu) \times L^p(\omega)$ to $L^p(\mu)$ if and only if there are positive functions h, k and m on X, Y and Z respectively such that

$$\int_{Y\times Z} H(x,y,z)\, k(y)^{p'}\, m(z)^{p'}\, d\nu(y)\, d\omega(z) \leq \left(Ah(x)\right)^{p'},$$

for μ-a.e. $x \in X$, and

$$\int_X H(x,y,z)\, h(x)^p\, d\mu(x) \leq \left(Bk(y)\, m(z)\right)^p,$$

for $\nu \times \omega$-a.e. $(y,z) \in Y \times Z$. Moreover, $\|T\|_{operator} \leq AB$.

Proof. The necessity (which we don't use) is a standard iteration argument. For the sufficiency, we have

$$\begin{aligned}
&\int_X |Tf(x)|^p\, d\mu(x) \\
\leq\ & \int_X \left(\int_{Y\times Z} H(x,y,z)\, k(y)^{p'}\, m(z)^{p'}\, d\nu(y)\, d\omega(z)\right)^{p/p'} \\
&\times \left(\int_{Y\times Z} H(x,y,z) \left(\frac{f(y)}{k(y)}\right)^p d\nu(y) \left(\frac{g(z)}{m(z)}\right)^p d\omega(z)\right) d\mu(x) \\
\leq\ & A^p \int_{Y\times Z} \left(\int_X H(x,y,z)\, h(x)^p\, d\mu(x)\right) \left(\frac{f(y)}{k(y)}\right)^p d\nu(y) \left(\frac{g(z)}{m(z)}\right)^p d\omega(z) \\
\leq\ & A^p B^p \int_{Y\times Z} k(y)^p\, m(z)^p \left(\frac{f(y)}{k(y)}\right)^p d\nu(y) \left(\frac{g(z)}{m(z)}\right)^p d\omega(z) \\
=\ & (AB)^p \int_Y f(y)^p\, d\nu(y) \int_Z g(z)^p\, d\omega(z)\,.
\end{aligned}$$

∎

This theorem can be used along with the estimates

$$\int_{\mathbb{D}} \frac{\left(1-|w|^2\right)^t}{|1-\overline{w}z|^{2+t+c}} dw \approx \begin{cases} C_t & \text{if} \quad c < 0, \ t > -1 \\ -C_t \log(1-|z|^2) & \text{if} \quad c = 0, \ t > -1 \\ C_t (1-|z|^2)^{-c} & \text{if} \quad c > 0, \ t > -1 \end{cases}, . \tag{2.23}$$

to prove the following Corollary which we will use later. It is proved as Theorem 2.10 in [21].

Corollary 8. *Define*

$$\begin{aligned} Tf(z) &= (1-|z|^2)^a \int_{\mathbb{D}} \frac{(1-|w|^2)^b}{(1-\overline{w}z)^{2+a+b}} f(w)\,dw, \\ Sf(z) &= (1-|z|^2)^a \int_{\mathbb{D}} \frac{(1-|w|^2)^b}{|1-\overline{w}z|^{2+a+b}} f(w)\,dw, \end{aligned}$$

where the kernel of S is the modulus of the kernel of T. Suppose that $t \in \mathbb{R}$ and $1 \le p < \infty$ and set

$$d\nu_t(z) = (1-|z|^2)^t dA.$$

Then T is bounded on $L^p(\mathbb{D}, d\nu_t)$ if and only if S is bounded on $L^p(\mathbb{D}, d\nu_t)$ if and only if

$$-pa < t+1 < p(b+1). \tag{2.24}$$

We now apply Theorem 2 to prove a lemma about a bilinear operator mapping $\ell^2(\mathcal{A}) \times \ell^2(\mathcal{B})$ to $L^2(\mathbb{D})$ where $\mathcal{A}$ and $\mathcal{B}$ are subsets of T which are well separated.

Lemma 5. *Suppose $\mathcal{A}$ and $\mathcal{B}$ are subsets of T, $h \in \ell^2(\mathcal{A})$ and $k \in \ell^2(\mathcal{B})$, and $\frac{1}{2} < \alpha < 1$. Suppose further that $\mathcal{A}$ and $\mathcal{B}$ satisfy the separation condition: $\forall \kappa \in \mathcal{A}$, $\gamma \in \mathcal{B}$ we have*

$$|\kappa - \gamma| \ge (1-|\gamma|^2)^\alpha. \tag{2.25}$$

Then the bilinear map of (h,k) to functions on the disk given by

$$T(h,b^*)(z) = \left(\sum_{\kappa\in\mathcal{A}} h(\kappa) \frac{(1-|\kappa|^2)^{1+s}}{|1-\overline{\kappa}z|^{2+s}}\right)\left(\sum_{\gamma\in\mathcal{B}} b^*(\gamma) \frac{(1-|\gamma|^2)^{1+s}}{|1-\overline{\gamma}z|^{1+s}}\right)$$

is bounded from $\ell^2(\mathcal{A}) \times \ell^2(\mathcal{B})$ to $L^2(\mathbb{D})$.

Proof. We will verify the hypotheses of the previous theorem. The kernel function here is

$$H(z,\kappa,\gamma) = \frac{(1-|\kappa|^2)^{1+s}}{|1-\overline{\kappa}z|^{2+s}} \frac{(1-|\gamma|^2)^{1+s}}{|1-\overline{\gamma}z|^{1+s}}, \ z \in \mathbb{D}, \kappa \in \mathcal{A}, \gamma \in \mathcal{B},$$

with Lebesgue measure on $\mathbb{D}$, and counting measure on $\mathcal{A}$ and $\mathcal{B}$. We will take as Schur functions

$$h(z) = \left(1-|z|^2\right)^{-\frac{1}{4}}, \ k(\kappa) = \left(1-|\kappa|^2\right)^{\frac{1}{4}} \text{ and } m(\gamma) = (1-|\gamma|^2)^{\frac{\varepsilon}{2}},$$

on $\mathbb{D}$, $\mathcal{A}$ and $\mathcal{B}$ respectively, where $\varepsilon > 0$ will be chosen sufficiently small later. We must then verify

$$\sum_{\kappa\in\mathcal{A}} \sum_{\gamma\in\mathcal{B}} \frac{\left(1-|\kappa|^2\right)^{\frac{3}{2}+s}}{|1-\overline{\kappa}z|^{2+s}} \frac{\left(1-|\gamma|^2\right)^{1+\varepsilon+s}}{|1-\overline{\gamma}z|^{1+s}} \le A^2 \left(1-|z|^2\right)^{-\frac{1}{2}}, \tag{2.26}$$

for $z \in \mathbb{D}$, and

$$\begin{aligned} &\int_{\mathbb{D}} \frac{\left(1-|\kappa|^2\right)^{1+s}}{|1-\overline{\kappa}z|^{2+s}} \frac{\left(1-|\gamma|^2\right)^{1+s}}{|1-\overline{\gamma}z|^{1+s}} \left(1-|z|^2\right)^{-\frac{1}{2}} dA \\ &\le B^2 \left(1-|\kappa|^2\right)^{\frac{1}{2}} \left(1-|\gamma|^2\right)^{\varepsilon}, \end{aligned} \tag{2.27}$$

for $\kappa \in \mathcal{A}$ and $\gamma \in \mathcal{B}$.

To prove (2.26) we write

$$\begin{aligned}
&\sum_{\kappa \in \mathcal{A}} \sum_{\gamma \in \mathcal{B}} \frac{\left(1-|\kappa|^2\right)^{\frac{3}{2}+s}}{|1-\overline{\kappa} z|^{2+s}} \frac{\left(1-|\gamma|^2\right)^{1+\varepsilon+s}}{|1-\overline{\gamma} z|^{1+s}} \\
= &\left(\sum_{\kappa \in \mathcal{A}} \frac{\left(1-|\kappa|^2\right)^{\frac{3}{2}+s}}{|1-\overline{\kappa} z|^{2+s}}\right)\left(\sum_{\gamma \in \mathcal{B}} \frac{\left(1-|\gamma|^2\right)^{1+\varepsilon+s}}{|1-\overline{\gamma} z|^{1+s}}\right).
\end{aligned}$$

Then from (2.23) we obtain

$$\sum_{\kappa \in \mathcal{A}} \frac{\left(1-|\kappa|^2\right)^{\frac{3}{2}+s}}{|1-\overline{\kappa} z|^{2+s}} \leq C \int_{\mathbb{D}} \frac{\left(1-|w|^2\right)^{-\frac{1}{2}+s}}{|1-\overline{w} z|^{2+s}} dw \leq C\left(1-|z|^2\right)^{-\frac{1}{2}}$$

and

$$\sum_{\gamma \in \mathcal{B}} \frac{\left(1-|\gamma|^2\right)^{1+\varepsilon+s}}{|1-\overline{\gamma} z|^{1+s}} \leq C \int_{\zeta \in V_G} \frac{\left(1-|\zeta|^2\right)^{-1+\varepsilon+s}}{\left|1-\overline{\zeta} z\right|^{1+s}} dA \leq C,$$

which yields (2.26).

The proof of (2.27) will use (2.25). We have

$$\begin{aligned}
&\int_{\mathbb{D}} \frac{\left(1-|\kappa|^2\right)^{1+s}}{|1-\overline{\kappa} z|^{2+s}} \frac{\left(1-|\gamma|^2\right)^{1+s}}{|1-\overline{\gamma} z|^{1+s}}\left(1-|z|^2\right)^{-\frac{1}{2}} dA \\
= &\int_{|z-\gamma^*| \leq 1-|\gamma|^2} + \int_{1-|\gamma|^2 \leq|z-\gamma^*| \leq \frac{1}{2}|\kappa-\gamma|} \\
&+\int_{|z-\kappa^*| \leq 1-|\kappa|^2} + \int_{1-|\kappa|^2 \leq|z-\kappa^*| \leq \frac{1}{2}|\kappa-\gamma|} + \int_{|z-\gamma^*|,|z-\kappa^*| \geq|\kappa-\gamma|} \ldots dA \\
= &\ I+II+III+IV+V.
\end{aligned}$$

By (2.25) $|\kappa-\gamma| \geq\left(1-|\gamma|^2\right)^\alpha$ and so

$$\begin{aligned}
I &\approx \frac{\left(1-|\kappa|^2\right)^{1+s}}{|\kappa-\gamma|^{2+s}} \int_{|z-\gamma^*| \leq 1-|\gamma|^2}\left(1-|z|^2\right)^{-\frac{1}{2}} dA \\
&\approx \frac{\left(1-|\kappa|^2\right)^{1+s}\left(1-|\gamma|^2\right)^{\frac{3}{2}}}{|\kappa-\gamma|^{2+s}} \leq C\left(1-|\kappa|^2\right)^{\frac{1}{2}}\left(1-|\gamma|^2\right)^{\frac{3}{2}(1-\alpha)}.
\end{aligned}$$

Similarly we have

$$\begin{aligned} II &\approx \frac{\left(1-|\kappa|^2\right)^{1+s}\left(1-|\gamma|^2\right)^{1+s}}{|\kappa-\gamma|^{2+s}} \int_{1-|\gamma|^2\leq|z-\gamma^*|\leq\frac{1}{2}|\kappa-\gamma|} \frac{\left(1-|z|^2\right)^{-\frac{1}{2}}}{|z-\gamma^*|^{1+s}} dA \\ &\approx \frac{\left(1-|\kappa|^2\right)^{1+s}\left(1-|\gamma|^2\right)^{1+s}}{|\kappa-\gamma|^{2+s}} \left(1-|\gamma|^2\right)^{\frac{1}{2}-s} \\ &= \frac{\left(1-|\kappa|^2\right)^{1+s}\left(1-|\gamma|^2\right)^{\frac{3}{2}}}{|\kappa-\gamma|^{2+s}} \leq C\left(1-|\kappa|^2\right)^{\frac{1}{2}}\left(1-|\gamma|^2\right)^{\frac{3}{2}(1-\alpha)}. \end{aligned}$$

Continuing to use $|\kappa-\gamma|\geq\left(1-|\gamma|^2\right)^{\alpha}$ we obtain

$$III \approx \frac{\left(1-|\kappa|^2\right)^{\frac{1}{2}}\left(1-|\gamma|^2\right)^{1+s}}{|\kappa-\gamma|^{1+s}} \leq C\left(1-|\kappa|^2\right)^{\frac{1}{2}}\left(1-|\gamma|^2\right)^{(1+s)(1-\alpha)},$$

and similarly,

$$IV \leq C\left(1-|\kappa|^2\right)^{\frac{1}{2}}\left(1-|\gamma|^2\right)^{\varepsilon},$$

for some $\varepsilon>0$. Finally

$$\begin{aligned} V &\approx \int_{|z-\gamma^*|,|z-\kappa^*|\geq|\kappa-\gamma|} \frac{\left(1-|\kappa|^2\right)^{1+s}}{|z-\kappa^*|^{2+s}} \frac{\left(1-|\gamma|^2\right)^{1+s}}{|z-\gamma^*|^{1+s}} \left(1-|z|^2\right)^{-\frac{1}{2}} dA \\ &\approx \frac{\left(1-|\kappa|^2\right)^{1+s}\left(1-|\gamma|^2\right)^{1+s}}{|\kappa-\gamma|^{\frac{3}{2}+2s}} \\ &\leq C\left(1-|\kappa|^2\right)^{\frac{1}{2}}\left(1-|\gamma|^2\right)^{(1+s)(1-\alpha)}. \end{aligned}$$

∎

2.4. **The Main Bilinear Estimate.** To complete the proof we will show that μ_b is a $\mathcal{D}$-Carleson measure by verifying Stegenga's condition (2.3); that is, we will show that for any finite collection of disjoint arcs $\{I_j\}_{j=1}^N$ in the circle $\mathbb{T}$ we have

$$\mu_b\left(\dot{\cup}_{j=1}^N T(I_j)\right) \leq C\ Cap_{\mathbb{D}}\left(\dot{\cup}_{j=1}^N I_j\right).$$

In fact we will see that it suffices to verify this for the single set $G=\dot{\cup}_{j=1}^N I_j$ described in (2.19). We will prove the inequality

$$\mu_b(V_G) \leq C\|T_b\|^2 Cap_{\mathbb{D}}(G). \tag{2.28}$$

Once we have this Corollary 6 yields

$$M=\frac{\int_{\mathbb{T}}\mu_b(T_\theta(G))\,d\sigma}{\int_{\mathbb{T}} Cap_\theta(G)\,d\sigma} \leq \frac{\mu_b(V_G)}{\int_{\mathbb{T}} Cap_\theta(G)\,d\sigma} \leq C\|T_b\|^2.$$

By Corollary 7 $\|\mu_b\|^2_{\mathcal{D}-Carleson}\approx M$ which then completes the proof of Theorem 1.

We now turn to the proof of the estimate (2.28). Let $\frac{1}{2}<\beta<\beta_1<\gamma<\alpha<1$ to be chosen later. We will obtain our estimate by using the boundedness of T_b on certain functions f and g in $\mathcal{D}$. The function f will be, approximately, $b'\chi_{V_G}$ and the function g will be constructed using an approximate extremal function of the type described in Subsection 2.3.4 and will be approximately equal to χ_{V_G}.

We have chosen G ; we now set

$$E=\{w_k^\alpha\}_{k=1}^{M_\alpha} \text{ and } F=\{w_\ell^\gamma\}_{\ell=1}^{M_\gamma}$$

where

$$G^\alpha=\cup_{k=1}^{M_\alpha}J_k^\alpha \text{ and } G^\gamma=\cup_{\ell=1}^{M_\gamma}J_\ell^\gamma.$$

Now construct a Bergman tree $\mathcal{T}$ for the disk $\mathbb{D}$ with uniform bounds (independent of G) such that $E\cup F\subset\mathcal{T}$ and both E and F are stopping times in $\mathcal{T}$. Note that every point w_k^α in E is contained in a unique successor set $S\left(w_\ell^\gamma\right)$ for a point w_ℓ^γ in F. Now define Φ as in (2.10) above, so that we have the estimates in Theorem 3 and Corollary 5. From Corollaries 6 and **??** we obtain

$$Cap\,(E,F)\le CCap_{\mathbb{D}}\;G. \tag{2.29}$$

We will use $g=\Phi^2$ and

$$f\left(z\right)=\Gamma_s\left(\frac{1}{\left(1+s\right)\overline{\zeta}}\chi_{V_G}b'\left(\zeta\right)\right)\left(z\right) \tag{2.30}$$

as our test functions in the bilinear inequality

$$\left|T_b\left(f,g\right)\right|\le\left\|T_b\right\|\left\|f\right\|_{\mathcal{D}}\left\|g\right\|_{\mathcal{D}}. \tag{2.31}$$

From (2.30) we have

$$f\left(z\right)=\int_{V_G}\frac{b'\left(\zeta\right)\left(1-\left|\zeta\right|^2\right)^s}{\left(1-\overline{\zeta}z\right)^{1+s}}\frac{dA}{\left(1+s\right)\overline{\zeta}},$$

so that

$$\begin{aligned} f'\left(z\right) &= \int_{V_G}\frac{b'\left(\zeta\right)\left(1-\left|\zeta\right|^2\right)^s}{\left(1-\overline{\zeta}z\right)^{2+s}}dA \\ &= b'\left(z\right)-\int_{\mathbb{D}\setminus V_G}\frac{b'\left(\zeta\right)\left(1-\left|\zeta\right|^2\right)^s}{\left(1-\overline{\zeta}z\right)^{2+s}}dA \\ &= b'\left(z\right)+\Lambda b'\left(z\right), \end{aligned}$$

where

$$\Lambda b'\left(z\right)=-\int_{\mathbb{D}\setminus V_G}\frac{b'\left(\zeta\right)\left(1-\left|\zeta\right|\right)^s}{\left(1-\overline{\zeta}z\right)^{2+s}}dA, \tag{2.32}$$

by the reproducing property of the generalized Bergman kernels $\frac{\left(1-\left|\zeta\right|^2\right)^s}{\left(1-\overline{\zeta}z\right)^{2+s}}$. Now if we plug f and $g=\Phi^2$ as above in $T_b\left(f,g\right)$ we obtain $T_b\left(f,g\right)=T_b\left(f,\Phi^2\right)=$

$T_b\left(f\Phi,\Phi\right)$ which we analyze as

$$
\begin{aligned}
(2.33)\qquad & T_b\left(f,\Phi^2\right)=T_b\left(f\Phi,\Phi\right) \\
&= \int_{\mathbb{D}}\left\{f'\left(z\right)\Phi\left(z\right)+2f\left(z\right)\Phi'\left(z\right)\right\}\Phi\left(z\right)\overline{b'\left(z\right)}dA+f\left(0\right)\Phi\left(0\right)^2\overline{b\left(0\right)} \\
&= f\left(0\right)\Phi\left(0\right)^2\overline{b\left(0\right)}+\int_{\mathbb{D}}\left|b'\left(z\right)\right|^2\Phi\left(z\right)^2dA \\
&\quad +2\int_{\mathbb{D}}\Phi\left(z\right)\Phi'\left(z\right)f\left(z\right)\overline{b'\left(z\right)}dA+\int_{\mathbb{D}}\Lambda b'\left(z\right)\overline{b'\left(z\right)}\Phi\left(z\right)^2dA \\
&= (1)+(2)+(3)+(4).
\end{aligned}
$$

Trivially, we have

$$
(2.34)\qquad \left|(1)\right|\leq C\left\|b\right\|_{\mathcal{D}}^2 Cap\left(E,F\right)\leq C\left\|T_b\right\|^2 Cap\left(E,F\right).
$$

Now we write

$$
\begin{aligned}
(2.35)\qquad (2) &= \int_{\mathbb{D}}\left|b'\left(z\right)\right|^2\Phi\left(z\right)^2dA \\
&= \left\{\int_{V_G}+\int_{V_G^\beta\setminus V_G}+\int_{\mathbb{D}\setminus V_G^\beta}\right\}\left|b'\left(z\right)\right|^2\Phi\left(z\right)^2dA \\
&= (2_A)+(2_B)+(2_C).
\end{aligned}
$$

The main term (2A) satisfies

$$
\begin{aligned}
(2.36)\qquad (2_A) &= \mu_b\left(V_G\right)+\int_{V_G}\left|b'\left(z\right)\right|^2\left(\Phi\left(z\right)^2-1\right)dA \\
&= \mu_b\left(V_G\right)+O\left(\left\|T_b\right\|^2 Cap\left(E,F\right)\right),
\end{aligned}
$$

by (2.14) and (2.1). For term (2B) we use (2.20) to obtain

$$
(2.37)\qquad \left|(2_B)\right|\leq C\mu_b\left(V_G^\beta\setminus V_G\right)\leq C\varepsilon\mu_b\left(V_G\right).
$$

Using (2.14) once more, we see that term (2C) satisfies

$$
(2.38)\qquad \left|(2_C)\right|\leq\int_{\mathbb{D}\setminus V_G^\beta}\left|b'\left(z\right)\right|^2\left(C_{\alpha,\beta,\rho}Cap\left(E,F\right)\right)^3dA\leq C\left\|T_b\right\|^2 Cap\left(E,F\right).
$$

Altogether, using (2.34), (2.35), (2.36), (2.37) and (2.38) in (2.33) we have

$$
(2.39)\quad \mu_b\left(V_G\right)\leq\left|T_b\left(f,\Phi^2\right)\right|+C\mu_b\left(V_G^\beta\setminus V_G\right)+C\left\|T_b\right\|^2 Cap\left(E,F\right)+\left|(3)\right|+\left|(4)\right|.
$$

We estimate (3) using Cauchy-Schwarz with $\varepsilon>0$ small as follows:

$$
\begin{aligned}
\left|(3)\right| &\leq 2\int_{\mathbb{D}}\left|\Phi\left(z\right)b'\left(z\right)\right|\left|\Phi'\left(z\right)f\left(z\right)\right|dA \\
&\leq \varepsilon\int_{\mathbb{D}}\left|\Phi\left(z\right)b'\left(z\right)\right|^2dA+\frac{C}{\varepsilon}\int_{\mathbb{D}}\left|\Phi'\left(z\right)f\left(z\right)\right|^2dA \\
&= (3_A)+(3_B).
\end{aligned}
$$

Using the decomposition and argument surrounding term (2) we obtain

$$\begin{aligned} |(3_A)| &\leq \varepsilon\left\{\int_{V_G}+\int_{V_G^{\beta}\setminus V_G}+\int_{\mathbb{D}\setminus V_G^{\beta}}\right\}\left|\Phi\left(z\right)b'\left(z\right)\right|^2 dA \\ &\leq C\varepsilon\left(\mu_b\left(V_G\right)+C\left\|T_b\right\|^2 Cap\left(E,F\right).\right) \end{aligned} \tag{2.40}$$

To estimate term (3_B) we use

$$\begin{aligned} |f\left(z\right)| &\leq \left|\Gamma_s\left(\frac{1}{(1+s)\,\overline{\zeta}}\chi_{V_G}b'\left(\zeta\right)\right)(z)\right| \\ &\leq \int_{V_G}\frac{\left(1-|\zeta|^2\right)^s}{\left|1-\overline{\zeta}z\right|^{1+s}}\left|b'\left(\zeta\right)\right|dA \\ &\approx \sum_{\gamma\in\mathcal{T}_1\cap V_G}\frac{\left(1-|\gamma|^2\right)^{1+s}}{\left|1-\overline{\gamma}z\right|^{1+s}}\int_{B_\gamma}\left|b'\left(\zeta\right)\right|\left(1-|\zeta|^2\right)d\lambda\left(\zeta\right) \\ &= \sum_{\gamma\in\mathcal{T}_1\cap V_G}\frac{\left(1-|\gamma|^2\right)^{1+s}}{\left|1-\overline{\gamma}z\right|^{1+s}}b^*\left(\gamma\right), \end{aligned}$$

where

$$\sum_{\gamma\in\mathcal{T}_1\cap V_G}b^*\left(\gamma\right)^2\approx\sum_{\gamma\in\mathcal{T}_1\cap V_G}\int_{B_\gamma}\left|b'\left(\zeta\right)\right|^2\left(1-|\zeta|^2\right)^2 d\lambda\left(\zeta\right)=\int_{V_G}\left|b'\left(\zeta\right)\right|^2 dA.$$

We now use the separation of $\mathbb{D}\setminus V_G^{\alpha}$ and V_G. The facts that $\mathcal{A}=supp\left(h\right)\subset\mathbb{D}\setminus V_G^{\alpha}$ and $\mathcal{B}=\mathcal{T}_1\cap V_G\subset V_G$ insure that (2.25) is satisfied. Hence we can use Lemma 5 and the representation of Φ in 2.10 to continue with

$$(3_B)=\int_{\mathbb{D}}\left|\Phi'\left(z\right)f\left(z\right)\right|^2 dA\leq C\left(\sum_{\kappa\in\mathcal{A}}h\left(\kappa\right)^2\right)\left(\sum_{\gamma\in\mathcal{B}}b^*\left(\gamma\right)^2\right),$$

We also have from (2.1) and Corollary 5 that

$$\left(\sum_{\kappa\in\mathcal{A}}h\left(\kappa\right)^2\right)\left(\sum_{\gamma\in\mathcal{B}}b^*\left(\gamma\right)^2\right)\leq CCap\left(E,F\right)\left\|T_b\right\|^2.$$

Altogether we then have

$$(3_B)\leq CCap\left(E,F\right)\left\|T_b\right\|^2, \tag{2.41}$$

and thus also

$$|(3)|\leq\varepsilon\int_{V_G}\left|b'\left(z\right)\right|^2+C\left\|T_b\right\|^2 Cap\left(E,F\right). \tag{2.42}$$

We begin our estimate of term (4) by

$$\begin{aligned} |(4)| &= \left|\int_{\mathbb{D}}\Lambda b'\left(z\right)\overline{b'\left(z\right)}\Phi\left(z\right)^2 dA\right| \\ &\leq \sqrt{\int_{\mathbb{D}}\left|b'\left(z\right)\Phi\left(z\right)\right|^2 dA}\sqrt{\int_{\mathbb{D}}\left|\Lambda b'\left(z\right)\Phi\left(z\right)\right|^2 dA} \end{aligned} \tag{2.43}$$

Now we claim the following estimate for $(4_A) = \|\Phi \Lambda b'\|_{L^2(\mathbb{D})}$:

$$\begin{aligned}
(4_A) &= \int_{\mathbb{D}} \left|\Phi(z)\Lambda b'(z)\right|^2 dA && (2.44)\\
&\leq C\mu_b\left(V_G^\beta \setminus V_G\right) + C\left\|T_b\right\|^2 Cap(E,F) && (2.45)\\
&\leq \varepsilon\mu_b(V_G) + C\left\|T_b\right\|^2 Cap(E,F).
\end{aligned}$$

Indeed, the second inequality follows from (2.20). From (2.32) we obtain

$$\begin{aligned}
(4_A) &= \int_{\mathbb{D}} |\Phi(z)|^2 \left|\left\{\int_{V_G^\beta \setminus V_G} + \int_{\mathbb{D}\setminus V_G^\beta}\right\} \frac{b'(\zeta)(1-|\zeta|)^s}{\left(1-\overline{\zeta}z\right)^{2+s}} dA\right|^2 dA \\
&\leq C\int_{\mathbb{D}} |\Phi(z)|^2 \left(\int_{V_G^\beta \setminus V_G} \frac{|b'(\zeta)|(1-|\zeta|)^s}{\left|1-\overline{\zeta}z\right|^{2+s}} dA\right)^2 dA \\
&\quad + C\int_{\mathbb{D}} |\Phi(z)|^2 \left|\int_{\mathbb{D}\setminus V_G^\beta} \frac{b'(\zeta)(1-|\zeta|)^s}{\left(1-\overline{\zeta}z\right)^{2+s}} dA\right|^2 dA \\
&= (4_{AA}) + (4_{AB}).
\end{aligned}$$

Corollary 8 shows that

$$\begin{aligned}
|(4_{AA})| &\leq \int_{\mathbb{D}} \left(\int_{V_G^\beta \setminus V_G} \frac{|b'(\zeta)|(1-|\zeta|)^s}{\left|1-\overline{\zeta}z\right|^{2+s}} dA\right)^2 dA \\
&\leq C\int_{V_G^\beta \setminus V_G} |b'(\zeta)|^2 dA = C\mu_b\left(V_G^\beta \setminus V_G\right).
\end{aligned}$$

We write the second integral as

$$\begin{aligned}
(4_{AB}) &= \left\{\int_{V_G^\gamma} + \int_{\mathbb{D}\setminus V_G^\gamma}\right\} |\Phi(z)|^2 \left|\int_{\mathbb{D}\setminus V_G^\beta} \frac{b'(\zeta)(1-|\zeta|)^s}{\left(1-\overline{\zeta}z\right)^{2+s}} dA\right|^2 dA \\
&= (4_{ABA}) + (4_{ABB}),
\end{aligned}$$

where by Corollary 8 again,

$$\begin{aligned}
|(4_{ABB})| &\leq C Cap(E,F)^2 \int_{\mathbb{D}} |b'(\zeta)|^2 dA \\
&\leq C\left\|T_b\right\|^2 Cap(E,F)^2 \\
&\leq C\left\|T_b\right\|^2 Cap(E,F).
\end{aligned}$$

Finally, with $\beta < \beta_1 < \gamma < \alpha < 1$, Corollary 8 shows that the term (4_{ABA}) satisfies the following estimate. Recall that $V_G^\gamma = \cup J_k^\gamma$ and $w_j^\gamma = z\left(J_k^\gamma\right)$. We set

$A_\ell = \left\{k : J_k^\gamma \subset J_\ell^{\beta_1}\right\}$ and define $\ell(k)$ by the condition $k \in A_{\ell(k)}$. Then

$$\begin{aligned}
|(4_{ABA})| &\leq C \int_{V_G^\gamma} \left(\int_{\mathbb{D}\setminus V_G^\beta} \frac{|b'(\zeta)|(1-|\zeta|)^s}{\left|1-\overline{\zeta}z\right|^{2+s}} dA \right)^2 dA \\
&\approx C \sum_k \int_{J_k^\gamma} |J_k^\gamma| \left(\frac{|b'(\zeta)|(1-|\zeta|)^s}{\left|1-\overline{\zeta}w_k^\gamma\right|^{2+s}} dA \right)^2 dA \\
&= C \sum_k \frac{|J_k^\gamma|}{\left|J_{\ell(k)}^{\beta_1}\right|} \left|J_{\ell(k)}^{\beta_1}\right| \int_{J_k^\gamma} \left(\frac{|b'(\zeta)|(1-|\zeta|)^s}{\left|1-\overline{\zeta}w_k^\gamma\right|^{2+s}} dA \right)^2 dA \\
&\approx C \sum_\ell \frac{\sum_{k\in A_\ell} |J_k^\gamma|}{\left|J_\ell^{\beta_1}\right|} \int_{J_\ell^{\beta_1}} \left(\int_{\mathbb{D}\setminus V_G^\beta} \frac{|b'(\zeta)|(1-|\zeta|)^s}{\left|1-\overline{\zeta}z\right|^{2+s}} dA \right)^2 dA \\
&\leq C \left|V_G^{\beta_1}\right|^{\varepsilon(\gamma-\beta_1)} \int_{V_G^{\beta_1}} \left(\int_{\mathbb{D}\setminus V_G^\beta} \frac{|b'(\zeta)|(1-|\zeta|)^s}{\left|1-\overline{\zeta}z\right|^{2+s}} dA \right)^2 dA \\
&\leq C \left|V_G^{\beta_1}\right|^{\varepsilon(\gamma-\beta_1)} \|b\|_{\mathcal{D}}^2 \leq C \|T_b\|^2 Cap(E,F).
\end{aligned}$$

This completes the proof of (2.44).

Now we can estimate term (4) by

$$\begin{aligned}
|(4)| &= \left| \int_{\mathbb{D}} \Lambda b'(z) \overline{b'(z)} \Phi(z)^2 \, dA \right| \qquad (2.46) \\
&\leq \sqrt{\int_{\mathbb{D}} |b'(z)\Phi(z)|^2 \, dA} \sqrt{\int_{\mathbb{D}} |\Lambda b'(z)\Phi(z)|^2 \, dA} \\
&\leq \sqrt{(3_A)/\varepsilon} \sqrt{(4_A)} \qquad (2.47) \\
&\leq \sqrt{C\mu_b(V_G) + C\|T_b\|^2 Cap(E,F)} \\
&\quad \times \sqrt{\varepsilon\mu_b(V_G) + C\|T_b\|^2 Cap(E,F)} \\
&\leq \sqrt{\varepsilon}\mu_b(V_G) + C\sqrt{\mu_b(V_G)}\sqrt{\|T_b\|^2 Cap(E,F)} \\
&\quad + C\|T_b\|^2 Cap(E,F),
\end{aligned}$$

using (2.44) and the estimate (2.40) for (3_A) already proved above. Finally, we estimate $T_b\left(f,\Phi^2\right) = T_b(f\Phi,\Phi)$by

$$|T_b(f\Phi,\Phi)| \leq \|T_b\| \|\Phi\|_{\mathcal{D}} \|\Phi f\|_{\mathcal{D}} \leq C\|T_b\| \sqrt{Cap(E,F)} \|\Phi f\|_{\mathcal{D}}.$$

Now

$$\begin{aligned}
\|\Phi f\|_{\mathcal{D}}^2 &\leq C \int |\Phi'(z) f(z)|^2 \, dA + C \int |\Phi(z) f'(z)|^2 \, dA \\
&\leq C|3_A| + C|3_B| + C \int |\Phi(z)\Lambda b'(z)|^2 \, dA \\
&\leq C\mu_b(V_G) + C\|T_b\|^2 Cap(E,F),
\end{aligned}$$

by (2.44) and the estimates (2.40) and (2.41) for (3_A) and (3_B). When we plug this into the previous estimate we get

$$
\begin{aligned}
(2.48)\left|T_b\left(f,\Phi^2\right)\right| &\leq C\left\|T_b\right\|\sqrt{Cap\left(E,F\right)}\sqrt{\mu_b\left(V_G\right)+\left\|T_b\right\|^2 Cap\left(E,F\right)} \\
&\leq C\sqrt{\left\|T_b\right\|^2 Cap\left(E,F\right)}\left(\sqrt{\mu_b\left(V_G\right)}+\left\|T_b\right\| Cap\left(E,F\right)^{\frac{1}{2}}\right).
\end{aligned}
$$

Using Proposition 3 and the estimates (2.42), (2.46) and (2.48) in (2.39) we obtain

$$
\begin{aligned}
\mu_b\left(V_G\right) &\leq \sqrt{\varepsilon}\mu_b\left(V_G\right)+C\left\|T_b\right\|^2 Cap\left(E,F\right) \\
&\quad +C\sqrt{\left\|T_b\right\|^2 Cap\left(E,F\right)}\sqrt{\mu_b\left(V_G\right)} \\
&\leq \sqrt{\varepsilon}\mu_b\left(V_G\right)+C\left\|T_b\right\|^2 Cap\left(E,F\right).
\end{aligned}
$$

Absorbing the first term on the right side, and using (2.29), we finally obtain

$$
\mu_b\left(V_G\right) \leq C\left\|T_b\right\|^2 Cap\left(E,F\right) \leq C\left\|T_b\right\|^2 Cap_{\mathbb{D}}G,
$$

which is (2.28).

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