UNIQUENESS IN THE CAUCHY PROBLEM FOR A CLASS OF HYPOELLIPTIC ULTRAPARABOLIC OPERATORS

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Abstract. We consider a class of hypoelliptic ultraparabolic operators in the form

$$\mathcal{L} = \sum_{j=1}^{m} X_j^2 + X_0 - \partial_t,$$

under the assumption that the vector fields $X_1, \dots, X_m$ and $X_0 - \partial_t$ are invariant with respect to a suitable homogeneous Lie group $\mathbb{G}$. We show that if u, v are two solutions of $\mathcal{L}u = 0$ on $\mathbb{R}^N \times]0, T[$ and $u(\cdot, 0) = \varphi$, then each of the following conditions: $|u(x,t) - v(x,t)|$ can be bounded by $M \exp(c|x|^2_{\mathbb{G}})$, or both u and v are non negative, implies $u \equiv v$. We use a technique which relies on a pointwise estimate of the fundamental solution of $\mathcal{L}$.

1. Introduction and main results

We consider a class of linear second order operators in $\mathbb{R}^{N+1}$ of the form

$$\mathcal{L} = \sum_{j=1}^{m} X_j^2 + X_0 - \partial_t, \tag{1.1}$$

where $1 \leq m \leq N$. In (1.1) the X_j's are smooth vector fields on $\mathbb{R}^N$, i.e. denoting $z = (x,t)$ the point in $\mathbb{R}^{N+1}$,

$$X_j(x) = \sum_{k=1}^{N} a_k^j(x) \partial_{x_k}, \qquad j = 0, \dots, m,$$

and any a_k^j is a C^∞ function. In the sequel we also consider the X_j's as vector fields in $\mathbb{R}^{N+1}$ and denote

$$Y = X_0 - \partial_t.$$

Our main assumption on the operator $\mathcal{L}$ is the invariance with respect to a homogeneous Lie group structure, and a controllability condition:

[H.1] there exists a homogeneous Lie group $\mathbb{G} = \left(\mathbb{R}^{N+1}, \circ, \delta_\lambda\right)$ such that
   i) $X_1, \dots, X_m, Y$ are left translation invariant on $\mathbb{G}$;
   ii) $X_1, \dots, X_m$ are δ_λ-homogeneous of degree one and Y is δ_λ-homogeneous of degree two;

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[H.2] for every $(x,t),(y,s) \in \mathbb{R}^{N+1}$ with $t > s$, there exists an absolutely continuous path $\gamma : [0,t-s] \to \mathbb{R}^{N+1}$ such that

$$\begin{cases} \gamma'(\tau) = \sum_{k=1}^{m} \omega_k(\tau) X_k(\gamma(\tau)) + Y(\gamma(\tau)), & \text{a.e. in } [0,t-s], \\ \gamma(0) = (x,t), \quad \gamma(t-s) = (y,s), \end{cases} \tag{1.2}$$

with $\omega_1, \dots, \omega_m \in L^\infty([0,t-s])$. We say that this curve is an $\mathcal{L}$-*admissible* path connecting (x,t) with (y,s), and in the sequel we will denote it by $\gamma((x,t),(y,s),\omega)$.

It is know that [H.1] implies that the coefficients a_k^j of the X_j's are polynomial functions, hence [H.2] yields

$$\text{rank Lie}\{X_1, \dots, X_m, Y\}(z) = N+1, \qquad \text{for every } z \in \mathbb{R}^{N+1}, \tag{1.3}$$

(see, for instance, [8] or [16, Chap. II, Sec. 8]). This is the well know Hörmander condition for the hypoellipticity of $\mathcal{L}$ (see [11]).

Operators of the form (1.1), verifying assumptions [H.1] and [H.2], have been studied by Kogoj and Lanconelli in [12], [13] and [14]. Note that in [12] and [14] the $\mathcal{L}$-admissibile path γ in [H.2] is supposed to satisfy $\gamma'(\tau) = \sum_{k=1}^{m} \omega_k(\tau) X_k(\gamma(\tau)) + \mu(\tau) Y(\gamma(\tau))$ for piecewise constant real functions $\omega_1, \dots, \omega_m, \mu$, with $\mu \geq 0$. However, even if this definition is slightly different from our one, the main results stated in these papers hold true also with our assumption. In [12] it is proved that $\mathcal{L}$ has a fundamental solution $\Gamma(\cdot, \zeta)$ which shares several properties of the fundamental solution of the heat equation (see Section 2 for the details).

In this paper we prove some uniqueness results for the Cauchy problem related to $\mathcal{L}$:

$$\begin{cases} \mathcal{L}u = 0 & \text{in } \mathbb{R}^N \times]0,T[, \\ u(\cdot, 0) = \varphi & \text{in } \mathbb{R}^N, \end{cases} \tag{1.4}$$

where $\varphi \in C(\mathbb{R}^N)$. Our main achievements are the following ones:

Theorem 1.1. *Assume that $\mathcal{L}$ satisfies conditions* [H.1]-[H.2], *and let $u, v \in C(\mathbb{R}^N \times [0,T])$ be two solutions of the Cauchy problem* (1.4). *If there exists a positive constant c such that*

$$\int_0^T \int_{\mathbb{R}^N} e^{-c|x|_{\mathbb{G}}^2} |u(x,t) - v(x,t)| \, \mathrm{d}x \, \mathrm{d}t < \infty \tag{1.5}$$

then $u \equiv v$.

(See Section 2 for the definition of the norm $|\cdot|_{\mathbb{G}}$).

Theorem 1.2. *Assume that $\mathcal{L}$ satisfies conditions* [H.1]-[H.2], *and let $u, v \in C(\mathbb{R}^N \times [0,T])$ be two solutions of the Cauchy problem* (1.4). *If both u and v are non negative, then $u \equiv v$.*

Our theorems 1.1 and 1.2 extend some classical uniqueness result for parabolic operators. We first quote the paper by Tychonoff [19], where it is shown that the Cauchy problem for the heat equation has an unique solution satisfying $u(x,t) \leq M e^{c|x|^2}$ for every $(x,t) \in \mathbb{R}^N \times]0,T]$. On the other hand, Widder in [20] proved that the unique non-negative solution of the heat equation in $\mathbb{R} \times [0,T]$ such that $u(\cdot,0) = 0$ is the null function.

With regard to more general parabolic operators, Krzyżański [15] showed that the Tychonoff condition ensures uniqueness for parabolic operators in non-divegence form with bounded and continuous coefficients. Serrin in [18] extended Widder's result to solutions of equations in the form $u_t = a(x)u_{x,x} + b(x)u_x + c(x)u$ with Hölder continuous and uniformly bounded coefficients. Aronson and Besala in [1] proved that, if u is a solution of a divergence form parabolic equation with measurable coefficients satisfying certain growth condition at infinity, the uniqueness of the homogeneous Cauchy problem is guaranteed by the following integral condition: $\int_0^T \int_{\mathbb{R}^N} e^{-c|x|^2} u^2(x,t)\,\mathrm{d}x\,\mathrm{d}t < \infty$. Moreover, the same authors in [2] proved that an hypothesis analogous to (1.5) yields the uniqueness for a class of uniformly parabolic operators $\sum_{i,j} a_{i,j}(x,t)\partial_{x_i x_j} + \sum_j b_j(x,t)\partial_{x_j} - \partial_t$ with locally Hölder continuous coefficients which grow at most linearly at infinity.

Concerning Kolmogorov-type operators, some results like our Theorem 1.1 and 1.2 have been obtained by Polidoro [17], Di Francesco and Pascucci [9], Di Francesco and Polidoro [10]. In [17] it is showed that there is only one solution which is in the Tychonoff class or non-negative to the operator $L = \mathrm{div}(A(z)D) + \langle x, BD\rangle - \partial_t$; here L is homogeneous with respect to a suitable Lie group structure, and the $a_{ij}(z)$'s are uniformly Hölder continuous with respect to the geometry of L. This results have been improved respectively in [9] and [10] for Kolmogorov equations in non-divergence form, assuming that the coefficients and their derivatives are bounded and Hölder continuous (in a certain sense), and removing the homogeneity assumption. In all these papers, the authors relied on pointwise estimates for the fundamental solution of the operator considered.

Finally, we quote the paper of Bonfiglioli, Lanconelli and Uguzzoni [3], where Tychonoff-type and Widder-type uniqueness theorems are extended for the heat operator $\mathcal{H} = \mathcal{L} - \partial_t$ related to the sub-Laplacian $\mathcal{L}$ on a stratified Carnot group.

We recall that Tychonoff constructed in [19] a non trivial solution u to the Cauchy problem for the heat equation such that $u(\cdot, 0) = 0$ and $u(x,t) \le M e^{c|x|^{2+\varepsilon}}$ in $\mathbb{R}^N \times]0,T]$. Since our results apply to heat operators, this example shows that the growth condition allowed in Theorem 1.1 cannot be improved by increasing the exponent of $|x|_{\mathbb{G}}$. Nevertheless, it seem possible to sharpen hypothesis (1.5) by using the value function V (see Remark 2.4 in Section 2).

This paper is organized as follows: in Section 2 we collect some preliminaries, and in Section 3 we give the proof of our main results, Theorem 1.1 and 1.2.

2. Notations and preliminary results

We say that a Lie group $\mathbb{G} = (\mathbb{R}^{N+1}, \circ)$ is *homogeneous* if on $\mathbb{G}$ there exists a family of dilations $\{\delta_\lambda\}_{\lambda>0}$ which is an automorphism of the group:

$$\delta_\lambda(z \circ \zeta) = (\delta_\lambda z) \circ (\delta_\lambda \zeta), \qquad \text{for all } z, \zeta \in \mathbb{R}^{N+1} \text{ and } \lambda > 0.$$

As we stated in the Introduction, [H.1] and [H.2] imply the Hörmander condition (1.3). From [H.1] and [H.2] it follows also that the composition law $\circ$ is euclidean in the "time" variable, i.e.

$$(x,t) \circ (\xi, \tau) = (S(x,t,\xi,\tau), t + \tau)$$

for a suitable C^∞ function S with value in $\mathbb{R}^N$. Moreover the dilation of the group induces a direct sum decomposition on $\mathbb{R}^N$

$$\mathbb{R}^N = V_1 \oplus \cdots \oplus V_k, \tag{2.1}$$

as follows. If $x = x^{(1)} + x^{(2)} + \cdots + x^{(k)}$ with $x^{(j)} \in V_j$, then $\delta_\lambda(x,t) = \big(D(\lambda)x, \lambda^2 t\big)$, where

$$D(\lambda)\big(x^{(1)} + x^{(2)} + \cdots + x^{(k)}\big) = \big(\lambda x^{(1)} + \lambda^2 x^{(2)} + \cdots + \lambda^k x^{(k)}\big). \tag{2.2}$$

The natural number

$$Q = 2 + \sum_{j=1}^{k} j \,\mathrm{dim} V_j$$

is usually called the *homogeneous dimension* of $\mathbb{G}$ with respect to δ_λ. We set

$$|x|_{\mathbb{G}} = \left(\sum_{j=1}^{k} \sum_{i=1}^{\mathrm{dim}V_j} \big(x_i^{(j)}\big)^{\frac{2k!}{j}} \right)^{\frac{1}{2k!}}, \qquad \|(x,t)\|_{\mathbb{G}} = \big(|x|_{\mathbb{G}}^{2k!} + |t|^{k!}\big)^{\frac{1}{2k!}},$$

and we observe that the above functions are δ_λ-homogeneous of degree 1, respectively on $\mathbb{R}^N$ and $\mathbb{R}^{N+1}$:

$$\big|\big(\lambda x^{(1)} + \cdots + \lambda^k x^{(k)}\big)\big|_{\mathbb{G}} = \lambda |x|_{\mathbb{G}}, \qquad \|\delta_\lambda(x,t)\|_{\mathbb{G}} = \lambda \|(x,t)\|_{\mathbb{G}},$$

for every $(x,t) \in \mathbb{R}^{N+1}$ and for any $\lambda > 0$. We define the quasi-distance in $\mathbb{G}$ as

$$d(z,\zeta) := \|\zeta^{-1} \circ z\|_{\mathbb{G}}, \qquad \text{for all } z, \zeta \in \mathbb{R}^{N+1}, \tag{2.3}$$

and we recall that, for a positive constant $\mathbf{c}$,

i) $d(z,\zeta) \le \mathbf{c}\, d(\zeta, z)$ for all $z, \zeta \in \mathbb{R}^{N+1}$;

ii) $d(z,\zeta) \le \mathbf{c}\, (d(z,z_1) + d(z_1,\zeta))$ for all $z, z_1, \zeta \in \mathbb{R}^{N+1}$.

Moreover,

$$d(\delta_\lambda z, \delta_\lambda \zeta) = \lambda\, d(z,\zeta), \qquad \text{for all } z, \zeta \in \mathbb{R}^{N+1},\ \lambda > 0.$$

Throughout the paper we shall write $d(z)$ instead of $d(0,z) = \|z^{-1}\|_{\mathbb{G}}$. Obviously, from *i)*, we have

$$\mathbf{c}^{-1}\|z\|_{\mathbb{G}} \le d(z) \le \mathbf{c}\|z\|_{\mathbb{G}}.$$

We also use the following notation for the $|\cdot|_{\mathbb{G}}$-ball of radius $r > 0$ centered at the origin:

$$B_r = \{x \in \mathbb{R}^N \mid |x|_{\mathbb{G}} < r\}.$$

We recall the following weak maximum principle on strips (see e.g. [6, Theorem 4.1] for an elementary proof based on suitable mean-value representation formulas).

Proposition 2.1. *Let $u \in C^2(\mathbb{R}^N \times]0,T[)$. If $\mathcal{L}u \ge 0$, $\limsup u \le 0$ both in $\mathbb{R}^N \times \{0\}$ and at infinity, then $u \le 0$ in the whole strip.*

We next collect some useful facts on the fundamental solution of the hypoelliptic operator $\mathcal{L}$. If $\Gamma(\cdot,\zeta)$ is the fundamental solution of $\mathcal{L}$ with pole in $\zeta \in \mathbb{R}^{N+1}$, then Γ is smooth out of the pole and has the following properties (see [12]):

i) for any $z \in \mathbb{R}^{N+1}$, $\Gamma(\cdot,z)$ and $\Gamma(z,\cdot)$ belong to $L^1_{\mathrm{loc}}(\mathbb{R}^{N+1})$;

ii) $\Gamma(z,\zeta) \ge 0$, and $\Gamma(x,t,\xi,\tau) > 0$ if, and only if, $t > \tau$;

iii) for every $\varphi \in C_0^\infty(\mathbb{R}^{N+1})$ and $x \in \mathbb{R}^N$ we have

$$\lim_{t \to \tau^+} \int_{\mathbb{R}^N} \Gamma(x,t,\xi,\tau)\, \varphi(\xi)\, \mathrm{d}\xi = \varphi(x);$$

iv) for every $\varphi \in C_0^\infty(\mathbb{R}^{N+1})$ and $z \in \mathbb{R}^{N+1}$ we have

$$\mathcal{L} \int_{\mathbb{R}^{N+1}} \Gamma(z,\zeta)\,\varphi(\zeta)\,\mathrm{d}\zeta = \int_{\mathbb{R}^{N+1}} \Gamma(z,\zeta)\,\mathcal{L}\varphi(\zeta)\,\mathrm{d}\zeta = -\varphi(z);$$

v) $\mathcal{L}\,\Gamma(\cdot,\zeta) = -\delta_\zeta$, where δ_ζ denotes the Dirac measure supported at $\{\zeta\}$;

vi) for every $(x,t),(\xi,\tau) \in \mathbb{R}^{N+1}$ such that $t > \tau$ we have

$$\int_{\mathbb{R}^N} \Gamma(x,t,\xi,\tau)\,\mathrm{d}\xi = 1, \qquad \int_{\mathbb{R}^N} \Gamma(x,t,\xi,\tau)\,\mathrm{d}x = 1;$$

vii) for every $(x,t),(\xi,\tau) \in \mathbb{R}^{N+1}$ and $s \in \mathbb{R}$ such that $\tau < s < t$ we have

$$\Gamma(x,t,\xi,\tau) = \int_{\mathbb{R}^N} \Gamma(x,t,y,s)\Gamma(y,s,\xi,\tau)\,\mathrm{d}y;$$

viii) the function $\Gamma^*(z,\zeta) := \Gamma(\zeta,z)$ is the fundamental solution of the adjoint $\mathcal{L}^*$ of $\mathcal{L}$.

Moreover, Γ is invariant with respect to the group operation and δ_λ-homogeneous of degree $2-Q$:

$$\begin{aligned} &\Gamma(z,\zeta) = \Gamma(\zeta^{-1}\circ z, 0) =: \Gamma(\zeta^{-1}\circ z), \qquad z,\zeta \in \mathbb{R}^{N+1}, z \neq \zeta; \\ &\Gamma(\delta_\lambda(z)) = \lambda^{2-Q}\Gamma(z), \qquad z \in \mathbb{R}^{N+1}\setminus\{0\},\ \lambda > 0. \end{aligned} \tag{2.4}$$

We recall then the following result related to the Cauchy problem for $\mathcal{L}$, obtained by Kogoj and Lanconelli (see [14, Proposition 2]):

Theorem 2.2. *Let $\varphi \in C(\mathbb{R}^N)$ satisfying $\varphi(x) = O(|x|^n)$ as $|x| \to \infty$, for some $n \in \mathbb{N}$. Then the function*

$$u(x,t) = \int_{\mathbb{R}^N} \Gamma(x,t,\xi,\tau)\varphi(\xi)\,\mathrm{d}\xi, \qquad x \in \mathbb{R}^N, t > \tau$$

is well defined and is a solution to the Cauchy problem

$$\begin{cases} \mathcal{L}u = 0 & \text{in } \mathbb{R}^N \times]\tau,\infty[, \\ \lim\limits_{(x,t)\to(y,\tau)} u(x,t) = \varphi(y), & \text{for every } y \in \mathbb{R}^N. \end{cases}$$

The main tools we shall employ in the proof of our results are the following pointwise estimates of Γ and its derivatives, proved by Kogoj and Lanconelli. There exists a positive constant C such that

$$\Gamma(x,t,\xi,\tau) \le \frac{C}{(t-\tau)^{\frac{Q-2}{2}}} \exp\left(-\frac{d^2\big((\xi,\tau)^{-1}\circ(x,t)\big)}{C(t-\tau)}\right), \tag{2.5}$$

for every $x,\xi \in \mathbb{R}^N$ and $t > \tau$ (see [12, (5.1)] and also [14, (2)]). Moreover, as a consequence of a Harnack-type inequality for non-negative solution to $\mathcal{L}u = 0$, for any $j = 1,\dots,m$ there exists $c_j > 0$ with

$$\big|X_j\Gamma(x,t,\xi,\tau)\big| \le c_j(t-\tau)^{-\frac{1}{2}}\,\Gamma\left((\xi,\tau)^{-1}\circ(x,t)\circ\left(0,-\frac{t-\tau}{2}\right)^{-1}\right), \tag{2.6}$$

for all $x,\xi \in \mathbb{R}^N$, $t > \tau$ (see (7) in [14]). We point out that the upper bound in (2.5) is not the best possible. Indeed, in [7] it is given a more precise asymptotic behavior of the exponent in terms of the value function V of the following optimal control problem related to the ordinary differential equation in (1.2):

Definition 2.3. Let $(x,t),(y,s)\in\mathbb{R}^{N+1}$, with $t>s$, and let $\gamma((x,t),(y,s),\omega)$ be any $\mathcal{L}$-admissible path connecting (x,t) with (y,s):

$$\gamma'(\tau)=\sum_{k=1}^{m}\omega_k(\tau)X_k(\gamma(\tau))+Y(\gamma(\tau)),\quad \gamma(0)=(x,t),\ \gamma(t-s)=(y,s).$$

We consider the set of functions $\omega_1,\dots,\omega_m$ as the *control* of the path γ, and the integral

$$\Phi(\omega)=\int_0^{t-s}\left(\omega_1^2(\tau)+\cdots+\omega_m^2(\tau)\right)d\tau$$

as its *cost*. We then define the *value function*

$$V(x,t,y,s)=\inf\left\{\Phi(\omega)\mid\gamma((x,t),(y,s),\omega)\text{ is an }\mathcal{L}\text{-admissible path}\right\}.$$

Hence, if V is locally Lipschitz continuous, for every $\varepsilon>0$ there exists a positive constant C_ε, only depending on the vector fields $X_1,\dots,X_m,Y$ and on ε, such that

$$\Gamma(x,t,0,0)\le\frac{C_\varepsilon}{t^{\frac{Q-2}{2}}}\exp\left(-\frac{1}{32}V((0,\varepsilon t)\circ(x,t)\circ(0,\varepsilon t),0,0)\right),\qquad\forall\,(x,t)\in\mathbb{R}^N\times\mathbb{R}^+.$$

(see [7, Theorem 1.6]). On the other hand, Theorem 1.2 of [5] provides a lower bound of Γ, also stated in terms of V: there exists two constants $C>0$ and $\theta\in]0,1[$ only depending on $\mathcal{L}$, such that

$$\Gamma(x,t,0,0)\ge\frac{1}{C\,t^{\frac{Q-2}{2}}}\exp\left(-CV(x,\theta^2t,0,0)\right),\qquad\forall\,(x,t)\in\mathbb{R}^N\times\mathbb{R}^+.\tag{2.7}$$

Remark 2.4. With these more careful Gaussian estimates at hands, maybe it is possible to improve Theorem 1.1. Indeed, by comparing (2.7) with (2.5), we deduce that

$$V(x,\theta^2t,0,0)\ge c_0\frac{|x|_{\mathbb{G}}^2}{t},\qquad x\in\mathbb{R}^N,\ t>0,$$

for a positive constant c_0. Thus, a growth condition on $u-v$ in terms of V of this kind:

$$\int_0^T\int_{\mathbb{R}^N}e^{-cV(x,\tau,0,0)}|u(x,t)-v(x,t)|\,\mathrm{d}x\,\mathrm{d}t<\infty,\qquad\tau\in\mathbb{R}^+,$$

is sharper than (1.5). However, working with V yields some technical problems, mainly due to the fact that we do not always know explicitly the expression of V. We plan to check on this in a forthcoming paper.

3. Proof of the main results

The purpose of this section is the proof of Theorems 1.1 and 1.2, our main results. We start with a lemma.

Lemma 3.1. *Let $u,v\in C(\mathbb{R}^N\times[0,T])$ be two solutions of the Cauchy problem* (1.4). *If we set $w=u-v$, then there exists a positive constant C such that*

$$|w(x,t)|\le C\int_0^t\int_{\mathbb{R}^N\setminus B_R}|w(\xi,\tau)|\left(\Gamma(x,t,\xi,\tau)+\sum_{j=1}^m\left|X_j\Gamma(x,t,\xi,\tau)\right|\right)\mathrm{d}\xi\,\mathrm{d}\tau,\tag{3.1}$$

for every $(x,t)\in\mathbb{R}^N\times]0,T[$ and $R>|x|_{\mathbb{G}}$.

Proof. Since the vector fields $X_j = \sum_{k=1}^N a_k^j \partial_{x_k}$ for $j = 0, \dots, m$ are δ_λ-homogeneous of a positive degree, the coefficient $a_k^j(x)$ does not depend of x_k, for any $k = 1, \dots, N$. As a consequence, the X_j's are divergence free, $X_j^* = -X_j$ and

$$X_j^2 = \mathrm{div}\,(A^j \nabla)$$

where A^j is the square matrix $(a_h^j a_k^j)_{h,k \le N}$ and $\nabla = (\partial_{x_1}, \dots \partial_{x_N})$. Thus the operator $\mathcal{L}$ takes the following form

$$\mathcal{L} = \mathrm{div}\,(A\nabla) + Y,$$

for the $N \times N$ symmetric matrix $A = (a_{h,k})_{h,k \le N} = \sum_{j=1}^m A^j$, and Y has null divergence in $\mathbb{R}^{N+1}$. Note that we can write $\mathcal{L}^* = \mathrm{div}\,(A\nabla) - Y$. Furthermore, it holds

$$\langle A(x)\xi, \xi\rangle = \sum_{j=1}^m \langle X_j(x), \xi\rangle^2, \qquad \text{for every } x, \xi \in \mathbb{R}^N. \tag{3.2}$$

We consider the following Green's identity:

$$\psi \mathcal{L} w - w \mathcal{L}^* \psi = \sum_{h,k=1}^N \partial_{x_h}\big(a_{h,k}(\psi \partial_{x_k} w - w \partial_{x_k}\psi)\big) + \sum_{k=1}^N \partial_{x_k}(a_k^0\, w\psi) - \partial_t(w\psi), \tag{3.3}$$

for any $w, \psi \in C_0^\infty(\mathbb{R}^{N+1})$.

Now, let $(x,t) \in \mathbb{R}\times]0,T[$ be fixed. For any $R > |x|_{\mathbb{G}}$ we consider $h_R \in C_0^\infty(B_{R+1})$, $0 \le h_R \le 1$, such that $h_R \equiv 1$ on B_R and with first and second order derivatives bounded uniformly w.r.t. R. We integrate the Green's identity (3.3) with $w = u - v$ and $\psi(\xi,\tau) = h_R(\xi)\Gamma(x,t,\xi,\tau)$ over the domain $\{\zeta \in \mathbb{R}^{N+1} \mid \xi \in B_{R+1}, 0 < \tau < t - \delta\}$, for some $\delta > 0$. Recalling that $\mathcal{L}w = 0$ and using the divergence theorem, we get

$$\begin{aligned}
&- \int_0^{t-\delta}\int_{B_{R+1}} w(\xi,\tau)\mathcal{L}^*\psi(\xi,\tau)\, \mathrm{d}\xi\, \mathrm{d}\tau \;= \\
&- \int_{B_{R+1}} w(\xi, t-\delta) h_R(\xi)\Gamma(x,t,\xi,t-\delta)\, \mathrm{d}\xi + \int_{B_{R+1}} w(\xi,0) h_R(\xi)\Gamma(x,t,\xi,0)\, \mathrm{d}\xi \\
&+ \sum_{h,k=1}^N \int_0^{t-\delta}\int_{\partial B_{R+1}} a_{h,k}(\xi)\big(h_R(\xi)\Gamma(x,t,\xi,\tau)\partial_{\xi_k} w(\zeta) - w(\zeta)\partial_{\xi_k}(h_R(\xi)\Gamma(x,t,\xi,\tau))\big)\nu_h\, \mathrm{d}\sigma(\zeta) \\
&+ \sum_{k=1}^N \int_0^{t-\delta}\int_{\partial B_{R+1}} a_k^0(\xi) w(\zeta) h_R(\xi)\Gamma(x,t,\xi,\tau)\nu_k\, \mathrm{d}\sigma(\zeta).
\end{aligned}$$

By hypothesis, the last three terms in the above equation are null. Hence, as δ tends to 0^+, by using the property *vi)* of Γ and (2.5), we obtain

$$w(x,t) = \lim_{\delta \to 0^+} \int_{B_{R+1}} w(\xi, t-\delta) h_R(\xi)\Gamma(x,t,\xi,t-\delta)\, \mathrm{d}\xi = \int_0^t \int_{B_{R+1}} w(\xi,\tau)\mathcal{L}^*\psi(\xi,\tau)\, \mathrm{d}\xi\, \mathrm{d}\tau.$$

Being $\mathcal{L}^*\Gamma(x,t,\xi,\tau) = 0$ and $\mathrm{supp}(\partial_{\xi_k} h_R) \subset B_{R+1} \setminus B_R$, we obtain

$$w(x,t) = \int_0^t \int_{\mathbb{R}^N \setminus B_R} w(\xi,\tau)\big(\Gamma(x,t,\xi,\tau)\mathcal{L}^* h_R(\xi) + 2\langle A(\xi)\nabla\Gamma(x,t,\xi,\tau), \nabla h_R(\xi)\rangle\big)\, \mathrm{d}\xi\, \mathrm{d}\tau.$$

Now (3.1) directly follows from the above equation, by using the Cauchy-Schwartz inequality and (3.2). The assertion is proved. □

As a simple corollary of the previous lemma, we have the following

Proposition 3.2. *Let $u, v \in C(\mathbb{R}^N \times [0,T])$ be two solutions of the Cauchy problem* (1.4). *If*

$$\int_0^t \int_{\mathbb{R}^N} |u(\xi,\tau) - v(\xi,\tau)| \left(\Gamma(x,t,\xi,\tau) + \sum_{j=1}^m \big| X_j \Gamma(x,t,\xi,\tau) \big| \right) \mathrm{d}\xi \, \mathrm{d}\tau < \infty \tag{3.4}$$

for every $(x,t) \in \mathbb{R}^N \times]0,T[$, then $u \equiv v$.

Proof. Condition (3.4) implies

$$\lim_{R\to\infty} \int_0^t \int_{\mathbb{R}^N \setminus B_R} |u(\xi,\tau) - v(\xi,\tau)| \left(\Gamma(x,t,\xi,\tau) + \sum_{j=1}^m \big| X_j \Gamma(x,t,\xi,\tau) \big| \right) \mathrm{d}\xi \, \mathrm{d}\tau = 0,$$

then, by (3.1), $u \equiv v$ in the strip $\mathbb{R} \times]0,T[$. This ends the proof. □

As we will see, the hypothesis of Theorem 1.1 is another condition which, like (3.4), together with Lemma 3.1 implies the uniqueness of the solution of the Cauchy problem (1.4).

Proof of Theorem 1.1. Let $u, v \in C(\mathbb{R}^N \times [0,T])$ be two solutions to the Cauchy problem (1.4). We first prove that $u \equiv v$ in a thin strip $\mathbb{R}^N \times]0,\varepsilon[$, where $\varepsilon \in]0, \min\{1,T\}]$ will be suitably chosen later.

Let $(x,t) \in \mathbb{R}^N \times]0,\varepsilon[$ be fixed. Aiming to use (3.1), we estimate the fundamental solution Γ with pole in $(\xi,\tau) \in \mathbb{R}^N \times]0,t[$ valued in (x,t), and every $X_j\Gamma(x,t,\xi,\tau)$'s.
By using the pseudo-triangular inequality for d, we get

$$\begin{aligned} d\big((\xi,\tau)^{-1} \circ (x,t)\big) &\geq \mathbf{c}^{-1} d\big(0, (\xi,\tau)^{-1}\big) - d\big((\xi,\tau)^{-1} \circ (x,t), (\xi,\tau)^{-1}\big) \\ &= \mathbf{c}^{-1} \|(\xi,\tau)\|_{\mathbb{G}} - \|(x,t)\|_{\mathbb{G}} \geq \mathbf{c}^{-1}|\xi|_{\mathbb{G}} - \|(x,t)\|_{\mathbb{G}} \geq (2\,\mathbf{c})^{-1}|\xi|_{\mathbb{G}}, \end{aligned}$$

if we take $|\xi|_{\mathbb{G}} \geq 2\mathbf{c} \sup_{0<t<T} \|(x,t)\|_{\mathbb{G}} =: R_1(x)$. Hence, by the Gaussian estimate (2.5),

$$\Gamma(x,t,\xi,\tau) \leq \frac{C}{(t-\tau)^{\frac{Q-2}{2}}} \exp\left(- \frac{|\xi|^2_{\mathbb{G}}}{4\mathbf{c}^2 C (t-\tau)} \right), \qquad \text{if } \ |\xi|_{\mathbb{G}} \geq R_1(x). \tag{3.5}$$

On the other hand, (2.6) and (2.5) imply that for any $j = 1, \dots, m$ there exists a positive constant c_j such that

$$\big| X_j \Gamma(x,t,\xi,\tau) \big| \leq \frac{c_j C}{(t-\tau)^{\frac{Q-1}{2}}} \exp\left(- \frac{d^2\Big((\xi,\tau)^{-1} \circ (x,t) \circ \big(0, -\frac{t-\tau}{2}\big)^{-1}\Big)}{C(t-\tau)} \right);$$

see also (2) in [14]. With the same argument as above, we have

$$\begin{aligned} d\Big((\xi,\tau)^{-1} \circ (x,t) \circ \big(0, -\tfrac{t-\tau}{2}\big)^{-1}\Big) &\geq \mathbf{c}^{-1} d\big(0, (\xi,\tau)^{-1} \circ (x,t)\big) - \Big\| \big(0, -\tfrac{t-\tau}{2}\big)^{-1} \Big\|_{\mathbb{G}} \\ &\geq \mathbf{c}^{-1}\big(\mathbf{c}^{-1}|\xi|_{\mathbb{G}} - \|(x,t)\|_{\mathbb{G}}\big) - \Big\| \delta_{\sqrt{(t-\tau)/2}}\big((0,-1)^{-1}\big) \Big\|_{\mathbb{G}} \\ &> \mathbf{c}^{-2}|\xi|_{\mathbb{G}} - \mathbf{c}^{-1}\|(x,t)\|_{\mathbb{G}} - d((0,-1))\sqrt{T/2} \ \geq (\sqrt{2}\,\mathbf{c})^{-2}\, |\xi|_{\mathbb{G}}, \end{aligned}$$

for $|\xi|_{\mathbb{G}} \geq R_1(x) + \mathbf{c}^2 d((0,-1))\sqrt{2\,T} =: R(x)$. It follows that, for every $j = 1, \dots, m$,

$$\big|X_j\Gamma(x,t,\xi,\tau)\big| \leq \frac{c_j C}{(t-\tau)^{\frac{Q-1}{2}}} \exp\left(-\frac{|\xi|_{\mathbb{G}}^2}{\sqrt{2}\,\mathbf{c}\,C(t-\tau)}\right), \qquad \text{if } |\xi|_{\mathbb{G}} \geq R(x). \tag{3.6}$$

As a consequence, from (3.1), (3.5) and (3.6),

$$\begin{aligned}|u(x,t)-v(x,t)| &\leq C\int_0^t\int_{\mathbb{R}^N\setminus B_R}|u(\xi,\tau)-v(\xi,\tau)|\left(\Gamma(x,t,\xi,\tau)+\sum_{j=1}^m\big|X_j\Gamma(x,t,\xi,\tau)\big|\right)\mathrm{d}\xi\,\mathrm{d}\tau\\ &\leq C_1\int_0^t\int_{\mathbb{R}^N\setminus B_R}|u(\xi,\tau)-v(\xi,\tau)|\frac{1}{(t-\tau)^{\frac{Q-1}{2}}}\exp\left(-\frac{C_2|\xi|_{\mathbb{G}}^2}{t-\tau}\right)\mathrm{d}\xi\,\mathrm{d}\tau,\end{aligned}$$

for every $R > \max\{|x|_{\mathbb{G}}, R(x)\}$, where C_1, C_2 are two positive constants only dependent on $\mathbf{c}$ and on the operator $\mathcal{L}$. Now set $\varepsilon = \min\left\{\frac{C_2}{2c}, 1, T\right\}$, where $c > 0$ is the constant in (1.5). Since the function $(\xi,\tau) \mapsto (t-\tau)^{-\frac{Q-1}{2}}\exp\left(-\frac{C_2|\xi|_{\mathbb{G}}^2}{2(t-\tau)}\right)$ is bounded on $(\mathbb{R}^N\setminus B_R)\times]0,T[$, by the choice of ε we get

$$|u(x,t)-v(x,t)| \leq C_3\int_0^t\int_{\mathbb{R}^N\setminus B_R} e^{-c|\xi|_{\mathbb{G}}^2}|u(\xi,\tau)-v(\xi,\tau)|\mathrm{d}\xi\,\mathrm{d}\tau, \qquad \forall\, R > \max\{|x|_{\mathbb{G}}, R(x)\}.$$

On the other hand, hypothesis (1.5) implies that

$$\lim_{R\to\infty}\int_0^t\int_{\mathbb{R}^N\setminus B_R} e^{-c|\xi|_{\mathbb{G}}^2}|u(\xi,\tau)-v(\xi,\tau)|\mathrm{d}\xi\,\mathrm{d}\tau = 0,$$

whence $u(x,t) = v(x,t)$ for every $(x,t) \in \mathbb{R}^N\times]0,\varepsilon[$. The thesis follows by repeating the previous argument finitely many times (note that ε depends only on $\mathbf{c}$, on the constant c in (1.5) and on the operator $\mathcal{L}$). □

In order to prove Theorem 1.2, we need a preliminary result.

Lemma 3.3. *Let $u \in C(\mathbb{R}^N \times [0,T])$ be a non-negative solution of $\mathcal{L}u = 0$ in $\mathbb{R}^N\times]0,T[$. Then*

$$u(x,t) \geq \int_{\mathbb{R}^N}\Gamma(x,t,\xi,\tau)u(\xi,\tau)\,\mathrm{d}\xi, \tag{3.7}$$

for every $(x,t) \in \mathbb{R}^N\times]0,T[$ and $0 < \tau < t$.

Proof. Fix $\tau \in]0,T[$. For every $n \in \mathbb{N}$ and $(x,t) \in \mathbb{R}^N\times]\tau,T[$, set

$$u_n(x,t,\tau) := \int_{\mathbb{R}^N}\Gamma(x,t,\xi,\tau)h\left(\frac{|\xi|_{\mathbb{G}}}{n}\right)u(\xi,\tau)\,\mathrm{d}\xi,$$

where $h \in C^\infty(\mathbb{R})$ is a fixed non-increasing cut-off function such that $h(s) = 1$ if $s \leq 1$ and $h(s) = 0$ if $s \geq 2$. By Theorem 2.2, we know that u_n is a solution to the Cauchy problem

$$\begin{cases}\mathcal{L}u_n(\cdot,\tau) = 0 & \text{in } \mathbb{R}^N\times]\tau,T[,\\ \lim\limits_{(x,t)\to(y,\tau)} u_n(x,t,\tau) = h\left(\frac{|y|_{\mathbb{G}}}{n}\right)u(y,\tau), & \text{for every } y \in \mathbb{R}^N.\end{cases}$$

Furthermore, by the estimate (2.5),

$$
\begin{aligned}
0 \le u_n(x,t,\tau) &\le \frac{C}{(t-\tau)^{\frac{Q-2}{2}}} \int_{\mathbb{R}^N} \exp\left(-\frac{d^2\big((\xi,\tau)^{-1}\circ(x,t)\big)}{C(t-\tau)}\right) h\left(\frac{|\xi|_{\mathbb{G}}}{n}\right) u(\xi,\tau)\,\mathrm{d}\xi \\
&\le \frac{C}{(t-\tau)^{\frac{Q-2}{2}}} \int_{B_{2n}} \exp\left(-\frac{d^2\big((\xi,\tau)^{-1}\circ(x,t)\big)}{C(t-\tau)}\right) u(\xi,\tau)\,\mathrm{d}\xi.
\end{aligned} \tag{3.8}
$$

Recalling the properties of the quasi-distance d, we obtain

$$
\begin{aligned}
d\big((\xi,\tau)^{-1}\circ(x,t)\big) &\ge \mathbf{c}^{-1} d\big((\xi,\tau)^{-1}\circ(x,t),0\big) \ge \mathbf{c}^{-2}\|(x,t)\|_{\mathbb{G}} - \mathbf{c}^{-1}\|(\xi,\tau)\|_{\mathbb{G}} \\
&\ge \mathbf{c}^{-2}|x|_{\mathbb{G}} - \mathbf{c}^{-1}\big((2n)^{2k!} + T^{k!}\big)^{\frac{1}{2k!}},
\end{aligned}
$$

so that, by (3.8),

$$
0 \le u_n(x,t,\tau) \le \max_{B_{2n}} u(\cdot,\tau) \frac{C\,\mathrm{meas}(B_{2n})}{(t-\tau)^{\frac{Q-2}{2}}} \exp\left(-\frac{\big(\mathbf{c}^{-1}|x|_{\mathbb{G}} - ((2n)^{2k!}+T^{k!})^{\frac{1}{2k!}}\big)^2}{\mathbf{c}^2 C(t-\tau)}\right) \longrightarrow 0
$$

as $|x|_{\mathbb{G}} \to \infty$. We now apply the weak maximum principle to the $\mathcal{L}$-harmonic function $v_n = u_n(\cdot,\tau) - u$ in the strip $\mathbb{R}^N \times]\tau,T[$. Indeed, we have $\lim_{(x,t)\to(y,\tau)} v_n(x,t) \le 0$ for every $y \in \mathbb{R}^N$ being $h \le 1$, and $\limsup v_n \le 0$ at infinity in the strip, recalling that $u \ge 0$. The maximum principle of Proposition 2.1 then gives

$$
0 \le u_n(x,t,\tau) \le u(x,t), \qquad \text{for every } (x,t) \in \mathbb{R}^N \times]\tau,T[.
$$

Letting n go to infinity, from the above inequality we obtain (3.7), since

$$
u_n(x,t,\tau) \nearrow \int_{\mathbb{R}^N} \Gamma(x,t,\xi,\tau) u(\xi,\tau)\,\mathrm{d}\xi \qquad \text{as } n \to \infty
$$

by monotone convergence. This accomplishes the proof. □

Proof of Theorem 1.2. We first show that, if $u \in C(\mathbb{R}^N \times [0,T])$ is a non-negative solution of $\mathcal{L}u = 0$ on the strip $\mathbb{R}^N \times]0,T[$ then

$$
\int_0^t \int_{\mathbb{R}^N} u(\xi,\tau) \Big(\Gamma(x,t,\xi,\tau) + \sum_{j=1}^m \big|X_j \Gamma(x,t,\xi,\tau)\big| \Big)\,\mathrm{d}\xi\,\mathrm{d}\tau < \infty \tag{3.9}
$$

for every $(x,t) \in \mathbb{R}^N \times \big]0, \frac{2}{3}T\big[$. Indeed, by integrating the inequality (3.7) with respect to $\tau \in]0,t[$, we obtain

$$
\int_0^t \int_{\mathbb{R}^N} u(\xi,\tau)\,\Gamma(x,t,\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau \le u(x,t)\,t < \infty, \qquad \text{for every } (x,t) \in \mathbb{R}^N \times]0,T[.
$$

On the other hand, from (2.6) it follows that

$$
\begin{aligned}
&\int_0^t \int_{\mathbb{R}^N} u(\xi,\tau) \sum_{j=1}^m \big|X_j\Gamma(x,t,\xi,\tau)\big|\,\mathrm{d}\xi\,\mathrm{d}\tau \\
&\le C \int_0^t \frac{1}{\sqrt{t-\tau}} \left(\int_{\mathbb{R}^N} u(\xi,\tau)\,\Gamma\Big((x,t)\circ\Big(0,-\frac{t-\tau}{2}\Big)^{-1},\xi,\tau\Big)\mathrm{d}\xi\right)\mathrm{d}\tau \qquad \text{(by (3.7))} \\
&\le C \int_0^t \frac{1}{\sqrt{t-\tau}}\, u\Big((x,t)\circ\Big(0,-\frac{t-\tau}{2}\Big)^{-1}\Big)\mathrm{d}\tau \le C\,C_{x,t}\,2\sqrt{t} < \infty,
\end{aligned}
$$

for every $(x,t) \in \mathbb{R}^N \times \left]0, \frac{2}{3}\,T\right[$. In the last but one inequality we have used that, since u is a continuous function on $\mathbb{R}^N \times [0,T]$, there exists a constant $C_{x,t} > 0$ such that

$$u\Big((x,t) \circ \Big(0, -\frac{t-\tau}{2}\Big)^{-1}\Big) \le \sup_{\tau \in [0,t]} u\Big(S(x,t,\tau), \frac{3t-\tau}{2}\Big) =: C_{x,t} < \infty.$$

Hence, (3.9) is proved.

We next conclude the proof of the theorem. Let $u, v \in C(\mathbb{R}^N \times [0,T])$ be two non-negative solutions to the Cauchy problem (1.4). As $|u-v| \le u+v$, the first part of the proof yields condition (3.4) for every $(x,t) \in \mathbb{R}^N \times \left]0, \frac{2}{3}\,T\right[$, whence $u \equiv v$ in $\mathbb{R}^N \times \left[0, \frac{2}{3}\,T\right]$ by Proposition 3.2. We repeate the above argument, and we find, at the n-th step,

$$u \equiv v \qquad \text{in } \mathbb{R}^N \times \left[0, \left(1 - \tfrac{1}{3^{n+1}}\right)T\right].$$

This proves the theorem. □

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