On the variety parametrizing completely decomposable polynomials

Enrique Arrondo and Alessandra Bernardi

Abstract: The purpose of this paper is to relate the variety parameterizing completely decomposable homogeneous polynomials of degree d in $n+1$ variables on an algebraically closed field, called $\mathrm{Split}_d(\mathbb{P}^n)$, with the Grassmannian of $n-1$ dimensional projective subspaces of $\mathbb{P}^{n+d-1}$. We compute the dimension of some secant varieties to $\mathrm{Split}_d(\mathbb{P}^n)$ and find a counterexample to a conjecture that wanted its dimension related to the one of the secant variety to $\mathbb{G}(n-1, n+d-1)$. Moreover by using an invariant embedding of the Veronse variety into the Plücker space, then we are able to compute the intersection of $\mathbb{G}(n-1, n+d-1)$ with $\mathrm{Split}_d(\mathbb{P}^n)$, some of its secant variety, the tangential variety and the second osculating space to the Veronese variety.

Introduction

A classic problem inspired by Waring problem in number theory is the following: which is the least integer s such that a general homogeneous polynomial of degree d in $n+1$ variables can be written as $L_1^d + \cdots + L_s^d$, where $L_1, \ldots, L_s$ are linear forms? In terms of algebraic geometry, this problem is equivalent to find the least s such that the s-th secant variety of the d-uple Veronese embedding of $\mathbb{P}^n$ is the whole ambient space. In general, it is interesting to find projective varieties with defective secant varieties, i.e. not having the expected dimension. This problem has been completely solved by J. Alexander and A. Hirschowitz (see [**AH**], or [**BO**] for a recent proof with a different approach), who found all the defective secant varieties to Veronese varieties. Our original problem can be rephrased in the language of tensors. Specifically, given an $(n+1)$-dimensional vector space W, which is the least integer s such that a general tensor in S^dW can be written as a sum of s completely decomposable symmetric tensors?

With this new language, it is natural to wonder about the same problem in the case of tensors not necessarily symmetric. For example, the case of tensors in $W_1 \otimes \cdots \otimes W_d$, yields the question of studying the smallest s-th secant variety of a Segre variety filling up the ambient space (see [**AOP1**], [**CGG3**], [**CGG4**] for some known results regarding this problem). Another interesting problem is the case in which the tensors are skew-symmetric or, geometrically, the study of the smallest s-th secant variety of a Grassmann variety filling up the ambient Plücker space. In this case, the only known examples of defective s-th secant varieties are: the third secant varieties to $\mathbb{G}(2,6)$ –which is also isomorphic to $\mathbb{G}(3,6)$– and to $\mathbb{G}(3,7)$ and the fourth secant varieties to $\mathbb{G}(3,7)$ and $\mathbb{G}(2,8)$ –which is also isomorphic to $\mathbb{G}(5,8)$– ([**CGG1**], [**McG**] and [**AOP2**]).

There is a particularly interesting numerical relation among the different types of tensors we just mentioned. Indeed, the dimension of the above S^dW is $\binom{n+d}{n}$, which coincides with the dimension of the space $\bigwedge^n W'$ of skew-symmetric tensors on an $(n+d)$-dimensional vector space W'. Therefore the projectivization of the space of homogeneous polyonomials of degree d in $n+1$ variables has the same dimension as the Plücker ambient space of the Grassmannian $\mathbb{G}(n-1, n+d-1)$. Moreover, this Grassmannian has dimension nd, which is also the dimension of the variety, which we will call $\mathrm{Split}_d(\mathbb{P}^n)$, parametrizing those polynomials that decompose as the product of d linear forms. These coincidences led Ehrenborg to formulate (see [**Eh**]) the following

Conjecture 0.1. (**Ehrenborg**) *The least positive integer s such that the s-th secant variety to $\mathbb{G}(n-1, n+d-1)$ fills up $\mathbb{P}^{\binom{n+d}{d}-1}$ is the same least $s \in \mathbb{N}$ such that the s-th secant variety to $\mathrm{Split}_d(\mathbb{P}^n)$ fills up $\mathbb{P}^{\binom{n+d}{d}-1}$.*

If this were true, defective secant varieties to Grassmannians would also produce defective secant varieties to Split varieties. It is easy to see that, if $d=2$, then the conjecture is true (Proposition 1.10). Unfortunately, the other possible defective cases coming from Grassmannians, namely the third secant varieties to $\mathrm{Split}_4(\mathbb{P}^3)$, to $\mathrm{Split}_3(\mathbb{P}^4)$ and to $\mathrm{Split}_4(\mathbb{P}^4)$, and the fourth secant varieties to $\mathrm{Split}_4(\mathbb{P}^4)$, $\mathrm{Split}_6(\mathbb{P}^3)$ and $\mathrm{Split}_3(\mathbb{P}^6)$, are not defective (Example 1.9). In particular, we get that Ehrenborg's conjecture is not true.

The starting point of this paper was to understand until which extent Ehrenborg's conjecture remains true and to find possible common cases of defectivity between Grassmannians and Split varieties. Since M.V. Catalisano, A.V. Geramita and A. Gimigliano conjecture in [**CGG1**] that the only defective secant varieties to Grassmannians are those listed above, one could conjecture that any Split variety with $d \neq 2$ has regular secant varieties (i.e. with the expected dimension). In fact, we were not able to find any defective case.

We thus turn to the core of Ehrenborg's conjecture and study what is behind the numerical coincidence. Our main idea is to identify the $(n+1)$-dimensional vector space W with S^nV, where V is a two-dimensional vector space. Then we use the well known isomorphism between $\bigwedge^d(S^{n+d-1}V)$ and $S^d(S^nV)$ (see [**Mu**]), which has a nice and classical interpretation. Precisely, the d-uple Veronese variety is naturally embedded in $\mathbb{G}(n-1, n+d-1)$ as the set of n-secant spaces to the rational normal curve in $\mathbb{P}^{n+d-1}$. This allows to consider $\mathbb{G}(n-1, n+d-1)$ and $\mathrm{Split}_d(\mathbb{P}^n)$ as subvarieties of the same projective space. Depending on the context, we will regard this space as the Plücker space of $\mathbb{G}(n-1, n+d-1)$ or the projective space parametrizing classes of homogeneous polynomials of degree d in $n+1$ variables.

With the point of view of homogeneous polynomials, we observe (Remark 3.2) that points of $\mathrm{Split}_d(\mathbb{P}^n)$ are characterized by belonging to certain osculating spaces to the Veronese variety. Hence, in order to completely understand $\mathrm{Split}_d(\mathbb{P}^n)$ we will need to first understand these osculating spaces.

The goal of this paper is to use the previous identification to compare $\mathrm{Split}_d(\mathbb{P}^n)$ –or any other variety related to it, like osculating spaces to the Veronese variety– with $\mathbb{G}(n-1, n+d-1)$. In particular, intersecting those varieties with $\mathbb{G}(n-1, n+d-1)$, we can regard the corresponding types of polynomials as $(n-1)$-dimensional linear subspaces of $\mathbb{P}^{n+d-1}$.

We start the paper with section 1, in which we introduce the preliminaries and give some first results about $\mathrm{Split}_d(\mathbb{P}^n)$ without still using its relation with $\mathbb{G}(n-1, n+d-1)$. More precisely, we prove the regularity of the secant varieties to $\mathrm{Split}_d(\mathbb{P}^n)$ in a certain range not depending on d (Proposition 1.8). We also include in this section our counterexample to Ehrenborg's conjecture.

In section 2, we first describe in coordinates the embeddings of the Veronese variety, $\mathrm{Split}_d(\mathbb{P}^n)$ and $\mathbb{G}(n-1, n+d-1)$ in the same projective space. This allows us to give a first general result about the intersection of $\mathrm{Split}_d(\mathbb{P}^n)$ and $\mathbb{G}(n-1, n+d-1)$ (Proposition 2.7), which we can improve in the case $d=3$ (Proposition 2.10). We use this geometric description to end with Example 2.11 (which we will need later on), showing that some particular elements of $\mathbb{G}(n-1, n+d-1)$ cannot be in $\mathrm{Split}_d(\mathbb{P}^n)$.

In section 3 we study the intersection between $\mathbb{G}(n-1, n+d-1)$ and the tangential variety to the d-uple Veronese variety. We arrive to the precise intersection in Corollary 3.11. Since this tangential variety parameterizes classes of homogeneous polynomials that can be written as $L^{d-1}M$ (where L and M are linear forms) we can also give a necessary condition on M for $L^{d-1}M$ to represent an element of $\mathbb{G}(n-1, n+d-1)$ (Proposition 3.12). As a consequence of the results of this section, we can compute the intersection of $\mathbb{G}(n-1, n+d-1)$ and $\mathrm{Split}_d(\mathbb{P}^n)$ when $d = 2$.

In order to compute the above intersection when $d = 3$, we will need to study first the intersection between $\mathbb{G}(n-1, n+d-1)$ and the second osculating space to the Veronese variety, to which we devote section 4 (see Theorem 4.3 for the precise result). With the result of this section, we eventually give in section 5 the intersection between $\mathbb{G}(n-1, n+d-1)$ and $\mathrm{Split}_d(\mathbb{P}^n)$ when $d = 3$ (Theorem 5.4).

We end this paper with an appendix in which we give various results about the intersection of $\mathbb{G}(n-1, n+d-1)$ with several secant varieties to the d-uple Veronese variety. In particular, we completely describe this intersection when $d = 2$ and for any secant variety. We include this appendix, even if sometimes we just sketch the proofs, because the results we got give an idea of how the techniques introduced in the paper can be useful.

We like to thank Silvia Abrescia for the many and useful conversations and Maria Virginia Catalisano for suggestions and ideas.

During the preparation of this work, the first author was supported by the Spanish project number MTM2006-04785; the second author was supported by MIUR and by funds from the University of Bologna.

1 Preliminaries and first results

Throughout all the paper, the symbol $\mathbb{P}^n$ will denote the projective space over an algebraically closed field K of characteristic zero, and we will fix a system of homogeneous coordinates $x_0, \dots, x_n$. We also write $\mathbb{G}(k, d+k)$ for the *Grassmannian of k-spaces in* $\mathbb{P}^{d+k}$ and $\vec{G}(k, V)$ for the *Grassmannian of k-spaces in* V.

We will indicate for brevity the polynomial ring $K[x_0, \dots, x_n]$ with R and its homogeneous part of degree d with R_d. With this notation, $\mathbb{P}(R_d)$ is naturally identified with the set of hypersurfaces of degree d in $\mathbb{P}^n$ and, in particular, $\mathbb{P}(R_1)$ is identified with $(\mathbb{P}^n)^*$.

Definition 1.1. The *Veronese variety* is the subset of $\mathbb{P}(R_d)$ parametrizing d-uple hyperplanes, i.e. classes of forms that are a d-th power of linear forms. We will write $\mathrm{Split}_d(\mathbb{P}^n)$ for the subset of hypersurfaces that are the union of d hyperplanes.

Remark 1.2. If we use as homogeneous coordinates for $\mathbb{P}(R_d)$ the coefficients of the monomials, the d-uple Veronese embedding

$$\begin{array}{rccl} \nu_d : & \mathbb{P}(R_1) & \hookrightarrow & \mathbb{P}(R_d) \quad = \mathbb{P}^{\binom{n+d}{d}-1} \\ & [L] & \mapsto & [L^d]. \end{array}$$

(whose image is the Veronese variety) can be written as

$$(u_0 : \ldots : u_n) \mapsto (u_0^d : u_0^{d-1}u_1 : u_0^{d-1}u_2 : \ldots : u_n^d).$$

Similarly, $\mathrm{Split}_d(\mathbb{P}^n)$ is the image of the finite map (of degree $d!$):

$$\begin{array}{rccc} \phi : & \mathbb{P}(R_1) \times \overset{d}{\cdots} \times \mathbb{P}(R_1) & \hookrightarrow & \mathbb{P}(R_d) \\ & ([L_1], \dots, [L_d]) & \mapsto & [L_1 \cdots L_d] \end{array}$$

which sends the point $([u_0^{(1)}, \dots, u_n^{(1)}], \dots, [u_0^{(d)}, \dots, u_n^{(d)}])$ to the point whose coordinates form the canonical basis of the space V of symmetric forms of $K[u_0^{(1)}, \dots, u_n^{(1)}; \dots; u_0^{(d)}, \dots, u_n^{(d)}]$ of multidegree $(1, \dots, 1)$. Hence, $\mathrm{Split}_d(\mathbb{P}^n)$ has dimension nd and it is the image of $\mathbb{P}(R_1) \times \cdots \times \mathbb{P}(R_1)$ under the linear subsystem $V \subset H^0(\mathcal{O}_{\mathbb{P}(R_1)\times\cdots\times\mathbb{P}(R_1)}(1, \dots, 1))$ of symmetric forms. When $d = 2$, $\mathrm{Split}_2(\mathbb{P}^n)$ can also be regarded as the set of classes of $(n+1) \times (n+1)$ symmetric matrices of rank at most two.

Definition 1.3. If $X \subset \mathbb{P}^N$ is a projective variety of dimension n then it's *s-th Secant Variety* is defined as follows:

$$\mathrm{Sec}_{s-1}(X) := \overline{\bigcup_{P_1,\dots,P_s \in X} < P_1, \dots, P_s >}.$$

Its expected dimension is

$$\mathrm{expdim}(\mathrm{Sec}_{s-1}(X)) = \min\{N, sn + s - 1\}$$

but this is not always equal to $\dim(\mathrm{Sec}_{s-1}(X))$ in fact there are many exceptions. When $\delta_{s-1} = \mathrm{expdim}(\mathrm{Sec}_{s-1}(X)) - \dim(\mathrm{Sec}_{s-1}(X)) > 0$ we will say that $\mathrm{Sec}_{s-1}(X)$ is *defective* and δ_{s-1} is called *defect.*

Before starting the study on the dimension of secant varieties of Split varieties we need to introduce some important instruments classically utilized to study secant varieties.

Definition 1.4. If $X \subset \mathbb{P}^N$ is an irreducible projective variety, an *m-fat point* (or an *m-th point*) on X is the $(m-1)$-th infinitesimal neighborhood of a smooth point $P \in X$ and it will be denoted by mP (i.e. it is the projective scheme mP defined by the ideal sheaf $\mathcal{I}_{P,X}^m \subset \mathcal{O}_X$).

If $\dim(X) = n$ then an m-fat point mP on X is a 0-dimensional scheme of length $\binom{m-1+n}{n}$. If Z is the union of the $(m-1)$-th infinitesimal neighborhoods in X of s generic smooth points on X, we will say for short that Z is the union of s generic m-fat points on X.

The most useful (and classical) theorem for the computation of the dimension of a secant variety of a projective variety is the so called *Terracini's Lemma.*

Theorem 1.5. (**Terracini's Lemma**) *Let X be an irreducible variety in $\mathbb{P}^N$, and let $P_1, \dots, P_s$ be s generic points on X. Then, the projectivized tangent space to $\mathrm{Sec}_{s-1}(X)$ at a generic point $Q \in < P_1, \dots, P_s >$ is the linear span in $\mathbb{P}^N$ of the tangent spaces $T_{P_i}(X)$ to X at P_i, $i = 1, \dots, s$, i.e.*

$$T_Q(\mathrm{Sec}_{s-1}(X)) = < T_{P_1}(X), \dots, T_{P_s}(X) > .$$

Proof. For a proof see [**Te**] or [**Ad**]. □

From Terracini's Lemma we immediately get a way of checking the defectivity of secant varieties. We include the precise result for $\mathrm{Split}_d(\mathbb{P}^n)$, although the same technique works for arbitrary varieties with a generically finite map to a projective space.

Corollary 1.6. *The secant variety Sec_{s-1} $(\mathrm{Split}_d(\mathbb{P}^n))$ is not defective if and only if s general fat points on $\mathbb{P}(R_1) \times \overset{d}{\dots} \times \mathbb{P}(R_1)$ impose $\min\{s(dn+1), \binom{n+d}{d}\}$ independent conditions to the linear system V of symmetric forms of multidegree $(1, \dots, 1)$ in $K[u_0^{(1)}, \dots, u_n^{(1)}; \dots; u_0^{(d)}, \dots, u_n^{(d)}]$.*

Proof. By Terracini's Lemma, $\dim(\mathrm{Sec}_{s-1}(\mathrm{Split}_d(\mathbb{P}^n))) = \dim(< T_{P_1}(\mathrm{Split}_d(\mathbb{P}^n)), \dots, T_{P_s}(\mathrm{Split}_d(\mathbb{P}^n)) >)$, with $P_1, \dots, P_s$ general points of $\mathrm{Split}_d(\mathbb{P}^n)$. Since the hyperplanes of $\mathbb{P}^{\binom{n+d}{d}-1}$ containing $T_{P_i}(\mathrm{Split}_d(\mathbb{P}^n))$ are those containing the fat point $2P_i$, it follows that $\dim(\mathrm{Sec}_{s-1}(\mathrm{Split}_d(\mathbb{P}^n))) = \binom{n+d}{d} - 1 - h^0(\mathcal{I}_Z(1))$, where Z is the scheme union of the fat points $2P_1, \dots, 2P_s$.

On the other hand, by Remark 1.2, $\mathrm{Split}_d(\mathbb{P}^n)$ is the image of $\mathbb{P}(R_1) \times \overset{d}{\dots} \times \mathbb{P}(R_1)$ by the finite map ϕ determined by V. Therefore $h^0(\mathcal{I}_Z(1))$ is the dimension of the space of forms in V vanishing on $\phi^{-1}(Z)$. By the symmetry of the forms of V, it is enough to take preimages $P'_1, \dots, P'_s$ of $P_1, \dots, P_s$ by ϕ, and $h^0(\mathcal{I}_Z(1))$ is still the dimension of the forms of V vanishing at $2P'_1, \dots, 2P'_s$. The result follows now at once. $\square$

From this corollary, we can prove directly the non-defectivity of several secant varieties to $\mathrm{Split}_d(\mathbb{P}^n)$. We start from a technical result.

Lemma 1.7. *Let $Q_1, \dots, Q_d, P_1, \dots, P_n \in \mathbb{P}(R_1) = \mathbb{P}^n$ be a set of points in general position. Then there exist $dn+1$ symmetric forms $F, F_{ij} \in K[u_0^{(1)}, \dots, u_n^{(1)}; \dots; u_0^{(d)}, \dots, u_n^{(d)}]$, with $i = 1, \dots, n$ and $j = 1, \dots, d$, of multidegree $(1, \dots, 1)$, such that:*

(i) $F(Q_1, \dots, Q_d) \neq 0$ while $F(P_i, A_2, \dots, A_d) = 0$ for any $i = 1, \dots, n$ and any $A_2, \dots, A_d \in \mathbb{P}(R_1)$.

(ii) $F_{ij}(P_k, A_2, \dots, A_d) = 0$ for any $i, k = 1, \dots, n$, $j = 1, \dots, d$, $k \neq i$ and $A_2, \dots, A_d \in \mathbb{P}(R_1)$.

(iii) $F, F_{11}, \dots, F_{nd}$ are independent modulo I^2, where $I \subset K[u_0^{(1)}, \dots, u_n^{(1)}; \dots; u_0^{(d)}, \dots, u_n^{(d)}]$ is the multihomogeneous ideal of $(Q_1, \dots, Q_d)$ in $\mathbb{P}(R_1) \times \dots \times \mathbb{P}(R_1)$.

Proof. For any linear form $L \in K[u_0, \dots, u_n]$, we will denote by $\tilde{L}$ for the symmetrized form

$$\tilde{L} := L(u_0^{(1)}, \dots, u_n^{(1)}) \cdot L(u_0^{(2)}, \dots, u_n^{(2)}) \cdots L(u_0^{(d)}, \dots, u_n^{(d)}).$$

Since the points are in general position we can take a linear form $L \in K[u_0, \dots, u_n]$ vanishing at $P_1, \dots, P_n$ and not vanishing at any $Q_1, \dots, Q_d$. We thus take $F = \tilde{L}$, which satisfies (i).

Similarly, for any $i = 1, \dots, n$ and $j = 1, \dots, d$, we can find $L_{ij} \in K[u_0, \dots, u_n]$ vanishing at $P_1, \dots, P_{i-1}, P_{i+1}, \dots, P_n, Q_j$, and we take $F_{ij} = \tilde{L}_{ij}$, and clearly (ii) holds.

Finally, to prove (iii), assume that there is a linear combination $\lambda F + \lambda_{11} F_{11} + \dots + \lambda_{nd} F_{nd} \in I^2$. Evaluating at the point $(Q_1, \dots, Q_d)$, we get $\lambda = 0$. On the other hand, taking an arbitrary point $U \in \mathbb{P}(R_1)$ of coordinates $[u_0, \dots, u_n]$, and evaluating at $(Q_1, \dots, Q_{j-1}, Q_{j+1}, \dots, Q_d, U)$ we get, for any $j = 1, \dots, d$, that the linear form

$$\Sigma_{i=1}^n \lambda_{ij} F_{ij}(Q_1, \dots, Q_{j-1}, Q_{j+1}, \dots, Q_d, U) \in K[u_1, \dots, u_n]$$

is in the square of the ideal of Q_i in $\mathbb{P}(R_1)$. This clearly implies that this linear form is identically zero. Morevover, evaluating it at each P_i, with $i = 1, \dots, n$, we get $\lambda_{ij} = 0$, which completes the proof. $\square$

Proposition 1.8. *If $d > 2$ and $3(s-1) \leq n$, then $\mathrm{Sec}_{s-1}(\mathrm{Split}_d(\mathbb{P}^n))$ is not defective.*

Proof. It is enough to apply Corollary 1.6. We thus take s general points $A_1, \ldots, A_s \in \mathbb{P}(R_1) \times \overset{d}{\ldots} \times \mathbb{P}(R_1)$ and need to show that the evaluation map $\varphi : V \to H^0(\mathcal{O}_Z)$ is surjective, where Z is the scheme union of the fat points $2A_1, \ldots, 2A_s$.

For each $i = 1, \ldots, s$, we write $A_i = (Q_{i1}, \ldots, Q_{id})$. Since $n \geq 3(s-1)$ and $d > 2$, we can pick $P_{i1}, \ldots, P_{in} \in \mathbb{P}^n$ in general position and such that they contain the points Q_{j1}, Q_{j2}, Q_{j3} for any $j = 1, \ldots, i-1, i+1, \ldots, s$. Applying Lemma 1.7, we can find symmetric forms $F_i, F_{i1}, \ldots, F_{i,nd} \in V$ such that the image of them under the evaluation map φ maps surjectively to $H^0(\mathcal{O}_{2A_i})$. Also, the properties (i) and (ii) of the lemma imply, together with our choice of $P_{i1}, \ldots, P_{in} \in \mathbb{P}^n$, that these forms map to zero in any direct summand $\mathcal{O}_{2A_j}$ of $H^0(\mathcal{O}_Z)$. Since this is true for any i, the surjectivity of φ follows. □

We finish this section discussing Ehrenborg's conjecture.

Example 1.9. It is a known result (see for example [**CGG1**]) that $\mathrm{Sec}_{3-1}(\mathbb{G}(2,6))$ has defect $\delta_2 = 1$, i.e one expects that $\mathrm{Sec}_2(\mathbb{G}(2,6)) = \mathbb{P}^{34}$ but $\dim(\mathrm{Sec}_2(\mathbb{G}(2,6))) = 33$; we need $\mathrm{Sec}_3(\mathbb{G}(2,6))$ in order to fill up $\mathbb{P}^{34}$. However, it is not true that the least integer s such that $\mathrm{Sec}_{s-1}(\mathrm{Split}_4(\mathbb{P}^3))$ fills up the ambient space is 4 too; in fact $\mathrm{Sec}_2(\mathrm{Split}_4(\mathbb{P}^3)) = \mathbb{P}^{34}$ (we checked this using the previous techniques, and making computations with [**CoCoA**]).

In the same way, we can also prove that the third secant varieties to $\mathrm{Split}_3(\mathbb{P}^4)$ and to $\mathrm{Split}_4(\mathbb{P}^4)$, and the fourth secant varieties to $\mathrm{Split}_4(\mathbb{P}^4)$, $\mathrm{Split}_6(\mathbb{P}^3)$ and $\mathrm{Split}_3(\mathbb{P}^6)$, are not defective

The only case for which we are able to prove that Ehrenborg's conjecture is true is for $d = 2$.

Proposition 1.10. *The dimensions of* $\mathrm{Sec}_{s-1}(\mathbb{G}(1, n+1))$ *and* $\mathrm{Sec}_{s-1}(\mathrm{Split}_2(\mathbb{P}^n))$ *are equal.*

Proof. The embedding of $\mathbb{G}(1, n+1)$ into $\mathbb{P}^{\binom{n+2}{2}-1} \simeq \mathbb{P}(R_2) = \mathbb{P}(K[x_0, \ldots, x_n]_2)$ allows to look at the Grassmannian as the set of quadrics whose representative $(n+2) \times (n+2)$ matrices are skewsymmetric and of rank at most 2. Therefore $\mathrm{Sec}_{s-1}(\mathbb{G}(1, n+1)) \simeq \{M \in M_{n+2}(K) \mid M = -M^T,\ \mathrm{rk}(M) \leq 2s\}$, then $\mathrm{codim}(\mathrm{Sec}_{s-1}(\mathbb{G}(1, n+1))) = \binom{n+2-2s}{2}$.

In the same way $\mathrm{Split}_2(\mathbb{P}^n) \simeq \{M \in M_{n+1}(K) \mid M = M^T,\ \mathrm{rk}(M) \leq 2\}$; therefore $\mathrm{Sec}_{s-1}(\mathrm{Split}_2(\mathbb{P}^n)) \simeq \{M \in M_{n+1}(K) \mid M \text{ is symmetric and } \mathrm{rk}(M) \leq 2s\}$, then $\mathrm{codim}(\mathrm{Sec}_{s-1}(\mathrm{Split}_d(\mathbb{P}^n)) = \binom{n+2-2s}{2} = \mathrm{codim}(\mathrm{Sec}_{s-1}(\mathbb{G}(1, n+1)))$. □

2 Veronese varieties and Grassmannians

In this section we want to study the other problem inspired to us by Ehrenborg's conjecture: the "intersection" between $\mathbb{G}(n-1, n+d-1)$ and $\mathrm{Split}_d(\mathbb{P}^n)$. To do this, we will need to identify the ambient spaces of both varieties.

We collect first in a lemma the main results (written in an intrinsic way) of a classical construction that we will need in the sequel.

Lemma 2.1. *Consider the map* $\phi_{n,d} : \mathbb{P}(K[t_0, t_1]_n) \to \vec{G}(d, K[t_0, t_1]_{n+d-1})$ *that sends the class of* $p_0 \in K[t_0, t_1]_n$ *to the d-dimensional subspace of* $K[t_0, t_1]_{n+d-1}$ *of forms of the type* $p_0 q$*, with* $q \in K[t_0, t_1]_{d-1}$*. Then the following hold:*

(i) The image of $\phi_{n,d}$*, after the Plücker embedding of* $\vec{G}(d, K[t_0, t_1]_{n+d-1})$*, is the n-dimensional d-th Veronese variety.*

(ii) Identifying $\vec{G}(d, K[t_0,t_1]_{n+d-1})$ with the Grassmann variety of subspaces of dimension $n-1$ in $\mathbb{P}(K[t_0,t_1]^*_{n+d-1})$, the above Veronese variety is the set V of n-secant spaces to a rational normal curve $\Sigma \subset \mathbb{P}(K[t_0,t_1]^*_{n+d-1})$.

(iii) For any $p \in K[t_0,t_1]_s$, with $s < n$, there is a commutative diagram

$$\begin{array}{ccc} \mathbb{P}(K[t_0,t_1]_{n-s}) & \stackrel{\phi_{n-s,d}}{\longrightarrow} & \vec{G}(d, K[t_0,t_1]_{n+d-s-1}) \\ \downarrow & & \downarrow \\ \mathbb{P}(K[t_0,t_1]_n) & \stackrel{\phi_{n,d}}{\longrightarrow} & \vec{G}(d, K[t_0,t_1]_{n+d-1}) \end{array}$$

where the vertical arrows are inclusions naturally induced by the multiplication by p.

(iv) When identifying $\vec{G}(d, K[t_0,t_1]_{n+d-1})$ with the Grassmann variety of subspaces of dimension $n-1$ in $\mathbb{P}(K[t_0,t_1]^*_{n+d-1})$, the image by $\phi_{n,d}$ of $\mathbb{P}(K[t_0,t_1]_{n-s}) \subset \mathbb{P}(K[t_0,t_1]_n)$ as in (iii) is the set of n-secants to Σ containing the subscheme $Z \subset \Sigma$ defined by the zeros of p.

Proof. Write $p_0 = u_0t_0^n + u_1t_0^{n-1}t_1 + \cdots + u_nt_1^n$. Then a basis of the subspace of $K[t_0,t_1]_{n+d-1}$ of forms of the type p_0q is given by:

$$\begin{cases} u_0t_0^{n+d-1} + \cdots + u_nt_0^{d-1}t_1^n \\ \quad u_0t_0^{n+d-2}t_1 + \cdots + u_nt_0^{d-2}t_1^{n+1} \\ \qquad \ddots \\ \qquad\quad u_0t_0^nt_1^{d-1} + \cdots + u_nt_1^{n+d-1}. \end{cases}$$

The coordinates of these elements with respect to the basis $\{t_0^{n+d-1}, t_0^{n+d-2}t_1, \ldots, t_1^{n+d-1}\}$ of $K[t_0,t_1]_{n+d-1}$ are thus given by the rows of the matrix

$$\begin{pmatrix} u_0 & u_1 & \ldots & u_n & 0 & \ldots & 0 & 0 \\ 0 & u_0 & u_1 & \ldots & u_n & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & u_0 & u_1 & \ldots & u_n & 0 \\ 0 & \ldots & 0 & 0 & u_0 & \ldots & u_{n-1} & u_n \end{pmatrix}.$$

The standard Plücker coordinates of the subspace $\phi_{n,d}([p_0])$ are the maximal minors of this matrix. It is known (see for example [**AP**]), these minors form a basis of $K[u_0,\ldots,u_n]_d$, so that the image of ϕ is indeed a Veronese variety, which proves (i).

To prove (ii), we still recall some standard facts from [**AP**]. Take homogeneous coordinates $z_0,\ldots,z_{n+d-1}$ in $\mathbb{P}(K[t_0,t_1]^*_{n+d-1})$ corresponding to the dual basis of $\{t_0^{n+d-1}, t_0^{n+d-2}t_1, \ldots, t_1^{n+d-1}\}$. Consider $\Sigma \subset \mathbb{P}(K[t_0,t_1]^*_{n+d-1})$ the standard rational normal curve with respect to these coordinates. Then, the image of $[p_0]$ by $\phi_{n,d}$ is precisely the n-secant space to Σ spanned by the divisor on Σ induced by the zeros of p_0. This completes the proof of (ii).

Part (iii) comes directly from the definitions. Finally, in order to prove (iv), observe that (iii) implies that the image by $\phi_{n,d}$ of $\mathbb{P}(K[t_0,t_1]_{n-s}) \subset \mathbb{P}(K[t_0,t_1]_n)$ is the subset of subspaces of $K[t_0,t_1]_{n+d-1}$ all of whose elements are divisible by some pp_0 with $p_0 \in K[t_0,t_1]_{n-s}$, in particular divisible by p. The proof of (ii) implies that the corresponding subspace in $\mathbb{P}(K[t_0,t_1]^*_{n+d-1})$ contains the subscheme $Z \subset \Sigma$ defined by the zeros of p. $\square$

Remark 2.2. In the above proof we used coordinates to describe the curve Σ, because it will be useful for us later on. However, it can be described also in an intrinsic way. Specifically, the elements of $\mathbb{P}(K[t_0,t_1]^*_{n+d-1})$ are linear forms $K[t_0,t_1]_{n+d-1}\to K$ up to multiplication by a constant. Then Σ is nothing but the set of classes of linear forms of the type $F\mapsto F(a_0,a_1)$ for some $a_0,a_1\in K$.

Remark 2.3. In order to relate our Veronese variety V with the standard Veronese variety, we will identify R_1 with $K[t_0,t_1]_n$ by assigning to any $L=u_0x_0+\cdots+u_nx_n\in R_1$ the homogeneous form $L(t_0^n,t_0^{n-1}t_1,\ldots,t_1^n)=u_0t_0^n+u_1t_0^{n-1}t_1+\cdots+u_nt_1^n\in K[t_0,t_1]_n$. If we just write $\mathbb{P}^{n+d-1}$ instead of $\mathbb{P}(K[t_0,t_1]^*_{n+d-1})$, the map $\phi:\mathbb{P}(R_1)\to\mathbb{G}(n-1,n+d-1)$ sends the class of the linear form to the subspace of $\mathbb{P}^{n+d-1}$ defined (in the above coordinates) as the intersection of the hyperplanes:

$$\begin{cases} u_0z_0+\cdots+u_nz_n=0\\ \quad u_0z_1+\cdots+u_nz_{n+1}=0\\ \qquad\ddots\\ \qquad\quad u_0z_{d-1}+\cdots+u_nz_{n+d-1}=0\end{cases}. \tag{1}$$

By this reason, we will use from now on Plücker coordinates, but in a way that is dual to the standard one. Specifically, for any projective space $\mathbb{P}^{d+k}$ with homogenous coordinates $z_0,\ldots,z_{d+k}$, if $\Lambda\subset\mathbb{P}^{d+k}$ is the space defined by the linearly independent equations

$$\begin{gathered} u_{1,0}z_0+\cdots+u_{1,d+k}z_{d+k}=0\\ \vdots\\ u_{d,0}z_0+\cdots+u_{d,d+k}z_{d+k}=0\end{gathered}$$

for each $0\le i_1<\cdots<i_d\le d+k$ we define $p_{i_1\cdots i_d}$ to be the determinant

$$p_{i_1\cdots i_d}:=\begin{vmatrix} u_{1,i_1} & \cdots & u_{1,i_d}\\ \vdots & & \vdots\\ u_{d,i_1} & \cdots & u_{d,i_d}\end{vmatrix}.$$

In this way, the Plücker embedding is described as follows:

$$\begin{array}{rcl} p:\mathbb{G}(k,n) & \hookrightarrow & \mathbb{P}^{\binom{n+1}{k+1}-1}\\ \Lambda & \mapsto & \{\{p_{i_1\cdots i_d}\}\mid 0\le i_1<\cdots<i_d\le d+k\}\end{array} \tag{2}$$

Example 2.4. After Remark 2.3, we are implicitly identifying $\mathbb{P}(R_d)$ with the Plücker ambient space of $\mathbb{G}(n-1,n+d-1)$. When using the standard coordinates in each of these varieties (the coefficients of the polynomial and Plücker coordinates, respectively), this identification should be made explicit for any concrete case. For example, let us make explicit such an identification in the case $n=2,d=3$. In this case, the map $\phi_{2,3}$ assigns to any linear form $u_0x_0+u_1x_1+u_2x_2$ the line of $\mathbb{P}^4$ given as intersection of the hyperplanes

$$\begin{cases} u_0z_0 & +u_1z_1 & +u_2z_2 & & & =0\\ & u_0z_1 & +u_1z_2 & +u_2z_3 & & =0\\ & & u_0z_2 & +u_1z_3 & +u_2z_4 & =0\end{cases}$$

so that it has Plücker coordinates

$$\begin{aligned}
p_{012} &= u_0^3 \\
p_{013} &= u_0^2 u_1 \\
p_{014} &= u_0^2 u_2 \\
p_{023} &= u_0 u_1^2 - u_0^2 u_2 \\
p_{024} &= u_0 u_1 u_2 \\
p_{034} &= u_0 u_2^2 \\
p_{123} &= u_1^3 - 2u_0 u_1 u_2 \\
p_{124} &= u_1^2 u_2 - u_0 u_2^2 \\
p_{134} &= u_1 u_2^2 \\
p_{234} &= u_2^3.
\end{aligned}$$

Since the Veronese embedding $\mathbb{P}(R_1) \to \mathbb{P}(R_3)$ is defined by $u_0x_0 + u_1x_1 + u_2x_2 \mapsto (u_0x_0 + u_1x_1 + u_2x_2)^3$, the above relations show that an element of the ambient Plücker space is naturally identified with the polynomial

$$\begin{aligned}
&p_{012}x_0^3 + 3p_{013}x_0^2x_1 + 3p_{014}x_0^2x_2 + 3(p_{023} + p_{014})x_0x_1^2 + 6p_{024}x_0x_1x_2 + \\
&+3p_{034}x_0x_2^2 + (p_{123} + 2p_{024})x_1^3 + 3(p_{034} + p_{124})x_1^2x_2 + 3p_{134}x_1x_2^2 + p_{234}x_2^3.
\end{aligned} \tag{3}$$

After the identification of Remark 2.3, we can restate Lemma 2.1 in terms of polynomials in $K[x_0, \dots, x_n]$.

Lemma 2.5. *Let $p := a_0t_0^s + a_1t_0^{s-1}t_1 + \cdots + a_st_1^s \in K[t_0, t_1]_s$ and set, for $j = 1, \dots, n-s+1$, the linear forms*

$$\begin{array}{rccccccccccc}
N_0 := & a_0x_0 & + & a_1x_1 & + & \cdots & + & a_sx_s & & & & \\
N_1 := & & & a_0x_1 & + & a_1x_2 & + & \cdots & + & a_sx_{s+1} & & \\
\vdots & & & & \ddots & & & & & & \ddots & \\
N_{n-s} := & & & & & a_0x_{n-s} & + & a_1x_{n-s+1} & + & \cdots & + & a_sx_n.
\end{array}$$

Then, in the set up of Lemma 2.1, and identifying $\mathbb{P}(K[t_0, t_1]_n)$ with $\mathbb{P}(R_1)$, the inclusion $\mathbb{P}(K[t_0, t_1]_{n-s}) \subset \mathbb{P}(K[t_0, t_1]_n)$ is identified with $\mathbb{P}(K[N_0, \dots, N_{n-s}]_1) \subset \mathbb{P}(R_1)$ and its image by $\phi_{n,d}$ in $\mathbb{G}(n-1, n+d-1)$ is the locus

$$G' := \{\Lambda \in \mathbb{G}(n-1, n+d-1) \mid \Lambda \cap \Sigma \supseteq Z\}$$

where $Z \subset \Sigma$ is the subscheme defined by the zeros of p. Moreover, diagram (iii) of Lemma 2.1 can be written as

$$\begin{array}{ccc}
\mathbb{P}(K[N_0, \dots, N_{n-s}]_1) & \xrightarrow{\phi_{n-s,d}} & \mathbb{G}(n-s-1, n+d-s-1) \\
\downarrow & & \downarrow \\
\mathbb{P}(K[x_0, \dots, x_n]_1) & \xrightarrow{\phi_{n,d}} & \mathbb{G}(n-1, n+d-1)
\end{array}$$

where $\mathbb{P}^{n+d-s-1}$ is identified with the projection of $\mathbb{P}^{n+d-1}$ from $< Z >$, and the natural map $\mathbb{G}(n-s-1, n+d-s-1) \to \mathbb{G}(n-1, n+d-1)$ is identified with the inclusion of G'.

Proof. It is enough to recall that the subspace $\mathbb{P}(K[t_0, t_1]_{n-s}) \subset \mathbb{P}(K[t_0, t_1]_n)$ corresponds to the subspace of polynomials in $K[t_0, t_1]_n$ divisible by p. These polynomials take the form $(a_0t_0^s + a_1t_0^{s-1}t_1 + \cdots + a_st_1^s)(b_0t_0^{n-s} + b_1t_0^{n-s-1}t_1 + \cdots + b_{n-s}t_1^{n-s})$, which, as elements of R_1, are precisely those of the form $b_0N_0 + \cdots + b_{n-s}N_{n-s}$. The rest of the statement is obtained directly from Lemma 2.1. $\square$

Remark 2.6. When $s = n$, there is only one form N_0 and G' is just one point of $\mathbb{G}(n-1, n+d-1)$, which is precisely the point of V corresponding to $[N_0^d]$.

When $s = n-1$, the set G' is a projective space of dimension d, so it is the whole $\mathbb{P}(K[N_0, N_1]_d)$. This case allows to give some first relation between $\mathrm{Split}_d(\mathbb{P}^n)$ and $\mathbb{G}(n-1, n+d-1)$, as we do in the following proposition.

Proposition 2.7. *The intersection* $\mathrm{Split}_d(\mathbb{P}^n) \cap \mathbb{G}(n-1, n+d-1)$ *contains the locus of* $(n-1)$*-linear spaces that are* $(n-1)-$*secant to* Σ.

Proof. If Λ is an $(n-1)$-secant space to Σ, then it contains a subscheme $Z \subset \Sigma$ of length $n-1$. Hence, Lemma 2.5, implies that Λ, as an element of $\mathbb{P}(R_d)$, comes from a homogeneous form in $K[N_0, N_1]_d$, so that it necessarily splits. $\square$

At this point of the discussion it becomes interesting to investigate if the previous corollary describes only an inclusion or an equality. Let us see that, at least for $d = 3$, the intersection contains another component. We start with the case $n = 2$.

Example 2.8. In the set up of Example 2.4, consider the class of the polynomial $x_1(x_0 - x_1)(x_1 - x_2)$. This clearly gives an element in $\mathbb{P}^9$ that is in $\mathrm{Split}_3(\mathbb{P}^2)$. With the identification given in (3), it corresponds to the element of Plücker coordinates

$$[p_{012}, p_{013}, p_{014}, p_{023}, p_{024}, p_{034}, p_{123}, p_{124}, p_{134}, p_{234}] = [0, 0, 0, 2, -1, 0, -4, 2, 0, 0].$$

This point is in $\mathbb{G}(1, 4)$, and corresponds precisely to the line of equations $z_0 - 2z_1 = z_2 = z_4 - 2z_3 = 0$, which does not meet the standard rational normal curve $\Sigma \subset \mathbb{P}^4$. The geometric interpretation of this line is that it is the intersection of the following three hyperplanes:

- $z_2 = 0$, the span of the of the tangent lines of Σ at the points $[1, 0, 0, 0, 0]$ and $[0, 0, 0, 0, 1]$,
- $z_0 - 2z_1 + z_2 = 0$, the span of the of the tangent lines of Σ at the points $[1, 0, 0, 0, 0]$ and $[1, 1, 1, 1, 1]$,
- $z_2 - 2z_3 + z_4 = 0$, the span of the of the tangent lines of Σ at the points $[0, 0, 0, 0, 1]$ and $[1, 1, 1, 1, 1]$.

Since Σ is a homogeneous variety, we get that, for any choice of different points $y_1, y_2, y_3 \in \Sigma$, the intersection of $< T_{y_1}\Sigma, T_{y_2}\Sigma > \cap < T_{y_1}\Sigma, T_{y_3}\Sigma > \cap < T_{y_2}\Sigma, T_{y_3}\Sigma >$ is an element of $\mathbb{G}(1, 4)$ that is also in $\mathrm{Split}_3(\mathbb{P}^2)$.

The above example can be generalized to any n, showing that $\mathrm{Split}_3(\mathbb{P}^n) \cap \mathbb{G}(n-1, n+2)$ contains not only the $(n+2)$-dimensional subvariety given in Proposition 2.7, but also another $(n+1)$-dimensional subvariety (we will see in Theorem 5.4 that the intersection consists exactly of those two components). We introduce first a notation that we will use throughout the paper.

Notation 2.9. If Σ is a smooth curve, we will write $\{r_1 y_1, \ldots, r_k y_k\}$ or $r_1 y_1 + \cdots + r_k y_k$ to denote the subscheme of Σ supported on the different points $y_1, \ldots, y_k \in \Sigma$ with respective multiplicities $r_1, \ldots, r_k$.

Proposition 2.10. *For any* $n \geq 2$, *the intersection of* $\mathrm{Split}_3(\mathbb{P}^n)$ *and* $\mathbb{G}(n-1, n+2)$ *contains the set*

$$\{< Z + 2y_1 + 2y_2 > \cap < Z + 2y_1 + 2y_3 > \cap < Z + 2y_2 + 2y_3 > \ \mid Z \subset \Sigma,\ \mathrm{length}(Z) = n-2,\ y_1, y_2, y_3 \in \Sigma\}.$$

Proof. Fix a subscheme $Z \subset \Sigma$ of length $n-2$ and let $\Lambda \in \mathbb{G}(n-1, n+2)$ be a subspace that can be written as

$$\Lambda = < Z + 2y_1 + 2y_2 > \cap < Z + 2y_1 + 2y_3 > \cap < Z + 2y_2 + 2y_3 > .$$

In particular Λ contains Z, so that it is contained in the set G' of Lemma 2.5. Consider the projection of $\mathbb{P}^{n+2}$ to $\mathbb{P}^4$ from $< Z >$. In this way, Σ becomes a rational normal curve $\Sigma' \subset \mathbb{P}^4$, while Λ becomes a line $\Lambda' \subset \mathbb{P}^4$ that can be written as

$$\Lambda' = < 2y'_1 + 2y'_2 > \cap < 2y'_1 + 2y'_3 > \cap < 2y'_2 + 2y'_3 >$$

where each $y'_i \in \Sigma'$ is the image of y_i. By Example 2.8, the line Λ' is an element of $\mathrm{Split}_3(\mathbb{P}^2)$. With the identifications of Lemma 2.5, this should be interpreted as follows. The set G' is identified with $\mathbb{G}(1, 4)$, whose Plücker ambient space is $\mathbb{P}(K[N_0, N_1, N_2]_3)$, so that the line Λ' is represented by a polynomial $F \in K[N_0, N_1, N_2]_3$ that factor into three linear forms. Hence, regarding $\Lambda \in G' \subset \mathbb{G}(n-1, n+2)$ as an element of its ambient Plücker space $\mathbb{P}(K[x_0, \ldots, x_n]_d)$, it is represented by the same polynomial $F \in K[x_0, \ldots, x_n]_d$. Therefore $\Lambda \in \mathrm{Split}_3(\mathbb{P}^n)$. $\square$

Example 2.11. In the same way as in Proposition 2.10, it is possible to prove that certain elements of $\mathbb{G}(n-1, n+2)$ are not in $\mathrm{Split}_3(\mathbb{P}^n)$. In particular, we will need later on (see Lemma 5.2) to check that, given different points $y_1, \ldots, y_k$ on the rational normal curve $\Sigma \subset \mathbb{P}^{n+2}$ and nonnegative integers $r_1, \ldots, r_k$ such that $r_1 + \cdots + r_k = n$, the linear subspaces

$$< (r_1+2)y_1, r_2y_2, r_3y_3 \ldots, r_ky_k > \cap < r_1y_1, (r_2+2)y_2, r_3y_3, \ldots, r_ky_k > \cap < (r_1-2)y_1, (r_2+4)y_2, r_3y_3, \ldots, r_ky_k >$$

$$< (r_1+2)y_1, r_2y_2, r_3y_3, \ldots, r_ky_k > \cap < r_1y_1, (r_2+1)y_2, (r_3+1)y_3, r_4y_4, \ldots, r_ky_k > \cap$$
$$\cap < (r_1-2)y_1, (r_2+3)y_2, (r_3+1)y_3, r_4y_4, \ldots, r_ky_k >$$

$$< (r_1+2)y_1, r_2y_2, \ldots, r_ky_k > \cap < r_1y_1, (r_2+2)y_2, r_3y_3, \ldots, r_ky_k > \cap < (r_1-2)y_1, (r_2+3)y_2, (r_3+1)y_3, r_4y_4, \ldots, r_k >$$

$$< (r_1+2)y_1, r_2y_2, \ldots, r_ky_k > \cap < r_1y_1, (r_2+2)y_2, r_3y_3, \ldots, r_ky_k > \cap < (r_1-1)y_1, (r_2-1)y_2, (r_3+4)y_3, r_4y_4, \ldots, r_k >$$

$$< (r_1+2)y_1, r_2y_2, \ldots, r_ky_k > \cap < r_1y_1, (r_2+2)y_2, r_3y_3, \ldots, r_ky_k > \cap$$
$$\cap < (r_1-1)y_1, (r_2-1)y_2, (r_3+3)y_3, (r_4+1)y_4, r_5y_5, \ldots, r_k >$$

have dimension $n-1$ and, as elements of $\mathbb{G}(n-1, n+2)$, they are not in $\mathrm{Split}_3(\mathbb{P}^n)$. To prove that, we first observe that all those subspaces always contain a finite subscheme $Z \subset \Sigma$ of length $n-2$, namely $< (r_1-2)y_1, r_2y_2, r_3y_3 \ldots, r_ky_k >$ in the first three cases and $< (r_1-1))y_1, (r_2-1)y_2, r_3y_3 \ldots, r_ky_k >$ in the last two cases. Hence, projecting from Z, we are reduced to the case $n = 2$ and we need to check that, given points y_1, y_2, y_3, y_4 in the rational normal curve in $\mathbb{P}^4$, the subspaces

$$< 4y_1 > \cap < 2y_1, 2y_2 > \cap < 4y_2 >$$

$$< 4y_1 > \cap < 2y_1, y_2, y_3 > \cap < 3y_2, y_3 >$$

$$< 4y_1 > \cap < 2y_1, 2y_2 > \cap < 3y_2, y_3 >$$

$$< 3y_1, y_2 > \cap < y_1, 3y_2 > \cap < 4y_3 >$$

$$< 3y_1, y_2 > \cap < y_1, 3y_2 > \cap < 3y_3, y_4 >$$

are lines and that, as elements of $\mathbb{G}(1,4)$, they are not in $\mathrm{Split}_3(\mathbb{P}^2)$. By the homogeneity of Σ, we can assume $y_1 = [1,0,0,0,0]$, $y_2 = [0,0,0,0,1]$, $y_3 = [1,1,1,1,1]$ and $y_4 = [1,\lambda,\lambda^2,\lambda^3,\lambda^4]$ with $\lambda \neq 0,1$. With this choice, the above five spaces become respectively the lines

$$z_4 = z_2 = z_0 = 0$$

$$z_4 = z_2 - z_3 = z_0 - z_1 = 0$$

$$z_4 = z_2 = z_0 - z_1 = 0$$

$$z_3 = z_1 = z_0 - 4z_1 + 6z_2 - 4z_3 + z_4 = 0$$

$$z_3 = z_1 = \lambda z_0 + (-3\lambda - 1)z_1 + (3\lambda + 3)z_2 + (-\lambda - 3)z_3 + z_4 = 0$$

with Plücker coordinates $[p_{012}, p_{013}, p_{014}, p_{023}, p_{024}, p_{034}, p_{123}, p_{124}, p_{134}, p_{234}]$ equal to

$$[0,0,0,0,-1,0,0,0,0,0]$$

$$[0,0,0,0,-1,1,0,1,-1,0]$$

$$[0,0,0,0,-1,0,0,1,0,0]$$

$$[0,-1,0,0,0,0,6,0,-1,0]$$

$$[0,-\lambda,0,0,0,0,3\lambda+3,0,-1,0]$$

Using (3), we get respective polynomials

$$-2x_1(3x_0x_2 + x_1^2)$$

$$-6x_0x_1x_2 + 3x_0x_2^2 - 2x_1^3 + 6x_1^2x_2 - 3x_1x_2^2$$

$$-x_1(6x_0x_2 + 2x_1^2 - 3x_1x_2)$$

$$-3x_1(x_0^2 - 2x_1^2 + x_2^2)$$

$$-3x_1(\lambda x_0^2 - (\lambda+1)x_1^2 + x_2^2).$$

Since none of the above polynomials split into linear factors, they do not represent points in $\mathrm{Split}_3(\mathbb{P}^2)$.

3 Tangential varieties to Veronese varieties and Grassmannians

We want to devote the rest of the paper to understand the intersection of $\mathrm{Split}_d(\mathbb{P}^n)$ and $\mathbb{G}(n-1, n+d-1)$. The strategy will be to relate the algebraic properties of polynomials with the geometry of subspaces in $\mathbb{P}^{n+d-1}$ (where we have the rational normal curve Σ defining V, thus giving the connection between the two approches). The main idea is that a polynomial representing a point in $\mathrm{Split}_d(\mathbb{P}^n)$ is characterized by having many linear factors. This is translated in terms of geometry by means of osculating spaces, and we will devote this section to the first case, the tangential varities.

We recall first the background for this theory.

Notation 3.1. Denote with $O_x^k(X)$ the k-th osculating space to a projective variety X at the point $x \in X$ (observe that $O_x^0(X) = x$ and $O_x^1(X) = T_x(X)$).

Remark 3.2. We recall from [**BCGI**] that, for any $[L^d] \in V$, the elements of $O^k_{[L^d]}(V)$ are precisely those represented by forms of the type $L^{d-k}F$ where $F \in R_k$. Therefore any point of $\mathrm{Split}_d(\mathbb{P}^n)$, which can be written as $[L_1^{m_1} \cdots L_t^{m_t}]$ with $L_1, \dots, L_t \in R_1$ different linear forms and $m_1, \dots, m_t$ positive integers with $\sum_{i=1}^t m_i = d$, can be obtained as the only point in the intersection $O^{d-m_1}_{[L_1^d]}(V) \cap \cdots \cap O^{d-m_t}_{[L_t^d]}(V)$. Hence we have an equality

$$\mathrm{Split}_d(\mathbb{P}^n) = \bigcup_{\substack{\sum_{i=1}^t m_i = d \\ \Lambda_1, \dots, \Lambda_t \in V}} O^{d-m_1}_{\Lambda_1}(V) \cap \cdots \cap O^{d-m_t}_{\Lambda_t}(V)$$

where the subspaces $\Lambda_1, \dots, \Lambda_t$ are assumed to be different. In the particular case $d = 3$, we can simply write

$$\mathrm{Split}_3(\mathbb{P}^n) = \tau(V) \bigcup \Big(\bigcup_{\Lambda_1, \Lambda_2, \Lambda_3 \in V} O^2_{\Lambda_1}(V) \cap O^2_{\Lambda_2}(V) \cap O^2_{\Lambda_3}(V) \Big)$$

because any form of degree three containing a square necessarily splits.

The above remark is saying that, in order to understand the intersection of $\mathrm{Split}_d(\mathbb{P}^n)$, with $\mathbb{G}(n-1, n+d-1)$, it is enough to understand the intersection of the osculating spaces to V. A first geometric result in this direction is the following.

Proposition 3.3. *Let Λ be a point in the osculating space $O^k_{\Lambda_0}(V)$ with $k < d$. If we regard Λ_0 as an n-secant linear subspace to the rational normal curve $\Sigma \subset \mathbb{P}^{n+d-1}$, then Λ_0 contains the points (counted with multiplicity) of the intersection $\Lambda \cap \Sigma$.*

Proof. Let $L \in R_1$ be a linear form such that $\Lambda_0 = [L^d]$. Since $\lambda \in O^k(V)$ with $k < d$, Remark 3.2 implies that Λ is represented by a form of the type $L^{d-k}M$.

On the other hand, let $Z \subset \Sigma$ be the schematic intersection of Λ and Σ and set $s = \mathrm{length}(Z)$. Let $p \in K[t_0, t_1]_s$ be the polynomial whose scheme of zeros in $\mathbb{P}^1$ corresponds to $Z \subset \Sigma$. By Lemma 2.5, the Plücker ambient space of the set G' of $(n-1)$-dimensional subspaces containing Z is $\mathbb{P}(K[N_0, \dots, N_{n-s}]_d)$, for some linear forms $N_0, \dots, N_{n-s} \in K[x_0, \dots, x_n]$.

Putting both things together, we get $L^{d-k}M \in K[N_0, \dots, N_{n-s}]$. Since $d - k > 0$, necessarily $L \in K[N_0, \dots, N_{n-s}]$. Again by Lemma 2.5, this implies that Λ_0 is in G, i.e. it contains Z, as wanted. $\square$

We introduce next the main tool that we will use to study the osculating spaces to V and their intersection with $\mathbb{G}(n-1, n+d-1)$.

Definition 3.4. Consider the incidence variety

$$I := \{(\Lambda, y) \in \mathbb{G}(n-1, n+d-1) \times \Sigma \mid \mathrm{length}_y(\Lambda \cap \Sigma) \geq r\} \subset \mathbb{G}(n-1, n+d-1) \times \Sigma.$$

Fix $\Lambda_0 \in \mathbb{G}(n-1, n+d-1)$ such that the intersection between Λ_0 and Σ in $\mathbb{P}^{n+d-1}$ is a zero-dimensional scheme whose support at a point $y \in \Lambda_0 \cap \Sigma$ has length r. Let π_1 be the projection from I to $\mathbb{G}(n-1, n+d-1)$. We denote by $Z_y \subset \mathbb{P}^{\binom{n+d}{d}-1}$ the image by π_1 of a neighborhood of I near (Λ_0, y).

Remark 3.5. Let $\Lambda_0 \in V$ be a point corresponding to a subspace $\Lambda_0 \subset \mathbb{P}^{n+d-1}$ meeting Σ at points $y_1, \dots, y_k$ with respective multiplicities $r_1, \dots, r_k$ (hence $r_1 + \cdots + r_k = n$). With the above notation, each

$Z_i := Z_{y_i}$ is smooth at Λ_0 and a neighbourhood of V near Λ_0 is given by the intersection $Z_1 \cap \cdots \cap Z_k$. Therefore

$$T_{\Lambda_0}(V) = \bigcap_{i=1}^{k} T_{\Lambda_0}(Z_i).$$

The same equality does not hold for arbitrary osculating spaces, in which we only have one inclusion:

$$\bigcap_{i=1}^{k} O^s_{\Lambda_0}(Z_i) \subset O^s_{\Lambda_0}(V)$$

for any s. Hence, in order to study tangent or osculating spaces to the Veronese variety V we will study first those spaces for the Z_i.

We devote the rest of the section to the tangent spaces to the Grassmannian, while we will see in later sections that the inclusion we have for second osculating spaces is enough if $d = 3$. The first step will be to compute the intersection of $\mathbb{G}(n-1, n+d-1)$ with the tangent spaces to each of the above neighborhoods.

Theorem 3.6. *Let $\Lambda \in \mathbb{G}(n-1, n+d-1)$ meeting Σ at a zero-dimensional scheme whose support at a point $y \in \Lambda_0 \cap \Sigma$ has length r. If Z_y is as in Definition 3.4, then the intersection between the tangent space to Z_y in Λ_0 and the Grassmannian $\mathbb{G}(n-1, n+d-1)$ is*

$$T_{\Lambda_0}(Z_y) \cap \mathbb{G}(n-1, n+d-1) =$$

$$= \{\Lambda \in \mathbb{G}(n-1, n+d-1) \mid \Lambda \supset O^{r-1}_x(\Sigma),\ \dim(\Lambda \cap \Lambda_0) \geq n-2\} \cup$$
$$\cup \{\Lambda \in \mathbb{G}(n-1, n+d-1) \mid O^{r-2}_x(\Sigma) \subset \Lambda \subset < \Lambda_0, O^r_x(\Sigma) >\}.$$

Proof. Let the map $\mathbb{P}^1 \to \mathbb{P}^{n+d-1}$ defined by $(t_0, t_1) \mapsto (t_0^{n+d-1}, t_0^{n+d-2}t_1, \ldots t_1^{n+d-1})$ be a parameterization of Σ; without loss of generality we may assume that $y = [1, 0, \ldots, 0] \in \Sigma$ and that $a_1, \ldots, a_r \in K$ are such that $\nu_{n+d-1}((t_1^r + a_1 t_1^{r-1} t_0 + \cdots + a_{r-1} t_1 t_0^{r-1} + a_r t_0^r)^*) = y$. Hence $\Lambda_0 \in \mathbb{G}(n-1, n+d-1)$ is defined in $\mathbb{P}^{n+d-1}$ by the equations $z_r = \cdots = z_{r+d-1} = 0$. We will study the affine tangent space $\hat{T}_{\Lambda_0}(Z_y)$ in the affine chart of the Plücker coordinates $\{p_{r,\ldots,r+d-1} \neq 0\}$. Observe that in this affine chart we have a system of coordinates given by $\{p_{r,\ldots,\hat{\imath},\ldots,r+d-1,j}\}$, with $i \in \{r, \ldots, r+d-1\}$ and $j \notin \{r, \ldots, r+d-1\}$, while the other Plücker coordinates are homogeneous forms of degree at least two in these coordinates. Let H_i for $i = 1, \ldots, n+d-r$ be the hyperplane of $\mathbb{P}^{n+d-1}$ defined by the equation

$$H_i : a_r z_{i-1} + a_{r-1} z_i + \cdots + a_1 z_{r+i-2} + z_{r+i-1} = 0.$$

Hence Z_y is described by

$$\begin{cases} H_1 + \mu_{1,d+1} H_{d+1} + \cdots + \mu_{1,n+d-r} H_{n+d-r} = 0 \\ \vdots \\ H_d + \mu_{d,d+1} H_{d+1} + \cdots + \mu_{d,n+d-r} H_{n+d-r} = 0 \end{cases} \tag{4}$$

with $\mu_{i,j} \in K$ for $i = 1, \ldots, d$ and $j = d+1, \ldots, n+d-r$.
We want to write the matrix of the coefficients of the previous system since it will be the matrix whose

$d \times d$ minors will give Plücker coordinates of Z_y. Actually we will be interested only in $T_{\Lambda_0}(Z_y)$ hence we can write such a matrix modulo all the terms of degree bigger or equal then 2:

$$A := \left(\begin{array}{ccccccc|cccc|ccc} a_r & a_{r-1} & \cdots & \cdots & \cdots & a_2 & a_1 & 1 & 0 & \cdots & 0 & \mu_{1,d+1} & \cdots & \mu_{1,n+d-r} \\ & a_r & a_{r-1} & & & & a_2 & a_1 & 1 & & & & & \\ & & \ddots & \ddots & & & \vdots & \vdots & \ddots & \ddots & 0 & \vdots & & \vdots \\ & & & a_r & a_{r-1} & \cdots & a_d & a_{d-1} & \cdots & a_1 & 1 & \mu_{d,d+1} & \cdots & \mu_{d,n+d-r} \end{array} \right). \tag{5}$$

With the above system of coordinates, an affine parametrization of $Z_y \subset \mathbb{G}(n-1, n+d-1)$ at Λ_0 is given by $p_{r,\dots,\hat{i},\dots,r+d-1,j} = \pm A_{i,j}$+quadratic terms, so that the other Plücker coordinates are at least quadratic in the parameters $a_k, \mu_{l,m}$ of Z. Therefore an affine parameterization of $T_{\Lambda_0}(Z_y) \subset \mathbb{P}^{\binom{n+d}{d}-1}$ is given by

$$\left\{ \begin{array}{rcll} p_{r,\dots,\hat{i},\dots,r+d-1,j} & = & \pm A_{i,j} & \\ p_{i_1,\dots,i_d} & = & 0 & \text{otherwise} \end{array} \right. \tag{6}$$

with the same parameters $a_k, \mu_{l,m}$ as Z_y.

Therefore, the first part of (6) shows that, if an element of $T_{\Lambda_0}(Z_y)$ belongs also to $\mathbb{G}(n-1, n+d-1)$, it should correspond to the linear subspace defined by the matrix

$$B := \left(\begin{array}{ccccccc|cccc|ccc} a_r & a_{r-1} & \cdots & \cdots & \cdots & a_2 & a_1 & 1 & 0 & \cdots & 0 & \mu_{1,d+1} & \cdots & \mu_{1,n+d-r} \\ & a_r & a_{r-1} & & & & a_2 & 0 & 1 & & & & & \\ & & \ddots & \ddots & & & \vdots & \vdots & \ddots & \ddots & 0 & \vdots & & \vdots \\ & & & a_r & a_{r-1} & \cdots & a_d & 0 & \cdots & 0 & 1 & \mu_{d,d+1} & \cdots & \mu_{d,n+d-r} \end{array} \right). \tag{7}$$

On the other hand, the second part of (6) implies that the submatrix of B obtained by removing the central identity block has rank at most one. Hence $a_r = \cdots = a_2 = 0$, and depending on the vanishing of a_1 or not, B takes one of the following forms:

$$B_1 = \left(\begin{array}{ccc|ccc|ccc} 0 & \cdots & 0 & 1 & & 0 & \mu_{1,d+1} & \cdots & \mu_{1,n+d-r} \\ \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & & 1 & \mu_{d,d+1} & \cdots & \mu_{d,n+d-r} \end{array} \right)$$

with the last block of rank at most one, or

$$B_2 = \left(\begin{array}{cccc|ccc|ccc} 0 & \cdots & 0 & a_1 & 1 & & 0 & \mu_{1,d+1} & \cdots & \mu_{1,n+d-r} \\ & & & & & & & 0 & \cdots & 0 \\ \vdots & & \vdots & & & \ddots & & & \cdots & \\ 0 & \cdots & 0 & 0 & 0 & & 1 & 0 & \cdots & 0 \end{array} \right).$$

Now observe that, reciprocally, the matrices of the type B_1 and B_2 represent linear subspaces satisfying the equations (6), so that they are in $T_{\Lambda_0}(V)$. On the other hand, matrices of type B_1 correspond to linear subspaces $\Lambda \in \mathbb{G}(n-1, n+d-1)$ such that $\Lambda \supset O_x^{r-1}(\Sigma)$ and $\dim(\Lambda \cap \Lambda_0) \geq n-2$, while matrices of type B_2 correspond to linear subspaces $\Lambda \in \mathbb{G}(n-1, n+d-1)$ such that $O_x^{r-2}(\Sigma) \subset \Lambda \subset < \Lambda_0, O_x^r(\Sigma) >$. $\square$

With this result in mind, we can now compute the intersection of $\mathbb{G}(n-1, n+d-1)$ with the tangential variety to V. In the statement, we will use the following notation, which we will often repeat along the paper.

Notation 3.7. Given linear subspaces $A \subset B \in \mathbb{P}^{n+d-1}$ of respective dimensions $n-2, n$, we will write $F(A,B)$ to denote the pencil of subspaces $\Lambda \in \mathbb{G}(n-1, n+d-1)$ such that $A \subset \Lambda \subset B$.

Theorem 3.8. *Let $\Lambda_0 \in \mathbb{G}(n-1, n+d-1)$ such that the intersection between Λ_0 and Σ in $\mathbb{P}^{n-1}$ is a zero-dimensional scheme with support on $\{y_1, \dots, y_k\} \subset \Sigma$ and degree n such that each point y_i has multiplicity r_i and $\sum_{i=1}^{k} r_i = n$ (obviously $1 \leq k \leq n$). Then*

$$T_{\Lambda_0}(V) \cap \mathbb{G}(n-1, n+d-1) = \bigcup_{i=1}^{k} F(< O_{y_1}^{r_1-1}(\Sigma), \dots, O_{y_i}^{r_i-2}(\Sigma), \dots, O_{y_k}^{r_k-1}(\Sigma) >, < O_{y_i}^{r_i}(\Sigma), \Lambda_0 >). \quad (8)$$

Proof. With the notation of Remark 3.5, Theorem 3.6 shows that, for each $i = 1, \dots, k$:

$$T_{\Lambda_0}(Z_i) \cap \mathbb{G}(n-1, n+d-1) =$$

$$= \{\Lambda \in \mathbb{G}(n-1, n+d-1) \mid \Lambda \supset O_{y_i}^{r-1}(\Sigma),\ \dim(\Lambda \cap \Lambda_0) \geq n-2\} \cup$$

$$\cup \{\Lambda \in \mathbb{G}(n-1, n+d-1) \mid O_{y_i}^{r-2}(\Sigma) \subset \Lambda \subset < \Lambda_0, O_{y_i}^{r}(\Sigma) >\}.$$

Let us call for brevity $\mathcal{A}_i := \{\Lambda \in \mathbb{G}(n-1, n+d-1) \mid \Lambda \supset O_{y_i}^{r-1}(\Sigma),\ \dim(\Lambda \cap \Lambda_0) \geq n-2\}$ and $\mathcal{B}_i := \{\Lambda \in \mathbb{G}(n-1, n+d-1) \mid O_{y_i}^{r-2}(\Sigma) \subset \Lambda \subset < \Lambda_0, O_{y_i}^{r}(\Sigma) >\}$. By Remark 3.5 we have that

$$T_{\Lambda_0}(V) \cap \mathbb{G}(n-1, n+d-1) = \left(\bigcap_{i=1}^{k} T_{\Lambda_0}(Z_i) \right) \cap \mathbb{G}(n-1, n+d-1).$$

Then

$$T_{\Lambda_0}(V) \cap \mathbb{G}(n-1, n+d-1) = \bigcap_{i=1}^{k} \mathcal{A}_i \cup \mathcal{B}_i$$

Now it is sufficient to observe that all these intersections are equal to Λ_0 except for $\mathcal{A}_1 \cap \cdots \cap \hat{\mathcal{A}}_i \cap \cdots \cap \mathcal{A}_k \cap \mathcal{B}_i$, for all $i = 1, \dots, k$, that is $\{\Lambda \in \mathbb{G}(n-1, n+d-1) \mid\ < O_{y_1}^{r_1-1}(\Sigma), \dots, O_{y_{i-1}}^{r_{i-1}-1}(\Sigma), O_{y_i}^{r_i-2}(\Sigma), O_{y_{i+1}}^{r_{i+1}-1}(\Sigma), \dots, O_{y_k}^{r_k-1}(\Sigma) > \subset \Lambda \subset < O_{y_i}^{r_i}(\Sigma), \Lambda_0 >\}$ from which we have the statement. □

Remark 3.9. Observe that if $\sharp\{y_1, \dots, y_k\} = \deg(\Lambda_0 \cap \Sigma) = n$ then (8) becomes:

$$T_{\Lambda_0}(V) \cap \mathbb{G}(n-1, n+d-1) = \bigcup_{i=1}^{n} F(< y_1, \dots, \hat{y}_i, \dots, y_n >, < y_1, \dots, l_i, \dots, y_n >)$$

where $l_i = T_{y_i}(\Sigma)$.
On the other hand, if $\mathrm{length}(\Lambda_0 \cap \Sigma) = n$ and $y_1 = \cdots = y_k$ then (8) becomes:

$$T_{\Lambda_0}(V) \cap \mathbb{G}(n-1, n+d-1) = F(O_{y_1}^{n-2}(\Sigma), O_{y_1}^{n}(\Sigma)).$$

Definition 3.10. Let $X \subset \mathbb{P}^N$ be a projective, reduced and irreducible variety. Let $X_0 \subset X$ be the dense subset of regular points of X. We define the tangential variety to X as

$$\tau(X) := \overline{\bigcup_{P \in X_0} T_P(X)}.$$

Corollary 3.11. *The intersection between tangential variety to Veronese variety $V = \nu_d(\mathbb{P}^n)$ and the Grassmannian $\mathbb{G}(n-1, n+d-1)$ is*

$$\tau(V) \cap \mathbb{G}(n-1, n+d-1) =$$

$$= \bigcup_{\Lambda=<r_1y_1,\dots,r_ky_k>\in V} \left(\bigcup_{i=1}^{k} F(< O_{y_1}^{r_1-1}(\Sigma), \dots, O_{y_i}^{r_i-2}(\Sigma), \dots, O_{y_k}^{r_k-1}(\Sigma) >, < O_{y_i}^{r_i}(\Sigma), \Lambda >) \right). \tag{9}$$

□

Observe that, when $d = 2$, we have $\tau(V) = \mathrm{Sec}_1(V) = \mathrm{Split}_2(\mathbb{P}^n)$, so that the above corollary also gives the intersection of $\mathbb{G}(n-1, n+1)$ with $\mathrm{Sec}_1(V)$ and $\mathrm{Split}_2(\mathbb{P}^n)$.

Since elements of the tangent space to V at $[L^d]$ take the form $[L^{d-1}M]$, one can wonder whether it is possible to give some information about the linear form M. We conclude this section answering that question.

Proposition 3.12. *Let $[L_0^d] \in V$ be an element corresponding to an n-secant subspace $\Lambda_0 \subset \mathbb{P}^{n+d-1}$ to Σ. Then, if $\Lambda \in T_{\Lambda_0}(V) \cap \mathbb{G}(n-1, n+d-1)$ is given by $[L_0^{d-1}L_1]$, the point $[L_1^d] \in V$ corresponds to a linear space $\Lambda_1 \subset \mathbb{P}^{n+d-1}$ sharing with Λ_0 a subscheme of Σ of length $n-1$.*

Proof. By Theorem 3.8, we have that Λ shares with Λ_0 a subscheme $Z \subset \Sigma$ of length $n-1$. On the other hand, the fact that Λ corresponds to $L_0^{d-1}L_1$ implies (see Remark 3.2) that $\Lambda \in O_{[L_1^d]}^{d-1}(V)$. Hence, by Proposition 3.3, it follows that Λ_1 contains Z, as wanted. □

4 Second osculating space to the Veronese Variety and the Grassmannian

We devote this section to study the intersection with the Grassmanniann of the second osculating space to the Veronese variety. As we have seen, in the case of the first osculating space (i.e. the tangential variety), the computations were difficult to manage. In fact, the case of the second osculating space is maybe the last handy case with these techniques, although only the case $d = 3$ seems to be treatable.

Theorem 4.1. *Let $\Lambda_0 \in \mathbb{G}(n-1, n+2)$ such that the intersection $\Lambda_0 \cap \Sigma \subset \mathbb{P}^{n+2}$ is a zero-dimensional scheme whose support contains $x \in \Sigma$ with multiplicity r. Let Z_y be as in Definition 3.4 with $d = 3$. Then the intersection between the second osculating space to Z_y in Λ_0 and the Grassmannian $\mathbb{G}(n-1, n+2)$ satisfies*

$$O_{\Lambda_0}^2(Z_y) \cap \mathbb{G}(n-1, n+2) \subseteq \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$$

where

$$\mathcal{A} = \{\Lambda \in \mathbb{G}(n-1, n+2) \mid \Lambda \subseteq < \Lambda_0, O_x^{r+1}(\Sigma) >, \dim(\Lambda \cap O_x^{r-1}(\Sigma)) \geq r-2\}$$

$$\mathcal{B} = \{\Lambda \in \mathbb{G}(n-1, n+2) \mid O_x^{r-2}(\Sigma) \subseteq \Lambda, \dim(\Lambda \cap O_x^{r}(\Sigma)) \geq r-1, \dim(\Lambda \cap < \Lambda_0, O_x^{r}(\Sigma) >) \geq n-2\}$$

$$\mathcal{C} = \{\Lambda \in \mathbb{G}(n-1, n+2) \mid O_x^{r-1}(\Sigma) \subseteq \Lambda, \dim(\Lambda \cap \Lambda_0) \geq n-3\}.$$

Proof. As in Theorem 3.6 we give a parameterization of Σ around the point $x := [1, 0, \dots, 0]$, and we give the descripition of Z_y via the system (4), that, in this case for $d = 3$, becomes

$$\begin{cases} H_1 + \mu_{1,4}H_4 + \cdots + \mu_{1,n+3-r}H_{n+3-r} = 0 \\ H_2 + \mu_{2,4}H_4 + \cdots + \mu_{2,n+3-r}H_{n+3-r} = 0 \\ H_3 + \mu_{3,4}H_4 + \cdots + \mu_{3,n+3-r}H_{n+3-r} = 0 \end{cases}$$

Next we have to consider the matrix A defined in (5), but now we have to keep the terms of degree two. Depending on whether $r \geq 3$ or $r = 1, 2$ the form of the matrix is different, so that we will distinguish the three cases.

CASE $r \geq 3$**:** In this case the matrix A takes the form:

$$A = \left(\begin{array}{ccc|ccc|} a_r & a_{r-1} & a_{r-2} & a_{r-3} + \mu_{1,4}a_r & \cdots & a_1 + \sum_{i=4}^r \mu_{1,i}a_i \\ 0 & a_r & a_{r-1} & a_{r-2} + \mu_{2,4}a_r & \cdots & a_2 + \sum_{i=4}^r \mu_{2,i}a_i \\ 0 & 0 & a_r & a_{r-1} + \mu_{3,4}a_r & \cdots & a_3 + \sum_{i=4}^r \mu_{3,i}a_i \end{array} \right. \tag{10}$$

$$\left| \begin{array}{ccc|ccc} 1 + \sum_{i=3}^r \mu_{1,i+1}a_i & 0 + \sum_{i=2}^r \mu_{1,i+2}a_i & 0 + \sum_{i=1}^r \mu_{1,i+3}a_i & \mu_{1,4} + \sum_{i=1}^{n-r-1} \mu_{1,i+4}a_i & \cdots & \mu_{1,n+3-r} \\ a_1 + \sum_{i=3}^r \mu_{2,i+1}a_i & 1 + \sum_{i=2}^r \mu_{2,i+2}a_i & 0 + \sum_{i=1}^r \mu_{2,i+3}a_i & \mu_{2,4} + \sum_{i=1}^{n-r-1} \mu_{2,i+4}a_i & \cdots & \mu_{2,n+3-r} \\ a_2 + \sum_{i=3}^r \mu_{3,i+1}a_i & a_1 + \sum_{i=2}^r \mu_{3,i+2}a_i & 1 + \sum_{i=1}^r \mu_{3,i+3}a_i & \mu_{3,4} + \sum_{i=1}^{n-r-1} \mu_{3,i+4}a_i & \cdots & \mu_{3,n+3-r} \end{array} \right).$$

(We apologize with the reader but the matrix A is too big to be written on only one line: it is a $(3 \times (n + 3 - r))$ size and we write first the firsts r columns and secondly the others.)

From this matrix, and proceeding as in the proof of Theorem 3.6, one could get an affine parametrization of $O^2_{\Lambda_x}(V) \subset \mathbb{P}^{\binom{n+3}{3}-1}$ in the affine open set $p_{r,r+1,r+2} \neq 0$. However, such a parameterization becomes too complicated, so that we just write the part that we need to get the result:

- $p_{j,r+1,r+2} = \begin{cases} a_{r-j}, & j = 0, 1, 2 \\ a_{r-j} + \overline{a_{r-j+3}\mu_{1,4}} + \cdots + \overline{a_r\mu_{1,j+1}}, & j = 3, \dots, r-1 \\ \mu_{1,j-r+1} + \overline{a_1\mu_{1,j-r+2}} + \cdots + \overline{a_{n-j+2}\mu_{1,n-r+3}}, & j = r+3, \dots, n+2, \end{cases}$

- $-p_{j,r,r+2} = \begin{cases} -\overline{a_r a_1}, & j = 0 \\ -\overline{a_{r-j}a_1} + a_{r-j+1}, & j = 1, 2 \\ -\overline{a_{r-j}a_1} + a_{r-j+1} + \overline{a_{r-j+3}\mu_{2,4}} + \cdots + \overline{a_r\mu_{2,j+1}}, & j = 3, \dots, r-1 \\ -\overline{a_1\mu_{1,j-r+1}} + \mu_{2,j-r+1} + \overline{a_1\mu_{2,j-r+2}} + \cdots + \overline{a_{n-j+2}\mu_{2,n-r+3}}, & j = r+3, \dots, n+2, \end{cases}$

- $p_{j,r,r+1} = \begin{cases} -\overline{a_r a_2}, & j = 0 \\ -\overline{a_{r-1}a_2} - \overline{a_r a_1}, & j = 1 \\ -\overline{a_{r-2}a_2} - \overline{a_{r-1}a_1} + a_r, & j = 2 \\ -\overline{a_{r-j}a_2} - \overline{a_{r-j+1}a_1} + a_{r-j+2} + \overline{a_{r-j+3}\mu_{3,4}} + \cdots + \overline{a_r\mu_{3,j+1}}, & j = 3, \dots, r-1 \\ -\overline{a_2\mu_{1,j-r+1}} - \overline{a_1\mu_{2,j-r+1}} + \mu_{3,j-r+1} + \overline{a_1\mu_{3,j-r+2}} + \cdots + \overline{a_{n-j+2}\mu_{3,n-r+3}}, & j = r+3, \dots, n+2. \end{cases}$

- $p_{0,r-1,r+2} = \overline{a_r a_2} = -p_{0,r,r+1}$;

- $p_{1,r-1,r+2} = \overline{a_{r-1}a_2} - \overline{a_r a_1} = -p_{1,r,r+1} - 2p_{0,r,r+2}$;

- $p_{2,r-1,r+2} = \overline{a_{r-2}a_2} - \overline{a_{r-1}a_1} = -p_{2,r,r+1} - 2p_{1,r,r+2} - p_{0,r+1,r+2}$;

- $p_{0,1,r+1} = 0$;

- $p_{0,i,r+1} = -\overline{a_r a_{r-i+2}} = -p_{0,i-1,r+2}$, for $i = 3, \dots, r-1$;

- $p_{1,2,r+1} = -\overline{a_r a_{r-1}} = -p_{0,2,r+2}$;
- $p_{1,i,r+1} = -\overline{a_{r-1}a_{r-i+2}} = -p_{1,i-1,r+2} + p_{0,i+1,r+1} = -p_{1,i-1,r+2} - p_{0,i-1,r+2}$, for $i = 3, \dots, r-1$;
- $p_{2,3,r+1} = -\overline{a_{r-2}a_{r-1}} + \overline{a_r a_{r-3}} = -p_{1,3,r+2}$;
- $p_{2,i,r+1} = -\overline{a_{r-2}a_{r-i+2}} + \overline{a_r a_{r-i}} = -p_{2,i-1,r+2} + p_{1,i+1,r+1} - p_{0,i+2,r+1}$, for $i = 4, \dots, r-1$;
- $p_{0,i,r} = 0$ for all $i = 1, \dots, n+3-r$;
- $p_{1,i,r} = p_{0,i,r+1} = p_{0,i-1,r+2}$ for $i = 2, \dots, r-1$;
- $p_{2,i,r} = p_{1,i-1,r+2}$ for $i = 3, \dots, r+2$;
- $p_{i,j,r+2} = p_{i+1,j+1,r}$ for $i = 0, \dots, r-3$ and $j = 1, \dots, r-2$;
- $p_{r-i,r-2,r+2} - p_{r-i,r-1,r+1} = \overline{a_{i+2}a_1} - \overline{a_{i+1}a_2} = -p_{r-i-1,r-1,r+2}$, for $i = 3, \dots, r-1$.

where the bars denote new parameters corresponding to terms of degree two in the parametrization of Z_y.

Now it is needless to say that applying all these relations at the matrix B (defined as in the proof of Theorem 3.6) is a complete mess... At the end of the game we succeed with a matrix that can be only of one of the following forms:

$$B' = \left(\begin{array}{ccc|ccccc|ccc|ccc} 0 & 0 & 0 & 0 & \cdots & 0 & * & * & 1 & 0 & 0 & * & \cdots & * \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 & 0 & * & \cdots & * \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 1 & * & \cdots & * \end{array}\right)$$

with the condition that the rank of the submatrix obtained omitting the third block is 2;
or

$$B'' = \left(\begin{array}{ccc|ccc|ccc|ccc} 0 & * & * & * & \cdots & * & 1 & 0 & 0 & * & \cdots & * \\ 0 & * & * & * & \cdots & * & 0 & 1 & 0 & * & \cdots & * \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & * & \cdots & * \end{array}\right)$$

with the conditions that at least one of the element of the second row in the firsts two blocks is different from zero, the submatrix maid by the first two blocks has rank 1 and that that one obtained by omitting the third block has rank 2.

B' **case:** Observe that $p_{r-2,r+1,r+2+i} = \overline{a_2} \cdot B_{3,r+2+i}$ for $i = 1, \dots, n+1$. From the parameterization we get:

1. $p_{r-2,r+1,r+2+i} = \overline{a_2\mu_{3,r+2+i}} - \overline{a_4\mu_{1,r+2+i}}$, for $i = 1, \dots, n+1$;
2. $p_{r-1,r,r+2+i} = \overline{a_2\mu_{3,r+2+i}} - \overline{a_3\mu_{2,r+2+i}}$ for $i = 1, \dots, n+1$; since it is equal to that we know from the description of B' that is zero;
3. $p_{r-3,r+2,r+2+i} = \overline{a_3\mu_{2,r+2+i}} - \overline{a_4\mu_{1,r+2+i}}$ for $i = 1, \dots, n+1$ that we know from the description of B' that is zero;

hence $p_{r-2,r+1,r+2+i} = 0$ for $i = 1, \dots, n+1$. Therefore or $\overline{a_2} = 0$ or $B_{3,r+2+i} = 0$ for all $i = 1, \dots, n+1$. Then we get the following three subcases:

$$B'_I := \left(\begin{array}{ccc|ccccc|ccc|ccc} 0 & 0 & 0 & 0 & \cdots & 0 & a_2 & a_1 & 1 & 0 & 0 & * & \cdots & * \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 & 0 & * & \cdots & * \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 \end{array}\right),$$

$$B'_{II} := \left(\begin{array}{ccc|ccccc|ccc|ccc} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & a_1 & 1 & 0 & 0 & * & \cdots & * \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 & 0 & * & \cdots & * \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 1 & * & \cdots & * \end{array}\right)$$

with the condition that the rank of the submatrix obtained considering only the last two rows of the last block is 1;
and

$$B'_{III} := \left(\begin{array}{ccc|ccccc|ccc|ccc} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 & 0 & * & \cdots & * \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 & 0 & * & \cdots & * \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 1 & * & \cdots & * \end{array}\right)$$

with the condition that the last block has rank 2.

B'' **case:** Let i be the least index such that $B''_{2,i}$ is different from zero, $i = 2, \ldots, r-1$. Observe that $p_{i,r,r+2+j} = B''_{2,i} \cdot B''_{3,r+2+j}$ for all $j = 1, \ldots, n+1$. As previously we get from the parameterization that

1. $p_{i,r,r+2+j} = \overline{a_{r-i+1}\mu_{3,r+2+j}} - \overline{a_{r-i+2}\mu_{2,r+2+j}}$ for $j = 1, \ldots, n+1$;
2. $p_{i-1,r+1,r+2+j} = \overline{a_{r-i+1}\mu_{3,r+2+j}} - \overline{a_{r-i+3}\mu_{1,r+2+j}}$ that we know from the form of B'' that is zero for all $j = 1, \ldots, n+1$;
3. $p_{i-2,r+2,r+2+j} = \overline{a_{r-i+2}\mu_{2,r+2+j}} - \overline{a_{r-i+3}\mu_{1,r+2+j}}$ that again we know from the form of B'' that is zero for all $j = 1, \ldots, n+1$.

Hence $p_{i,r,r+2+j} = B''_{2,i} \cdot B''_{3,r+2+j} = 0$ and, since $B''_{2,i}$ is different from zero, we get that $B''_{3,r+2+j} = 0$ for all $j = 1, \ldots, n+1$. Therefore B'' becomes:

$$B'' = \left(\begin{array}{ccc|ccc|ccc|ccc} 0 & * & * & * & \cdots & * & 1 & 0 & 0 & * & \cdots & * \\ 0 & * & * & * & \cdots & * & 0 & 1 & 0 & * & \cdots & * \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 & \cdots & 0 \end{array}\right)$$

with the condition that the first two blocks have rank 1.

It is not difficult to see that the case B'_I is contained in the case B'', hence the only remaining meaningful cases are B'_{II}, B'_{III} and B'' that describe respectively the sets $\mathcal{C}$, $\mathcal{B}$, and $\{\Lambda \in \mathbb{G}(n-1, n+2) \mid x \in \Lambda \subseteq < \Lambda_0, O_x^{r+1}(\Sigma) >, \dim(\Lambda \cap O_x^{r-1}(\Sigma)) \geq r-2\}$, which is clearly contained in $\mathcal{A}$.

CASE $r = 2$**:** The analogous of the matrix A defined in (5) now is:

$$A = \left(\begin{array}{cc|ccc|} a_2 & a_1 & 1 & 0 + \mu_{1,4}a_2 & 0 + \mu_{1,4}a_1 + \mu_{1,5}a_2 \\ 0 & a_2 & a_1 & 1 + \mu_{2,4}a_2 & 0 + \mu_{2,4}a_1 + \mu_{2,5}a_2 \\ 0 & 0 & a_2 & a_1 + \mu_{3,4}a_2 & 1 + \mu_{3,4}a_1 + \mu_{3,5}a_2 \end{array}\right.$$

$$\left|\begin{array}{ccccc} \mu_{1,4} + \mu_{1,5}a_1 + \mu_{1,6}a_2 & \cdots & \mu_{1,n} + \mu_{1,n+1}a_1 + \mu_{1,n+2}a_2 & \mu_{1,n+1} + \mu_{1,n+2}a_1 & \mu_{1,n+2} \\ \mu_{2,4} + \mu_{2,5}a_1 + \mu_{2,6}a_2 & \cdots & \mu_{2,n} + \mu_{2,n+1}a_1 + \mu_{2,n+2}a_2 & \mu_{2,n+1} + \mu_{2,n+2}a_1 & \mu_{2,n+2} \\ \mu_{3,4} + \mu_{3,5}a_1 + +\mu_{3,6}a_2 & \cdots & \mu_{3,n} + \mu_{3,n+1}a_1 + \mu_{3,n+2}a_2 & \mu_{3,n+1} + \mu_{3,n+2}a_1 & \mu_{3,n+2} \end{array}\right).$$

With the usual notation, the affine parameterization of $O^2_{\Lambda_0}(Z_y)$ yield that the matrix B takes the form

$$B = \left(\begin{array}{cc|ccc|ccc} a_2 & a_1 & 1 & 0 & 0 & * & \ldots & * \\ \overline{a_2a_1} & \overline{a_1^2 - a_2} & 0 & 1 & 0 & * & \ldots & * \\ -\overline{a_2^2} & -2\overline{a_2a_1} & 0 & 0 & 1 & * & \ldots & * \end{array}\right)$$

We also write the following relevant parts of the affine parameterization of $O^2_{\Lambda_0}(Z_y)$

1. $p_{0,1,2} = p_{0,1,3} = 0$,

2. $p_{0,1,4} = \overline{a_2^2} = -p_{0,2,3}$,

3. $p_{i,j,k} = 0$ if $i, j, k \neq 2, 3, 4$,

4. $p_{0,3,i} = \overline{a_2\mu_{3,i-1}} = -p_{1,2,i}$ for $i = 5, \dots, n+2$

Equalities 1. are precisely the vanishing of two of the three minors of the left block of B. If it were $\overline{a_2^2} \neq 0$, also the third minor would be zero, i.e. $p_{0,1,4} = 0$. Thus equality 2. implies $\overline{a_2^2} = 0$. Hence we have $\overline{a_2^2} = 0$ in any case. Since $p_{0,1,2} = 0$, also $\overline{a_2 a_1} = 0$. Since also $p_{0,1,3} = 0$, either a_2 or $\overline{a_1^2} - a_2$ are zero. With these vanishings in mind, equations 3. say also that the submatrix of B after removing the central identity block has rank at most two. This yields three possibilities for B. One of them corresponds exactly to the set $\mathcal{C}$, while each the other two cases splits, using equations 4., into two different possibilities, which are inside the sets $\mathcal{A}$, $\mathcal{B}$ or $\mathcal{C}$.

CASE $r = 1$: The analogous of the matrix A defined in (5) now is:

$$A = \left(\begin{array}{c|ccc|ccc} a_1 & 1 & 0 & 0+\mu_{1,4}a_1 & \mu_{1,4}+\mu_{1,5}a_1 & \cdots & \mu_{1,n+2} \\ 0 & a_1 & 1 & 0+\mu_{2,4}a_1 & \mu_{2,4}+\mu_{2,5}a_1 & \cdots & \mu_{2,n+2} \\ 0 & 0 & a_1 & 1+\mu_{3,4}a_1 & \mu_{3,4}+\mu_{3,5}a_1 & \cdots & \mu_{3,n+2} \end{array}\right).$$

From this, we obtain our result as above. $\square$

Remark 4.2. The statement of Theorem 4.1 can be improved. For example, when $r = 1$ we know that equality holds, even for arbitray d, although we preferred to write only the part we need.

Theorem 4.3. *Let $\Lambda_0 \in \mathbb{G}(n-1, n+2)$ such that the intersection between Λ_0 and Σ in $\mathbb{P}^{n-1}$ is a zero-dimensional scheme with support on $\{y_1, \dots, y_k\} \subset \Sigma$ and degree n such that each point y_i has multiplicity r_i and $\sum_{i=1}^k r_i = n$ (obviously $1 \leq k \leq n$). Then, for any $\Lambda \in O^2_{\Lambda_0}(V) \cap \mathbb{G}(n-1, n+2)$, there are two possibilities:*

1. *if $\dim(<\Lambda, \Lambda_0>) = n+1$ then there exist:*
   (a) *$y_{i_1}, y_{i_2} \in \Lambda_0 \cap \Sigma$ such that $\Lambda \cap \Sigma = \{r_1y_1, \dots, (r_{i_1}-1)y_{i_1}, \dots, (r_{i_1}-1)y_{i_2}, \dots, r_ky_k\}$ and $\Lambda \cap \Lambda_0 = <\Lambda \cap \Sigma>$;*
   (b) *$Q'_1 \in O^{r_{i_1}}_{y_{i_1}}(\Sigma), Q'_2 \in O^{r_{i_2}}_{y_{i_2}}(\Sigma)$ such that $(r_{i_1}+1)y_{i_1} \in <\Lambda, Q'_1>, (r_{i_2}+1)y_{i_2} \in <\Lambda, Q'_2>$*

2. *if $\dim(<\Lambda, \Lambda_0>) = n$ then*
   (a) *either $\Lambda \in T_{\Lambda_0}V$;*
   (b) *or there exist $y_{i_1}, y_{i_2} \in \Lambda_0 \cap \Sigma$ such that $<r_1y_1, \dots, \widehat{r_{i_1}y_{i_1}}, \dots, \widehat{r_{i_2}y_{i_2}}, \dots, r_ky_k> \subset \Lambda$ and $<\Lambda, \Lambda_0> = <\Lambda_0, (r_{i_1}+2)y_{i_1}> \cap <\Lambda_0, (r_{i_2}+2)y_{i_2}>$.*
   (c) *or there exists $y_i \in \Lambda_0 \cap \Sigma$ such that $<r_1y_1, \dots, \widehat{r_iy_i}, \dots, r_ky_k> \subset \Lambda \subset <\Lambda_0, (r_i+2)y_i>$.*

Proof. For each $i = 1, \dots, k$, let $\mathcal{A}_i, \mathcal{B}_i, \mathcal{C}_i \subset \mathbb{G}(n-1, n+2)$ be the sets defined in the statement of Theorem 4.1 for the point $y_i \in \Sigma$. By Remark 3.5 and Theorem 4.1, we have

$$O^2_{\Lambda_0}(V) \cap \mathbb{G}(n-1, n+2) \subset \bigcap_{i=1}^{k} (\mathcal{A}_i \cup \mathcal{B}_i \cup \mathcal{C}_i). \tag{11}$$

It is clear from (11) that if $\Lambda \in O^2_{\Lambda_0}(V) \cap \mathbb{G}(n-1, n+2)$ the dimension of $< \Lambda, \Lambda_0 >$ is either n or $n+1$.

1. Assume that $\dim(< \Lambda, \Lambda_0 >) = n+1$. We always have $O^{r_i-1}_{y_i}(\Sigma) \subset < \Lambda, \Lambda_0 >$. Moreover, if $\Lambda \in \mathcal{A}_i$, then $< \Lambda, \Lambda_0 > = < \Lambda_0, O^{r_i+1}_{y_i}(\Sigma) >$ hence $O^{r_i+1}_{y_i}(\Sigma) \subset < \Lambda, \Lambda_0 >$. Also, if $\Lambda \in \mathcal{B}_i$, then $< \Lambda, \Lambda_0 > = < \Lambda, \Lambda_0, O^{r_i}_{y_i}(\Sigma) >$ hence $O^{r_i}_{y_i}(\Sigma) \subset < \Lambda, \Lambda_0 >$. Since in $< \Lambda, \Lambda_0 >$ there are at most $n+2$ points of Σ (counted with multiplicity), then it follows that an intersection of k sets of the form $\mathcal{A}_i, \mathcal{B}_j, \mathcal{C}_k$ is larger that $\{\Lambda_0\}$ only if is of the type $\mathcal{C}_1 \cap \cdots \cap \widehat{\mathcal{C}_{i_1}} \cap \cdots \cap \widehat{\mathcal{C}_{i_2}} \cap \cdots \cap \mathcal{C}_k \cap \mathcal{B}_{i_1} \cap \mathcal{B}_{i_2}$ or $\mathcal{C}_1 \cap \cdots \cap \widehat{\mathcal{C}_i} \cap \cdots \cap \mathcal{C}_k \cap \mathcal{A}_i$. The latter is not possible because otherwise $\Lambda \cap \Lambda_0$ would contain all the $r_j y_j$ with $j \neq i$ and also a hyperplane of $< r_i y_i >$, and hence its dimension would be at least $n-2$, which would imply that $\dim(< \Lambda, \Lambda_0 >) < n+1$, contrary to our hypothesis.

   Assume for simplicity $i_1 = 1, i_2 = 2$. Now clearly $< (r_1-1)y_1, (r_2-1)y_2, r_3y_3, \dots, r_ky_k > \subset \Lambda$ and there exist $Q'_1 \in O^{r_1}_{y_1}(\Sigma)$ and $Q'_2 \in O^{r_2}_{y_2}(\Sigma)$ such that $< (r_1+1)y_1, (r_2-1)y_2, r_3y_3, \dots, r_ky_k > \subset < \Lambda, Q'_1 >, < (r_1-1)y_1, (r_2+1)y_2, r_3y_3, \dots, r_ky_k > \subset < \Lambda, Q'_2 >$. $\Lambda_x = < x, r_1y_1, (r_2-1)y_2, r_3y_3, \dots, r_ky_k >$, it follows from Corollary 3.11 that $\Lambda \in T_{\Lambda_x}V$. Hence Λ should belong to an infinite number of tangent space to V, and this is absurd. Now it remains to show that $\Lambda \cap \Sigma$ is not bigger than $\{(r_1-1)y_1, (r_2-1)y_2, r_3y_3, \dots, r_ky_k\}$. Since $\dim(\Lambda \cap \Lambda_0) < n-2$ it cannot happen that r_1y_1 or r_2y_2 belong to Λ. Then it is sufficient to show that, for example, $(r_3+1)y_3 \notin \Lambda$ (if we allow $r_3 = 0$ then we are considering the case $y_3 \notin \Lambda_0$). Suppose for contradiction that $(r_1-1)y_1, (r_2-1)y_2, (r_3+1)y_3, r_4y_4, \dots, r_ky_k \in \Lambda$. Hence from Corollary 3.11 that $\Lambda \in T_{\Lambda_1}V$ where $\Lambda_1 = < r_1y_1, (r_2-1)y_2, (r_3+1)y_3, r_4y_4 \dots, r_ky_k >$. Analogously $\Lambda \in T_{\Lambda_2}V$ where $\Lambda_2 = < (r_1-1)y_1, r_2y_2, (r_3+1)y_3, r_4y_4 \dots, r_ky_k >$. Since Λ corresponds to a degree three form, it is not possible Λ belongs to two different tangent spaces because the elements of the tangent spaces corresponds to a form containing a double factor.

2. Assume now that $\dim(< \Lambda, \Lambda_0 >) = n$. Then the projection $\pi : \mathbb{P}^{n+2} \to \mathbb{P}^2$ from Λ_0 sends Λ in a point P of $\mathbb{P}^2$. Under this projection Σ is sent to a conic Q and the image P_i of each $y_i \in \Sigma$ is obtained by projecting $< (r_i+1)y_i >$.
   If $\Lambda \in \mathcal{A}_i$ for some $i = 1, \dots, k$, then $\Lambda \subset < \Lambda_0, (r_i+2)y_i >$ and hence P belongs to the tangent line in P_i to Q.
   If instead $\Lambda \in \mathcal{B}_i \backslash \mathcal{C}_i$ for some $i = 1, \dots, k$, then $\dim(\Lambda \cap < (r_i+1)y_i >) \geq r_i - 1$ and, since $\dim(\Lambda \cap \Lambda_0) \geq n-3$, then $< r_iy_i >$ is not contained in Λ. Hence there exist $P' \in \Lambda \cap < (r_i+1)y_i > \backslash < r_iy_i >$. Since $P' \in \Lambda$, then $\pi(P') = P$, while since $P' \in < (r_i+1)y_i > \backslash < r_iy_i >$, also $\pi(P') = P_i$, so that $P = P_i$.
   From this description it is clear that intersections involving either three $\mathcal{A}_i$'s or one $(\mathcal{B}_j \backslash \mathcal{C}_j)$'s and one $\mathcal{A}_i$'s or two $\mathcal{B}_j \backslash \mathcal{C}_j$'s are empty. Let us study the remaining cases.

   (a) Assume first, after reordering, that $\Lambda \in \mathcal{B}_1 \cap \mathcal{C}_2 \cap \cdots \cap \mathcal{C}_k$. By definition $< (r_1-1)y_1, r_2y_2, \dots, r_ky_k > \subset \Lambda$ and there exists $Q' \in < (r_1+1)y_1 >$ such that $(r_1+1)y_1 \in < Q', \Lambda >$ hence $\Lambda \subset < Q', \Lambda > = < (r_1+1)y_1, r_2y_2, \dots, r_ky_k >$. By Corollary 3.11, $\Lambda \in T_{\Lambda_0}(V)$.

(b) Assume now, after reordering, $\Lambda \in \mathcal{A}_1 \cap \mathcal{A}_2 \cap \mathcal{C}_3 \cap \cdots \cap \mathcal{C}_k$. By definition $< \Lambda, \Lambda_0 > \subseteq < \Lambda_0, (r_1+2)y_1 > \cap < \Lambda_0, (r_2+2)y_2 >$ and this is an equality because both the spaces on the left and the right hand side have the same dimension n.

(c) The last case $\Lambda \in \mathcal{C}_1 \cap \cdots \cap \mathcal{A}_i \cap \cdots \cap \mathcal{C}_k$ is trivial by definition.

□

5 Split Variety and the Grassmannian

Proposition 5.1. *Let $\Lambda \in O^2_{\Lambda_1}(V) \cap O^2_{\Lambda_2}(V) \cap \mathbb{G}(n-1, n+2)$ for some $\Lambda_1, \Lambda_2 \in V$, and assume* $\dim(< \Lambda, \Lambda_1 >) = \dim(< \Lambda, \Lambda_2 >) = n+1$. *Then there exist $s_1y_1, \ldots, s_ky_k \in \Sigma$ with $\sum_{i=1}^k s_i = n-2$ such that $\Lambda_1 =< (s_1+1)y_1, (s_2+1)y_2, s_3y_3, s_4y_4 \ldots, s_ky_k >$, $\Lambda_2 =< (s_1+1)y_1, s_2y_2, (s_3+1)y_3, \ldots, s_ky_k >$ and*

$$\Lambda =< (s_1+2)y_1, (s_2+2)y_2, s_3y_3, s_4y_4 \ldots, s_ky_k > \cap$$

$$\cap < (s_1+2)y_1, s_2y_2, (s_3+2)y_3, \ldots, s_ky_k > \cap < s_1y_1, (s_2+2)y_2, (s_3+2)y_3, \ldots, s_ky_k > .$$

Proof. From Theorem 4.3 we can derive $\Lambda \cap \Lambda_1 =< \Lambda \cap \Sigma >= \Lambda \cap \Lambda_2$ and $\Lambda \cap \Sigma = \{s_1y_1, \ldots, s_ky_k\}$ with $\sum_{i=1}^k s_i = n-2$ (the s_i's do not have to be necessarily different from zero). Moreover we know that Λ_1, Λ_2 can be obtained form $< \Lambda \cap \Sigma >$ increasing two s_i's by 1.

We show now that the s_i's we have to increase do not correspond to four different y_i's. Assume for contradiction, up to reordering, that $\Lambda_1 =< (s_1+1)y_1, (s_2+1)y_2, s_3y_3, s_4y_4, s_5y_5, \ldots, s_ky_k >$, $\Lambda_2 =< s_1y_1, s_2y_2, (s_3+1)y_3, (s_4+1)y_4, s_5y_5, \ldots, s_ky_k >$. By Theorem 4.3 there exist $Q_1' \in O^{s_1+1}_{y_1}(\Sigma)$, $Q_1'' \in O^{s_3+1}_{y_3}(\Sigma)$ such that $(s_1+2)y_1 \in< \Lambda, Q_1' >$ and $(s_3+2)y_3 \in< \Lambda, Q_1'' >$, hence the $(n+1)$-dimensional subspace $< \Lambda, Q_1', Q_1'' >$ contains the following $n+4$ points of Σ: $(s_1+2)y_1, (s_2+1)y_2, (s_3+2)y_3, (s_4+1)y_4, s_5y_5, \ldots, s_ky_k$, which is clearly a contradiction. Hence we can assume, up to reordering,

$$\Lambda_1 =< (s_1+1)y_1, (s_2+1)y_2, s_3y_3, s_4y_4, \ldots, s_ky_k >$$

$$\Lambda_2 =< (s_1+1)y_1, s_2y_2, (s_3+1)y_3, s_4y_4, \ldots, s_ky_k > .$$

By Theorem 4.3, there exists $Q_1' \in< (s_1+2)y_1 >$ such that $< (s_1+2)y_1 > \subset < \Lambda, Q_1' >$. Since Λ is a hyperplane in $< \Lambda, Q_1' >$, we can find $R_1' \in \Lambda \cap < (s_1+2)y_1 > \setminus < s_1y_1 >$. Analogously, we can find $R_2' \in \Lambda \cap < (s_2+2)y_2 > \setminus < s_2y_2 >$ and $R_3' \in \Lambda \cap < (s_3+2)y_3 > \setminus < s_3y_3 >$.

We claim that $< s_1y_1, \ldots, s_ky_k, R_1', R_2' >$ has dimension $n-1$. Indeed, if $s_1y_1, \ldots, s_ky_k, R_1', R_2'$ were dependent, the projection from $< s_1y_1, \ldots, s_ky_k >$ would produce a rational normal curve in $\mathbb{P}^4$ in which the tangent lines at the image of y_1 and y_2 would meet at the image of R_1' (which would have the same image as R_2'), but this is impossible. As a consequence of the claim, $\Lambda =< s_1y_1, \ldots, s_ky_k, R_1', R_2' >$, so that it is contained in $< (s_1+2)y_1, (s_2+2)y_2, s_3y_3, s_4y_4, \ldots, s_ky_k >$.

Analogously, $\Lambda \subset< (s_1+2)y_1, s_2y_2, (s_3+2)y_3, s_4y_4, \ldots, s_ky_k >$ and $\Lambda \subset< s_1y_1, (s_2+2)y_2, (s_3+2)y_3, \ldots, s_ky_k >$. Therefore

$$\Lambda \subset< (s_1+2)y_1, (s_2+2)y_2, s_3y_3, s_4y_4 \ldots, s_ky_k > \cap$$

$$\cap < (s_1+2)y_1, s_2y_2, (s_3+2)y_3, \ldots, s_ky_k > \cap < s_1y_1, (s_2+2)y_2, (s_3+2)y_3, \ldots, s_ky_k > .$$

We actually have an equality, since otherwise the usual projection from $< s_1y_1,\dots,s_ky_k >$ would produce a rational normal curve $\Sigma'\subset\mathbb{P}^4$, with points y'_1,y'_2,y'_3 such that the intersection $< T_{y'_1}\Sigma',T_{y'_2}\Sigma' > \cap < T_{y'_1}\Sigma',T_{y'_3}\Sigma' > \cap < T_{y'_2}\Sigma',T_{y'_3}\Sigma' >$ is more than a line. But since Σ' is homogeneous, the same would be true for any choice of three points of Σ', which is not true, as we showed in Example 2.8. $\square$

Lemma 5.2. *Let* $\Lambda\in(O^2_{\Lambda_1}(V)\backslash T_{\Lambda_1}(V))\cap(O^2_{\Lambda_2}(V)\backslash T_{\Lambda_2}(V))\cap\mathbb{G}(n-1,n+2)$ *for some* $\Lambda_1,\Lambda_2\in V$*, and assume* $\dim(<\Lambda,\Lambda_1>)=\dim(<\Lambda,\Lambda_2>)=n$*. Then* Λ_1 *and* Λ_2 *have* $n-1$ *points of* Σ *in common (counted with multiplicity).*

Proof. We assume for contradiction that Λ_1 and Λ_2 have at most $n-2$ points of Σ in common. Therefore $<\Lambda_1,\Lambda_2>$ contains at least $n+2$ points of Σ. This implies $\dim(<\Lambda_1,\Lambda_2>)\ge n+1$. On the other hand, since $\dim(<\Lambda,\Lambda_1>)=\dim(<\Lambda,\Lambda_2>)=n$, it follows that $\dim(<\Lambda,\Lambda_1,\Lambda_2>)\le n+1$. As a consequence, $\dim(<\Lambda_1,\Lambda_2>)=n+1$, $\Lambda\subset<\Lambda_1,\Lambda_2>$ and Λ_1 and Λ_2 share exactly $n-2$ points of Σ.

We will write $\Lambda_1=< r_1y_1,\dots,r_ky_k >$, with $r_1+\dots+r_k=n$. Since Λ_1 and Λ_2 share $n-2$ points of Σ, then Λ_2 is obtained by substracting two points to $r_1y_1,\dots,r_ky_k$ and adding two more, maybe just substracting or adding some multiplicities to the points. To simplify the notation, we will include the points of $\Lambda_2\setminus\Lambda_1$ in $y_1,\dots,y_k$, so that maybe some r_i (two at most) can be zero. From Theorem 4.3 we know that the possible cases for Λ_1 and Λ_2 are those described in 2b) and 2c).

We exclude first the possibility that Λ_1 is in case 2c) of Theorem 4.3. Otherwise, up to reordering $r_2y_2,\dots,r_k,y_k\in\Lambda$ and $\Lambda\subset<(r_1+2)y_1,r_2y_2,r_3y_3,\dots,y_k>$. Using Proposition 3.3, we get $r_2y_2,\dots,r_k,y_k\in\Lambda_2$. We have now two possibilities (after probably reordering $y_1,\dots,y_k$) for Λ_2, namely

$$<(r_1-2)y_1,(r_2+2)y_2,r_3y_3,r_4y_4,\dots,r_ky_k>$$

$$<(r_1-2)y_1,(r_2+1)y_2,(r_3+1)y_3,r_4y_4,\dots,r_ky_k>.$$

This gives the following respective possibilities for $<\Lambda_1,\Lambda_2>$:

$$< r_1y_1,(r_2+2)y_2,r_3y_3,r_4y_4,\dots,r_ky_k>$$

$$< r_1y_1,(r_2+1)y_2,(r_3+1)y_3,r_4y_4,\dots,r_ky_k>.$$

Observe that it cannot be $(r_2+1)y_2\in\Lambda$, since Proposition 3.3 would imply $(r_2+1)y_2\in\Lambda_1$. Therefore, by part 2. of Theorem 4.3 taking $\Lambda_0=\Lambda_2$, we have $\Lambda\subset<\Lambda_2,(r_2+4)y_2>$ or $\Lambda\subset<\Lambda_2,(r_2+3)y_2>$, depending on the two possibilities for Λ_2. Having also in mind the inclusion $\Lambda\subset<\Lambda_1,\Lambda_2>$, we get that Λ is contained in one of the following (corresponding to the two possibilities for Λ_2):

$$<(r_1+2)y_1,r_2y_2,r_3y_3\dots,r_ky_k>\cap<r_1y_1,(r_2+2)y_2,r_3y_3,\dots,r_ky_k>\cap<(r_1-2)y_1,(r_2+4)y_2,r_3y_3,\dots,r_ky_k>$$

$$<(r_1+2)y_1,r_2y_2,r_3y_3,\dots,r_ky_k>\cap<r_1y_1,(r_2+1)y_2,(r_3+1)y_3,r_4y_4,\dots,r_ky_k>\cap$$
$$\cap<(r_1-2)y_1,(r_2+3)y_2,(r_3+1)y_3,r_4y_4,\dots,r_ky_k>$$

which is a contradiction by Example 2.11 (since Λ is in two different osculating spaces to V, it necessarily belongs to $\mathrm{Split}_3(\mathbb{P}^n)$).

We are thus reduced to the possibility that Λ_1 is in case 2b) of Theorem 4.3. Therefore, up to reordering, $r_3y_3,\dots,r_ky_k\in\Lambda$ and $\Lambda\subset<(r_1+2)y_1,r_2y_2,r_3y_3,\dots,r_ky_k>\cap<r_1y_1,(r_2+2)y_2,r_3y_3,\dots,r_ky_k>$. By

Proposition 3.3, it follows that $r_3y_3, \dots, r_ky_k \in \Lambda_2$. Hence there are four possibilities (after probably reordering $y_1, \dots, y_k$) for Λ_2, namely

$$< (r_1-2)y_1, (r_2+2)y_2, r_3y_3, r_4y_4, r_5y_5, \dots, r_ky_k >$$
$$< (r_1-2)y_1, (r_2+1)y_2, (r_3+1)y_3, r_4y_4, r_5y_5, \dots, r_ky_k >$$
$$< (r_1-1)y_1, (r_2-1)y_2, (r_3+2)y_3, r_4y_4, r_5y_5, \dots, r_ky_k >$$
$$< (r_1-1)y_1, (r_2-1)y_2, (r_3+1)y_3, (r_4+1)y_4, r_5y_5, \dots, r_ky_k > .$$

As before, Proposition 3.3 implies that it cannot be $(r_2+1)y_2 \in \Lambda$ or $(r_3+1)y_3 \in \Lambda$. Hence, by part 2. of Theorem 4.3 applied for $\Lambda_0 = \Lambda_2$ in the four possibilities above we have, respectively,

$$\Lambda \subset < \Lambda_2, (r_2+4)y_2 > = < (r_1-2)y_1, (r_2+4)y_2, r_3y_3, r_4y_4, \dots, r_k >$$
$$\Lambda \subset < \Lambda_2, (r_2+3)y_2 > = < (r_1-2)y_1, (r_2+3)y_2, (r_3+1)y_3, r_4y_4, \dots, r_k >$$
$$\Lambda \subset < \Lambda_2, (r_3+4)y_3 > = < (r_1-1)y_1, (r_2-1)y_2, (r_3+4)y_3, r_4y_4, \dots, r_k >$$
$$\Lambda \subset < \Lambda_2, (r_3+3)y_3 > = < (r_1-1)y_1, (r_2-1)y_2, (r_3+3)y_3, (r_4+1)y_4, r_5y_5, \dots, r_k > .$$

Since we also have $\Lambda \subset < (r_1+2)y_1, r_2y_2, r_3y_3, \dots, r_ky_k > \cap < r_1y_1, (r_2+2)y_2, r_3y_3, \dots, r_ky_k >$, we get a contradiction from Example 2.11. □

Proposition 5.3. *Let* $\Lambda \in (O^2_{\Lambda_1}(V)\backslash T_{\Lambda_1}(V)) \cap (O^2_{\Lambda_2}(V)\backslash T_{\Lambda_2}(V)) \cap \mathbb{G}(n-1, n+2)$ *for some* $\Lambda_1, \Lambda_2 \in V$*, and assume* $\dim(< \Lambda, \Lambda_1 >) = \dim(< \Lambda, \Lambda_2 >) = n$*. If* Λ_1 *and* Λ_2 *do have* $n-1$ *points of* Σ *in common, also* Λ *contains those points.*

Proof. Since, by hypothesis, the intersection of Λ_1 and Λ_2 has dimension $n-2$, and also the intersection of Λ with each of them has dimension $n-2$, it follows that there are two possibilities:

–Either Λ contains the intersection of Λ_1, Λ_2, hence their $n-1$ common points of Σ.

–Or Λ is contained in the n-dimensional span of Λ_1, Λ_2. By Theorem 4.3, in any case there exists $y_1 \in \Sigma \cap \Lambda_1$ such that $\Lambda \subset < \Lambda_1, (r_1+2)y_1 >$, where r_1 is the intersection multiplicity at y_1 of Σ and Λ_1. Hence $\Lambda \subset < \Lambda_1, (r_1+2)y_1 > \cap < \Lambda_1, \Lambda_2 >$. Since $\Lambda \neq \Lambda_1$, necessarily $< \Lambda_1, (r_1+2)y_1 >$ contains $< \Lambda_1, \Lambda_2 >$, in particular the point of $\Lambda_1 \cap \Sigma$ that is not in Λ_2. Since the hyperplane $< \Lambda_1, (r_1+2)y_1 > \subset \mathbb{P}^{n+2}$ cannot $n+3$ different point of Σ, it follows that $(r_1+1)y_1 \in \Lambda_2$. We cannot have another $y_1' \neq y_1$ in $\Sigma \cap \Lambda_1$ such that $\Lambda \subset < \Lambda_1, (r_1'+2)y_1' >$, because the same reasoning would show $(r_1'+1)y_1' \in \Lambda_2$, which contradicts the fact that Λ_1 and Λ_2 share $n-1$ points of Σ. Therefore Λ_1 is in case 2.(c) of Theorem 4.3. The same reasoning for Λ_2 shows that there exists $y_2 \in \Sigma \cap \Lambda_2$ with multiplicity r_2 and such that $(r_2+1)y_2 \in \Lambda_1$. Moreover, Λ_2 is also in case 2.(c) of Theorem 4.3. But then, using again the part 2.(c) of Theorem 4.3, we deduce that Λ should contain $(r_1+1)y_1, (r_2+1)y_2$ and the other $n-r_1-r_2$ common points of Σ, which is a contradiction. □

Theorem 5.4. *The intersection between* $\mathrm{Split}_3(\mathbb{P}^n)$ *and* $\mathbb{G}(n-1, n+2)$ *is*

$$\mathrm{Split}_3(\mathbb{P}^n) \cap \mathbb{G}(n-1, n+2) = X_{n+1} \cup X_{n+2}$$

where

$$X_{n+1} = \{< Z+2y_1+2y_2 > \cap < Z+2y_1+2y_3 > \cap < Z+2y_2+2y_3 > \mid Z \subset \Sigma,\ \mathrm{length}(Z) = n-2,\ y_1, y_2, y_3 \in \Sigma\}$$
$$X_{n+2} = \{\Lambda \subset \mathbb{G}(n-1, n+3) \mid \mathrm{length}(\Lambda \cap \Sigma) \geq n-1\}.$$

Proof. We have $X_{n+1} \subset \mathrm{Split}_3(\mathbb{P}^n)$ by Proposition 2.10 and $X_{n+2} \subset \mathrm{Split}_3(\mathbb{P}^n)$ by Corollary 2.7. Hence $X_{n+1} \cup X_{n+2} \subset \mathrm{Split}_3(\mathbb{P}^n) \cap \mathbb{G}(n-1, n+2)$.

Reciprocally, let $\Lambda \in \mathrm{Split}_3(\mathbb{P}^n) \cap \mathbb{G}(n-1, n+2)$. By Remark 3.2, either $\Lambda \in \tau(V) \cap \mathbb{G}(n-1, n+2)$ or $\Lambda \in O^2_{\Lambda_1}(V) \cap O^2_{\Lambda_2}(V) \cap O^2_{\Lambda_3}(V)$ for different subspaces $\Lambda_1, \Lambda_2, \Lambda_3 \in \mathbb{G}(n-1, n+2)$. In the first case, by Corollary 3.11, Λ contains at least $n-1$ points of Σ, so that $\Lambda \in X_{n+2}$. We will thus assume $\Lambda \notin \tau(V)$ and $\Lambda \in O^2_{\Lambda_1}(V) \cap O^2_{\Lambda_2}(V) \cap O^2_{\Lambda_3}(V)$. Theorem 4.3 implies that the span of Λ with each Λ_i has dimension $n+1$ or n. Hence for at least two of the subspaces, say Λ_1, Λ_2, the dimensions of $< \Lambda, \Lambda_1 >$ and $< \Lambda, \Lambda_2 >$ are the same. We study separately the different possibilities:

If $\dim(< \Lambda, \Lambda_1 >) = \dim(< \Lambda, \Lambda_2 >) = n+1$, by Proposition 5.1, we have $\Lambda \in X_{n+1}$.

If $\dim(< \Lambda, \Lambda_1 >) = \dim(< \Lambda, \Lambda_2 >) = n$, by Lemma 5.2 it follows that Λ_1, Λ_2 have $n-1$ points of Σ in common, so that we are done by Proposition 5.3.

□

6 Appendix

In this appendix we want to explore the following problem: is it possible to detect when the s-th secant variety to $\mathrm{Split}_d(\mathbb{P}^n)$ fills up the whole ambient space by just detecting when its intersection with $\mathbb{G}(n-1, n+d-1)$ is the whole Grassmannian?

To test the validity of this method, one could replace $\mathrm{Split}_d(\mathbb{P}^n)$ with $\nu_d(\mathbb{P}^n)$, for which the dimensions of all secant varieties are known (see [**AH**]). We will see that in fact, the method perfectly works for $d = 2$ and any secant variety, and give some partial answer for any d and the second secant variety.

Proposition 6.1. *The intersection between the Grassmannian $\mathbb{G}(n-1, n+1)$ and the variety $\mathrm{Sec}_{r-1}(\nu_2(\mathbb{P}^n))$ is to the set of all $(n-1)$-spaces of $\mathbb{P}^{n+1}$ that are $(n-r+1)$-secant to the rational normal curve $\Sigma \subset \mathbb{P}^{n+1}$.*

Proof. Assume first that a subspace $\Lambda \subset \mathbb{P}^{n+1}$ contains a subscheme $Z \subset \Sigma$ of length $n-r+1$. By Lemma 2.5, we can find linear forms $N_0, \ldots, N_{r-1} \in K[X_0, \ldots, X_n]$ such that Λ, as an element of $\mathbb{P}(K[X_0, \ldots, X_n]_2)$ lies in $\mathbb{P}(K[N_0, \ldots, N_{r-1}]_2)$. But now the r-th secant variety of $\nu_2(\mathbb{P}(K[N_0, \ldots, N_{r-1}]_1)$ is the whole $\mathbb{P}(K[N_0, \ldots, N_{r-1}]_2)$. Thus necessarily Λ belongs to $\mathrm{Sec}_{r-1}(\nu_2(\mathbb{P}^n))$.

We just sketch the proof of the other inclusion (although the case $r = 2$ is an immediate consequence of Corollary 3.11). The main idea for the proof is that, since $d = 2$, the Plücker space of $\mathbb{G}(n-1, n+1)$ can be identified with the space of classes of skew-symmetric matrices of order $n+2$, while the space of homogeneous polynomials of degree two in $n+1$ variables can be regarded as the space of symmetric matrices of order $n+1$. In this language, one can write down explicitly the identification of these two spaces. Specifically, to any skew-symmetric matrix

$$A = \begin{pmatrix} 0 & p_{0,1} & \cdots & p_{0,n+1} \\ -p_{0,1} & 0 & \cdots & p_{1,n+1} \\ \vdots & & \ddots & \vdots \\ -p_{0,n+1} & -p_{1,n+1} & \cdots & 0 \end{pmatrix}.$$

the corresponding symmetric matrix is

$$Q = \begin{pmatrix} p_{0,1} & p_{0,2} & p_{0,3} & \cdots & p_{0,n+1} \\ p_{0,2} & p_{1,2}+p_{0,3} & p_{1,3}+p_{0,4} & \cdots & p_{1,n+1} \\ p_{0,3} & p_{1,3}+p_{0,4} & p_{2,3}+p_{1,4}+p_{0,5} & \cdots & p_{2,n+1} \\ \vdots & \vdots & & & \vdots \\ p_{0,n+1} & p_{1,n+1} & \cdots & \cdots & p_{n,n+1} \end{pmatrix}.$$

Take then $\Lambda \in \mathbb{G}(n-1,n+1)$ represented by a rank-two matrix A as above. If it belongs to $\mathrm{Sec}_{r-1}(\nu_2(\mathbb{P}^n))$, this means that the corresponding matrix Q has rank at most r. It is then possible to verify that this is equivalent to the fact that the system

$$A \begin{pmatrix} t_0^{n+1} \\ t_0^n t_1 \\ \vdots \\ t_1^{n+1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

admits at least $n-r+1$ solutions in $\mathbb{P}^1$, counted with multiplicity. It follows that A describes an $(n-1)$-space of $\mathbb{P}^{n+1}$ that is $(n-r+1)$-secant to Σ. □

Corollary 6.2. *The intersection between* $\mathrm{Sec}_{s-1}(\mathrm{Split}_2(\mathbb{P}^n))$ *and* $\mathbb{G}(n-1,n+1)$ *is set-theoretically the locus* $\{\Lambda \in \mathbb{G}(n-1,n+1) \mid \Lambda$ *is* $(n-2s+1)-$*secant to* $\nu_{n+1}(\mathbb{P}^1)\}$.

Proof. This is a consequence of the previous proposition and of the observation that, since $\mathrm{Split}_2(\mathbb{P}^n) = \{Q \in M_{n+1}(K)$ s.t. Q is symmetric and $\mathrm{rk}(Q) = 2\}$ and the elements of $\mathrm{Split}_2(\mathbb{P}^n)$ are of the form $[L_1 \cdot L_2]$ with $L_1, L_2 \in R_1$, then $\mathrm{Sec}_{s-1}(\mathrm{Split}_2(\mathbb{P}^n)) = \{[L_1L_2 + \cdots + L_{2s-1}L_{2s}] \in \mathbb{P}(R_2) \mid L_i \in R_1$ for $i = 1, \ldots, 2s\}$ is the set of all symmetric matrices of $M_{n+1}(K)$ of rank at most $2s$. □

Remark 6.3. Observe that, the previous results show that the technique proposed at the beginning of this appendix works for $\nu_2(\mathbb{P}^n)$ and $\mathrm{Split}_2(\mathbb{P}^n)$. Indeed, $\mathrm{Sec}_{r-1}(\nu_2(\mathbb{P}^n)) = \mathbb{P}^{\frac{n(n+3)}{2}}$ if and only if $r \geq n+1$, which is equivalent (by Proposition 6.1) to $\mathrm{Sec}_{r-1}(\nu_2(\mathbb{P}^n)) \cap \mathbb{G}(n-1,n+1) = \mathbb{G}(n-1,n+1)$. Similarly, $\mathrm{Sec}_{s-1}(\mathrm{Split}_2(\mathbb{P}^n)) = \mathbb{P}^{\frac{n(n+3)}{2}}$ if and only if $s \geq \frac{n+1}{2}$ (because $\mathrm{Sec}_{s-1}(\mathrm{Split}_2(\mathbb{P}^n))$ can be interpreted as the space of symmetric matrices of rank at most $2s$) and this is equivalent (by Corollary 6.2) to $\mathrm{Sec}_{s-1}(\mathrm{Split}_2(\mathbb{P}^n)) \cap \mathbb{G}(n-1,n+1) = \mathbb{G}(n-1,n+1)$.

We end by presenting some generalizations of Proposition 6.1. We need some preliminary results.

Lemma 6.4. *Let* $\Lambda_1, \Lambda_2 \in \nu_d(\mathbb{P}^n)$ *such that the line spanned by them is contained in* $\mathbb{G}(n-1,n+d-1)$. *Then* Λ_1 *and* Λ_2 *share at least* $n-1$ *points of* Σ.

Proof. Since the line spanned by Λ_1, Λ_2 is contained in $\mathbb{G}(n-1,n+d-1)$, they belong to a pencil of subspaces. Hence the span of Λ_1, Λ_2 in $\mathbb{P}^{n+d-1}$ is a linear space of dimension n. The hypothesis $\Lambda_1, \Lambda_2 \in \nu_d(\mathbb{P}^n)$, implies that Λ_1, Λ_2 contain each n points of Σ. Since $< \Lambda_1, \Lambda_2 >$ can contain at most $n+1$ points of Σ, the result follows readily. □

Proposition 6.5. *Let N_0, N_1 be two linear forms of $K[x_0, \dots, x_n]$; then $\mathbb{G}(n-1, n+2) \cap \mathbb{P}(K[N_0, N_1]_3) = \{\Lambda \in \mathbb{G}(n-1, n+2) \mid \deg(\Lambda \cap \Sigma) \geq n-1\}$.*

Proof. Take $\Lambda \in \mathbb{G}(n-1, n+2)$. If $\Lambda \cap \Sigma$ contains a subscheme $Z \subset \Sigma$ of length $n-1$, Lemma 2.5 implies that there exist linear forms $N_0', N_1' \in K[x_0, \dots, x_n]$ such that $\mathbb{G}(n-1, n+2) \cap \mathbb{P}(K[N_0', N_1']_3) = \{\Lambda \in \mathbb{G}(n-1, n+2) \mid \Lambda \cap \Sigma \supset Z\}$. In particular, $N_0, N_1 \in K[N_0', N_1']$, so that $K[N_0, N_1] = K[N_0', N_1']$ and one of the wanted inclusions follows.

Reciprocally, assume $\Lambda \in \mathbb{P}(K[N_0, N_1]_3)$. Then we can consider the twisted cubic $C \subset \mathbb{P}(K[N_0, N_1]_3)$ defined by the classes of the type $(\alpha N_0 + \beta N_1)^3 \in K[N_0, N_1]_3$. If $\Lambda \in C$, in particular $\Lambda \in \nu_3(\mathbb{P}^n)$, so that it contains n points of Σ. If $\Lambda \notin C$, then it belongs to a bisecant (or tangent) line to Σ. This line is thus trisecant to $\mathbb{G}(n-1, n+2)$, hence it is contained in $\mathbb{G}(n-1, n+2)$. The other inclusion follows now from Lemma 6.4. □

Corollary 6.6. *If $M \in K[N_0, N_1]_3 \cap \mathbb{G}(n-1, n+2)$, with N_0, N_1 generic linear forms, then $M \in \nu_3(\mathbb{P}^n)$.*

Proof. If M is a binary form contained into the Grassmannian $\mathbb{G}(n-1, n+2)$, then by Proposition 6.5 the linear forms N_0, N_1 must be "special", i.e. they have at least $n-1$ roots in common. □

Lemma 6.7. *Let $A, B \in \nu_d(\mathbb{P}^n)$. If there exists a point $C \in \mathrm{Sec}_1(\nu_d(\mathbb{P}^n)) \cap \mathbb{G}(n-1, n+d-1)$ such that $C \in < A, B > \smallsetminus \nu_d(\mathbb{P}^n)$, then $< A, B > \subset \mathbb{G}(n-1, n+d-1)$.*

Proof. The set of the three points $\{A, B, C\}$ is contained in the intersection $< A, B > \cap \mathbb{G}(n-1, n+d-1)$. Since the Grassmannian is an intersection of quadrics, it cannot exist a point $D \in < A, B >$ but $D \notin \mathbb{G}(n-1, n+d-1)$ then $< A, B > \subset \mathbb{G}(n-1, n+d-1)$. □

Proposition 6.8. *The intersection between $\mathrm{Sec}_1(\nu_d(\mathbb{P}^n))$ and $\mathbb{G}(n-1, n+d-1)$ is contained in $\{\Lambda \in \mathbb{G}(n-1, n+d-1) \mid \deg(\Lambda \cap \Sigma) \geq n-1\}$.*

Proof. Let us take a point $A \in \mathrm{Sec}_1(\nu_d(\mathbb{P}^n) \cap \mathbb{G}(n-1, n+d-1)) \smallsetminus \nu_d(\mathbb{P}^n)$, then there exist $\pi_1, \pi_2 \in \nu_d(\mathbb{P}^n)$ such that $A \in < \pi_1, \pi_2 >$. Since $\nu_d(\mathbb{P}^n)$ is the locus of the $(n-1)$-spaces of $\mathbb{P}^{n+d-1}$ that are n-secant to Σ, there exist $P_1, \dots, P_n, Q_1, \dots, Q_n \in \Sigma$ such that $\pi_1 =< P_1, \dots, P_n >$ and $\pi_2 =< Q_1, \dots, Q_n >$. Therefore $< \pi_1, \pi_2 > \subset (\mathrm{Sec}_1(\nu_d(\mathbb{P}^n)) \subset \mathrm{Split}_d(\mathbb{P}^n)$. By the Lemma 6.7 we have that $< \pi_1, \pi_2 > \subset \mathbb{G}(n-1, n+d-1)$. The span $< \pi_1, \pi_2 >$ parameterizes a pencil of $(n-1)$-spaces contained in $\mathbb{P}^n \subset \mathbb{P}^{n+d-1}$ and containing a $\mathbb{P}^{n-2}$. Then $P_1, \dots, P_n, Q_1, \dots, Q_n$ lie on a $\mathbb{P}^n$ instead of being generic in $< \Sigma >= \mathbb{P}^{n+d-1}$, hence $\sharp\{P_1, \dots, P_n, Q_1, \dots, Q_n\} = n+1$. □

Proposition 6.9. *Let $V = \nu_3(\mathbb{P}^n) \subset \mathbb{G}(n-1, n+2)$, then*

$$\mathrm{Sec}_1(V) \cap \mathbb{G}(n-1, n+2) = \{\Lambda \in \mathbb{G}(n-1, n+2) \mid \deg(\Sigma \cap \Lambda) \geq n-1\}.$$

Proof. Proposition 6.8 presents one inclusion. Let's then prove that $\{\Lambda \in \mathbb{G}(n-1, n+2) \mid \deg(\Sigma \cap \Lambda) \geq n-1\} \subseteq \mathrm{Sec}_1(\nu_3(\mathbb{P}^n)) \cap \mathbb{G}(n-1, n+2)$.

Let $\Lambda \in \mathbb{G}(n-1, n+2)$ be a subspace containing a subscheme $Z \subset \Sigma$ of length $n-1$.

Consider the projection $\pi : \mathbb{P}^{n+2} \to \mathbb{P}^3$ from $< Z > \subset \mathbb{P}^{n+2}$. Observe that all $\widetilde{\Lambda} \in \mathbb{G}(n-1, n+2)$ that intersect Σ in degree n are sent by π in the rational normal cubic $\Sigma' \subset \mathbb{P}^3$, and $\pi(\Lambda) = Q$ does not belong

to such a cubic.
A line $L \in \mathbb{P}^3$ passing through Q can be or tangent or bisecant to the cubic.

If L is the tangent line to Σ' at a point y', consider $y \in \Sigma$ the point of Σ whose image is y. Then $< Z > \subset \Sigma \subset < Z + 2y >$, so that $\Lambda \in \tau(V)$.

If it is bisecant consider the $\mathbb{P}^n$ obtained as $\pi^{-1}(L) = H \subset \mathbb{P}^{n+2}$. Since L intersects the rational normal cubic in two points, then H contains two $\mathbb{P}^{n-1}$'s, say Λ_1 and Λ_2, that intersect Σ in degree n, therefore from one side we can assume that H is spanned by them, from the other side H can intersect Σ at most in degree $n+1$, hence Λ_1 and Λ_2 have a 0-dimensional scheme of degree $n-1$ on Σ in common.

Therefore we have found that an element $\Lambda \in \{\Lambda \in \mathbb{G}(n-1, n+2) \mid \deg(\Sigma \cap \Lambda) \geq n-1\}$ belongs to a pencil of $\mathbb{P}^{n-1}$'s, that is a line in the Grassmannian and in particular such a line is spanned by two points belonging to $\mathbb{G}(n-1, n+2) \cap V$, therefore $\Lambda \in \mathrm{Sec}_1(V) \cap \mathbb{G}(n-1, n+2)$. □

References

[**Ad**] B. Ådlandsvik. *Varieties with an extremal number of degenerate higher secant varieties.* J. Reine Angew. Math. **392** (1988), 16-26.

[**AH**] J. Alexander, A. Hirschowitz. *Polynomial interpolation in several variables.* J. of Alg. Geom.**4** (1995), 201-222.

[**AOP1**] H. Abo, G. Ottaviani, C. Peterson, *Induction for secant varieties of Segre varieties.* Trans. Amer. Math. Soc. **361** (2009), no. 2, 767-792.

[**AOP2**] H. Abo, G. Ottaviani, C. Peterson, *Non-Defectivity of Grassmannians of planes.* preprint http://arxiv.org/abs/0901.2601.

[**AP**] E. Arrondo, R. Paoletti, *Characterization of Veronese varieties via projections in Grassmannians*, Projective varieties with unexpected properties (eds. Ciliberto, Geramita, Mir-Roig, Ranestad), De Gruyter (2005), 1-12.

[**Ba**] E. Ballico; *On the secant varieties to the tangent developable of a Veronese variety*, Preprint.

[**BCGI**] A. Bernardi, M.V. Catalisano, A. Gimigliano e M. Idà; *Osculating varieties of Veronese Varieties and their higher secant varieties.* Canadian Journal of Math Vol. **59** (3), 2007 pp. 488-502.

[**BO**] M.C Brambilla, G. Ottaviani; *On the Alexander-Hirschowitz theorem.* J. Pure Appl. Algebra **212** (2008), no. 5, 1229-1251.

[**CGG1**] M. V. Catalisano, A. V. Geramita, A. Gimigliano, *Secant varieties of Grassmann Varieties*, Proc. Am. Math. Soc, **130**, (2002), 975-985.

[**CGG2**] M. V. Catalisano, A. V. Geramita, A. Gimigliano, *On the secant varieties to the tangential varieties of a Veronesean*, Proc. Am. Math. Soc, **130**, (2002), no. 4, 975-985.

[**CGG3**] M. V. Catalisano, A. V. Geramita, A. Gimigliano, *Higher Secant Varieties of Segre and Segre-Veronese varieties*, preprint arXiv:math.AG/0309399 (2003), To appear on J. of Pure and Appl. Algebra.

[**CGG4**] M. V. Catalisano, A. V. Geramita, A. Gimigliano, *Rank of tensors, secant varieties of Segre varieties and fat points*, Lin. Alg. Appl., **355**, (2002), 263-285.

[**CLO**] D. Cox, J. Little, D. O' Shea; *Ideal, varieties and algorithms.*, Springer.

[**CoCoA**] A. Capani, G. Niesi, L. Robbiano *CoCoA, A system for doing Computations in Commutative Algebra*, Available via anonymous ftp from: cocoa.dima.unige.it.

[**Eh**] R. Ehrenborg, *On Apolarity and Generic Canonical Forms*, Journal of Algebra **213**, 167-194 (1999).

[**McG**] B. Mc Gillivray *A probabilistic algorithm for the secant defect of Grassmann varieties*, Linear Algebra and its Applications, **418** (2006), 708-718.

[**Mu**] F. D. Murnaghan, *The Theory of Group Representations* The Johns Hopkins Press, Baltimore, 1938.

[**Se**] B. Segre, *Bertini forms and hessian matrices*, J. of the London Math. Soc., **26**, (1951), 164-176.

[**Te**] A.Terracini. *Sulle V_k per cui la varietà degli S_h $(h+1)$-seganti ha dimensione minore dell'ordinario.* Rend. Circ. Mat. Palermo **31** (1911), 392-396.

[**W**] K.Wakeford. On canonical forms. Proc. London Math. Soc. (2) **18** (1919/20). 403-410.