A CENTRAL LIMIT THEOREM AND ITS APPLICATIONS TO MULTICOLOR RANDOMLY REINFORCED URNS

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Abstract. Let (X_n) be a sequence of integrable real random variables, adapted to a filtration $(\mathcal{G}_n)$. Define

$$C_n = \sqrt{n}\,\Big\{\frac{1}{n}\sum_{k=1}^{n} X_k - E(X_{n+1} \mid \mathcal{G}_n)\Big\} \quad \text{and} \quad D_n = \sqrt{n}\,\big\{E(X_{n+1} \mid \mathcal{G}_n) - Z\big\}$$

where Z is the a.s. limit of $E(X_{n+1} \mid \mathcal{G}_n)$ (assumed to exist). Conditions for $(C_n, D_n) \longrightarrow \mathcal{N}(0, U) \times \mathcal{N}(0, V)$ stably are given, where U, V are certain random variables. In particular, under such conditions, one obtains

$$\sqrt{n}\,\Big\{\frac{1}{n}\sum_{k=1}^{n} X_k - Z\Big\} = C_n + D_n \longrightarrow \mathcal{N}(0, U + V) \quad \text{stably.}$$

This CLT has natural applications to Bayesian statistics and urn problems. The latter are investigated, by paying special attention to multicolor randomly reinforced generalized Polya urns.

1. Introduction and motivations

As regards asymptotics in urn models, there is not a unique reference framework. Rather, there are many (ingenious) disjoint ideas, one for each class of problems. Well known examples are martingale methods, exchangeability, branching processes, stochastic approximation, dynamical systems and so on; see [16].

Those limit theorems which unify various urn problems, thus, look of some interest.

In this paper, we focus on the CLT. While thought for urn problems, our CLT is stated for an arbitrary sequence (X_n) of real random variables. Accordingly, it potentially applies to every urn situation, but it has generally a broader scope. Suppose $E|X_n| < \infty$ and define $Z_n = E\big(X_{n+1} \mid \mathcal{G}_n\big)$ where $(\mathcal{G}_n)$ is some filtration which makes (X_n) adapted. Under various assumptions, one obtains $Z_n \overset{a.s.,L_1}{\longrightarrow} Z$ for some random variable Z. Define further $\overline{X}_n = \frac{1}{n}\sum_{k=1}^{n} X_k$ and

$$C_n = \sqrt{n}\,\big(\overline{X}_n - Z_n\big), \quad D_n = \sqrt{n}\,\big(Z_n - Z\big), \quad W_n = \sqrt{n}\,\big(\overline{X}_n - Z\big).$$

The limit distribution of C_n, D_n or W_n is a main goal in various fields, including Bayesian statistics, discrete time filtering, gambling and urn problems. See [2], [3], [5], [6], [7], [8], [9] and references therein. In fact, suppose the next observation X_{n+1} is to be predicted basing on the available information $\mathcal{G}_n$. If the predictor Z_n

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cannot be evaluated in closed form, one needs some estimate $\widehat{Z}_n$ and C_n reduces to the scaled error when $\widehat{Z}_n = \overline{X}_n$. And $\overline{X}_n$ is a sound estimate of Z_n under some distributional assumptions on (X_n), for instance when (X_n) is exchangeable, as it is usual in Bayesian statistics. Similarly, D_n and W_n are of interest provided Z is regarded as a random parameter. In this case, Z_n is the Bayesian estimate (of Z) under quadratic loss and $\overline{X}_n$ can be often viewed as the the maximum likelihood estimate. Note also that, in the trivial case where (X_n) is i.i.d. and $\mathcal{G}_n = \sigma(X_1, \ldots, X_n)$, one obtains $C_n = W_n = \sqrt{n}\,(\overline{X}_n - EX_1)$ and $D_n = 0$. As to urn problems, X_n could be the indicator of $\{$black ball at time $n\}$ in a multicolor urn. Then, Z_n becomes the proportion of black balls in the urn at time n and $\overline{X}_n$ the observed frequency of black balls at time n.

Our main result (Theorem 2) provides conditions for

$$(C_n, D_n) \longrightarrow \mathcal{N}(0, U) \times \mathcal{N}(0, V) \quad \text{stably} \tag{1}$$

where U, V are certain random variables and $\mathcal{N}(0, L)$ is the Gaussian kernel with mean 0 and variance L. A nice consequence is that

$$W_n = C_n + D_n \longrightarrow \mathcal{N}(0, U + V) \quad \text{stably}.$$

Stable convergence, in the sense of Aldous and Renyi, is a strong form of convergence in distribution; the definition is recalled in Section 2.

To check the conditions for (1), it is fundamental to know something about the convergence rate of

$$Z_{n+1} - Z_n = E\big(X_{n+2} \mid \mathcal{G}_{n+1}\big) - E\big(X_{n+1} \mid \mathcal{G}_n\big),$$
$$E\big(Z_{n+1} - Z_n \mid \mathcal{G}_n\big) = E\big(X_{n+2} - X_{n+1} \mid \mathcal{G}_n\big).$$

If (X_n) is conditionally identically distributed with respect to $(\mathcal{G}_n)$, in the sense of [5], then (Z_n) is a $(\mathcal{G}_n)$-martingale and thus only $Z_{n+1} - Z_n$ plays a role. This happens in particular if (X_n) is exchangeable and $\mathcal{G}_n = \sigma(X_1, \ldots, X_n)$.

To illustrate how the CLT works, three applications are given: r-step predictions, Poisson-Dirichlet sequences, and *randomly reinforced generalized Polya urns*. We next describe the latter, the main of such applications, and we refer to Subsections 4.1 and 4.2 for the remaining two.

An urn contains black and red balls. At each time $n \geq 1$, a ball is drawn and then replaced together with a random number of balls of the same color. Say that B_n black balls or R_n red balls are added to the urn according to whether $X_n = 1$ or $X_n = 0$, where X_n is the indicator of $\{$black ball at time $n\}$. Suppose

$$B_n \geq 0, \quad R_n \geq 0, \quad EB_n = ER_n \quad \text{for all } n,$$
$$\sup_n E\big\{(B_n + R_n)^u\big\} < \infty \quad \text{for some } u > 2,$$
$$m := \lim_n EB_n > 0, \quad q := \lim_n EB_n^2, \qquad s := \lim_n ER_n^2.$$

Letting $\mathcal{G}_n = \sigma(X_1, B_1, R_1, \ldots, X_n, B_n, R_n)$, suppose also that (B_{n+1}, R_{n+1}) is independent of $\mathcal{G}_n \vee \sigma(X_{n+1})$. Then, as shown in Corollary 7, the conditions for (1) are satisfied with

$$U = Z(1 - Z)\,\big(\frac{(1 - Z)q + Zs}{m^2} - 1\big) \quad \text{and} \quad V = Z(1 - Z)\,\frac{(1 - Z)q + Zs}{m^2}.$$

Corollary 7 improves the existing result on this type of urns, obtained in [2], under two respects. First, Corollary 7 implies convergence of the pairs (C_n, D_n)

and not only of D_n. Hence, one also gets $W_n \longrightarrow \mathcal{N}(0, U+V)$ stably. Second, unlike [2], neither the sequence $((B_n, R_n))$ is identically distributed nor the random variables $B_n + R_n$ have compact support.

By just the same argument used for two color urns, multicolor versions of Corollary 7 are easily manufactured. To our knowledge, results of this type were not available so far. Briefly, for a d-color urn, let $X_{n,j}$ be the indicator of $\{$ball of color j at time $n\}$ where $n \geq 1$ and $1 \leq j \leq d$. Suppose $A_{n,j}$ balls of color j are added in case $X_{n,j} = 1$. The random variables $A_{n,j}$ are requested exactly the same conditions asked above to B_n and R_n. Then,

$$\big(\mathbf{C_n},\, \mathbf{D_n}\big) \longrightarrow \mathcal{N}_d(0, \mathbf{U}) \times \mathcal{N}_d(0, \mathbf{V}) \quad \text{stably},$$

where $\mathbf{C_n}$ and $\mathbf{D_n}$ are the vectorial versions of C_n and D_n while $\mathbf{U}$, $\mathbf{V}$ are certain random covariance matrices; see Corollary 10.

A last note is the following. In the previous urn, the n-th reinforce matrix is

$$\mathbf{A_n} = \operatorname{diag}\big(A_{n,1}, \ldots, A_{n,d}\big).$$

Since $EA_{n,1} = \ldots = EA_{n,d}$, the leading eigenvalue of the mean matrix $E\mathbf{A_n}$ has multiplicity greater than 1. Even if significant for applications, this particular case (the leading eigenvalue of $E\mathbf{A_n}$ is not simple) is typically neglected; see [4], [11], [12], and page 20 of [16]. Our result, and indeed the result in [2], contribute to fill this gap.

2. Stable convergence

Stable convergence has been introduced by Renyi in [18] and subsequently investigated by various authors. In a sense, it is intermediate between convergence in distribution and convergence in probability. We recall here basic definitions. For more information, we refer to [1], [7], [10] and references therein.

Let $(\Omega, \mathcal{A}, P)$ be a probability space and S a metric space. A *kernel* on S (or a *random probability measure* on S) is a measurable collection $N = \{N(\omega) : \omega \in \Omega\}$ of probability measures on the Borel σ-field on S. Measurability means that

$$N(\omega)(f) = \int f(x)\, N(\omega)(dx)$$

is $\mathcal{A}$-measurable, as a function of $\omega \in \Omega$, for each bounded Borel map $f : S \to \mathbb{R}$.

Let (Y_n) be a sequence of S-valued random variables and N a kernel on S. Both (Y_n) and N are defined on $(\Omega, \mathcal{A}, P)$. Say that Y_n converges *stably* to N in case

$$P\big(Y_n \in \cdot \mid H\big) \to E\big(N(\cdot) \mid H\big) \quad \text{weakly}$$
$$\text{for all } H \in \mathcal{A} \text{ such that } P(H) > 0.$$

Clearly, if $Y_n \to N$ stably, then Y_n converges in distribution to the probability law $E\big(N(\cdot)\big)$ (just let $H = \Omega$). Moreover, when S is separable, it is not hard to see that $Y_n \overset{P}{\to} Y$ if and only if Y_n converges stably to the kernel $N = \delta_Y$.

We next mention a strong form of stable convergence, introduced in [7], to be used later on. Let $\mathcal{F}_n \subset \mathcal{A}$ be a sub-σ-field, $n \geq 1$. Say that Y_n converges to N *stably in strong sense*, with respect to the sequence $(\mathcal{F}_n)$, in case

$$E\big(f(Y_n) \mid \mathcal{F}_n\big) \overset{P}{\longrightarrow} N(f) \quad \text{for each } f \in C_b(S)$$

where $C_b(S)$ denotes the set of real bounded continuous functions on S.

Finally, we state a simple but useful fact as a lemma.

Lemma 1. *Suppose that S is a separable metric space and*
C_n and D_n are S-valued random variables on $(\Omega, \mathcal{A}, P)$, $n \geq 1$;
M and N are kernels on S defined on $(\Omega, \mathcal{A}, P)$;
$(\mathcal{G}_n : n \geq 1)$ is an (increasing) filtration satisfying

$$\sigma(C_n) \subset \mathcal{G}_n \quad \text{and} \quad \sigma(D_n) \subset \mathcal{G}_\infty \quad \text{for all } n, \text{ where } \mathcal{G}_\infty = \sigma(\cup_n \mathcal{G}_n).$$

If $C_n \to M$ stably and $D_n \to N$ stably in strong sense, with respect to $(\mathcal{G}_n)$, then

$$(C_n, D_n) \longrightarrow M \times N \quad \text{stably.}$$

(Here, $M \times N$ is the kernel on $S \times S$ such that $\big(M \times N\big)(\omega) = M(\omega) \times N(\omega)$ for all ω).

Proof. By standard arguments, since S is separable and $\sigma(C_n, D_n) \subset \mathcal{G}_\infty$, it suffices to prove that $E\big\{I_H\, f_1(C_n)\, f_2(D_n)\big\} \to E\big\{I_H\, M(f_1)\, N(f_2)\big\}$ whenever $H \in \cup_n \mathcal{G}_n$ and $f_1,\, f_2 \in C_b(S)$. Let $L_n = E\big(f_2(D_n) \mid \mathcal{G}_n\big) - N(f_2)$. Since $H \in \cup_n \mathcal{G}_n$, there is k such that $H \in \mathcal{G}_n$ for $n \geq k$. Thus,

$$\begin{aligned} E\big\{I_H\, f_1(C_n)\, f_2(D_n)\big\} &= E\big\{I_H\, f_1(C_n)\, E\big(f_2(D_n) \mid \mathcal{G}_n\big)\big\} \\ &= E\big\{I_H\, f_1(C_n)\, N(f_2)\big\} + E\big\{I_H\, f_1(C_n)\, L_n\big\} \quad \text{for all } n \geq k. \end{aligned}$$

Finally, $|E\big\{I_H\, f_1(C_n)\, L_n\big\}| \leq \sup|f_1|\, E|L_n| \to 0$, since $D_n \to N$ stably in strong sense, and $E\big\{I_H\, f_1(C_n)\, N(f_2)\big\} \to E\big\{I_H\, M(f_1)\, N(f_2)\big\}$ as $C_n \to M$ stably. □

3. Main result

In the sequel, $(X_n : n \geq 1)$ is a sequence of real random variables on the probability space $(\Omega, \mathcal{A}, P)$ and $(\mathcal{G}_n : n \geq 0)$ an (increasing) filtration. We assume $E|X_n| < \infty$ and we let

$$Z_n = E(X_{n+1} \mid \mathcal{G}_n) \quad \text{and} \quad \overline{X}_n = \frac{1}{n}\sum_{k=1}^n X_k.$$

In case $\sup_n EX_n^2 < \infty$ and

$$E\Big\{\big(E(Z_{n+1} \mid \mathcal{G}_n) - Z_n\big)^2\Big\} = o(n^{-3}), \tag{2}$$

the sequence (Z_n) is an uniformly integrable quasi-martingale; see e.g. page 532 of [13]. Accordingly,

$$Z_n \overset{a.s., L_1}{\longrightarrow} Z$$

for some real random variable Z. Define

$$C_n = \sqrt{n}\,\big(\overline{X}_n - Z_n\big), \quad D_n = \sqrt{n}\,\big(Z_n - Z\big).$$

Let $\mathcal{N}(a, b)$ denote the one-dimensional Gaussian law with mean a and variance $b \geq 0$ (where $\mathcal{N}(a, 0) = \delta_a$). Note that $\mathcal{N}(0, L)$ is a kernel on $\mathbb{R}$ for each real non negative random variable L. We are now in a position to state our main result.

Theorem 2. *Suppose $\sigma(X_n) \subset \mathcal{G}_n$ for each $n \geq 1$, (X_n^2) is uniformly integrable and condition (2) holds. Let us consider the following conditions*

- **(a)** $\frac{1}{\sqrt{n}}\, E\big\{\max_{1 \leq k \leq n} k\,|Z_{k-1} - Z_k|\big\} \longrightarrow 0,$
- **(b)** $\frac{1}{n}\sum_{k=1}^n \big\{X_k - Z_{k-1} + k(Z_{k-1} - Z_k)\big\}^2 \overset{P}{\longrightarrow} U,$

(c) $\sqrt{n}\,E\big\{\sup_{k\geq n}|Z_{k-1}-Z_k|\,\big\}\longrightarrow 0$,

(d) $n\sum_{k\geq n}(Z_{k-1}-Z_k)^2\overset{P}{\longrightarrow}V$,

where U and V are real non negative random variables. Then, $C_n\to\mathcal{N}(0,U)$ stably under (a)-(b), and $D_n\to\mathcal{N}(0,V)$ stably in strong sense, with respect to $(\mathcal{G}_n)$, under (c)-(d). In particular,

$$(C_n,D_n)\longrightarrow\mathcal{N}(0,U)\times\mathcal{N}(0,V)\quad\textit{stably under (a)-(b)-(c)-(d).}$$

Proof. Since $\sigma(C_n)\subset\mathcal{G}_n$ and Z can be taken $\mathcal{G}_\infty$-measurable, Lemma 1 applies. Thus, it suffices to prove that $C_n\to\mathcal{N}(0,U)$ stably and $D_n\to\mathcal{N}(0,V)$ stably in strong sense.

"$C_n\to\mathcal{N}(0,U)$ stably". Suppose conditions (a)-(b) hold. First note that

$$\sqrt{n}\,C_n=n\,\overline{X}_n-n\,Z_n=\sum_{k=1}^n X_k+\sum_{k=1}^n\big((k-1)Z_{k-1}-kZ_k\big)$$
$$=\sum_{k=1}^n\big\{X_k-Z_{k-1}+k(Z_{k-1}-Z_k)\big\}.$$

Letting

$$Y_{n,k}=\frac{X_k-Z_{k-1}+k\big(E(Z_k\mid\mathcal{G}_{k-1})-Z_k\big)}{\sqrt{n}}\quad\text{and}\quad Q_n=\frac{1}{\sqrt{n}}\sum_{k=1}^n k\big(Z_{k-1}-E(Z_k\mid\mathcal{G}_{k-1})\big),$$

it follows that $C_n=\sum_{k=1}^n Y_{n,k}+Q_n$. By (2),

$$E|Q_n|\leq\frac{1}{\sqrt{n}}\sum_{k=1}^n k\sqrt{E\big\{\big(Z_{k-1}-E(Z_k\mid\mathcal{G}_{k-1})\big)^2\big\}}=\frac{1}{\sqrt{n}}\sum_{k=1}^n \mathrm{o}(k^{-1/2})\longrightarrow 0.$$

Hence, it suffices to prove that $\sum_{k=1}^n Y_{n,k}\to\mathcal{N}(0,U)$ stably. Letting $\mathcal{F}_{n,k}=\mathcal{G}_k$, $k=1,\ldots,n$, one obtains $E\big(Y_{n,k}\mid\mathcal{F}_{n,k-1}\big)=0$ a.s.. Thus, by Corollary 7 of [7], $\sum_{k=1}^n Y_{n,k}\to\mathcal{N}(0,U)$ stably whenever

$$\text{(i)}\ \ E\big\{\max_{1\leq k\leq n}|Y_{n,k}|\big\}\longrightarrow 0;\quad\text{(ii)}\ \ \sum_{k=1}^n Y_{n,k}^2\overset{P}{\longrightarrow}U.$$

As to (i), first note that

$$\sqrt{n}\max_{1\leq k\leq n}|Y_{n,k}|\leq\max_{1\leq k\leq n}|X_k-Z_{k-1}|+\sum_{k=1}^n k\,|E(Z_k\mid\mathcal{G}_{k-1})-Z_{k-1}|+\max_{1\leq k\leq n}k\,|Z_{k-1}-Z_k|.$$

Since (X_n^2) is uniformly integrable, $((X_n-Z_{n-1})^2)$ is uniformly integrable as well, and this implies $\frac{1}{n}\,E\big\{\max_{1\leq k\leq n}(X_k-Z_{k-1})^2\big\}\longrightarrow 0$. By condition (2),

$$\frac{1}{\sqrt{n}}\sum_{k=1}^n k\,E\Big|E(Z_k\mid\mathcal{G}_{k-1})-Z_{k-1}\Big|=\frac{1}{\sqrt{n}}\sum_{k=1}^n\mathrm{o}(k^{-1/2})\longrightarrow 0.$$

Thus, (i) follows from condition (a).

As to (ii), write

$$\sum_{k=1}^n Y_{n,k}^2 = \frac{1}{n}\sum_{k=1}^n \big(X_k - Z_{k-1} + k(Z_{k-1} - Z_k)\big)^2 + \frac{1}{n}\sum_{k=1}^n k^2\big(E(Z_k \mid \mathcal{G}_{k-1}) - Z_{k-1}\big)^2 +$$
$$+ \frac{2}{n}\sum_{k=1}^n \big(X_k - Z_{k-1} + k(Z_{k-1} - Z_k)\big)\, k\, \big(E(Z_k \mid \mathcal{G}_{k-1}) - Z_{k-1}\big)$$
$$= R_n + S_n + T_n \quad \text{say.}$$

Then, $R_n \xrightarrow{P} U$ by (b) and $E|S_n| = ES_n \to 0$ by (2). Further $T_n \xrightarrow{P} 0$, since

$$\frac{T_n^2}{4} \le \frac{1}{n}\sum_{k=1}^n \big(X_k - Z_{k-1} + k(Z_{k-1} - Z_k)\big)^2 \cdot \frac{1}{n}\sum_{k=1}^n k^2\big(E(Z_k \mid \mathcal{G}_{k-1}) - Z_{k-1}\big)^2 = R_n\, S_n.$$

Hence, (ii) holds, and this concludes the proof of $C_n \to \mathcal{N}(0, U)$ stably.

"$D_n \to \mathcal{N}(0, V)$ stably in strong sense". Suppose conditions (c)-(d) hold. We first recall a known result; see Example 6 of [7]. Let (L_n) be a $(\mathcal{G}_n)$-martingale such that $L_n \xrightarrow{a.s.,L_1} L$ for some real random variable L. Then,

$$\sqrt{n}\,\big(L_n - L\big) \longrightarrow \mathcal{N}(0, V) \quad \text{stably in strong sense with respect to } (\mathcal{G}_n),$$

provided

$$\text{(c*)}\ \ \sqrt{n}\, E\big\{\sup_{k\ge n}|L_{k-1} - L_k|\,\big\} \longrightarrow 0; \quad \text{(d*)}\ \ n\sum_{k\ge n}(L_{k-1} - L_k)^2 \xrightarrow{P} V.$$

Next, define $L_0 = Z_0$ and

$$L_n = Z_n - \sum_{k=0}^{n-1}\big(E(Z_{k+1} \mid \mathcal{G}_k) - Z_k\big).$$

Then, (L_n) is a $(\mathcal{G}_n)$-martingale. Also, $L_n \xrightarrow{a.s.,L_1} L$ for some L, as (Z_n) is an uniformly integrable quasi martingale. In particular, $L_n - L$ can be written as $L_n - L = \sum_{k\ge n}(L_k - L_{k+1})$ a.s.. Similarly, $Z_n - Z = \sum_{k\ge n}(Z_k - Z_{k+1})$ a.s.. It follows that

$$E\Big|D_n - \sqrt{n}(L_n - L)\Big| = \sqrt{n}\, E\Big|(Z_n - Z) - (L_n - L)\Big|$$
$$= \sqrt{n}\, E\Big|\sum_{k\ge n}\big\{(Z_k - L_k) - (Z_{k+1} - L_{k+1})\big\}\Big|$$
$$\le \sqrt{n}\sum_{k\ge n} E\Big|Z_k - E(Z_{k+1} \mid \mathcal{G}_k)\Big| = \sqrt{n}\sum_{k\ge n} \mathrm{o}(k^{-3/2}) \longrightarrow 0.$$

Thus, $D_n \to \mathcal{N}(0, V)$ stably in strong sense if and only if $\sqrt{n}(L_n - L) \to \mathcal{N}(0, V)$ stably in strong sense, and to conclude the proof it suffices to check conditions (c*)-(d*). In turn, (c*)-(d*) are a straightforward consequence of conditions (2), (c), (d) and

$$L_{k-1} - L_k = \big(Z_{k-1} - Z_k\big) + \big(E(Z_k \mid \mathcal{G}_{k-1}) - Z_{k-1}\big).$$

□

Some remarks on Theorem 2 are in order.

In real problems, one of the quantities of main interest is

$$W_n = \sqrt{n}\,(\overline{X}_n - Z).$$

And, under the assumptions of Theorem 2, one obtains

$$W_n = C_n + D_n \longrightarrow \mathcal{N}(0, U+V) \quad \text{stably}.$$

Condition (2) trivially holds when (X_n) is conditionally identically distributed, in the sense of [5], with respect to the filtration $(\mathcal{G}_n)$. In this case, in fact, (Z_n) is even a $(\mathcal{G}_n)$-martingale. In particular, (2) holds if (X_n) is exchangeable and $\mathcal{G}_n = \sigma(X_1, \ldots, X_n)$.

Under (c), condition (a) can be replaced by

(a*) $\sup_n \frac{1}{n}\sum_{k=1}^n k^2 E\big\{(Z_{k-1} - Z_k)^2\big\} < \infty$.

Indeed, (a*) and (c) imply (a) (we omit calculations). Note that, for proving $C_n \to \mathcal{N}(0, U)$ stably under (a*)-(b)-(c), one can rely on more classical versions of the martingale CLT, such as Theorem 3.2 of [10].

To check conditions (b) and (d), the following simple lemma can help.

Lemma 3. *Let (Y_n) be a $(\mathcal{G}_n)$-adapted sequence of real random variables. If $\sum_{n=1}^\infty \frac{EY_n^2}{n^2} < \infty$ and $E\big(Y_{n+1} \mid \mathcal{G}_n\big) \xrightarrow{a.s.} Y$, for some random variable Y, then*

$$n\sum_{k\geq n} \frac{Y_k}{k^2} \xrightarrow{a.s.} Y \quad \text{and} \quad \frac{1}{n}\sum_{k=1}^n Y_k \xrightarrow{a.s.} Y.$$

Proof. Let $L_n = \sum_{k=1}^n \frac{Y_k - E\big(Y_k \mid \mathcal{G}_{k-1}\big)}{k}$. Then, L_n is a $(\mathcal{G}_n)$-martingale such that

$$\sup_n EL_n^2 \leq 4\sum_k \frac{EY_k^2}{k^2} < \infty.$$

Thus, L_n converges a.s. and Abel summation formula yields

$$n\sum_{k\geq n} \frac{Y_k - E\big(Y_k \mid \mathcal{G}_{k-1}\big)}{k^2} \xrightarrow{a.s.} 0.$$

Since $E\big(Y_{n+1} \mid \mathcal{G}_n\big) \xrightarrow{a.s.} Y$ and $n\sum_{k\geq n}\frac{1}{k^2} \longrightarrow 1$, it follows that

$$n\sum_{k\geq n}\frac{Y_k}{k^2} = n\sum_{k\geq n}\frac{Y_k - E\big(Y_k \mid \mathcal{G}_{k-1}\big)}{k^2} + n\sum_{k\geq n}\frac{E\big(Y_k \mid \mathcal{G}_{k-1}\big)}{k^2} \xrightarrow{a.s.} Y.$$

Similarly, Kroneker lemma and $E\big(Y_{n+1} \mid \mathcal{G}_n\big) \xrightarrow{a.s.} Y$ yield

$$\frac{1}{n}\sum_{k=1}^n Y_k = \frac{1}{n}\sum_{k=1}^n E(Y_k \mid \mathcal{G}_{k-1}) + \frac{1}{n}\sum_{k=1}^n k\,\frac{Y_k - E\big(Y_k \mid \mathcal{G}_{k-1}\big)}{k} \xrightarrow{a.s.} Y.$$

□

Our last comment needs a formal remark.

Remark 4. As regards D_n, a natural question is whether

$$E\big(f(D_n)\mid\mathcal{G}_n\big)\overset{a.s.}{\longrightarrow}\mathcal{N}(0,V)(f)\quad\text{for each } f\in C_b(\mathbb{R}).\tag{3}$$

This is a strengthening of $D_n\to\mathcal{N}(0,V)$ stably in strong sense, as $E\big(f(D_n)\mid\mathcal{G}_n\big)$ is requested to converge a.s. and not only in probability. Let (X_n) be a (non necessarily $(\mathcal{G}_n)$-adapted) sequence of integrable random variables. Then, for (3) to be true, it is enough that (Z_n) is uniformly integrable and

$$\sum_{k\geq 1}\sqrt{k}\,E\Big|\,E(Z_k\mid\mathcal{G}_{k-1})-Z_{k-1}\Big|<\infty,$$

$$E\big\{\sup_{k\geq 1}\sqrt{k}\,|Z_{k-1}-Z_k|\big\}<\infty,\quad n\sum_{k\geq n}(Z_{k-1}-Z_k)^2\overset{a.s.}{\longrightarrow}V.$$

The proof is essentially the same as that of Theorem 2, up to using Theorem 2.2 of [8] instead of Example 6 of [7].

4. Applications

This section is split into four subsections, arranged in increasing order of length.

4.1. **r-step predictions.** Suppose we are requested to make conditional forecasts on a sequence of events $A_n\in\mathcal{G}_n$. To fix ideas, for each n, we aim to predict

$$A_n^*=\big(\cap_{j\in J}A_{n+j}\big)\cap\big(\cap_{j\in J^c}A_{n+j}^c\big)$$

conditionally on $\mathcal{G}_n$, where J is a given subset of $\{1,\ldots,r\}$ and $J^c=\{1,\ldots,r\}\setminus J$. Letting $X_n=I_{A_n}$, the predictor can be written as

$$Z_n^*=E\big\{\prod_{j\in J}X_{n+j}\prod_{j\in J^c}(1-X_{n+j})\mid\mathcal{G}_n\big\}.$$

In the spirit of Section 1, when Z_n^* cannot be evaluated in closed form, one needs to estimate it. Under some assumptions, in particular when (X_n) is exchangeable and $\mathcal{G}_n=\sigma(X_1,\ldots,X_n)$, a reasonable estimate of Z_n^* is $\overline{X}_n^h(1-\overline{X}_n)^{r-h}$ where $h=\text{card}(J)$. Usually, under such assumptions, one also has $Z_n\overset{a.s.}{\longrightarrow}Z$ and $Z_n^*\overset{a.s.}{\longrightarrow}Z^h(1-Z)^{r-h}$ for some random variable Z. So, it makes sense to define

$$C_n^*=\sqrt{n}\,\big\{\overline{X}_n^h(1-\overline{X}_n)^{r-h}-Z_n^*\big\},\quad D_n^*=\sqrt{n}\,\big\{Z_n^*-Z^h(1-Z)^{r-h}\big\}.$$

Next result is a straightforward consequence of Theorem 2.

Corollary 5. *Let (X_n) be a $(\mathcal{G}_n)$-adapted sequence of indicators satisfying* (2). *If conditions (a)-(b)-(c)-(d) of Theorem 2 hold, then*

$$(C_n^*,D_n^*)\longrightarrow\mathcal{N}(0,\sigma^2U)\times\mathcal{N}(0,\sigma^2V)\quad\textit{stably, where}$$
$$\sigma^2=\big\{h\,Z^{h-1}(1-Z)^{r-h}-(r-h)\,Z^h(1-Z)^{r-h-1}\big\}^2.$$

Proof. We just give a sketch of the proof. Let $f(x)=x^h(1-x)^{r-h}$. Basing on (c), it can be shown that $\sqrt{n}\,E\Big|\,Z_n^*-f(Z_n)\Big|\longrightarrow 0$. Thus, C_n^* can be replaced by $\sqrt{n}\,\big\{f(\overline{X}_n)-f(Z_n)\big\}$ and D_n^* by $\sqrt{n}\,\big\{f(Z_n)-f(Z)\big\}$. By the mean value theorem,

$$\sqrt{n}\,\big\{f(\overline{X}_n)-f(Z_n)\big\}=\sqrt{n}\,f'(M_n)\,(\overline{X}_n-Z_n)=f'(M_n)\,C_n$$

where M_n is between $\overline{X}_n$ and Z_n. By (2), $Z_n \xrightarrow{a.s} Z$ and $\overline{X}_n \xrightarrow{a.s} Z$. Hence, $f'(M_n) \xrightarrow{a.s} f'(Z)$ as f' is continuous. By Theorem 2, $C_n \to \mathcal{N}(0,U)$ stably. Thus,

$$\sqrt{n}\,\big\{f(\overline{X}_n) - f(Z_n)\big\} \longrightarrow f'(Z)\,\mathcal{N}(0,U) = \mathcal{N}(0,\sigma^2 U) \quad \text{stably.}$$

By a similar argument, it can be seen that $\sqrt{n}\,\big\{f(Z_n) - f(Z)\big\} \longrightarrow \mathcal{N}(0,\sigma^2 V)$ stably in strong sense. An application of Lemma 1 concludes the proof. □

4.2. **Poisson-Dirichlet sequences.** Let $\mathcal{Y}$ be a finite set and (Y_n) a sequence of $\mathcal{Y}$-valued random variables satisfying

$$P\big(Y_{n+1} \in A \mid Y_1,\ldots,Y_n\big) = \frac{\sum_{y\in A}(S_{n,y}-\alpha)\,I_{\{S_{n,y}\neq 0\}} + \big(\theta + \alpha\sum_{y\in\mathcal{Y}} I_{\{S_{n,y}\neq 0\}}\big)\,\nu(A)}{\theta+n}$$

a.s. for all $A \subset \mathcal{Y}$ and $n \geq 1$. Here, $0 \leq \alpha < 1$ and $\theta > -\alpha$ are constants, ν is the probability distribution of Y_1 and $S_{n,y} = \sum_{k=1}^{n} I_{\{Y_k = y\}}$.

Sequences (Y_n) of this type play a role in various frameworks, mainly in population-genetics. They can be regarded as a generalization of those exchangeable sequences directed by a two parameter Poisson-Dirichlet process; see [17]. For $\alpha = 0$, (Y_n) reduces to a classical Dirichlet sequence (i.e., an exchangeable sequence directed by a Dirichlet process). But, for $\alpha \neq 0$, (Y_n) may even fail to be exchangeable.

From the point of view of Theorem 2, however, the only important thing is that $P\big(Y_{n+1} \in \cdot \mid Y_1,\ldots,Y_n\big)$ can be written down explicitly. Indeed, the following result is available.

Corollary 6. *Let $\mathcal{G}_n = \sigma(Y_1,\ldots,Y_n)$ and $X_n = I_A(Y_n)$, where $A \subset \mathcal{Y}$. Then, condition (2) holds (so that $Z_n \xrightarrow{a.s.} Z$) and*

$$(C_n, D_n) \longrightarrow \delta_0 \times \mathcal{N}\big(0, Z(1-Z)\big) \quad \textit{stably.}$$

Proof. Let $Q_n = -\alpha\sum_{y\in A} I_{\{S_{n,y}\neq 0\}} + \big(\theta + \alpha\sum_{y\in\mathcal{Y}} I_{\{S_{n,y}\neq 0\}}\big)\,\nu(A)$. Since

$$Z_n = P\big(Y_{n+1}\in A \mid Y_1,\ldots,Y_n\big) = \frac{n\,\overline{X}_n + Q_n}{\theta+n} \quad \text{and} \quad |Q_n| \leq c$$

for some constant c, then $C_n \xrightarrow{a.s.} 0$. By Lemma 1 and Theorem 2, thus, it suffices to check conditions (2), (c) and (d) with $V = Z(1-Z)$. On noting that

$$Z_{n+1} - Z_n = \frac{X_{n+1} - Z_n}{\theta+n+1} + \frac{Q_{n+1} - Q_n}{\theta+n+1},$$

condition (c) trivially holds. Since $S_{n+1,y} = S_{n,y} + I_{\{Y_{n+1}=y\}}$, one obtains

$$Q_{n+1} - Q_n = -\alpha\,\nu(A^c)\sum_{y\in A} I_{\{S_{n,y}=0\}} I_{\{Y_{n+1}=y\}} + \alpha\,\nu(A)\sum_{y\in A^c} I_{\{S_{n,y}=0\}} I_{\{Y_{n+1}=y\}}.$$

It follows that

$$E\big\{|Q_{n+1}-Q_n| \mid \mathcal{G}_n\big\} \leq 2\sum_{y\in\mathcal{Y}} I_{\{S_{n,y}=0\}}\,P\big(Y_{n+1}=y \mid \mathcal{G}_n\big) \leq \frac{d}{\theta+n} \quad \text{a.s.}$$

for some constant d, and this implies

$$\Big|E\big(Z_{n+1}\mid\mathcal{G}_n\big) - Z_n\Big| = \frac{\Big|E\big(Q_{n+1}-Q_n \mid \mathcal{G}_n\big)\Big|}{\theta+n+1} \leq \frac{d}{(\theta+n)^2} \quad \text{a.s..}$$

Hence, condition (2) holds. To check (d), note that $\sum_k k^2 E\big\{(Z_{k-1}-Z_k)^4\big\} < \infty$. Since $Z_k \overset{a.s.}{\longrightarrow} Z$ (by (2)) one also obtains

$$E\big\{(X_k - Z_{k-1})^2 \mid \mathcal{G}_{k-1}\big\} = Z_{k-1} - Z_{k-1}^2 \overset{a.s.}{\longrightarrow} Z(1-Z),$$
$$E\big\{(Q_k - Q_{k-1})^2 \mid \mathcal{G}_{k-1}\big\} + 2\,E\big\{(X_k - Z_{k-1})\,(Q_k - Q_{k-1}) \mid \mathcal{G}_{k-1}\big\} \overset{a.s.}{\longrightarrow} 0.$$

Thus, $k^2 E\big\{(Z_{k-1}-Z_k)^2 \mid \mathcal{G}_{k-1}\big\} \overset{a.s.}{\longrightarrow} Z(1-Z)$. Letting $Y_k = k^2(Z_{k-1}-Z_k)^2$ and $Y = Z(1-Z)$, Lemma 3 implies

$$n \sum_{k\geq n} (Z_{k-1}-Z_k)^2 = n \sum_{k\geq n} \frac{Y_k}{k^2} \overset{a.s.}{\longrightarrow} Z(1-Z).$$

□

As it is clear from the previous proof, all conditions of Remark 4 are satisfied. Therefore, D_n meets condition (3) with $V = Z(1-Z)$.

4.3. **Two color randomly reinforced generalized Polya urns.** An urn contains $b > 0$ black balls and $r > 0$ red balls. At each time $n \geq 1$, a ball is drawn and then replaced together with a random number of balls of the same color. Say that B_n black balls or R_n red balls are added to the urn according to whether $X_n = 1$ or $X_n = 0$, where X_n is the indicator of {black ball at time n}.

Urns of this type have some history: see [2], [3], [5], [8], [15], [16] and references therein.

To model such urns, we assume X_n, B_n, R_n random variables on the probability space $(\Omega, \mathcal{A}, P)$ such that

(∗) $X_n \in \{0,1\}, \quad B_n \geq 0, \quad R_n \geq 0,$

(B_n, R_n) independent of $\big(X_1, B_1, R_1, \ldots, X_{n-1}, B_{n-1}, R_{n-1}, X_n\big)$,

$$Z_n = P\big(X_{n+1} = 1 \mid \mathcal{G}_n\big) = \frac{b + \sum_{k=1}^n B_k X_k}{b + r + \sum_{k=1}^n \big(B_k X_k + R_k(1-X_k)\big)} \quad \text{a.s.,}$$

for each $n \geq 1$, where

$$\mathcal{G}_0 = \{\emptyset, \Omega\}, \quad \mathcal{G}_n = \sigma\big(X_1, B_1, R_1, \ldots, X_n, B_n, R_n\big).$$

In the particular case $B_n = R_n$, in Example 3.5 of [5], it is shown that C_n converges stably to a Gaussian kernel whenever $EB_1^2 < \infty$ and $B_n \sim B_1$ for all n. Further, in Corollary 4.1 of [8], D_n is shown to satisfy condition (3). The latter result on D_n is extended to $B_n \neq R_n$ in [2], under the assumptions that $B_1 + R_1$ has compact support, $EB_1 = ER_1$, and $(B_n, R_n) \sim (B_1, R_1)$ for all n.

Basing on Theorem 2, condition (3) can be shown to hold more generally. Indeed, it is fundamental that $EB_n = ER_n$ for all n and the three sequences (EB_n), (EB_n^2), (ER_n^2) approach a limit. But identity in distribution of (B_n, R_n) can be dropped and compact support of $B_n + R_n$ can be replaced by a moment condition such as

$$\sup_n E\big\{(B_n + R_n)^u\big\} < \infty \quad \text{for some } u > 2. \tag{4}$$

Under these conditions, not only D_n meets (3), but the pairs (C_n, D_n) converge stably as well. In particular, one obtains stable convergence of $W_n = C_n + D_n$ which is of potential interest in urn problems.

Corollary 7. *In addition to* $(*)$ *and* (4)*, suppose* $EB_n = ER_n$ *for all* n *and*

$$m := \lim_n EB_n > 0, \quad q := \lim_n EB_n^2, \quad s := \lim_n ER_n^2.$$

Then, condition (2) *holds (so that* $Z_n \xrightarrow{a.s.} Z$*) and*

$$(C_n, D_n) \longrightarrow \mathcal{N}(0,U) \times \mathcal{N}(0,V) \quad \text{stably, where}$$

$$U = Z(1-Z)\,\Big(\frac{(1-Z)q + Zs}{m^2} - 1\Big) \quad \text{and} \quad V = Z(1-Z)\,\frac{(1-Z)q + Zs}{m^2}.$$

In particular, $W_n = C_n + D_n \longrightarrow \mathcal{N}(0, U+V)$ *stably. Moreover,* D_n *meets condition* (3)*, that is,* $E\big(f(D_n) \mid \mathcal{G}_n\big) \xrightarrow{a.s.} \mathcal{N}(0,V)(f)$ *for each* $f \in C_b(\mathbb{R})$*.*

It is worth noting that, arguing as in [2] and [15], one obtains $P(Z = z) = 0$ for all z. Thus, $\mathcal{N}(0,V)$ is a non degenerate kernel. In turn, $\mathcal{N}(0,U)$ is non degenerate unless $q = s = m^2$, and this happens if and only if both B_n and R_n converge in probability (necessarily to m). In the latter case ($q = s = m^2$), $C_n \xrightarrow{P} 0$ and condition (3) holds with $V = Z(1-Z)$. Thus, in a sense, randomly reinforced urns behave as classical Polya urns (i.e., those urns with $B_n = R_n = m$) whenever the reinforcements converge in probability.

The proof of Corollary 7 is deferred to the Appendix as it needs some work. Here, to point out the underlying argument, we sketch such a proof under the superfluous but simplifying assumption that $B_n \vee R_n \leq c$ for all n and some constant c. Let

$$S_n = b + r + \sum_{k=1}^{n} \big(B_k X_k + R_k(1 - X_k)\big).$$

After some algebra, $Z_{n+1} - Z_n$ can be written as

$$\begin{aligned} Z_{n+1} - Z_n &= \frac{(1-Z_n)\,X_{n+1}\,B_{n+1} \;-\; Z_n\,(1-X_{n+1})\,R_{n+1}}{S_{n+1}} \\ &= \frac{(1-Z_n)\,X_{n+1}\,B_{n+1}}{S_n + B_{n+1}} \;-\; \frac{Z_n\,(1-X_{n+1})\,R_{n+1}}{S_n + R_{n+1}}. \end{aligned}$$

By $(*)$ and $EB_{n+1} = ER_{n+1}$,

$$\begin{aligned} E\big(Z_{n+1} - Z_n \mid \mathcal{G}_n\big) &= Z_n(1-Z_n)\,E\Big\{\frac{B_{n+1}}{S_n + B_{n+1}} - \frac{R_{n+1}}{S_n + R_{n+1}} \mid \mathcal{G}_n\Big\} \\ &= Z_n(1-Z_n)\,E\Big\{\frac{B_{n+1}}{S_n + B_{n+1}} - \frac{B_{n+1}}{S_n} - \frac{R_{n+1}}{S_n + R_{n+1}} + \frac{R_{n+1}}{S_n} \mid \mathcal{G}_n\Big\} \\ &= Z_n(1-Z_n)\,E\Big\{-\frac{B_{n+1}^2}{S_n(S_n + B_{n+1})} + \frac{R_{n+1}^2}{S_n(S_n + R_{n+1})} \mid \mathcal{G}_n\Big\} \quad \text{a.s..} \end{aligned}$$

Thus, $\Big|E\big(Z_{n+1} \mid \mathcal{G}_n\big) - Z_n\Big| \leq \frac{EB_{n+1}^2 + ER_{n+1}^2}{S_n^2}$ a.s.. Since $\sup_n \big(EB_n^2 + ER_n^2\big) < \infty$ and $E(S_n^{-p}) = \mathrm{O}(n^{-p})$ for all $p > 0$ (as shown in Lemma 11) then

$$E\big\{|E(Z_{n+1} \mid \mathcal{G}_n) - Z_n|^p\big\} = \mathrm{O}(n^{-2p}) \quad \text{for all } p > 0.$$

In particular, condition (2) holds and $\sum_k \sqrt{k}\,E\Big|E(Z_k \mid \mathcal{G}_{k-1}) - Z_{k-1}\Big| < \infty$.

To conclude the proof, in view of Lemma 1, Theorem 2 and Remark 4, it suffices to check conditions (a), (b) and

$$\text{(i)}\;\; E\big\{\sup_{k\geq 1} \sqrt{k}\,|Z_{k-1} - Z_k|\big\} < \infty; \quad \text{(ii)}\;\; n \sum_{k \geq n} (Z_{k-1} - Z_k)^2 \xrightarrow{a.s.} V.$$

Conditions (a) and (i) are straightforward consequences of $|Z_{n+1}-Z_n|\leq\frac{c}{S_n}$ and $E(S_n^{-p})=\mathrm{O}(n^{-p})$ for all $p>0$. Condition (b) follows from the same argument as (ii). And to prove (ii), it suffices to show that $E(Y_{n+1}\mid\mathcal{G}_n)\xrightarrow{a.s.}V$ where $Y_n=n^2(Z_{n-1}-Z_n)^2$; see Lemma 3. Write $(n+1)^{-2}E(Y_{n+1}\mid\mathcal{G}_n)$ as

$$Z_n(1-Z_n)^2E\big\{\frac{B_{n+1}^2}{(S_n+B_{n+1})^2}\mid\mathcal{G}_n\big\}\;+\;Z_n^2(1-Z_n)E\big\{\frac{R_{n+1}^2}{(S_n+R_{n+1})^2}\mid\mathcal{G}_n\big\}.$$

Since $\frac{S_n}{n}\xrightarrow{a.s.}m$ (by Lemma 11) and $B_{n+1}\leq c$, then

$$n^2E\big\{\frac{B_{n+1}^2}{(S_n+B_{n+1})^2}\mid\mathcal{G}_n\big\}\leq n^2E\big\{\frac{B_{n+1}^2}{S_n^2}\mid\mathcal{G}_n\big\}=n^2\frac{EB_{n+1}^2}{S_n^2}\xrightarrow{a.s.}\frac{q}{m^2}\quad\text{and}$$

$$n^2E\big\{\frac{B_{n+1}^2}{(S_n+B_{n+1})^2}\mid\mathcal{G}_n\big\}\geq n^2E\big\{\frac{B_{n+1}^2}{(S_n+c)^2}\mid\mathcal{G}_n\big\}=n^2\frac{EB_{n+1}^2}{(S_n+c)^2}\xrightarrow{a.s.}\frac{q}{m^2}.$$

Similarly, $n^2E\big\{\frac{R_{n+1}^2}{(S_n+R_{n+1})^2}\mid\mathcal{G}_n\big\}\xrightarrow{a.s.}\frac{s}{m^2}$. Since $Z_n\xrightarrow{a.s.}Z$, it follows that

$$E(Y_{n+1}\mid\mathcal{G}_n)\xrightarrow{a.s.}\;Z(1-Z)^2\frac{q}{m^2}+Z^2(1-Z)\frac{s}{m^2}=V.$$

This concludes the (sketch of the) proof.

Remark 8. In order to $(C_n,D_n)\longrightarrow\mathcal{N}(0,U)\times\mathcal{N}(0,V)$ stably, some of the assumptions of Corollary 7 can be stated in a different form. We mention two (independent) facts.

First, condition (4) can be weakened into uniform integrability of $(B_n+R_n)^2$.

Second, (B_n,R_n) independent of $\mathcal{G}_{n-1}\vee\sigma(X_n)$ can be replaced by the following four conditions:

(i) (B_n,R_n) conditionally independent of X_n given $\mathcal{G}_{n-1}$;

(ii) Condition (4) holds for some $u>4$;

(iii) There are an integer n_0 and a constant $l>0$ such that

$$E\big(B_n\wedge n^{1/4}\mid\mathcal{G}_{n-1}\big)\geq l\text{ and }E\big(R_n\wedge n^{1/4}\mid\mathcal{G}_{n-1}\big)\geq l\;\text{ a.s. whenever }n\geq n_0;$$

(iv) There are random variables m, q, s such that

$$E\big(B_n\mid\mathcal{G}_{n-1}\big)=E\big(R_n\mid\mathcal{G}_{n-1}\big)\xrightarrow{P}m,\quad E\big(B_n^2\mid\mathcal{G}_{n-1}\big)\xrightarrow{P}q,\quad E\big(R_n^2\mid\mathcal{G}_{n-1}\big)\xrightarrow{P}s.$$

Even if in a different framework, conditions similar to (i)-(iv) are in [4].

4.4. **The multicolor case.** To avoid technicalities, we firstly investigated two color urns, but the results in Subsection 4.3 extend to the multicolor case.

An urn contains $a_j>0$ balls of color $j\in\{1,\ldots,d\}$ where $d\geq 2$. Let $X_{n,j}$ denote the indicator of {ball of color j at time n}. In case $X_{n,j}=1$, the ball which has been drawn is replaced together with $A_{n,j}$ more balls of color j. Formally, we assume $\big\{X_{n,j},\,A_{n,j}:n\geq1,\,1\leq j\leq d\big\}$ random variables on the probability space $(\Omega,\mathcal{A},P)$ satisfying

$$(**)\quad X_{n,j}\in\{0,1\},\quad \textstyle\sum_{j=1}^d X_{n,j}=1,\quad A_{n,j}\geq0,$$

$$(A_{n,1},\ldots,A_{n,d})\text{ independent of }\big(A_{k,j},\,X_{k,j},\,X_{n,j}:1\leq k<n,\,1\leq j\leq d\big),$$

$$Z_{n,j}=P\big(X_{n+1,j}=1\mid\mathcal{G}_n\big)=\frac{a_j+\sum_{k=1}^nA_{k,j}X_{k,j}}{\sum_{i=1}^da_i+\sum_{k=1}^n\sum_{i=1}^dA_{k,i}X_{k,i}}\quad\text{a.s.},$$

$$\text{where}\quad\mathcal{G}_0=\{\emptyset,\Omega\},\quad\mathcal{G}_n=\sigma\big(A_{k,j},\,X_{k,j}:1\leq k\leq n,\,1\leq j\leq d\big).$$

Note that

$$Z_{n+1,j} - Z_{n,j} = (1 - Z_{n,j})\,\frac{A_{n+1,j}\,X_{n+1,j}}{S_n + A_{n+1,j}} \;-\; Z_{n,j}\sum_{i\neq j}\frac{A_{n+1,i}\,X_{n+1,i}}{S_n + A_{n+1,i}}$$

$$\text{where}\;\; S_n = \sum_{i=1}^{d} a_i + \sum_{k=1}^{n}\sum_{i=1}^{d} A_{k,i}X_{k,i}.$$

In addition to $(**)$, as in Subsection 4.3, we ask the moment condition

$$\sup_n E\Big\{\big(\sum_{j=1}^{d} A_{n,j}\big)^u\Big\} < \infty \quad \text{for some } u > 2. \tag{5}$$

Further, it is fundamental that

$$EA_{n,j} = EA_{n,1} \quad \text{for each } n \geq 1 \text{ and } 1 \leq j \leq d, \text{ and} \tag{6}$$
$$m := \lim_n EA_{n,1} > 0, \quad q_j := \lim_n EA_{n,j}^2 \quad \text{for each } 1 \leq j \leq d.$$

Fix $1 \leq j \leq d$. Since $EA_{n,i} = EA_{n,1}$ for all n and i, the same calculation as in Subsection 4.3 yields

$$\Big| E\big(Z_{n+1,j} \mid \mathcal{G}_n\big) - Z_{n,j}\Big| \leq \frac{\sum_{i=1}^{d} EA_{n+1,i}^2}{S_n^2} \quad \text{a.s..}$$

Also, $E(S_n^{-p}) = \mathrm{O}(n^{-p})$ for all $p > 0$; see Remark 12. Thus,

$$E\big\{|E\big(Z_{n+1,j} \mid \mathcal{G}_n\big) - Z_{n,j}|^p\big\} = \mathrm{O}(n^{-2p}) \quad \text{for all } p > 0. \tag{7}$$

In particular, $Z_{n,j}$ meets condition (2) so that $Z_{n,j} \xrightarrow{a.s.} Z_{(j)}$ for some random variable $Z_{(j)}$. Define

$$C_{n,j} = \sqrt{n}\,\big(\frac{1}{n}\sum_{k=1}^{n} X_{k,j} \;-\; Z_{n,j}\big) \quad \text{and} \quad D_{n,j} = \sqrt{n}\,\big(Z_{n,j} - Z_{(j)}\big).$$

Next result is quite expected at this point.

Corollary 9. *Suppose conditions $(**)$, (5), (6) hold and fix $1 \leq j \leq d$. Then,*

$$\big(C_{n,j},\, D_{n,j}\big) \longrightarrow \mathcal{N}(0, U_j) \times \mathcal{N}(0, V_j) \quad \textit{stably, where}$$

$$U_j = V_j - Z_{(j)}(1 - Z_{(j)}) \quad \textit{and} \quad V_j = \frac{Z_{(j)}}{m^2}\,\big\{\, q_j\,(1 - Z_{(j)})^2 \,+\, Z_{(j)}\sum_{i\neq j} q_i\, Z_{(i)}\,\big\}.$$

Moreover, $E\big(f(D_{n,j}) \mid \mathcal{G}_n\big) \xrightarrow{a.s.} \mathcal{N}(0, V_j)(f)$ for each $f \in C_b(\mathbb{R})$, that is, $D_{n,j}$ meets condition (3)*.*

Proof. Just repeat the proof of Corollary 7 with $X_{n,j}$ in the place of X_n. □

A vectorial version of Corollary 9 can be obtained with slight effort. Let $\mathcal{N}_d(0, \Sigma)$ denote the d-dimensional Gaussian law with mean vector 0 and covariance matrix Σ and

$$\mathbf{C_n} = \big(C_{n,1}, \ldots, C_{n,d}\big), \quad \mathbf{D_n} = \big(D_{n,1}, \ldots, D_{n,d}\big).$$

Corollary 10. *Suppose conditions (**), (5), (6) hold. Then,*

$$\big(\mathbf{C_n},\,\mathbf{D_n}\big)\longrightarrow\mathcal{N}_d(0,\mathbf{U})\times\mathcal{N}_d(0,\mathbf{V})\quad\textit{stably,}$$

where $\mathbf{U}$, $\mathbf{V}$ *are the* $d\times d$ *matrices with entries* $U_{j,j}=U_j$, $V_{j,j}=V_j$, *and*

$$U_{i,j}=V_{i,j}+Z_{(i)}Z_{(j)},\quad V_{i,j}=\frac{Z_{(i)}Z_{(j)}}{m^2}\,\big\{\sum_{h=1}^d q_hZ_{(h)}-q_i-q_j\big\}\quad\textit{for } i\neq j.$$

Moreover, $E\big(f(\mathbf{D_n})\mid\mathcal{G}_n\big)\overset{a.s.}{\longrightarrow}\mathcal{N}_d(0,\mathbf{V})(f)$ *for each* $f\in C_b(\mathbb{R}^d)$.

Proof. Given a linear functional $\phi:\mathbb{R}^d\to\mathbb{R}$, it suffices to see that

$$\phi(\mathbf{C_n})\longrightarrow\mathcal{N}_d(0,\mathbf{U})\circ\phi^{-1}\quad\text{stably, and}$$

$$E\big(g\circ\phi(\mathbf{D_n})\mid\mathcal{G}_n\big)\overset{a.s.}{\longrightarrow}\mathcal{N}_d(0,\mathbf{V})(g\circ\phi)\quad\text{for each } g\in C_b(\mathbb{R}).$$

To this purpose, note that

$$\phi(\mathbf{C_n})=\sqrt{n}\,\Big\{\,\frac{1}{n}\sum_{k=1}^n\phi(X_{k,1},\dots,X_{k,d})\;-\;E\big(\phi(X_{n+1,1},\dots,X_{n+1,d})\mid\mathcal{G}_n\big)\,\Big\},$$

$$\phi(\mathbf{D_n})=\sqrt{n}\,\big\{\,E\big(\phi(X_{n+1,1},\dots,X_{n+1,d})\mid\mathcal{G}_n\big)\;-\;\phi(Z_{(1)},\dots,Z_{(d)})\,\big\},$$

and repeat again the proof of Corollary 7 with $\phi(X_{n,1},\dots,X_{n,d})$ in the place of X_n. □

A nice consequence of Corollary 10 is that

$$\mathbf{W_n}=\mathbf{C_n}+\mathbf{D_n}\longrightarrow\mathcal{N}_d(0,\mathbf{U}+\mathbf{V})\quad\text{stably}$$

provided conditions (**)-(5)-(6) hold, where $\mathbf{W_n}=\big(W_{n,1},\dots,W_{n,d}\big)$ and $W_{n,j}=\sqrt{n}\,\big(\frac{1}{n}\sum_{k=1}^nX_{k,j}\,-\,Z_{(j)}\big)$.

Finally, we briefly mention a possible development of the above material. Suppose condition (6) is turned into

$$EA_{n,j}=EA_{n,1}\quad\text{whenever } n\geq 1 \text{ and } 1\leq j\leq d_0,$$

$$\liminf_n\big(EA_{n,1}-EA_{n,j}\big)>0\quad\text{whenever } j>d_0,$$

$$m:=\lim_n EA_{n,1}>0,\quad q_j:=\lim_n EA_{n,j}^2\quad\text{whenever } 1\leq j\leq d_0,$$

for some integer $1\leq d_0\leq d$. Roughly speaking, this means that some colors (those labelled from d_0+1 to d) are dominated by the others. So far, we dealt with $d_0=d$ but the case $d_0<d$ is not unusual in applications. The main trouble is that condition (7) may fail when $d_0<d$. It is still possible to get a CLT but one should decide how to handle dominated colors. There are essentially two options.

One is to make assumptions on dominated colors. A classical assumption is

$$\limsup_n\frac{EA_{n,j}}{EA_{n,1}}<\frac{1}{2}\quad\text{for each } j>d_0.$$

Under this condition, using some ideas from [15], an analogous of Corollary 9 can be proved for $(C_{n,j},\,D_{n,j})$ with $j=1,\dots,d_0$.

The other option is to neglect dominated colors, that is, to replace $Z_{n,j}$ and $\frac{1}{n}\sum_{k=1}^nX_{k,j}$ by

$$Z_{n,j}^*=\frac{a_j+\sum_{k=1}^nA_{k,j}X_{k,j}}{\sum_{i=1}^{d_0}a_i+\sum_{k=1}^n\sum_{i=1}^{d_0}A_{k,i}X_{k,i}}\quad\text{and}\quad M_{n,j}^*=\frac{\sum_{k=1}^nX_{k,j}}{1+\sum_{k=1}^n\sum_{i=1}^{d_0}X_{k,i}}.$$

Again, an analogous of Corollary 9 can be shown for

$$C^*_{n,j} = \sqrt{n}\,\big(M^*_{n,j} - Z^*_{n,j}\big) \quad \text{and} \quad D^*_{n,j} = \sqrt{n}\,\big(Z^*_{n,j} - Z_{(j)}\big), \quad j = 1, \ldots, d_0.$$

The case $d_0 < d$ will be deepened in a forthcoming paper.

APPENDIX

In the notation of Subsection 4.3, let $S_n = b + r + \sum_{k=1}^n \big(B_k X_k + R_k(1 - X_k)\big)$.

Lemma 11. *Under the assumptions of Corollary 7,*

$$\frac{n}{S_n} \longrightarrow \frac{1}{m} \quad \textit{a.s. and in } L_p \textit{ for all } p > 0.$$

Proof. Let $Y_n = B_n X_n + R_n(1 - X_n)$. By $(*)$ and $EB_{n+1} = ER_{n+1}$,

$$\begin{aligned} E\big(Y_{n+1} \mid \mathcal{G}_n\big) &= EB_{n+1}\, E\big(X_{n+1} \mid \mathcal{G}_n\big) \;+\; ER_{n+1}\, E\big(1 - X_{n+1} \mid \mathcal{G}_n\big) \\ &= Z_n\, EB_{n+1} \;+\; (1 - Z_n)\, EB_{n+1} = EB_{n+1} \overset{a.s.}{\longrightarrow} m. \end{aligned}$$

Since $m > 0$, Lemma 3 implies $\frac{n}{S_n} = \frac{1}{S_n/n} \overset{a.s.}{\longrightarrow} \frac{1}{m}$. To conclude the proof, it suffices to see that $E(S_n^{-p}) = \mathrm{O}(n^{-p})$ for all $p > 0$. Given $c > 0$, define

$$S_n^{(c)} = \sum_{k=1}^n \big\{X_k\big(B_k \wedge c - E(B_k \wedge c)\big) + (1 - X_k)\big(R_k \wedge c - E(R_k \wedge c)\big)\big\}.$$

By a classical martingale inequality (see e.g. Lemma 1.5 of [14])

$$P\big(|S_n^{(c)}| > x\big) \leq 2\,\exp\big(-x^2/2\,c^2\,n\big) \quad \text{for all } x > 0.$$

Since $EB_n = ER_n \longrightarrow m$ and both (B_n), (R_n) are uniformly integrable (as $\sup_n \big(EB_n^2 + ER_n^2\big) < \infty$), there are $c > 0$ and an integer n_0 such that

$$m_n := \sum_{k=1}^n \min\big\{E(B_k \wedge c),\, E(R_k \wedge c)\big\} > n\frac{m}{2} \quad \text{for all } n \geq n_0.$$

Fix one such $c > 0$ and let $l = m/4 > 0$. For every $p > 0$, one can write

$$\begin{aligned} E(S_n^{-p}) &= p \int_{b+r}^{\infty} t^{-p-1} P(S_n < t)\, dt \\ &\leq \frac{p}{(b+r)^{p+1}} \int_{b+r}^{b+r+n\,l} P(S_n < t)\, dt \;+\; p \int_{b+r+n\,l}^{\infty} t^{-p-1}\, dt. \end{aligned}$$

Clearly, $p \int_{b+r+n\,l}^{\infty} t^{-p-1}\, dt = (b + r + n\,l)^{-p} = \mathrm{O}(n^{-p})$. Further, for each $n \geq n_0$ and $t < b + r + n\,l$, since $m_n > n\,2\,l$ one obtains

$$\begin{aligned} P(S_n < t) &\leq P\big(S_n^{(c)} < t - b - r - m_n\big) \leq P\big(S_n^{(c)} < t - b - r - n\,2\,l\big) \\ &\leq P\big(|S_n^{(c)}| > b + r + n\,2\,l - t\big) \leq 2\,\exp\big(-(b + r + n\,2\,l - t)^2/2\,c^2\,n\big). \end{aligned}$$

Hence, $\int_{b+r}^{b+r+n\,l} P(S_n < t)\, dt \leq n\,2\,l\,\exp\big(-n\,\frac{l^2}{2\,c^2}\big)$ for every $n \geq n_0$, so that $E(S_n^{-p}) = \mathrm{O}(n^{-p})$. □

Remark 12. As in Subsection 4.4, let $S_n = \sum_{i=1}^d a_i + \sum_{k=1}^n \sum_{i=1}^d A_{k,i} X_{k,i}$. Under conditions $(**)$-(5)-(6), the previous proof still applies to such S_n. Thus, $\frac{n}{S_n} \longrightarrow \frac{1}{m}$ a.s. and in L_p for all $p > 0$.

Proof of Corollary 7. By Lemma 1, it is enough to prove $C_n \to \mathcal{N}(0,U)$ stably and D_n meets condition (3). Recall from Subsection 4.3 that

$$Z_{n+1} - Z_n = \frac{(1-Z_n)\,X_{n+1}\,B_{n+1} \;-\; Z_n\,(1-X_{n+1})\,R_{n+1}}{S_{n+1}}$$

$$\text{and}\quad E\big\{|E(Z_{n+1}\mid \mathcal{G}_n) - Z_n|^p\big\} = \,\mathrm{O}(n^{-2p}) \quad \text{for all } p>0.$$

In particular, condition (2) holds and $\sum_k \sqrt{k}\,E\Big|\,E(Z_k\mid\mathcal{G}_{k-1}) - Z_{k-1}\Big| < \infty$.

"D_n meets condition (3)". By (4) and Lemma 11,

$$E\big\{|Z_{k-1}-Z_k|^u\big\} \le E\Big\{\frac{(B_k+R_k)^u}{S_{k-1}^u}\Big\} = E\big\{(B_k+R_k)^u\big\}\,E(S_{k-1}^{-u}) = \mathrm{O}(k^{-u}).$$

Thus, $E\big\{\sup_k \sqrt{k}\,|Z_{k-1}-Z_k|\big\}^u \le \sum_k k^{\frac{u}{2}} E\big\{|Z_{k-1}-Z_k|^u\big\} < \infty$ as $u>2$. In view of Remark 4, it remains only to prove that

$$n\sum_{k\ge n}(Z_{k-1}-Z_k)^2 = n\sum_{k\ge n}\Big(\frac{(1-Z_{k-1})X_kB_k}{S_k} - \frac{Z_{k-1}(1-X_k)R_k}{S_k}\Big)^2$$
$$= n\sum_{k\ge n}\frac{(1-Z_{k-1})^2X_kB_k^2}{(S_{k-1}+B_k)^2} + n\sum_{k\ge n}\frac{Z_{k-1}^2(1-X_k)R_k^2}{(S_{k-1}+R_k)^2}$$

converges a.s. to $V = Z(1-Z)\,\frac{(1-Z)q+Zs}{m^2}$. It is enough to show that

$$n\sum_{k\ge n}\frac{(1-Z_{k-1})^2X_kB_k^2}{(S_{k-1}+B_k)^2} \xrightarrow{a.s.} Z(1-Z)^2\frac{q}{m^2} \;\text{ and }\; n\sum_{k\ge n}\frac{Z_{k-1}^2(1-X_k)R_k^2}{(S_{k-1}+R_k)^2} \xrightarrow{a.s.} Z^2(1-Z)\frac{s}{m^2}\,.$$

These two limit relations can be proved by exactly the same argument, and thus we just prove the first one. Let $U_n = B_n I_{\{B_n\le\sqrt{n}\}}$. Since $P(B_n>\sqrt{n}) \le n^{\frac{-u}{2}} EB_n^u$, condition (4) yields $P(B_n\ne U_n, \text{ i.o.}) = 0$. Hence, it suffices to show that

$$n\sum_{k\ge n}\frac{(1-Z_{k-1})^2X_kU_k^2}{(S_{k-1}+U_k)^2} \xrightarrow{a.s.} Z(1-Z)^2\frac{q}{m^2}. \tag{8}$$

Let $Y_n = n^2\frac{(1-Z_{n-1})^2X_nU_n^2}{(S_{n-1}+U_n)^2}$. Since (B_n^2) is uniformly integrable, $EU_n^2 \longrightarrow q$. Furthermore, $\frac{S_n}{n} \xrightarrow{a.s.} m$ and $Z_n \xrightarrow{a.s.} Z$. Thus,

$$E\big(Y_{n+1}\mid\mathcal{G}_n\big) \le (1-Z_n)^2(n+1)^2E\Big(\frac{X_{n+1}U_{n+1}^2}{S_n^2}\mid\mathcal{G}_n\Big)$$
$$= Z_n(1-Z_n)^2\frac{(n+1)^2}{S_n^2}\,EU_{n+1}^2 \xrightarrow{a.s.} Z(1-Z)^2\frac{q}{m^2} \quad\text{and}$$
$$E\big(Y_{n+1}\mid\mathcal{G}_n\big) \ge (1-Z_n)^2(n+1)^2E\Big(\frac{X_{n+1}U_{n+1}^2}{(S_n+\sqrt{n+1})^2}\mid\mathcal{G}_n\Big)$$
$$= Z_n(1-Z_n)^2\frac{(n+1)^2}{(S_n+\sqrt{n+1})^2}\,EU_{n+1}^2 \xrightarrow{a.s.} Z(1-Z)^2\frac{q}{m^2}.$$

By Lemma 3, for getting relation (8), it suffices that $\sum_n \frac{EY_n^2}{n^2} < \infty$. Since

$$\frac{EU_n^4}{n^2} \le \frac{E\big\{B_n^2 I_{\{B_n^2\le\sqrt{n}\}}\big\}}{n^{\frac{3}{2}}} + \frac{E\big\{B_n^2 I_{\{B_n^2>\sqrt{n}\}}\big\}}{n} \le \frac{EB_n^2}{n^{\frac{3}{2}}} + \frac{EB_n^u}{n^{1+\frac{u-2}{4}}}\,,$$

condition (4) implies $\sum_n \frac{EU_n^4}{n^2} < \infty$. By Lemma 11, $E(S_{n-1}^{-4}) = \mathrm{O}(n^{-4})$. Then,

$$\sum_n \frac{EY_n^2}{n^2} \leq \sum_n n^2 E\Big\{\frac{U_n^4}{S_{n-1}^4}\Big\} = \sum_n n^2 E(S_{n-1}^{-4})\, EU_n^4 \leq c \sum_n \frac{EU_n^4}{n^2} < \infty$$

for some constant c. Hence, condition (8) holds.

"$C_n \to \mathcal{N}(0,U)$ stably". By Theorem 2, it suffices to check conditions (a) and (b) with $U = Z(1-Z)\big(\frac{(1-Z)q+Zs}{m^2} - 1\big)$. As to (a), since $E\big\{|Z_{k-1} - Z_k|^u\big\} = \mathrm{O}(k^{-u})$,

$$\big(n^{-\frac{1}{2}}\, E\big\{\max_{1\leq k\leq n} k\,|Z_{k-1} - Z_k|\big\}\big)^u \leq n^{-\frac{u}{2}} \sum_{k=1}^n k^u E\big\{|Z_{k-1} - Z_k|^u\big\} \longrightarrow 0.$$

We next prove condition (b). After some algebra, one obtains

$$E\big\{(X_n - Z_{n-1})(Z_{n-1} - Z_n) \mid \mathcal{G}_{n-1}\big\} = -Z_{n-1}(1 - Z_{n-1})\, E\Big\{\frac{B_n}{S_{n-1} + B_n} \mid \mathcal{G}_{n-1}\Big\} +$$

$$+\, Z_{n-1}^2(1 - Z_{n-1})\, E\Big\{\frac{B_n}{S_{n-1} + B_n} - \frac{R_n}{S_{n-1} + R_n} \mid \mathcal{G}_{n-1}\Big\} \quad \text{a.s.}.$$

Arguing as in the first part of this proof ("D_n meets condition (3)"),

$$n\, E\Big\{\frac{B_n}{S_{n-1} + B_n} \mid \mathcal{G}_{n-1}\Big\} \xrightarrow{a.s.} 1 \quad \text{and} \quad n\, E\Big\{\frac{R_n}{S_{n-1} + R_n} \mid \mathcal{G}_{n-1}\Big\} \xrightarrow{a.s.} 1.$$

Thus, $n\, E\big\{(X_n - Z_{n-1})(Z_{n-1} - Z_n) \mid \mathcal{G}_{n-1}\big\} \xrightarrow{a.s.} -Z(1 - Z)$. Further,

$$E\big\{\big(X_n - Z_{n-1}\big)^2 \mid \mathcal{G}_{n-1}\big\} = Z_{n-1} - Z_{n-1}^2 \xrightarrow{a.s.} Z(1 - Z).$$

Thus, Lemma 3 implies

$$\frac{1}{n}\sum_{k=1}^n (X_k - Z_{k-1})^2 \,+\, \frac{2}{n}\sum_{k=1}^n k\,(X_k - Z_{k-1})\,(Z_{k-1} - Z_k) \xrightarrow{a.s.} -Z(1 - Z).$$

Finally, write $\frac{1}{n}\sum_{k=1}^n k^2(Z_{k-1} - Z_k)^2 = \frac{1}{n}\sum_{k=1}^n k^2\big\{\frac{(1-Z_{k-1})^2 X_k B_k^2}{(S_{k-1}+B_k)^2} + \frac{Z_{k-1}^2(1-X_k)R_k^2}{(S_{k-1}+R_k)^2}\big\}$. By Lemma 3 and the same truncation technique used in the first part of this proof, $\frac{1}{n}\sum_{k=1}^n k^2(Z_{k-1} - Z_k)^2 \xrightarrow{a.s.} V$. Squaring,

$$\frac{1}{n}\sum_{k=1}^n \big\{X_k - Z_{k-1} + k(Z_{k-1} - Z_k)\big\}^2 \xrightarrow{a.s.} V - Z(1 - Z) = U,$$

that is, condition (b) holds. This concludes the proof. □

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