# Computation of Regularization Parameters using the Fourier Coefficients.

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**Abstract**

In the solution of ill-posed problems by means of regularization methods, a crucial issue is the computation of the regularization parameter. In this work we focus on the Truncated Singular Value Decomposition (TSVD) and Tikhonov method and we define a method for computing the regularization parameter based on the behavior of Fourier coefficients. We compute a safe index for truncating the TSVD and consequently a value for the regularization parameter of the Tikhonov method. An extensive numerical experimentation is carried out on the Hansen's Regtool [3] test problems and the results confirm the effectiveness and robustness of the method proposed.

*Keywords: Singular Value Decomposition, Regularization methods, Tikhonov method, Ill-posed problems, Integral equations.*
*Classification: 65R30, 65R32, 65F22.*

## 1 Introduction

In this work discrete ill-posed problems are solved by means of regularization methods based on Singular Value Decomposition (SVD). By discrete ill-posed problems we intend a class of least squares problems:

$$\min_x \|Hf - g\|_2, \quad H \in \mathbb{R}^{m\times n}, g \in \mathbb{R}^m, \quad m \geq n$$

where the matrix $H$ is ill conditioned with smoothly decaying singular values. Such matrices are obtained by the discretization of ill-posed problems such as Fredholm first kind integral equations that model many imaging problems. These problems are very sensitive to small data perturbations such as noise present in the data $g$, usually represented by a random process, or errors in the matrix $H$.

A great variety of direct and iterative regularization methods can be found in the literature, they mainly replace the ill-posed problem with a nearby well

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posed one which is less sensitive to perturbations. One well known method is the Tikhonov method which computes a regularized solution $x_\lambda$ as the minimizer of the discrete Tikhonov functional:

$$\min_x \|Hf-g\|_2^2+\lambda\|Lf\|_2^2$$

where $\lambda > 0$ is the regularization parameter and $L$ is either the identity matrix or a well conditioned discrete approximation to some derivative operator.

For practical implementation of such methods it is necessary to compute a suitable value of the regularization parameter $\lambda$.

In this work, we focus on small-medium size problems where the Singular Value Decomposition (SVD) can be efficiently computed. Analyzing the behavior of the Fourier coefficients in terms of the Discrete Picard Condition [2], we define a rule for computing the index for truncating the singular values in the Truncated Singular Value Decomposition (TSVD) method, and consequently a value for the regularization parameter $\lambda$ of the Tikhonov method.

The effectiveness and robustness of our rule is evaluated by comparison one of the most widely used and successful method i.e. the Generalized Cross Validation method.

In section 2 the properties of the TSVD and Tikhonov methods are reported with respect to the value of the regularization parameter. In section 3 the rule proposed is reported and analyzed. Finally in section 4 an extensive numerical experimentation is carried out on the Hansen's Regtool [3] test problems.

## 2 The Discrete Picard Condition

In this section we recall the main properties of the regularization methods with respect to the SVD.

Let $f_{LS}$ be the minimal norm solution of the least squares problem:

$$\min_f \|Hf-g\|^2, \quad H\in\mathbb{C}^{m\times n},\ g\in\mathbb{C}^m,\ (m\geq n) \tag{1}$$

let $r$ be the rank of the matrix $H$ and let $H = USV^*$ be the SVD of H. It is possible to characterize $f_{LS}$ in terms of the SVD of the matrix $H$:

$$f_{LS}=\sum_{i=1}^{n}\frac{u_i^*g}{\sigma_i}v_i$$

where $\sigma_i$ are the singular values, $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ and $\sigma_{r+1} = \cdots = \sigma_n = 0$, $u_i$ $(i = 1,\dots,m)$ and $v_i$ $(i = 1,\dots,n)$ are the columns of the unitary matrices $U$ and $V$ respectively. When (1) is the discretization of an ill-posed problem (such as the Fredholm fist kind integral equation) $f_{LS}$ is dominated by the errors present in $g$ (data noise) and $H$, therefore it is necessary to introduce regularization methods for filtering out the components that cause the errors. A regularized solution can be computed as follows:

$$f_{reg}=\sum_{i=1}^{n}\Phi(\sigma_i)\frac{u_i^*g}{\sigma_i}v_i \tag{2}$$

where $\Phi$ is the *filter function* with values in the interval $[0, 1]$. By means of the *filter function* we characterize the following regularization methods:

- Truncated Singular Value Decomposition method (TSVD) where $\Phi \equiv \Phi_k$ is an ideal low pass filter:

$$\Phi_k(\sigma_i) = \begin{cases} 1 & \text{if } \sigma_i >= \sigma_k (\text{i.e. } \; i <= k) \\ 0 & \text{otherwise} \end{cases}$$

  where the threshold value $\sigma_k$ is defined by the *regularization parameter* $k$.

- Tikhonov regularization method (Tikh) where $\Phi \equiv \Phi_\alpha$ is a least squares smoothing filter:

$$\Phi_\alpha(\sigma_i) = \frac{\sigma_i^2}{\sigma_i^2 + \alpha^2}$$

  and the value of the *regularization parameter* $\alpha$ defines the behavior of the filter function as follows:

  – $\alpha << \sigma_i \;\Rightarrow\; \Phi_\alpha(\sigma_i) \approx 1$ i.e. the $i$-th term in (2) is almost unchanged;

  – $\alpha >> \sigma_i \;\Rightarrow\; \Phi_\alpha(\sigma_i) \approx 0$ i.e. the $i$-th term in (2) is damped to zero.

  The function $\Phi_\alpha$ behaves like a smooth low pass filter of the solution terms, damping the components relative to the singular values smaller than $\alpha$.

In both cases, the proper determination of the regularization parameter is a crucial issue: too small values of $\alpha$ or $\sigma_k$ (i.e. large values of $k$ in TSVD) produce a noisy solution dominated by the errors present in the system, while large values of $\alpha$ (i.e. small values of $k$ in TSVD) produce a smooth blurred solution.

The decay rate of the Fourier coefficients $|u_i^* g|$ with respect to the singular values $\sigma_i$ is a key point to determine the properties of the regularized solution $f_{reg}$.

The Discrete Picard Condition [2] is a property of the ratio between the Fourier coefficients $|u_i^* g|$ and the singular values $\sigma_i$, which is stated as follows.

> *If the Fourier coefficients $|u_i^* g|$ decay to zero faster than the singular values, then the regularized solution $f_{reg}$ has the same regularity properties as the exact solution $f$.*

The check of this property may be not easy since $|u_i^* g|$ may have a non monotonic behavior. A rule based on the moving geometric mean has been proposed in [2]. Further analysis of the Fourier coefficients concerns the terms that can be included in the regularized solution. As pointed out in [1] (pg. 70, 71):

> *the number of terms $k'$ that can be safely included in the solution $f_{reg}$ is such that:*
>
> $$k' \leq \min(i_H, i_g) \qquad (3)$$

*where $i_H$ is the index at which $\sigma_i$ begin to level off and $i_g$ is the index at which $|u_i^* g|$ begin to level off. The value $i_H$ is proportional to the error present in the matrix $H$ (i.e. model error) while the value $i_g$ is proportional to the errors present in the data $g$ (i.e. noise error).*

An example of this issue is shown in figure 1 reporting the plots of the singular values $\sigma_i$ (blue line), $|u_i^* g|$ (green crosses) and $|u_i^* g|/\sigma_i$ (red o) for the Foxgood (1(a), 1(b)) and Phillips (1(c), 1(d)) test problems taken from Hansens's Regtool, both in absence and presence of noise on the data $g$ (see section 4 for the details).

We observe that in the Foxgood test problem $25 \leq i_H \leq 35$ while $i_g \simeq 30$ for noiseless case and $i_g < 10$ at high noise values. For this test problem, the amount of noise present in the data determines the value $k'$ in (3) and we can observe a quite evident minimum value in the curve $|u_i^* g|/\sigma_i$.

In the Phillips test problem we can observe that the singular values do not level off at machine epsilon ($\sim 2.2e^{-16}$) but have a smooth decreasing behavior toward a minimum value $> 10^{-5}$, so we have $i_H \simeq n$ where $n$ is the dimension of the problem ($n = 100$ in this example). In the noiseless case, the coefficients $|u_i^* g|$ can be splitted into two successions: one that level off at machine epsilon (we can interpret this as numerically zero values) and the other that decreases faster than the singular values curve and fulfils the Picard condition. In this case the value of $k'$ in (3) is given by $i_H$ and practically no truncation is needed in the TSVD (figure 1(c)). In the case of high noise we observe that the values $|u_i^* g|$ settle down around the noise value corresponding to increasing values of the curve $|u_i^* g|/\sigma_i$ (figure 1(d)).

If the value at which the singular values level off is very small (machine epsilon) then the index $k'$ is determined by the value of $i_g$ (error data) otherwise the model error is dominant in the low noise cases.

# 3 The MinMax Rule

We define here a criterion for computing the values of the regularization parameter based on the terms that can be safely included in the $f_{reg}$ solution.

**Definition 1** (MinMax Rule)**.** Let $\varphi_i$ be the succession of the solution coefficients

$$\varphi_i = \frac{|u_i^* g|}{\sigma_i}, \quad i = 1, \dots, n$$

and let us separate the terms $\varphi_i$ in two sets:

$$\rho_1 = \{\varphi_i : \varphi_i \geq \sigma_i\}, \quad \rho_2 = \{\varphi_i : \varphi_i < \sigma_i\}$$

Let $K_1$ be the number of elements in $\rho_1$, $K_2$ be the number of elements in $\rho_2$ and $S_{min} = \min_i(\sigma_i)$, compute the index $k^*$ such that:

case I If $S_{min} \leq 1.e - 13$ or $K_2 = 0$ (i.e. the sequence $\rho_2$ is empty) then $k^*$ is the index relative to the minimum value in the sequence $\rho_1$.

case II Otherwise $k^*$ is the smallest index among the last 5 elements in $\rho_2$ such that

$$\frac{|u_{k^*+1}^* g|}{\sigma_{k^*+1}} > \sigma_{k^*+1}$$

Then $k^*$ is chosen as the regularization parameter of the TSVD method and

$$\alpha^* = \sigma(k^*)$$

is the regularization parameter for the Tikhonov method.

When the minimum singular value ($S_{min}$) is close to the machine epsilon (or the sequence $\rho_2$ is empty), than $\rho_1$ is used to determine the regularization index. In this case the data noise is assumed to be predominant with respect to the model errors and the sequence $\rho_1$ shows a quite evident minimum value.

An example of this case is reported in figure 2 relative to the Foxgood test problem. In this case only the sequence $\rho_1$ is computed since $K_2 = 0$ as shown in figure 2(a) for the noiseless case and in figure 2(c) for the high noise case. In the noiseless case $\rho_1$ has a spurious minimum (figure 2(a)) that causes the error to be greater to the optimal one (2(b)). In the noisy case (figure 2(c)) we observe that the computed value is very close to the optimal (figure 2(d)).

In the case II of the MinMax Rule the model errors are predominant and the greatest indices in $\rho_2$ are included in the TSVD provided that they are contiguous (i.e. the successive element does not belong to $\rho_1$). This situation is shown in figures 3 obtained with the Phillips test problem. In the noiseless the computed index is not close to the optimal one (figure 3(a)), but we can see that the relative error is constant so the solution is effectively as good as the optimal one (3(b)). In the noisy case we observe that the computed value is very close to the optimal (figures 3(c), 3(d)).

# 4 Numerical Experiments

In this section we report the results obtained by several test problems with the method (1) and the Generalized Cross Validation method implemented in Hansen's Regtool (`GCV` function) [3].

The first test is carried out on the 17 Regtools test problems relative to 1D integral equations. We subdivide the test problems into two groups named G1 and G2 respectively. The group G1 is constituted by the test problems with low errors present in the matrix (i.e. with singular values that level-off at machine epsilon or lower values). Table (1) reports the names of the test and the call to the specific matlab function used to generate the test. The group G2 is constituted by the test problems with quite large errors in the matrix (i.e. with singular values decay to a minimum value much larger than machine epsilon). The names of the test problems and function calls are reported in table (2).

Each problem defines the matrix $H$, the true solution $f_{true}$ and the r.h.s. $g$ of the least squares problem min $\|Hf_{true} - g\|$. White gaussian noise is added to the data $g$ with variance $\delta$ and the problem $min\|Hf - g^\delta\|$ is solved by means of the TSVD and Tikhonov methods. The noisy data $g^\delta$ are obtained as follows:

$$g^\delta = g + \delta\|g\|\underline{\eta}$$

where $\underline{\eta}$ is a unitary vector of gaussian distributed random values. All the tests are performed with dimension $n = 100$.

In tables 3,4 and 7 are reported the results obtained by the TSVD method: the first two columns are relative to the test problems and report the name of the test function (**Test** column) and the value of $\delta$ (**Noise** column). The results reported in columns **Best** are obtained by computing the optimal solution: the column *Index* is the number of terms used in the TSVD solution (2) and the column *Error* reports the relative error computed with respect to the solution $f_{true}$.

The solution obtained by MinMax Rule is reported in columns **MinMax_tsvd** where the columns *Index* and *Error* report the number of terms determined by the MinMax Rule and the relative error respectively. Analogously the columns **GCV_tsvd** report the number of terms (columns *Index*) and the relative error (column *Error*) computed by the `GCV` function.

Observing the last column of tables 3, 4, 5 and 6 (relative to the test problems G1) we can see that GCV presents several critical situations with errors greater than 10 while MinMax Rule has always maximum error $< 1$.

This behavior is not present in tables 7 and 8 showing that test problems like G2 are better than G1 for GCV method. Analyzing the errors we can see some cases where MinMax Rule and GCV have maximum error in the interval $[1, 6]$ for the TSVD method (table 7), while the GCV has better performance for the Tikhonov method (table 8).

The efficiency of each method $Ef$ can be evaluated as the ratio between the optimal error $E_b$ (i.e. the relative error obtained using the optimal regularization parameter) and the error computed by each method $E_m$:

$$Ef = \frac{E_b}{E_m}$$

The average behavior of each method with respect to the sets of test problems $G1$ and $G2$ is analyzed by computing the smallest efficiency value $Ef_{min}$ and the mean efficiency value $Ef_{mean}$ as reported in table 9. Moreover the same table reports the percentages of tests where each method reached the maximum efficiency value ($Ef = 1$) (column *Opt*%) and those where the method reported a relative error greater than 10 (column *Fail*%).

The MinMax rule performs better than GCV in terms of average efficiency for the tsvd method with both tests G1 and G2 and for the Tikhonov method with tests G1. The percentage of fails reported in these cases are 29 and 13 showing that GCV is not always able to compute an acceptable value for the regularization parameter.

For test problems G2 with the Tikhonov method we can see that GCV has better efficiency values with respect to the MinMax rule. Both methods are however robust for these test problems since they do not report failures.

The minimum efficiency value reported in table9 is relative to the test `ilaplace_ex1` in absence of noise, solved with GCV_tsvd method. The solution and errors are reported in figure4 show very bad reconstructions, the small value in the efficiency parameter is due to a very large value of the relative error obtained by the regularization method (figure 4(b)) and the computed regularization parameter is larger than the optimal.

The minimum efficiency value obtained by the MinMax rule is relative to the case test `heat` in absence of noise, solved tsvd method. In this case we can observe a good quality solution (figure 5(a)) since the small efficiency value is due to a very small value of the relative error in the optimal solution and the value of the regularization parameter is smaller than the optimal.

We can therefore conclude that gcv is a very efficient method for computing the tikhonov regularization parameter in test problems with predominant the errors in the matrix (G2 group) returning in many cases values very close to the optimal. In all the remaining cases MinMax rule is more robust, preventing failures and giving always good results.

## 5 Conclusions

A fast and robust method for computing the regularization parameter is proposed. The method is based on the properties of the Fourier coefficients obtained by computing the SVD. This strategy can be extended quite naturally to general regularization problems of the kind:

$$\min_f \|Hf - g\|^2 + \|Lf\|^2, \quad L \neq I$$

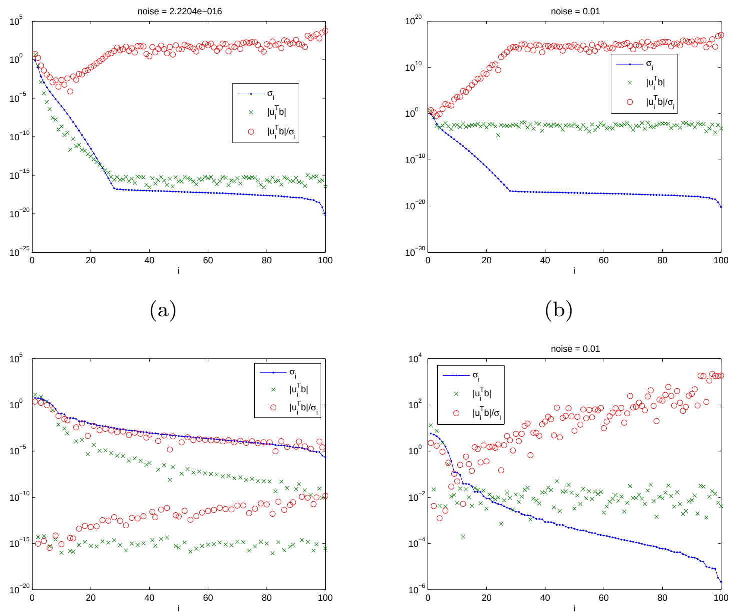
by means of the Generalized Singular Value Decomposition (GSVD).

## References


[1] P. C. Hansen, *Rank–deficient and discrete ill–posed problems*, SIAM, 1997.

[2] P.C.Hansen, *The discrete Picard condition for discrete ill–posed problems*, BIT **30** (1990), 658–672580.

[3] ______, *Regularization Tools: A Matlab package for analysis and solution of discrete ill–posed problems*, Numeric. Algor. **6** (1994), 1–35.




(a) (b) (c) (d)

Figure 1: (a)(b) Foxgood test problem: Coefficients, singular values and Fourier coefficients: (a) noiseless case (b) high noise = 1.e-2; (c),(d) Phillips test problem: Coefficients, singular values and Fourier coefficients: (c) noiseless case (d) high noise = 1.e-2;

| **Test** | `Regtool function` |
|---|---|
| Shaw | `shaw(n)` |
| Heat | `heat(n)` |
| Baart | `baart(n)` |
| iLaplace_ex1 | `i_laplace(n,1)` |
| iLaplace_ex2 | `i_laplace(n,2)` |
| iLaplace_ex3 | `i_laplace(n,3)` |
| Foxgood | `foxgood(n)` |
| Wing | `wing(n)` |
| Spikes | `spikes(n)` |
| Gravity_ex1 | `gravity(n,1)` |
| Gravity_ex2 | `gravity(n,2)` |
| Gravity_ex3 | `gravity(n,3)` |

Table 1: Group G1 test problems: $n = 100$

| **Test** | `Regtool function` |
|---|---|
| Phillips | `phillips(n)` |
| Deriv2_ex1 | `deriv2(n,1)` |
| Deriv2_ex2 | `deriv2(n,2)` |
| Deriv2_ex3 | `deriv2(n,3)` |

Table 2: Group G2 test problems: $n = 100$

| **Test** | **Noise** | **Best** | | **MinMax_tsvd** | | **GCV_tsvd** | |
|---|---|---|---|---|---|---|---|
| | | *Index* | *Error* | *Index* | *Error* | *Index* | *Error* |
| Shaw | 0.0e+000 | 19 | 3.53e-004 | 14 | 3.65e-003 | 45 | **1.58e+001** |
| Shaw | 1.0e-005 | 10 | 2.72e-002 | 8 | 4.72e-002 | 9 | 3.21e-002 |
| Shaw | 1.0e-003 | 7 | 4.87e-002 | 8 | 8.81e-002 | 99 | **5.38e+014** |
| Shaw | 1.0e-002 | 7 | 1.03e-001 | 5 | 1.47e-001 | 7 | 1.03e-001 |
| Heath | 0.0e+000 | 97 | 3.04e-011 | 56 | 1.04e-002 | 99 | **5.42e+006** |
| Heath | 1.0e-005 | 55 | 1.14e-002 | 50 | 1.35e-002 | 58 | 1.29e-002 |
| Heath | 1.0e-003 | 24 | 4.39e-002 | 30 | 4.58e-002 | 29 | 4.56e-002 |
| Heath | 1.0e-002 | 23 | 1.19e-001 | 21 | 1.27e-001 | 17 | 1.31e-001 |
| Baart | 0.0e+000 | 11 | 1.63e-002 | 10 | 1.93e-002 | 64 | 4.81e+000 |
| Baart | 1.0e-005 | 5 | 5.25e-002 | 5 | 5.25e-002 | 5 | 5.25e-002 |
| Baart | 1.0e-003 | 5 | 9.26e-002 | 5 | 9.26e-002 | 7 | **1.58e+003** |
| Baart | 1.0e-002 | 3 | 1.67e-001 | 3 | 1.67e-001 | 3 | 1.67e-001 |
| Foxgood | 0.0e+000 | 7 | 2.47e-004 | 13 | 8.27e-004 | 38 | **3.66e+001** |
| Foxgood | 1.0e-005 | 5 | 1.56e-003 | 5 | 1.56e-003 | 4 | 2.24e-003 |
| Foxgood | 1.0e-003 | 3 | 7.86e-003 | 3 | 7.86e-003 | 15 | **8.55e+004** |
| Foxgood | 1.0e-002 | 2 | 3.33e-002 | 3 | 9.20e-002 | 2 | 3.33e-002 |
| Wing | 0.0e+000 | 7 | 3.11e-001 | 3 | 5.95e-001 | 10 | 4.62e+000 |
| Wing | 1.0e-005 | 5 | 4.24e-001 | 3 | 5.95e-001 | 4 | 4.38e-001 |
| Wing | 1.0e-003 | 2 | 5.95e-001 | 3 | 5.95e-001 | 2 | 5.95e-001 |
| Wing | 1.0e-002 | 2 | 5.95e-001 | 3 | 6.30e-001 | 2 | 5.95e-001 |
| Spikes | 0.0e+000 | 21 | 3.70e-001 | 9 | 8.38e-001 | 27 | 7.78e+000 |
| Spikes | 1.0e-005 | 12 | 7.79e-001 | 9 | 8.38e-001 | 12 | 7.79e-001 |
| Spikes | 1.0e-003 | 10 | 8.15e-001 | 7 | 8.56e-001 | 99 | **3.56e+014** |
| Spikes | 1.0e-002 | 8 | 8.39e-001 | 9 | 8.41e-001 | 6 | 8.56e-001 |

Table 3: TSVD method with G1 test problems. Columns **Best**: regularization parameter and minimum relative error. Columns **MinMax_tsvd** regularization parameter and relative error of MinMax rule. Columns **GCV_tsvd** regularization parameter and relative error of `gcv` function.

| Test | Noise | Best | | MinMax_tsvd | | GCV_tsvd | |
|---|---|---|---|---|---|---|---|
| | | *Index* | *Error* | *Index* | *Error* | *Index* | *Error* |
| iLaplace_ex1 | 0.0e+000 | 25 | 4.07e-006 | 26 | 4.45e-006 | 98 | **1.45e+014** |
| iLaplace_ex1 | 1.0e-005 | 13 | 4.32e-002 | 14 | 5.04e-002 | 13 | 4.32e-002 |
| iLaplace_ex1 | 1.0e-003 | 9 | 1.08e-001 | 7 | 1.27e-001 | 16 | 1.20e+002 |
| iLaplace_ex1 | 1.0e-002 | 6 | 1.48e-001 | 7 | 1.73e-001 | 6 | 1.48e-001 |
| iLaplace_ex2 | 0.0e+000 | 31 | 7.14e-001 | 31 | 7.14e-001 | 30 | 7.16e-001 |
| iLaplace_ex2 | 1.0e-005 | 16 | 7.69e-001 | 15 | 7.71e-001 | 14 | 7.70e-001 |
| iLaplace_ex2 | 1.0e-003 | 10 | 7.95e-001 | 11 | 8.08e-001 | 16 | **6.81e+001** |
| iLaplace_ex2 | 1.0e-002 | 8 | 8.08e-001 | 7 | 8.15e-001 | 7 | 8.15e-001 |
| iLaplace_ex3 | 0.0e+000 | 24 | 1.40e-007 | 24 | 1.40e-007 | 23 | 2.42e-007 |
| iLaplace_ex3 | 1.0e-005 | 11 | 2.71e-003 | 11 | 2.71e-003 | 10 | 4.57e-003 |
| iLaplace_ex3 | 1.0e-003 | 8 | 2.00e-002 | 8 | 2.00e-002 | 16 | 1.37e+002 |
| iLaplace_ex3 | 1.0e-002 | 5 | 7.65e-002 | 8 | 1.56e-001 | 5 | 7.65e-002 |
| iLaplace_ex4 | 0.0e+000 | 25 | 7.29e-001 | 25 | 7.29e-001 | 37 | **4.01e+002** |
| iLaplace_ex4 | 1.0e-005 | 16 | 7.65e-001 | 15 | 7.67e-001 | 14 | 7.66e-001 |
| iLaplace_ex4 | 1.0e-003 | 10 | 7.91e-001 | 11 | 8.03e-001 | 16 | **6.49e+001** |
| iLaplace_ex4 | 1.0e-002 | 8 | 8.06e-001 | 8 | 8.06e-001 | 7 | 8.10e-001 |
| Gravity_ex1 | 0.0e+000 | 35 | 1.20e-006 | 36 | 1.59e-006 | 96 | **6.40e+001** |
| Gravity_ex1 | 1.0e-005 | 14 | 2.72e-003 | 15 | 3.91e-003 | 13 | 3.53e-003 |
| Gravity_ex1 | 1.0e-003 | 10 | 1.65e-002 | 10 | 1.65e-002 | 8 | 2.03e-002 |
| Gravity_ex1 | 1.0e-002 | 7 | 3.00e-002 | 8 | 4.42e-002 | 6 | 4.13e-002 |
| Gravity_ex2 | 0.0e+000 | 45 | 2.19e-003 | 27 | 4.39e-003 | 43 | 2.33e-003 |
| Gravity_ex2 | 1.0e-005 | 16 | 1.01e-002 | 14 | 1.03e-002 | 13 | 1.04e-002 |
| Gravity_ex2 | 1.0e-003 | 9 | 2.39e-002 | 10 | 2.46e-002 | 16 | 3.17e+000 |
| Gravity_ex2 | 1.0e-002 | 7 | 5.02e-002 | 10 | 1.83e-001 | 9 | 1.80e-001 |
| Gravity_ex3 | 0.0e+000 | 48 | 3.14e-002 | 36 | 3.79e-002 | 47 | 3.16e-002 |
| Gravity_ex3 | 1.0e-005 | 17 | 5.79e-002 | 16 | 5.98e-002 | 15 | 5.99e-002 |
| Gravity_ex3 | 1.0e-003 | 9 | 7.50e-002 | 10 | 7.52e-002 | 16 | 2.97e+000 |
| Gravity_ex3 | 1.0e-002 | 8 | 9.78e-002 | 10 | 1.86e-001 | 8 | 9.78e-002 |

Table 4: TSVD method with G1 test problems. Columns **Best**: regularization parameter and minimum relative error. Columns **MinMax_tsvd** regularization parameter and relative error of MinMax rule. Columns **GCV_tsvd** regularization parameter and relative error of `gcv` function.

| **Test** | **noise** | **Best** | | **MinMax_tikh** | | **GCV_tikh** | |
|---|---|---|---|---|---|---|---|
| | | value | Error | value | Error | value | Error |
| Shaw | 0.0e+000 | 5.44e-011 | 5.33e-004 | 6.43e-008 | 3.62e-003 | 1.12e-012 | 3.17e-004 |
| Shaw | 1.0e-005 | 4.34e-003 | 4.55e-002 | 4.34e-003 | 4.55e-002 | 1.79e-004 | 3.12e-002 |
| Shaw | 1.0e-003 | 4.34e-003 | 4.86e-002 | 4.34e-003 | 4.86e-002 | 1.22e-003 | 1.41e-001 |
| Shaw | 1.0e-002 | 5.90e-002 | 1.19e-001 | 5.90e-002 | 1.19e-001 | 1.94e-002 | 6.32e-002 |
| Heath | 0.0e+000 | 1.25e-006 | 1.59e-003 | 7.82e-005 | 8.89e-003 | 3.49e-014 | 3.06e-011 |
| Heath | 1.0e-005 | 1.01e-004 | 1.01e-002 | 1.32e-004 | 1.08e-002 | 4.16e-005 | 1.29e-002 |
| Heath | 1.0e-003 | 1.34e-003 | 3.89e-002 | 9.98e-004 | 4.27e-002 | 7.72e-004 | 5.14e-002 |
| Heath | 1.0e-002 | 3.10e-003 | 1.11e-001 | 3.10e-003 | 1.11e-001 | 2.75e-003 | 1.21e-001 |
| Baart | 0.0e+000 | 4.56e-012 | 1.94e-002 | 4.56e-012 | 1.94e-002 | 1.23e-014 | 3.37e-001 |
| Baart | 1.0e-005 | 2.36e-004 | 7.41e-002 | 2.36e-004 | 7.41e-002 | 3.63e-005 | 5.66e-002 |
| Baart | 1.0e-003 | 2.36e-004 | 1.46e-001 | 2.36e-004 | 1.46e-001 | 1.72e-007 | **1.20e+003** |
| Baart | 1.0e-002 | 7.16e-002 | 2.17e-001 | 7.16e-002 | 2.17e-001 | 1.17e-002 | 1.31e-001 |
| Foxgood | 0.0e+000 | 2.77e-006 | 3.14e-004 | 2.72e-008 | 7.92e-004 | 3.08e-015 | 2.61e-001 |
| Foxgood | 1.0e-005 | 2.57e-004 | 2.42e-003 | 2.57e-004 | 2.42e-003 | 2.19e-004 | 2.62e-003 |
| Foxgood | 1.0e-003 | 6.60e-003 | 1.44e-002 | 6.60e-003 | 1.44e-002 | 1.09e-009 | **7.75e+004** |
| Foxgood | 1.0e-002 | 6.60e-003 | 6.19e-002 | 6.60e-003 | 6.19e-002 | 1.00e-002 | 4.97e-002 |
| Wing | 0.0e+000 | 3.71e-007 | 3.55e-001 | 1.11e-003 | 5.95e-001 | 1.70e-015 | 3.14e-001 |
| Wing | 1.0e-005 | 1.11e-003 | 5.95e-001 | 1.11e-003 | 5.95e-001 | 1.84e-006 | 4.37e-001 |
| Wing | 1.0e-003 | 1.11e-003 | 5.94e-001 | 1.11e-003 | 5.94e-001 | 3.52e-011 | **5.79e+005** |
| Wing | 1.0e-002 | 1.11e-003 | 6.05e-001 | 1.11e-003 | 6.05e-001 | 2.00e-003 | 5.97e-001 |
| Spikes | 0.0e+000 | 4.41e-012 | 6.17e-001 | 1.02e-002 | 8.35e-001 | 6.82e-014 | 5.00e-001 |
| Spikes | 1.0e-005 | 2.42e-003 | 8.15e-001 | 1.02e-002 | 8.35e-001 | 2.72e-005 | 7.86e-001 |
| Spikes | 1.0e-003 | 3.75e-002 | 8.42e-001 | 1.22e-001 | 8.53e-001 | 2.10e-003 | 8.05e-001 |
| Spikes | 1.0e-002 | 1.02e-002 | 8.60e-001 | 1.02e-002 | 8.60e-001 | 7.06e-002 | 8.51e-001 |

Table 5: Tikhonov method with G1 test problems. Columns **Best**: regularization parameter and minimum relative error. Columns **MinMax_tikh** regularization parameter and relative error of MinMax rule. Columns **GCV_tikh** regularization parameter and relative error of `gcv` function.

| Test | noise | Best | | MinMax_tikh | | GCV_tikh | |
|---|---|---|---|---|---|---|---|
| | | value | Error | value | Error | value | Error |
| iLaplace_ex1 | 0.0e+000 | 3.52e-012 | 5.76e-006 | 3.52e-012 | 5.76e-006 | 7.80e-012 | 4.87e-006 |
| iLaplace_ex1 | 1.0e-005 | 2.76e-005 | 5.30e-002 | 2.76e-005 | 5.30e-002 | 6.16e-005 | 4.64e-002 |
| iLaplace_ex1 | 1.0e-003 | 2.51e-002 | 1.27e-001 | 2.51e-002 | 1.27e-001 | 2.18e-006 | **8.68e+001** |
| iLaplace_ex1 | 1.0e-002 | 2.51e-002 | 1.50e-001 | 2.51e-002 | 1.50e-001 | 3.04e-002 | 1.52e-001 |
| iLaplace_ex2 | 0.0e+000 | 1.87e-017 | 1.54e+000 | 1.87e-017 | 1.54e+000 | 9.02e-015 | 7.20e-001 |
| iLaplace_ex2 | 1.0e-005 | 9.57e-006 | 7.70e-001 | 9.57e-006 | 7.70e-001 | 1.62e-005 | 7.70e-001 |
| iLaplace_ex2 | 1.0e-003 | 5.89e-004 | 7.94e-001 | 5.89e-004 | 7.94e-001 | 2.18e-006 | **4.91e+001** |
| iLaplace_ex2 | 1.0e-002 | 2.51e-002 | 8.15e-001 | 2.51e-002 | 8.15e-001 | 1.25e-002 | 8.07e-001 |
| iLaplace_ex3 | 0.0e+000 | 9.58e-011 | 1.31e-007 | 9.58e-011 | 1.31e-007 | 2.05e-012 | 1.56e-005 |
| iLaplace_ex3 | 1.0e-005 | 5.89e-004 | 4.20e-003 | 5.89e-004 | 4.20e-003 | 4.58e-004 | 4.44e-003 |
| iLaplace_ex3 | 1.0e-003 | 1.02e-002 | 3.16e-002 | 1.02e-002 | 3.16e-002 | 2.18e-006 | **9.90e+001** |
| iLaplace_ex3 | 1.0e-002 | 2.51e-002 | 1.11e-001 | 1.02e-002 | 1.62e-001 | 2.68e-002 | 1.09e-001 |
| iLaplace_ex4 | 0.0e+000 | 1.91e-011 | 7.29e-001 | 1.91e-011 | 7.29e-001 | 9.02e-015 | 2.06e+000 |
| iLaplace_ex4 | 1.0e-005 | 9.57e-006 | 7.66e-001 | 9.57e-006 | 7.66e-001 | 1.37e-005 | 7.66e-001 |
| iLaplace_ex4 | 1.0e-003 | 5.89e-004 | 7.93e-001 | 5.89e-004 | 7.93e-001 | 2.18e-006 | **4.68e+001** |
| iLaplace_ex4 | 1.0e-002 | 1.02e-002 | 8.02e-001 | 1.02e-002 | 8.02e-001 | 1.06e-002 | 8.03e-001 |
| Gravity_ex1 | 0.0e+000 | 2.47e-010 | 1.41e-006 | 1.22e-010 | 1.54e-006 | 5.08e-010 | 1.36e-006 |
| Gravity_ex1 | 1.0e-005 | 1.07e-003 | 3.71e-003 | 1.07e-003 | 3.71e-003 | 2.02e-003 | 3.11e-003 |
| Gravity_ex1 | 1.0e-003 | 3.03e-002 | 1.92e-002 | 3.03e-002 | 1.92e-002 | 4.20e-004 | 2.03e+000 |
| Gravity_ex1 | 1.0e-002 | 1.12e-001 | 4.22e-002 | 1.12e-001 | 4.22e-002 | 1.18e-001 | 4.05e-002 |
| Gravity_ex2 | 0.0e+000 | 3.45e-012 | 2.34e-003 | 2.77e-007 | 4.17e-003 | 1.24e-012 | 2.27e-003 |
| Gravity_ex2 | 1.0e-005 | 2.10e-003 | 1.04e-002 | 2.10e-003 | 1.04e-002 | 1.88e-003 | 1.04e-002 |
| Gravity_ex2 | 1.0e-003 | 3.03e-002 | 2.16e-002 | 3.03e-002 | 2.16e-002 | 4.20e-004 | 2.17e+000 |
| Gravity_ex2 | 1.0e-002 | 5.86e-002 | 7.82e-002 | 3.03e-002 | 1.54e-001 | 8.33e-002 | 4.30e-002 |
| Gravity_ex3 | 0.0e+000 | 4.00e-013 | 3.20e-002 | 5.02e-010 | 3.75e-002 | 1.01e-013 | 3.16e-002 |
| Gravity_ex3 | 1.0e-005 | 5.41e-004 | 5.83e-002 | 5.41e-004 | 5.83e-002 | 4.57e-004 | 5.80e-002 |
| Gravity_ex3 | 1.0e-003 | 3.03e-002 | 7.14e-002 | 3.03e-002 | 7.14e-002 | 4.20e-004 | 2.04e+000 |
| Gravity_ex3 | 1.0e-002 | 3.03e-002 | 1.75e-001 | 3.03e-002 | 1.75e-001 | 1.11e-001 | 1.05e-001 |

Table 6: Tikhonov method with G1 test problems. Columns **Best**: regularization parameter and minimum relative error. Columns **MinMax_tikh** regularization parameter and relative error of MinMax rule. Columns **GCV_tikh** regularization parameter and relative error of `gcv` function.

| **Test** | **Noise** | **Best** | | **MinMax_tsvd** | | **GCV_tsvd** | |
|---|---|---|---|---|---|---|---|
| | | *Index* | *Error* | *Index* | *Error* | *Index* | *Error* |
| Phillips | 0.0e+000 | 99 | 7.57e-004 | 84 | 7.57e-004 | 99 | 7.57e-004 |
| Phillips | 1.0e-005 | 21 | 1.67e-003 | 24 | 2.24e-003 | 21 | 1.67e-003 |
| Phillips | 1.0e-003 | 11 | 1.22e-002 | 11 | 1.22e-002 | 13 | 3.25e-002 |
| Phillips | 1.0e-002 | 9 | 2.17e-002 | 10 | 2.71e-002 | 7 | 2.50e-002 |
| Deriv2_ex1 | 0.0e+000 | 100 | 3.43e-012 | 100 | 3.43e-012 | 73 | 6.50e-002 |
| Deriv2_ex1 | 1.0e-005 | 100 | 4.84e-002 | 86 | 5.80e-002 | 62 | 8.07e-002 |
| Deriv2_ex1 | 1.0e-003 | 25 | 1.81e-001 | 23 | 1.84e-001 | 99 | 4.46e+000 |
| Deriv2_ex1 | 1.0e-002 | 11 | 2.47e-001 | 26 | 1.04e+000 | 12 | 3.26e-001 |
| Deriv2_ex2 | 0.0e+000 | 100 | 8.29e-006 | 100 | 8.29e-006 | 73 | 6.04e-002 |
| Deriv2_ex2 | 1.0e-005 | 97 | 5.21e-002 | 64 | 7.43e-002 | 61 | 7.66e-002 |
| Deriv2_ex2 | 1.0e-003 | 25 | 1.78e-001 | 32 | 2.52e-001 | 99 | 4.83e+000 |
| Deriv2_ex2 | 1.0e-002 | 11 | 2.33e-001 | 32 | 1.61e+000 | 12 | 3.34e-001 |
| Deriv2_ex3 | 0.0e+000 | 99 | 5.37e-004 | 86 | 5.38e-004 | 99 | 5.37e-004 |
| Deriv2_ex3 | 1.0e-005 | 33 | 2.72e-003 | 30 | 2.92e-003 | 29 | 2.91e-003 |
| Deriv2_ex3 | 1.0e-003 | 7 | 1.86e-002 | 2 | 1.20e-001 | 99 | 5.63e+000 |
| Deriv2_ex3 | 1.0e-002 | 5 | 3.88e-002 | 2 | 1.20e-001 | 3 | 4.87e-002 |

Table 7: Tsvd method with G2 test problems. Columns **Best**: regularization parameter and minimum relative error. Columns **MinMax_tsvd** regularization parameter and relative error of MinMax rule. Columns **GCV_tsvd** regularization parameter and relative error of `gcv` function.

| **Test** | **noise** | **Best** | | **MinMax_tikh** | | **GCV_tikh** | |
|---|---|---|---|---|---|---|---|
| | | value | Error | value | Error | value | Error |
| Phillips | 0.0e+000 | 4.26e-005 | 7.56e-004 | 4.26e-005 | 7.56e-004 | 2.26e-006 | 7.57e-004 |
| Phillips | 1.0e-005 | 5.92e-003 | 2.04e-003 | 5.13e-003 | 2.20e-003 | 4.07e-003 | 2.63e-003 |
| Phillips | 1.0e-003 | 9.48e-002 | 1.23e-002 | 9.48e-002 | 1.23e-002 | 2.71e-002 | 3.60e-002 |
| Phillips | 1.0e-002 | 1.18e-001 | 5.31e-002 | 1.18e-001 | 5.31e-002 | 1.18e-001 | 5.32e-002 |
| Deriv2_ex1 | 0.0e+000 | 8.33e-006 | 3.00e-002 | 8.33e-006 | 3.00e-002 | 8.49e-006 | 3.07e-002 |
| Deriv2_ex1 | 1.0e-005 | 8.36e-006 | 4.45e-002 | 9.58e-006 | 4.64e-002 | 8.49e-006 | 4.47e-002 |
| Deriv2_ex1 | 1.0e-003 | 1.83e-004 | 1.76e-001 | 1.83e-004 | 1.76e-001 | 1.55e-004 | 1.87e-001 |
| Deriv2_ex1 | 1.0e-002 | 5.91e-004 | 2.79e-001 | 1.42e-004 | 1.41e+000 | 8.08e-004 | 2.41e-001 |
| Deriv2_ex2 | 0.0e+000 | 8.33e-006 | 2.81e-002 | 8.33e-006 | 2.81e-002 | 8.49e-006 | 2.87e-002 |
| Deriv2_ex2 | 1.0e-005 | 8.49e-006 | 4.40e-002 | 1.84e-005 | 5.98e-002 | 8.49e-006 | 4.40e-002 |
| Deriv2_ex2 | 1.0e-003 | 1.31e-004 | 2.19e-001 | 9.11e-005 | 3.00e-001 | 1.70e-004 | 1.88e-001 |
| Deriv2_ex2 | 1.0e-002 | 5.91e-004 | 2.88e-001 | 9.11e-005 | 2.75e+000 | 9.76e-004 | 2.31e-001 |
| Deriv2_ex3 | 0.0e+000 | 9.58e-006 | 3.91e-004 | 9.58e-006 | 3.91e-004 | 8.49e-006 | 4.09e-004 |
| Deriv2_ex3 | 1.0e-005 | 1.05e-004 | 3.13e-003 | 1.05e-004 | 3.13e-003 | 6.60e-005 | 4.85e-003 |
| Deriv2_ex3 | 1.0e-003 | 4.04e-003 | 3.18e-002 | 2.53e-002 | 1.19e-001 | 6.38e-004 | 3.09e-002 |
| Deriv2_ex3 | 1.0e-002 | 1.12e-002 | 7.55e-002 | 2.53e-002 | 1.20e-001 | 2.51e-003 | 4.92e-002 |

Table 8: Tikhonov method with G2 test problems. Columns **Best**: regularization parameter and minimum relative error. Columns **MinMax_tikh** regularization parameter and relative error of MinMax rule. Columns **GCV_tikh** regularization parameter and relative error of `gcv` function.

| **Test** | **Method** | $E_{min}$ | $E_{mean}$ | $Opt\%$ | $Fail\%$ |
|---|---|---|---|---|---|
| G1 | **MinMax_tsvd** | 3.54e-009 | 0.80 | 21 | 0 |
| | **GCV_tsvd** | 2.22e-020 | 0.59 | 23 | 27 |
| | **MinMax_tikh** | 7.21e-004 | 0.80 | 19 | 0 |
| | **GCV_tikh** | 5.34e-008 | 0.69 | 19 | 13 |
| G2 | **MinMax_tsvd** | 1.4e-001 | 0.72 | 19 | 0 |
| | **GCV_tsvd** | 5.48e-011 | 0.55 | 19 | 0 |
| | **MinMax_tikh** | 1.72e-002 | 0.57 | 13 | 0 |
| | **GCV_tikh** | 1.68e-002 | 0.70 | 56 | 0 |

Table 9: Efficiency values on the different test problems: $Ef_{min}$ is the minimum efficiency value; $Ef_{mean}$ is the mean efficiency value; $Opt\%$ is the percentage of tests where each method reached the maximum efficiency value and $Fail\%$ is percentage of cases with a relative error greater than 10 .

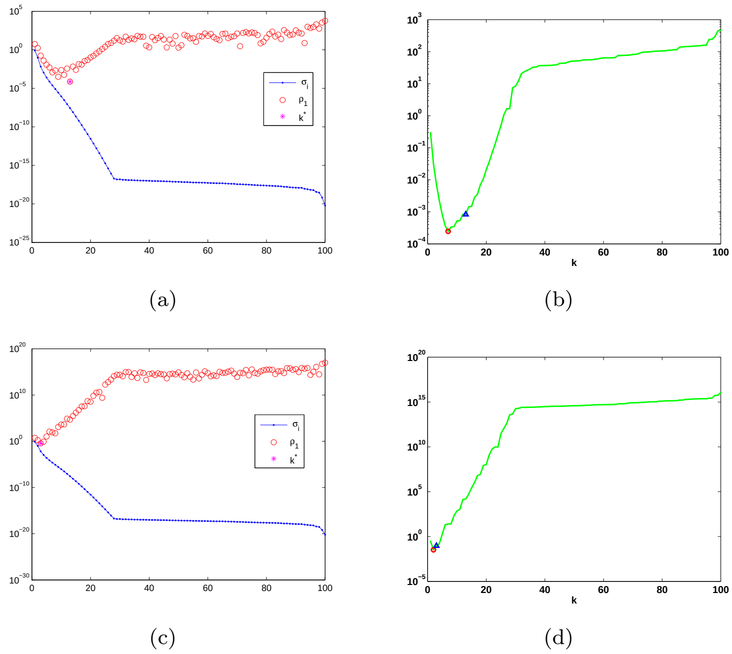
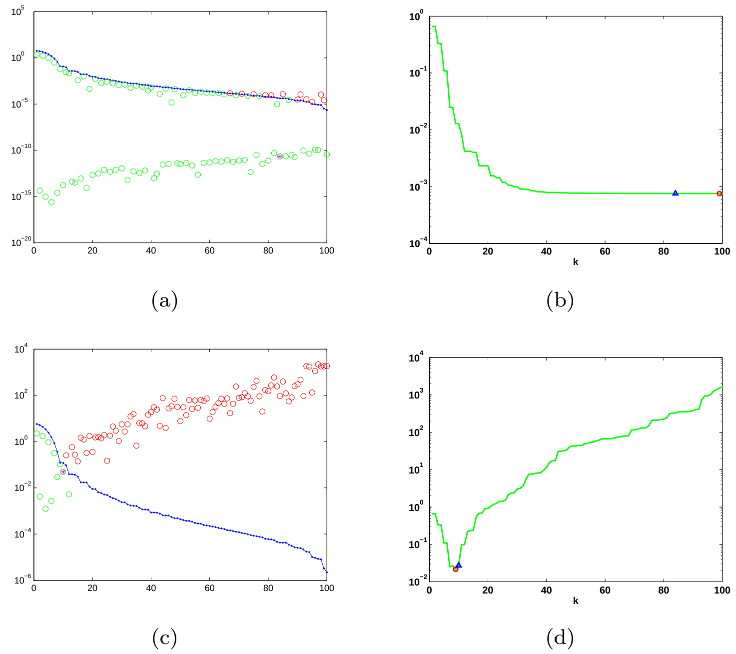
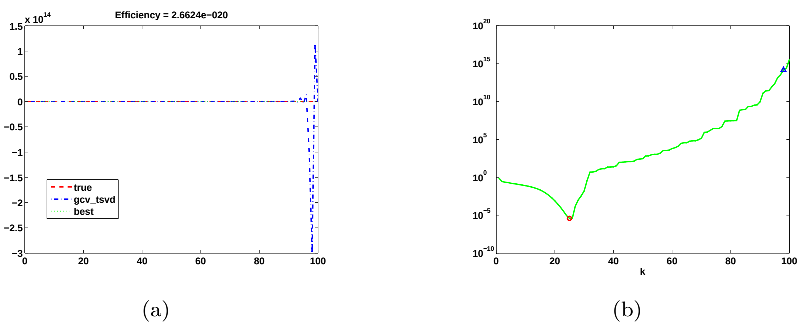



Figure 2: Foxgood test problem (a), (c) $\rho_1$ succession (red), $\rho_2$ succession (green);(b) (d) Relative error: read o (optimal solution), blue triangle (TSVD MinMax solution); (a)(b) noiseless case (c)(d) high noise = 1.e-2



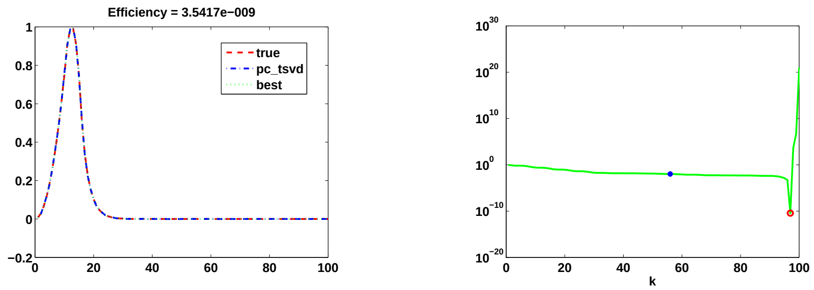
(a) (b) (c) (d)

Figure 3: Phillips test problem (a), (c) $\rho_1$ succession (red), ($\rho_2$ succession (green) is empty); (b) (d) Relative error: red o (optimal solution), blue triangle (TSVD MinMax solution); (a)(b) noiseless case (c)(d) high noise = 1.e-2




(a) (b)

Figure 4: Worse efficiency values for gcv method: (a) Solution of ilaplace_ex1 test problem, no noise (b) Error curve for different values of the regularization parameters. Red dot: optimal parameter. Blue triangle: solution computed by tsvd with GCV parameter.




(a) (b)

Figure 5: Worse efficiency values for MinMax rule: (a) Solution of heat test problem, no noise. (b) Error curve for different values of the regularization parameters. Red dot: optimal parameter. Blue triangle: solution computed by tsvd with regularization parameter given by MinMax rule.