MULTIDIMENSIONAL PERSISTENT HOMOLOGY IS STABLE

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Abstract. Multidimensional persistence studies topological features of shapes by analyzing the lower level sets of vector-valued functions. The rank invariant completely determines the multidimensional analogue of persistent homology groups. We prove that multidimensional rank invariants are stable with respect to function perturbations. More precisely, we construct a distance between rank invariants such that small changes of the function imply only small changes of the rank invariant. This result can be obtained by assuming the function to be just continuous. Multidimensional stability opens the way to a stable shape comparison methodology based on multidimensional persistence.

Introduction

The study of the topology of data is attracting more and more attention from the mathematical community. This challenging subject of research is motivated by the large amount of scientific contexts where it is required to deal with qualitative geometric information. Indeed, the topological approach allows us to greatly reduce the complexity of the data by focusing the analysis just on their relevant part. This research area is widely discussed in [2, 4].

In this line of research, persistent homology has been revealed to be a key mathematical method for studying the topology of data, with applications in an increasing number of fields, ranging from shape description (e.g., [6, 7, 22, 27]) to data simplification [14] and hole detection in sensor networks [12]. Recent surveys on the topic include [13, 19, 28]. Persistent homology describes topological events occurring through the filtration of a topological space X. Filtrations are usually expressed by real functions $\varphi : X \to \mathbb{R}$. The main idea underlying this approach is that the most important piece of information enclosed in geometrical data is usually the one that is "*persistent*" with respect to small changes of the function defining the filtration.

The analysis of persistent topological events in the lower level sets of the functions (e.g., creation, merging, cancellation of connected components, tunnels, voids) is important for capturing a global description of the data under study. These events can be encoded in the form of a parameterized version of the Betti numbers, called a *rank invariant* [5] (already known in the literature as a *size function* for the 0th homology [16, 21, 27]).

Until recently, research on persistence has mainly focused on the use of scalar functions for describing filtrations. The extent to which this theory can be generalized to a situation in which two or more functions characterize the data is currently under investigation [1, 5]. This generalization to vector-valued functions is usually known as the *Multidimensional Persistence Theory*, where the adjective multidimensional refers to the fact that filtrating functions are vector-valued, and has no

2000 *Mathematics Subject Classification.* Primary 55N05; Secondary 55U99, 68T10.

Key words and phrases. Multidimensional persistence, persistence diagram, size function, Čech homology, foliation.

Research carried out under the auspices of INdAM-GNSAGA. The last author partially carried out the research within the activity of ARCES "E. De Castro", University of Bologna.

connections with the dimensionality of the space under study. The use of vector-valued filtrating functions, introduced in [18], would enable the analysis of richer data structures.

One of the most important open questions in current research about multidimensional persistent homology concerns the *stability problem*. In plain words, we need to determine how the computation of invariants in this theory is affected by the unavoidable presence of noise and approximation errors. Indeed, it is clear that any data acquisition is subject to perturbation and, if persistent homology were not stable, then distinct computational investigations of the same object could produce completely different results. Obviously, this would make it impossible to use such a mathematical setting in real applications.

In this paper we succeed in solving this problem and giving a positive answer about the stability of multidimensional persistent homology (Theorem 1.3). More precisely, we prove that the rank invariants of nearby vector-valued filtrating functions are "close to each other" in the sense expressed by a suitable matching distance. We point out that, until now, the stability of persistent homology has been studied only for scalar-valued filtrating functions.

Our stability result for the multidimensional setting requires us to use some recently developed ideas to investigate *Multidimensional Size Theory* [1]. The proof of multidimensional stability is obtained by reduction to the one-dimensional case, via an appropriate foliation in half-planes of the domain of the multidimensional rank invariants, and the definition of a family of suitable and possibly non-tame continuous scalar-valued filtrating functions. Our result follows by proving the stability property in this one-dimensional case. We observe that the results obtained in [9], for tame scalar functions, and in [8], under some finiteness assumptions, cannot be applied here. Indeed, we underline that our stability result requires the functions only to be continuous. Our generalization of one-dimensional stability from tame to continuous functions is a positive answer to the question risen in [9].

Our main point is that all the proofs carried out in [17] to analyze size functions via diagrams of points, the so called persistence diagrams, as well as those in [10, 11] to prove the stability of size functions, can be developed in a completely parallel way for rank invariants associated with continuous functions. As a consequence, we do not repeat the technical details of all the proofs, whenever the arguments are completely analogous to those used in the literature about size functions. For the same reason, we refer the reader to [1] (see also [3]) for the proofs of the Reduction Theorem 3.2 and Theorem 3.3 (Stability w.r.t. Function Perturbations) used to deduce our main result about the stability of multidimensional persistent homology.

Finally, we emphasize that throughout this paper we work with Čech homology over a field. In the framework of persistence, Čech homology has already been considered by Robbins in [24, 25]. In our opinion, a strong motivation for using Čech homology in studying persistent homology groups is that, having the continuity axiom, it ensures that the rank invariant can be completely described by a persistence diagram, unlike the singular and simplicial theories that guarantee that such a description is complete only outside a set of vanishing measure, as explained in Section 2.2.1. We point out that the Čech approach to homology theory is currently being investigated for computational purposes [23].

1. Basic Definitions and the Main Result

In this paper, each considered space is assumed to be triangulable, i.e. there is a finite simplicial complex with homeomorphic underlying space. In particular, triangulable spaces are always compact and metrizable.

The following relations $\preceq$ and $\prec$ are defined in $\mathbb{R}^n$: for $\vec{u} = (u_1, \ldots, u_n)$ and $\vec{v} = (v_1, \ldots, v_n)$, we say $\vec{u} \preceq \vec{v}$ (resp. $\vec{u} \prec \vec{v}$) if and only if $u_i \leq v_i$ (resp. $u_i < v_i$) for every index $i = 1, \ldots, n$. Moreover, $\mathbb{R}^n$ is endowed with the usual max-norm: $\|(u_1, u_2, \ldots, u_n)\|_\infty = \max_{1 \leq i \leq n} |u_i|$.

We shall use the following notations: Δ^+ will be the open set $\{(\vec{u}, \vec{v}) \in \mathbb{R}^n \times \mathbb{R}^n : \vec{u} \prec \vec{v}\}$. Given a triangulable space X, for every n-tuple $\vec{u} = (u_1, \ldots, u_n) \in \mathbb{R}^n$ and for every function $\vec{\varphi} : X \to \mathbb{R}^n$, we shall denote by $X\langle \vec{\varphi} \preceq \vec{u} \rangle$ the set $\{x \in X : \varphi_i(x) \leq u_i,\ i = 1, \ldots, n\}$.

The definition below extends the concept of the persistent homology group to a multidimensional setting.

Definition 1.1. Let $\pi_k^{(\vec{u},\vec{v})} : \check{H}_k(X\langle \vec{\varphi} \preceq \vec{u} \rangle) \to \check{H}_k(X\langle \vec{\varphi} \preceq \vec{v} \rangle)$ be the homomorphism induced by the inclusion map $\pi^{(\vec{u},\vec{v})} : X\langle \vec{\varphi} \preceq \vec{u} \rangle \hookrightarrow X\langle \vec{\varphi} \preceq \vec{v} \rangle$ with $\vec{u} \preceq \vec{v}$, where $\check{H}_k$ denotes the kth Čech homology group. If $\vec{u} \prec \vec{v}$, the image of $\pi_k^{(\vec{u},\vec{v})}$ is called the *multidimensional kth persistent homology group of* $(X, \vec{\varphi})$ *at* $(\vec{u}, \vec{v})$, and is denoted by $\check{H}_k^{(\vec{u},\vec{v})}(X, \vec{\varphi})$.

In other words, the group $\check{H}_k^{(\vec{u},\vec{v})}(X, \vec{\varphi})$ contains all and only the homology classes of cycles born before $\vec{u}$ and still alive at $\vec{v}$.

For details about Čech homology, the reader can refer to [15].

In what follows, we shall work with coefficients in a field $\mathbb{K}$, so that homology groups are vector spaces, and hence torsion-free. Therefore, they can be completely described by their rank, leading to the following definition (cf. [5]).

Definition 1.2 (***k*th rank invariant**)**.** Let X be a triangulable space, and $\vec{\varphi} : X \to \mathbb{R}^n$ a continuous function. Let $k \in \mathbb{Z}$. The *kth rank invariant of the pair* $(X, \vec{\varphi})$ *over a field* $\mathbb{K}$ is the function $\rho_{(X,\vec{\varphi}),k} : \Delta^+ \to \mathbb{N} \cup \{\infty\}$ defined as

$$\rho_{(X,\vec{\varphi}),k}(\vec{u}, \vec{v}) = \operatorname{rank} \pi_k^{(\vec{u},\vec{v})}.$$

By the rank of a homomorphism we mean the dimension of its image. We shall prove in Lemma 2.1 that, under our assumptions on X and $\vec{\varphi}$, the value ∞ is never attained.

The main goal of this paper is to prove the following result.

Theorem 1.3 (Multidimensional Stability Theorem)**.** *Let X be a triangulable space. For every $k \in \mathbb{Z}$, there exists a distance D_{match} on the set $\{\rho_{(X,\vec{\varphi}),k} \,|\, \vec{\varphi} : X \to \mathbb{R}^n$ continuous$\}$ such that*

$$D_{match}\left(\rho_{(X,\vec{\varphi}),k}, \rho_{(X,\vec{\psi}),k}\right) \leq \max_{x \in X} \|\vec{\varphi}(x) - \vec{\psi}(x)\|_\infty.$$

The construction of D_{match} will be given in the course of the proof of the theorem.

1.1. **Idea of the proof.** Since the proof of Theorem 1.3 is quite technical, for the reader's convenience, we now summarize it in its essential ideas. We inform the reader that most of the intermediate results needed to prove Theorem 1.3, as well as the overall flow of the proof, perfectly match well-established results developed in Size Theory. For these reasons, whenever this happens, details will be skipped, providing the reader with suitable references. However, in the present paper, terminology will stick to that of Persistence Theory as much as possible.

The key idea is that a foliation in half-planes of Δ^+ can be given, such that the restriction of the multidimensional rank invariant to these half-planes turns out to be a one-dimensional rank invariant in two scalar variables. This approach implies that the comparison of two multidimensional rank invariants can be performed leaf by leaf by measuring the distance of appropriate one-dimensional rank invariants.

Therefore the stability of multidimensional persistence is a consequence of the one-dimensional persistence stability.

As is well known, in the case of tame functions, the one-dimensional persistence stability is obtained by considering the bottleneck distance between persistence diagrams, i.e. finite collections of points (with multiplicity) lying in $\mathbb{R}^2$ above the diagonal [9]. We show that the same approach works also for continuous functions, not necessarily tame. We recall that a filtrating function is said to be a tame function if it has a finite number of homological critical values, and the homology groups of all the lower level sets are finitely generated. The main technical problem in working with continuous functions, instead of tame functions, is that persistence diagrams may have infinitely many points, accumulating onto the diagonal. This difficulty is overcome following the same arguments used in [11] for proving stability of size functions.

2. Basic properties of the rank invariant and one-dimensional stability for continuous functions

The main result of this section is the stability of the one-dimensional rank invariant for continuous functions (Theorem 2.19). It generalizes the main theorem in [9] that requires continuous tame functions. Its proof relies on a number of basic simple properties of rank invariants that are completely analogous to those used in [11, 17] to prove the stability of size functions.

2.1. **Properties of the multidimensional rank invariant.** The next Lemma 2.1 ensures that the multidimensional kth rank invariant (Definition 1.2) is finite even dropping the tameness condition requested in [9]. We underline that the rank invariant finiteness is not obvious from the triangulability of the space. Indeed, the lower level sets with respect to a continuous function are not necessarily triangulable spaces.

Lemma 2.1 (Finiteness). *Let X be a triangulable space, and $\vec{\varphi}: X \to \mathbb{R}^n$ a continuous function. Then, for every $(\vec{u}, \vec{v}) \in \Delta^+$, it holds that $\rho_{(X,\vec{\varphi}),k}(\vec{u}, \vec{v}) < +\infty$.*

Proof. Since X is triangulable, we can assume that it is the support of a simplicial complex K and that a distance d is defined on X.

Let us fix $(\vec{u}, \vec{v}) \in \Delta^+$, and choose a real number $\varepsilon > 0$ such that, setting $\vec{\varepsilon} = (\varepsilon, \dots, \varepsilon)$, $\vec{u} + 2\vec{\varepsilon} \prec \vec{v}$.

We now show that there exist a function $\vec{\psi}: X \to \mathbb{R}^n$, a subdivision K'' of K, and a triangulation L'' of $\vec{\psi}(X)$, such that (i) the triple $(\vec{\psi}, K'', L'')$ is simplicial, and (ii) $\max_{x \in X} \|\vec{\varphi}(x) - \vec{\psi}(x)\|_\infty < \varepsilon$.

Indeed, by the uniform continuity of each component φ_i of $\vec{\varphi}$, there exists a real number $\delta > 0$ such that, for $i = 1, \dots, n$, $|\varphi_i(x) - \varphi_i(x')| < \varepsilon$, for every $x, x' \in X$ with $d(x, x') < \delta$. We take a subdivision K' of K such that $\mathrm{mesh}(K') < \delta$, and define $\vec{\psi}(x) = \vec{\varphi}(x)$ for every vertex x of K'. Next, we consider the linear extension of $\vec{\psi}$ to the other simplices of K'. In this way, $\vec{\psi}$ is linear on each simplex of K'.

Since $\vec{\psi}$ is piecewise linear, $\vec{\psi}(X)$ is the underlying space of a simplicial complex L'. By taking suitable subdivisions K'' of K' and L'' of L', $\vec{\psi}$ also sends simplices into simplices and therefore $(\vec{\psi}, K'', L'')$ is simplicial (cf. [26, Thm. 2.14]). This proves (i).

To see (ii), let us consider a point x belonging to a simplex in K', of vertices $v_1, \dots, v_r$. Since $x = \sum_{i=1}^r \lambda_i \cdot v_i$, with $\lambda_1, \dots, \lambda_r \geq 0$ and $\sum_{i=1}^r \lambda_i = 1$, and $\vec{\psi}$ is linear on each simplex, it follows that $\left\|\vec{\varphi}(x) - \vec{\psi}(x)\right\|_\infty = \left\|\vec{\varphi}(x) - \sum_{i=1}^r \lambda_i \cdot\right.$

$\vec{\psi}(v_i)\big\|_\infty = \big\|\vec{\varphi}(x) - \sum_{i=1}^r \lambda_i \cdot \vec{\varphi}(v_i)\big\|_\infty = \big\| \sum_{i=1}^r \lambda_i \cdot \vec{\varphi}(x) - \sum_{i=1}^r \lambda_i \cdot \vec{\varphi}(v_i)\big\|_\infty \leq \sum_{i=1}^r \lambda_i \big\|\vec{\varphi}(x) - \vec{\varphi}(v_i)\big\|_\infty < \varepsilon$.

We now prove that, since $(\vec{\psi}, K'', L'')$ is simplicial, it holds that $\check{H}_k(X\langle\vec{\psi} \preceq \vec{u}+\vec{\varepsilon}\rangle)$ is finitely generated. Indeed, since the intersection between a simplex and a half-space is triangulable, there exists a subdivision L''' of L'' such that $\vec{\psi}(X) \cap \{\vec{x} \in \mathbb{R}^n : \vec{x} \preceq \vec{u}+\vec{\varepsilon}\}$ is triangulated by a subcomplex of L'''. By [26, Lemma 2.16], there is a subdivision K''' of K'' such that $(\vec{\psi}, K''', L''')$ is simplicial. It follows that $X\langle\vec{\psi} \preceq \vec{u}+\vec{\varepsilon}\rangle$ is triangulable, and hence $\check{H}_k(X\langle\vec{\psi} \preceq \vec{u}+\vec{\varepsilon}\rangle)$ is finitely generated.

Since $\vec{u} + 2\vec{\varepsilon} \prec \vec{v}$ and $\max_{x\in X} \|\vec{\varphi}(x) - \vec{\psi}(x)\|_\infty < \varepsilon$, we have the inclusions $X\langle\vec{\varphi} \preceq \vec{u}\rangle \xrightarrow{i} X\langle\vec{\psi} \preceq \vec{u}+\vec{\varepsilon}\rangle \xrightarrow{j} X\langle\vec{\varphi} \preceq \vec{v}\rangle$, inducing the homomorphisms $\check{H}_k(X\langle\vec{\varphi} \preceq \vec{u}\rangle) \xrightarrow{i_k} \check{H}_k(X\langle\vec{\psi} \preceq \vec{u}+\vec{\varepsilon}\rangle) \xrightarrow{j_k} \check{H}_k(X\langle\vec{\varphi} \preceq \vec{v}\rangle)$. By recalling that $\check{H}_k(X\langle\vec{\psi} \preceq \vec{u}+\vec{\varepsilon}\rangle)$ is finitely generated, and since $\mathrm{rank}(j_k \circ i_k) \leq \mathrm{rank}(j_k)$, we obtain the claim. □

The following Lemma 2.2 and Lemma 2.3 generalize to the multidimensional setting analogous results valid for $n = 1$. We omit the trivial proof of Lemma 2.2.

Lemma 2.2 (Monotonicity). *$\rho_{(X,\vec{\varphi}),k}(\vec{u},\vec{v})$ is non-decreasing in the variable $\vec{u}$ and non-increasing in the variable $\vec{v}$.*

Lemma 2.3 (Diagonal Jump). *Let X, Y be two triangulable spaces, and $f : X \to Y$ a homeomorphism. Let $\vec{\varphi} : X \to \mathbb{R}^n$, $\vec{\psi} : Y \to \mathbb{R}^n$ be continuous functions such that $\max_{x\in X} \|\vec{\varphi}(x) - \vec{\psi} \circ f(x)\|_\infty \leq h$. Then, setting $\vec{h} = (h, \ldots, h)$, for every $(\vec{u},\vec{v}) \in \Delta^+$, we have $\rho_{(X,\vec{\varphi}),k}(\vec{u}-\vec{h}, \vec{v}+\vec{h}) \leq \rho_{(Y,\vec{\psi}),k}(\vec{u},\vec{v})$.*

Proof. Since $\|\vec{\varphi} - \vec{\psi} \circ f\|_\infty \leq h$, we have the following commutative diagram

$$\begin{tikzcd}
\check{H}_k(X\langle\vec{\varphi} \preceq \vec{u}-\vec{h}\rangle) \arrow[r, "i_k"] \arrow[d] & \check{H}_k(X\langle\vec{\varphi} \preceq \vec{v}+\vec{h}\rangle) \\
\check{H}_k(Y\langle\vec{\psi} \preceq \vec{u}\rangle) \arrow[r, "j_k"] & \check{H}_k(Y\langle\vec{\psi} \preceq \vec{v}\rangle) \arrow[u]
\end{tikzcd}$$

where i_k and j_k are induced by inclusions, and the vertical homomorphisms are induced by restrictions of f and f^{-1}, respectively. Thus the claim follows by observing that $\mathrm{rank}\ i_k \leq \mathrm{rank}\ j_k$. □

2.2. **Properties of the one-dimensional rank invariant.** Now we confine ourselves to the case $n = 1$. Therefore, for the sake of simplicity, the symbols $\vec{\varphi}, \vec{u}, \vec{v}$ will be replaced by φ, u, v, respectively. We remark that Δ^+ reduces to be the set $\{(u, v) \in \mathbb{R}^2 : u < v\}$. Moreover, we use the following notations: $\Delta = \partial\Delta^+$, $\Delta^* = \Delta^+ \cup \{(u,\infty) : u \in \mathbb{R}\}$, and $\bar{\Delta}^* = \Delta^* \cup \Delta$. Finally, we write $\|\varphi\|_\infty$ for $\max_{x\in X} |\varphi(x)|$.

2.2.1. *Right-continuity of the one-dimensional rank invariant.* In what follows we shall prove that, using Čech homology, the one-dimensional rank invariant is right-continuous with respect to both its variables u and v, i.e. $\lim_{u\to\bar{u}^+} \rho_{(X,\varphi),k}(u, v) = \rho_{(X,\varphi),k}(\bar{u}, v)$ and $\lim_{v\to\bar{v}^+} \rho_{(X,\varphi),k}(u, v) = \rho_{(X,\varphi),k}(u, \bar{v})$. This property will be necessary to completely characterize the rank invariant by a persistence diagram, a descriptor whose definition will be recalled later in this section. In the absence of right-continuity, persistence diagrams describe rank invariants only almost everywhere, thus justifying the use of Čech homology in this context.

The next example shows that the right-continuity in the variable u does not always hold when persistent homology groups are defined using simplicial or singular homology, even under the tameness assumption.

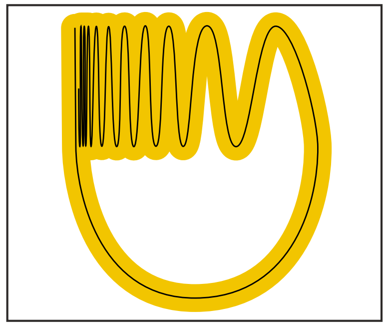


FIGURE 1. A lower level set $X\langle\varphi \le u\rangle$, for a sufficiently small $u > 0$, as considered in Example 2.4, corresponds to a dilation (shaded) of our Warsaw circle.

Example 2.4. Let X be a closed rectangle of $\mathbb{R}^2$ containing a Warsaw circle (see Figure 1). Let also $\varphi : X \to \mathbb{R}$ be the Euclidean distance from the Warsaw circle.
It is easy to see that φ is tame on X (with respect to both singular and Čech homology). Moreover, the rank of the singular persistent homology group $H_1^{(u,v)}(X,\varphi)$ is equal to 1 for $v > u > 0$ and v sufficiently small, whereas it is equal to 0 when $u = 0$, showing that singular persistent homology is not right-continuous in the variable u.

Analogously, it is possible to show that simplicial or singular homology do not ensure the right-continuity in the variable v (see Appendix A).

Let us fix two real numbers $\bar{u} < \bar{v}$ and, for $\bar{u} < u' \le u'' < \bar{v}$, consider the following commutative diagram

$$\begin{tikzcd} \check{H}_k(X\langle\varphi \le u'\rangle) \arrow[r, "\pi_k^{(u',u'')}"] \arrow[d, "\pi_k^{(u',\bar{v})}"'] & \check{H}_k(X\langle\varphi \le u''\rangle) \arrow[d, "\pi_k^{(u'',\bar{v})}"'] \\ \check{H}_k(X\langle\varphi \le \bar{v}\rangle) \arrow[r, "id"] & \check{H}_k(X\langle\varphi \le \bar{v}\rangle). \end{tikzcd} \tag{1}$$

By recalling that $\check{H}_k^{(u,v)}(X,\varphi) = \mathrm{im}\ \pi_k^{(u,v)}$, from the above diagram (1), it is easy to see that each $\pi_k^{(u',u'')}$ induces the inclusion map $\sigma_k^{(u',u'')} : \check{H}_k^{(u',\bar{v})}(X,\varphi) \to \check{H}_k^{(u'',\bar{v})}(X,\varphi)$. The following Lemma 2.5 states that, for every $u'' \ge u' > \bar{u}$, with u'' sufficiently close to $\bar{u}$, the maps $\sigma_k^{(u',u'')}$ are all isomorphisms.

Lemma 2.5. *Let $(\bar{u},\bar{v}) \in \Delta^+$, and let $\sigma_k^{(u',u'')} : \check{H}_k^{(u',\bar{v})}(X,\varphi) \to \check{H}_k^{(u'',\bar{v})}(X,\varphi)$ be the inclusion of vector spaces induced by the map $\pi_k^{(u',u'')}$. Then there exists $\hat{u}$, with $\bar{u} < \hat{u} < \bar{v}$, such that the maps $\sigma_k^{(u',u'')}$ are isomorphisms for every u', u'' with $\bar{u} < u' \le u'' \le \hat{u}$.*

Proof. We observe that the maps $\sigma_k^{(u',u'')}$ are injective by construction (see diagram (1)). Moreover, by the Finiteness Lemma 2.1 and the Monotonicity Lemma 2.2, there exists $\hat{u}$, with $\bar{u} < \hat{u} < \bar{v}$, such that $\rho_{(X,\varphi),k}(u',\bar{v}) := \mathrm{rank}\ \check{H}_k^{(u',\bar{v})}(X,\varphi)$ is finite and equal to $\rho_{(X,\varphi),k}(u'',\bar{v}) := \mathrm{rank}\ \check{H}_k^{(u'',\bar{v})}(X,\varphi)$ whenever $\bar{u} < u' \le u'' \le \hat{u}$. This implies that $\sigma_k^{(u',u'')}$ are isomorphisms. $\square$

Analogously, by considering the commutative diagram

$$\begin{tikzcd} \check{H}_k(X\langle\varphi \le \bar{u}\rangle) \arrow[r, "id"] \arrow[d, "\pi_k^{(\bar{u},v')}"'] & \check{H}_k(X\langle\varphi \le \bar{u}\rangle) \arrow[d, "\pi_k^{(\bar{u},v'')}"'] \\ \check{H}_k(X\langle\varphi \le v'\rangle) \arrow[r, "\pi_k^{(v',v'')}"] & \check{H}_k(X\langle\varphi \le v''\rangle), \end{tikzcd}$$

we obtain induced maps $\tau_k^{(v',v'')}: \check{H}_k^{(\bar{u},v')}(X,\varphi) \to \check{H}_k^{(\bar{u},v'')}(X,\varphi)$, and prove that they are isomorphisms whenever v', v'' are sufficiently close to $\bar{v}$, with $\bar{v} < v' \le v''$.

Lemma 2.6. *Let $(\bar{u},\bar{v}) \in \Delta^+$, and let $\tau_k^{(v',v'')}: \check{H}_k^{(\bar{u},v')}(X,\varphi) \to \check{H}_k^{(\bar{u},v'')}(X,\varphi)$ be the homomorphism of vector spaces induced by the map $\pi_k^{(v',v'')}$. Then there exists $\hat{v} > \bar{v}$ such that the homomorphisms $\tau_k^{(v',v'')}$ are isomorphisms for every v', v'' with $\bar{v} < v' \le v'' \le \hat{v}$.*

Proof. The proof is essentially the same as that of Lemma 2.5, after observing that the maps $\tau_k^{(v',v'')}$ are surjections between vector spaces of the same finite dimension. □

Lemma 2.7 (Right-Continuity). *$\rho_{(X,\varphi),k}(u,v)$ is right-continuous with respect to both the variables u and v.*

Proof. In order to prove that $\lim_{u\to\bar{u}^+} \rho_{(X,\varphi),k}(u,v) = \rho_{(X,\varphi),k}(\bar{u},v)$, by the Monotonicity Lemma 2.2, it will suffice to show that $\check{H}_k^{(\bar{u},\bar{v})}(X,\varphi) \cong \check{H}_k^{(\hat{u},\bar{v})}(X,\varphi)$, where $\hat{u}$ is taken as in Lemma 2.5. To this end, we consider the following sequence of isomorphisms

$$\begin{aligned} \check{H}_k^{(\bar{u},\bar{v})}(X,\varphi) &= \operatorname{im} \pi_k^{(\bar{u},\bar{v})} \cong \operatorname{im} \varprojlim \pi_k^{(u',\bar{v})} \\ &\cong \varprojlim \operatorname{im} \pi_k^{(u',\bar{v})} = \varprojlim \check{H}_k^{(u',\bar{v})}(X,\varphi) \cong \check{H}_k^{(\hat{u},\bar{v})}(X,\varphi). \end{aligned}$$

Let us now show how these equivalences can be obtained.

By the continuity of Čech Theory (cf. [15, Thm. X, 3.1]), it holds that $\operatorname{im} \pi_k^{(\bar{u},\bar{v})} \cong \operatorname{im} \varprojlim \pi_k^{(u',\bar{v})}$, where $\varprojlim \pi_k^{(u',\bar{v})}$ is the inverse limit of the inverse system of homomorphisms $\pi_k^{(u',\bar{v})}: \check{H}_k(X\langle\varphi \le u'\rangle) \to \check{H}_k(X\langle\varphi \le \bar{v}\rangle)$ over the directed set $\{u' \in \mathbb{R} : \bar{u} < u' \le \hat{u}\}$ decreasingly ordered.

Moreover, since the inverse limit of vector spaces is an exact functor, it preserves epimorphisms and hence images. Therefore, it holds that $\operatorname{im} \varprojlim \pi_k^{(u',\bar{v})} \cong \varprojlim \operatorname{im} \pi_k^{(u',\bar{v})} = \varprojlim \check{H}_k^{(u',\bar{v})}(X,\varphi)$, where the last inverse limit is taken with respect to the inverse system $\left(\check{H}_k^{(u',\bar{v})}(X,\varphi), \sigma_k^{(u',u'')}\right)$ over the directed set $\{u' \in \mathbb{R} : \bar{u} < u' \le \hat{u}\}$ decreasingly ordered, and $\sigma_k^{(u',u'')}$ are the maps introduced in Lemma 2.5.

Finally, $\varprojlim \check{H}_k^{(u',\bar{v})}(X,\varphi) \cong \check{H}_k^{(\hat{u},\bar{v})}(X,\varphi)$. Indeed, $\varprojlim \check{H}_k^{(u',\bar{v})}(X,\varphi)$ is the inverse limit of a system of isomorphic vector spaces by Lemma 2.5.

Analogously for the variable v, applying Lemma 2.6. □

2.2.2. *Stability of the one-dimensional rank invariant.* The following Lemma 2.8 and Lemma 2.9 can be proved in the same way as the analogous results holding when $k = 0$ (see [11]).

Lemma 2.8. *The following statements hold:*

(*i*) *For every $u < \min \varphi$, $\rho_{(X,\varphi),k}(u,v) = 0$.*

(*ii*) *For every* $v \geq \max \varphi$, $\rho_{(X,\varphi),k}(u,v)$ *is equal to the number of classes in* $\check{H}_k(X)$, *having at least one representative in* $X\langle\varphi \leq u\rangle$.

Since, for $u_1 \leq u_2 < v_1 \leq v_2$, the number of homology classes born between u_1 and u_2 and still alive at v_1 is certainly not smaller than those still alive at v_2, we have the next result.

Lemma 2.9 (Jump Monotonicity). *Let* u_1, u_2, v_1, v_2 *be real numbers such that* $u_1 \leq u_2 < v_1 \leq v_2$. *It holds that*

$$\rho_{(X,\varphi),k}(u_2,v_1) - \rho_{(X,\varphi),k}(u_1,v_1) \geq \rho_{(X,\varphi),k}(u_2,v_2) - \rho_{(X,\varphi),k}(u_1,v_2).$$

Lemma 2.9 justifies the following definitions of multiplicity. Since we are working with continuous instead of tame functions, we adopt the definitions introduced in [17] rather than those of [9]. Although based on the same idea, the difference relies in the computation of multiplicity on a varying grid, instead of a fixed one. So we can work with an infinite number of (possibly accumulating) points of positive multiplicity. Due to the lack of a well-established terminology for points with positive multiplicity, we call them *cornerpoints*, which is normally used for size functions.

Definition 2.10 (Proper cornerpoint). For every point $p = (u,v) \in \Delta^+$, we define the number $\mu_k(p)$ as the minimum over all the positive real numbers ε, with $u+\varepsilon < v-\varepsilon$, of

$$\rho_{(X,\varphi),k}(u+\varepsilon,v-\varepsilon)-\rho_{(X,\varphi),k}(u-\varepsilon,v-\varepsilon)-\rho_{(X,\varphi),k}(u+\varepsilon,v+\varepsilon)+\rho_{(X,\varphi),k}(u-\varepsilon,v+\varepsilon).$$

The number $\mu_k(p)$ will be called the *multiplicity* of p for $\rho_{(X,\varphi),k}$. Moreover, we shall call a *proper cornerpoint for* $\rho_{(X,\varphi),k}$ any point $p \in \Delta^+$ such that the number $\mu_k(p)$ is strictly positive.

Definition 2.11 (Cornerpoint at infinity). For every vertical line r, with equation $u = \bar{u}$, $\bar{u} \in \mathbb{R}$, let us identify r with $(\bar{u}, \infty) \in \Delta^*$, and define the number $\mu_k(r)$ as the minimum over all the positive real numbers ε, with $\bar{u} + \varepsilon < 1/\varepsilon$, of

$$\rho_{(X,\varphi),k}\left(\bar{u}+\varepsilon, \frac{1}{\varepsilon}\right) - \rho_{(X,\varphi),k}\left(\bar{u}-\varepsilon, \frac{1}{\varepsilon}\right).$$

The number $\mu_k(r)$ will be called the *multiplicity* of r for $\rho_{(X,\varphi),k}$. When this finite number is strictly positive, we call r a *cornerpoint at infinity for* $\rho_{(X,\varphi),k}$.

The concept of cornerpoint allows us to introduce a representation of the rank invariant, based on the following definition [9].

Definition 2.12 (Persistence diagram). The *persistence diagram* $D_k(X,\varphi) \subset \bar{\Delta}^*$ is the multiset of all cornerpoints (both proper and at infinity) for $\rho_{(X,\varphi),k}$, counted with their multiplicity, union the points of Δ, counted with infinite multiplicity.

In order to show that persistence diagrams completely describe rank invariants, we give some technical results concerning cornerpoints.

The Monotonicity Lemma 2.2, and Lemmas 2.7 (Right-Continuity) and 2.9 (Jump Monotonicity) imply the following result, by the same arguments as in [17].

Proposition 2.13 (Propagation of Discontinuities). *If* $\bar{p} = (\bar{u}, \bar{v})$ *is a proper cornerpoint for* $\rho_{(X,\varphi),k}$, *then the following statements hold:*

(*i*) *If* $\bar{u} \leq u < \bar{v}$, *then* $\bar{v}$ *is a discontinuity point for* $\rho_{(X,\varphi),k}(u,\cdot)$;

(*ii*) *If* $\bar{u} < v < \bar{v}$, *then* $\bar{u}$ *is a discontinuity point for* $\rho_{(X,\varphi),k}(\cdot,v)$.

If $\bar{r} = (\bar{u}, \infty)$ *is a cornerpoint at infinity for* $\rho_{(X,\varphi),k}$, *then it holds that*

(*iii*) *If* $\bar{u} < v$, *then* $\bar{u}$ *is a discontinuity point for* $\rho_{(X,\varphi),k}(\cdot,v)$.

We observe that any open arcwise connected neighborhood in Δ^+ of a discontinuity point for $\rho_{(X,\varphi),k}$ contains at least one discontinuity point in the variable u or v. Moreover, as a consequence of the Jump Monotonicity Lemma 2.9, discontinuity points in the variable u propagate downwards, while discontinuity points in the variable v propagate rightwards. So, by applying the Finiteness Lemma 2.1, we obtain the proposition below.

Proposition 2.14. *Let $k \in \mathbb{Z}$. For every point $\bar{p} = (\bar{u}, \bar{v}) \in \Delta^+$, a real number $\varepsilon > 0$ exists such that the open set*

$$W_\varepsilon(\bar{p}) = \{(u,v) \in \mathbb{R}^2 : |u - \bar{u}| < \varepsilon, |v - \bar{v}| < \varepsilon, u \neq \bar{u}, v \neq \bar{v}\}$$

is contained in Δ^+, and does not contain any discontinuity point for $\rho_{(X,\varphi),k}$.

As a simple consequence of Lemma 2.8 and Proposition 2.13 (Propagation of Discontinuities), we have the following proposition.

Proposition 2.15 (Localization of Cornerpoints)**.** *If $\bar{p} = (\bar{u}, \bar{v})$ is a proper cornerpoint for $\rho_{(X,\varphi),k}$, then $\bar{p} \in \{(u,v) \in \Delta^+ : \min \varphi \le u < v \le \max \varphi\}$.*

By applying Proposition 2.13 and Proposition 2.14, and recalling the finiteness of $\rho_{(X,\varphi),k}$ (Finiteness Lemma 2.1), it is easy to prove the following result.

Proposition 2.16 (Local Finiteness of Cornerpoints)**.** *For each strictly positive real number ε, $\rho_{(X,\varphi),k}$ has, at most, a finite number of cornerpoints in $\{(u,v) \in \mathbb{R}^2 : u + \varepsilon < v\}$.*

We observe that it is easy to provide examples of persistence diagrams containing an infinite number of proper cornerpoints, accumulating onto the diagonal Δ.

The following Theorem 2.17 shows that persistence diagrams uniquely determine one-dimensional rank invariants (the inverse also holds by definition of persistence diagram). It is a consequence of the definitions of multiplicity (Definitions 2.10 and 2.11), together with the previous results about cornerpoints, and the Right-Continuity Lemma 2.7, in the same way as done in [17] for size functions. We remark that a similar result was given in [9], under the name of *k-triangle Lemma*. Our Representation Theorem differs from the k-triangle Lemma in two respects. Firstly, our hypotheses on the function φ are weaker. Secondly, the k-triangle Lemma focuses not on all the set Δ^+, but only on the points having coordinates that are not homological critical values.

Theorem 2.17 (Representation Theorem)**.** *For every $(\bar{u}, \bar{v}) \in \Delta^+$, we have*

$$\rho_{(X,\varphi),k}(\bar{u}, \bar{v}) = \sum_{\substack{(u,v) \in \Delta^* \\ u \le \bar{u},\, v > \bar{v}}} \mu_k((u,v)).$$

As a consequence of the Representation Theorem 2.17 any distance between persistence diagrams induces a distance between one-dimensional rank invariants. This justifies the following definition [9, 11].

Definition 2.18 (Matching distance)**.** Let X be a triangulable space endowed with continuous functions $\varphi, \psi : X \to \mathbb{R}$. The *matching distance* d_{match} between $\rho_{(X,\varphi),k}$ and $\rho_{(X,\psi),k}$ is equal to the bottleneck distance between $D_k(X,\varphi)$ and $D_k(Y,\psi)$, i.e.

$$d_{match}\left(\rho_{(X,\varphi),k}, \rho_{(X,\psi),k}\right) = \inf_\gamma \max_{p \in D_k(X,\varphi)} \|p - \gamma(p)\|_{\widetilde{\infty}}, \tag{2}$$

where γ ranges over all multi-bijections between $D_k(X,\varphi)$ and $D_k(X,\psi)$, and for every $p = (u,v), q = (u',v')$ in Δ^*,

$$\|p - q\|_{\widetilde{\infty}} = \min\left\{\max\{|u - u'|, |v - v'|\}, \max\left\{\frac{v-u}{2}, \frac{v'-u'}{2}\right\}\right\},$$

with the convention about points at infinity that $\infty - y = y - \infty = \infty$ when $y \neq \infty$, $\infty - \infty = 0$, $\frac{\infty}{2} = \infty$, $|\infty| = \infty$, $\min\{c, \infty\} = c$ and $\max\{c, \infty\} = \infty$.

In plain words, $\|\cdot\|_{\widetilde{\infty}}$ measures the pseudodistance between two points p and q as the minimum between the cost of moving one point onto the other and the cost of moving both points onto the diagonal, with respect to the max-norm and under the assumption that any two points of the diagonal have vanishing pseudodistance. We observe that in (2) we can write max instead of sup, because by Proposition 2.15 (Localization of Cornerpoints) proper cornerpoints belong to a compact subset of the closure of Δ^+.

We are now ready to give the one-dimensional stability theorem for the rank invariant with continuous functions. The proof relies on a cone construction. The rationale behind this construction is to directly apply the arguments used in [11] for size functions, eliminating cornerpoints at infinity, whose presence would require us to modify all the proofs.

This stability theorem is a different result from the one given in [9], weakening the tameness requirement to continuity, and actually solving one of the open problems posed in that work by the authors.

Theorem 2.19 (One-Dimensional Stability Theorem)**.** *Let X be a triangulable space, and $\varphi, \psi : X \to \mathbb{R}$ two continuous functions. Then $d_{match}(\rho_{(X,\varphi),k}, \rho_{(X,\psi),k}) \leq \|\varphi - \psi\|_\infty$.*

Proof. In what follows we can assume that X is connected. Indeed, if X has r connected components $C_1, \ldots, C_r$, then the claim can be proved by induction after observing that $D_k(X, \varphi) = \bigcup_{i=1}^r D_k(C_i, \varphi_{|C_i})$, for every $k \in \mathbb{Z}$.

For $k = 0$, the claim has been proved in [11, Thm. 25].

Let us now assume $k > 0$. Consider the cone on X, $\tilde{X} = (X \times I)/(X \times \{1\})$ (see Figure 2).

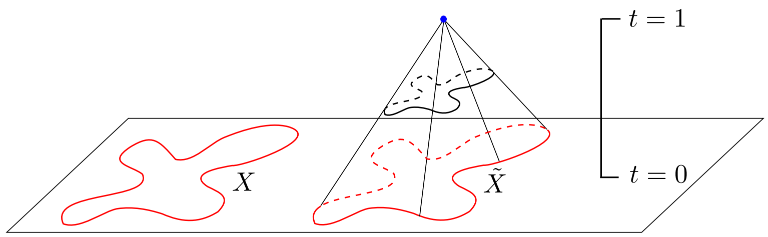


FIGURE 2. The cone construction used in the proof of Theorem 2.19. The cycles in the cone are trivial.

Since X is triangulable, so is $\tilde{X}$. We also consider the continuous function $\tilde{\varphi} : \tilde{X} \to \mathbb{R}$ taking the class of (x, t) to the value $\varphi(x) \cdot (1 - t) + M \cdot t$, where $M = 3 \cdot (\max|\varphi| + \max|\psi|) + 1$. This choice of M, besides guaranteeing that $(u, M) \in \Delta^+$ when $u \leq \max|\varphi|, \max|\psi|$, will be useful later.

By construction, it holds that

$$\rho_{(\tilde{X},\tilde{\varphi}),k}(u, v) = \begin{cases} \rho_{(X,\varphi),k}(u, v), & \text{if } v < M; \\ 0, & \text{if } v \geq M. \end{cases}$$

Indeed, it is well known that $\tilde{X}$ is contractible (see [20, Lemma 21.13]), explaining why $\rho_{(\tilde{X},\tilde{\varphi}),k}(u, v) = 0$ when $v \geq M$. The other part of the claim follows from the observation that, for every $v < M$, identifying $X\langle\varphi \leq v\rangle \times \{0\}$ with $X\langle\varphi \leq v\rangle$, the lower level set $X\langle\varphi \leq v\rangle$ is a strong deformation retract of $\tilde{X}\langle\tilde{\varphi} \leq v\rangle$. To see this, it is sufficient to consider the obvious retraction $r : (x,t) \mapsto x$ and the deformation retraction $H : \tilde{X}\langle\tilde{\varphi} \leq v\rangle \times I \to \tilde{X}\langle\tilde{\varphi} \leq v\rangle$, $H((x,t),s) = (x, t\cdot(1-s))$. This yields the following commutative diagram

$$\begin{tikzcd} \check{H}_k(\tilde{X}\langle\tilde{\varphi} \leq u\rangle) \arrow[r, "r_k"] \arrow[d, "\pi_k^{(u,v)}"'] & \check{H}_k(X\langle\varphi \leq u\rangle) \arrow[d, "\pi_k^{(u,v)}"'] \\ \check{H}_k(\tilde{X}\langle\tilde{\varphi} \leq v\rangle) \arrow[r, "r_k"] & \check{H}_k(X\langle\varphi \leq v\rangle), \end{tikzcd}$$

where the horizontal maps are isomorphisms, so that $\check{H}_k^{(u,v)}(\tilde{X},\tilde{\varphi}) \cong \check{H}_k^{(u,v)}(X,\varphi)$ when $v < M$.

Clearly, a point of Δ^+ is a proper cornerpoint for $\rho_{(X,\varphi),k}$ if and only if it is a proper cornerpoint for $\rho_{(\tilde{X},\tilde{\varphi}),k}$, with the ordinate strictly less than M. Moreover, a point (u,∞) of Δ^* is a cornerpoint at infinity for $\rho_{(X,\varphi),k}$ if and only if the point $(u,M) \in \Delta^+$ is a proper cornerpoint for $\rho_{(\tilde{X},\tilde{\varphi}),k}$. We remark that there are no cornerpoints (u,v) for $\rho_{(\tilde{X},\tilde{\varphi}),k}$ when $\max|\varphi| < v < M$. Analogously, we can construct $\tilde{\psi} : \tilde{X} \to \mathbb{R}$ out of ψ with the same properties.

Following the same technical steps as in [11, Thm. 25], simply substituting $\rho_{(X,\varphi),k}$ for $\rho_{(X,\varphi),0}$, it is possible to prove the inequality $d_{match}(\rho_{(\tilde{X},\tilde{\varphi}),k}, \rho_{(\tilde{X},\tilde{\psi}),k}) \leq \|\tilde{\varphi} - \tilde{\psi}\|_\infty$. To this end, we need to apply Lemma 2.3, and Propositions 2.13, 2.14, 2.15 and 2.16. Therefore, since $\|\tilde{\varphi} - \tilde{\psi}\|_\infty = \|\varphi - \psi\|_\infty$, it is sufficient to show that $d_{match}(\rho_{(X,\varphi),k}, \rho_{(X,\psi),k}) \leq d_{match}(\rho_{(\tilde{X},\tilde{\varphi}),k}, \rho_{(\tilde{X},\tilde{\psi}),k})$.

We can prove that a multi-bijection $\tilde{\gamma}$ between $D_k(\tilde{X},\tilde{\varphi})$ and $D_k(\tilde{X},\tilde{\psi})$ exists, with $d_{match}(\rho_{(\tilde{X},\tilde{\varphi}),k}, \rho_{(\tilde{X},\tilde{\psi}),k}) = \max_{\tilde{p}\in D_k(\tilde{X},\tilde{\varphi})} \|\tilde{p} - \tilde{\gamma}(\tilde{p})\|_{\widetilde{\infty}}$. This can be done by applying Proposition 2.16, as in [11, Thm. 28]. Such a $\tilde{\gamma}$ will be called optimal.

Since $\tilde{\gamma}$ is optimal, then $\tilde{\gamma}$ takes each point $(u,v) \in D_k(\tilde{X},\tilde{\varphi})$, with $v = M$, to a point $(u',v') \in D_k(\tilde{X},\tilde{\psi})$, with $v' = M$. Indeed, if it were not true, i.e. $\tilde{\gamma}((u,M)) = (u',v')$ with $v' < M$, then $v' \leq \max|\psi|$, and by the choice of M, we would have $\|(u,M) - \tilde{\gamma}((u,M))\|_{\widetilde{\infty}} \geq \|\varphi - \psi\|_\infty$, contradicting the fact that $d_{match}(\rho_{(\tilde{X},\tilde{\varphi}),k}, \rho_{(\tilde{X},\tilde{\psi}),k}) \leq \|\varphi - \psi\|_\infty$. Analogously for $\tilde{\gamma}^{-1}$, proving that $\tilde{\gamma}$ maps cornerpoints whose ordinate is smaller than M into cornerpoints whose ordinate is still below M.

Let us now consider an optimal multi-bijection $\tilde{\gamma}$ between $D_k(\tilde{X},\tilde{\varphi})$ and $D_k(\tilde{X},\tilde{\psi})$. We now show that there exists a multi-bijection γ between $D_k(X,\varphi)$ and $D_k(X,\psi)$, such that $\max_{p\in D_k(X,\varphi)} \|p - \gamma(p)\|_{\widetilde{\infty}} = \max_{\tilde{p}\in D_k(\tilde{X},\tilde{\varphi})} \|\tilde{p} - \tilde{\gamma}(\tilde{p})\|_{\widetilde{\infty}}$, thus proving that $d_{match}(\rho_{(X,\varphi),k}, \rho_{(X,\psi),k}) \leq d_{match}(\rho_{(\tilde{X},\tilde{\varphi}),k}, \rho_{(\tilde{X},\tilde{\psi}),k})$. Indeed, we can define $\gamma : D_k(X,\varphi) \to D_k(X,\psi)$ by setting $\gamma((u,v)) = \tilde{\gamma}((u,v))$ if $v < \infty$, and $\gamma((u,v)) = (u',v)$, where u' is the abscissa of the point $\tilde{\gamma}((u,M))$, if $v = \infty$. This concludes the proof. □

3. Proof of the Multidimensional Stability Theorem 1.3

We now provide the proof of the Multidimensional Stability Theorem 1.3. It will be deduced following the same arguments given in [1] to prove the stability of multidimensional size functions. Proofs that are still valid for $k > 0$ without any change will be omitted.

We start by recalling that the following parameterized family of half-planes in $\mathbb{R}^n \times \mathbb{R}^n$ is a foliation of Δ^+.

Definition 3.1 (Admissible pairs)**.** For every unit vector $\vec{l} = (l_1,\ldots,l_n)$ of $\mathbb{R}^n$ such that $l_i > 0$ for $i = 1,\ldots,n$, and for every vector $\vec{b} = (b_1,\ldots,b_n)$ of $\mathbb{R}^n$ such that $\sum_{i=1}^n b_i = 0$, we shall say that the pair $(\vec{l},\vec{b})$ is *admissible*. We shall denote the set of all admissible pairs in $\mathbb{R}^n \times \mathbb{R}^n$ by Adm_n. Given an admissible pair $(\vec{l}, \vec{b})$, we define the half-plane $\pi_{(\vec{l},\vec{b})}$ of $\mathbb{R}^n \times \mathbb{R}^n$ by the following parametric equations:

$$\begin{cases} \vec{u} = s\vec{l} + \vec{b} \\ \vec{v} = t\vec{l} + \vec{b} \end{cases}$$

for $s, t \in \mathbb{R}$, with $s < t$.

The key property of this foliation is that the restriction of $\rho_{(X,\vec{\varphi}),k}$ to each leaf can be seen as a particular one-dimensional rank invariant, as the following theorem states.

Theorem 3.2 (Reduction Theorem)**.** *Let $(\vec{l}, \vec{b})$ be an admissible pair, and $F^{\vec{\varphi}}_{(\vec{l},\vec{b})} : X \to \mathbb{R}$ be defined by setting*

$$F^{\vec{\varphi}}_{(\vec{l},\vec{b})}(x) = \max_{i=1,\dots,n} \left\{ \frac{\varphi_i(x) - b_i}{l_i} \right\} .$$

Then, for every $(\vec{u}, \vec{v}) = (s\vec{l} + \vec{b}, t\vec{l} + \vec{b}) \in \pi_{(\vec{l},\vec{b})}$ the following equality holds:

$$\rho_{(X,\vec{\varphi}),k}(\vec{u}, \vec{v}) = \rho_{(X,F^{\vec{\varphi}}_{(\vec{l},\vec{b})}),k}(s, t) .$$

As a consequence of the Reduction Theorem 3.2, we observe that the identity $\rho_{(X,\vec{\varphi}),k} \equiv \rho_{(X,\vec{\psi}),k}$ holds if and only if $d_{match}(\rho_{(X,F^{\vec{\varphi}}_{(\vec{l},\vec{b})}),k}, \rho_{(X,F^{\vec{\psi}}_{(\vec{l},\vec{b})}),k}) = 0$, for every admissible pair $(\vec{l}, \vec{b})$.

The next theorem gives a stability result on each leaf of the foliation. It represents an intermediate step toward the proof of the stability of the multidimensional rank invariant across the whole foliation.

Theorem 3.3 (Stability w.r.t. Function Perturbations)**.** *If X is triangulable and $\vec{\varphi}, \vec{\psi} : X \to \mathbb{R}^n$ are continuous functions, then for each admissible pair $(\vec{l}, \vec{b})$, it holds that*

$$d_{match}(\rho_{(X,F^{\vec{\varphi}}_{(\vec{l},\vec{b})}),k}, \rho_{(X,F^{\vec{\psi}}_{(\vec{l},\vec{b})}),k}) \leq \frac{\max_{x \in X} \|\vec{\varphi}(x) - \vec{\psi}(x)\|_\infty}{\min_{i=1,\dots,n} l_i} .$$

We are now ready to deduce the Multidimensional Stability Theorem 1.3.

Proof. (of Theorem 1.3) We set $D_{match}(\rho_{(X,\vec{\varphi}),k}, \rho_{(X,\vec{\psi}),k}) = \sup_{(\vec{l},\vec{b}) \in Adm_n} \min_i l_i \cdot d_{match}(\rho_{(X,F^{\vec{\varphi}}_{(\vec{l},\vec{b})}),k}, \rho_{(X,F^{\vec{\psi}}_{(\vec{l},\vec{b})}),k})$. Then, D_{match} is a distance on $\{\rho_{(X,\vec{\varphi}),k} \mid \vec{\varphi} : X \to \mathbb{R}^n \text{ continuous}\}$ that clearly proves the claim. □

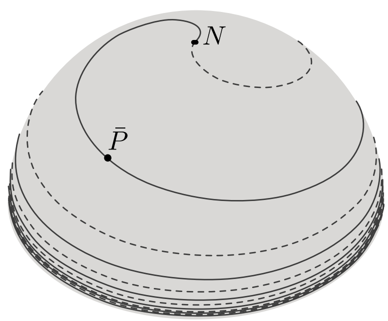
Roughly speaking, we have thus proved that small changes in the vector-valued filtrating function induce small changes in the associated multidimensional rank invariant, with respect to the distance D_{match}.

Acknowledgements. The authors thank Francesca Cagliari (University of Bologna) and Marco Grandis (University of Genoa) for their helpful advice. However, the authors are solely responsible for any errors.

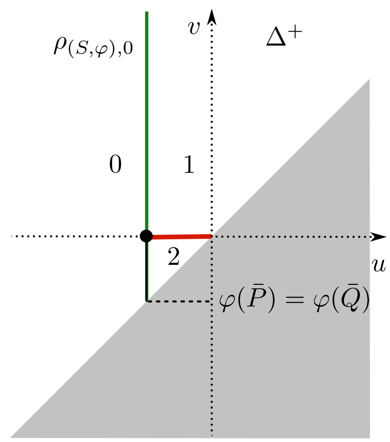
A. Appendix

The next example shows that the rank invariant is not right-continuous in the variable v when singular or simplicial homology are considered instead of Čech homology. We recall that an example concerning the right-continuity in the variable u has been given in Example 2.4.

Example A.1. Let $S \subset \mathbb{R}^3$ be a sphere parameterized by polar coordinates (α, β), $-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$ and $\beta \in [0, 2\pi)$. For every $\beta \in [0, 2\pi)$, consider on S the paths $\gamma^1_\beta : (-\frac{\pi}{2}, 0) \to S$ and $\gamma^2_\beta : (0, \frac{\pi}{2}) \to S$ defined by setting, for $i = 1, 2$, $\gamma^i_\beta(\alpha) = (\alpha', \beta')$ with $\alpha' = \alpha$ and $\beta' = (\beta + \cot \alpha) \bmod 2\pi$. We observe that each point of the set $S^* = \{(\alpha, \beta) \in S : \alpha \neq 0 \wedge |\alpha| \neq \frac{\pi}{2}\}$ belongs to the image of one and only one path γ^i_β. Such curves approach more and more a pole of the sphere on one side and the equator, winding an infinite number of times, on the other side (see, for instance, in Figure 3 (a), the paths $\gamma^2_{\frac{\pi}{2}}$ and $\gamma^2_{\frac{3\pi}{2}}$ lying in the northern hemisphere).



(a)



(b)

Figure 3. (a) Two of the paths covering the northern hemisphere considered in Example A.1. (b) The 0th rank invariant of the pair (S, φ). On the discontinuity points highlighted in bold red, the 0th rank invariant computed using singular homology takes value equal to 2, while using Čech homology, the value is equal to 1, showing the right-continuity in the variable v.

Then define the C^∞ function $\varphi^* : S^* \to \mathbb{R}$ that takes each point $P = \gamma^i_\beta(\alpha) \in S^*$ to the value $\exp\left(-\frac{1}{\alpha^2\left(\frac{\pi}{2} - |\alpha|\right)^2}\right) \sin(\beta)$. Now extend φ^* to a C^∞ function $\varphi : S \to \mathbb{R}$ in the unique possible way. In plain words, this function draws a ridge for $\beta \in (0, \pi)$, and a valley for $\beta \in (\pi, 2\pi)$. Moreover, observe that the points $\bar{P} \equiv (\frac{\pi}{4}, \frac{3\pi}{2})$ and $\bar{Q} \equiv (-\frac{\pi}{4}, \frac{3\pi}{2})$ of the sphere are the unique local minimum points of φ.

Let us now consider the 0th rank invariant of the pair (S, φ). Its graph is depicted in Figure 3 (b). The points $\bar{P}$ and $\bar{Q}$ belong to the same arcwise connected component of the lower level set $S\langle \varphi \leq \varepsilon \rangle$ for every $\varepsilon > 0$, whereas they do not for $\varepsilon = 0$, since the paths $\gamma^i_{\frac{\pi}{2}}$ $(i = 1, 2)$ are an "obstruction" to construct a continuous path from $\bar{P}$ to $\bar{Q}$. Hence, the singular rank invariant $\rho_{(S,\varphi),0}$ is not right-continuous in the second variable at $v = 0$, for any u with $\min \varphi < u < 0$.

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