A note on Harnack inequalities and propagation sets for a class of hypoelliptic operators

Chiara Cinti[*], Kaj Nyström[†] and Sergio Polidoro[‡]

Abstract: In this paper we are concerned with Harnack inequalities for non-negative solutions $u : \Omega \to \mathbb{R}$ to a class of second order hypoelliptic ultraparabolic partial differential equations in the form

$$\mathscr{L} u := \sum_{j=1}^{m} X_j^2 u + X_0 u - \partial_t u = 0$$

where Ω is any open subset of $\mathbb{R}^{N+1}$, and the vector fields $X_1, \dots, X_m$ and $X_0 - \partial_t$ are invariant with respect to a suitable homogeneous Lie group. Our main goal is the following result: for any fixed $(x_0, t_0) \in \Omega$ we give a geometric sufficient condition on the compact sets $K \subseteq \Omega$ for which the Harnack inequality

$$\sup_K u \le C_K \, u(x_0, t_0)$$

holds for all non-negative solutions u to the equation $\mathscr{L} u = 0$. We also compare our result with an abstract Harnack inequality from potential theory.

Keywords: Harnack inequality, hypoelliptic operators, potential theory

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[*]Dipartimento di Matematica, Università di Bologna, Piazza di Porta S. Donato 5, 40126 Bologna (Italy). E-mail: cinti@dm.unibo.it

[†]Department of Mathematics and Mathematical Statistics, Umeå University, 90187 Umeå (Sweden). E-mail: kaj.nystrom@math.umu.se

[‡]Dipartimento di Matematica Pura e Applicata, Università di Modena e Reggio Emilia, via Campi 213/b, 41115 Modena (Italy). E-mail: sergio.polidoro@unimore.it

1 Introduction

In this paper we consider second order partial differential operators in $\mathbb{R}^{N+1}$ of the form

$$\mathscr{L} = \sum_{j=1}^{m} X_j^2 + X_0 - \partial_t, \tag{1.1}$$

where $1 \le m \le N$. The X_j's are smooth vector fields on $\mathbb{R}^N$, *i.e.*, if we by $z = (x,t)$ denote points in $\mathbb{R}^{N+1}$, then

$$X_j(x) = \sum_{k=1}^{N} a_j^k(x)\partial_{x_k},$$

where a_j^k is a C^∞ function on $\mathbb{R}^N$, for $k \in \{1,\dots,N\}$. In the following we also often consider the X_j's as vector fields in $\mathbb{R}^{N+1}$. Furthermore, we define the vector field Y on $\mathbb{R}^{N+1}$ as

$$Y := X_0 - \partial_t.$$

A curve $\gamma : [0,T] \to \mathbb{R}^{N+1}$ is said to be $\mathscr{L}$-*admissible* if it is absolutely continuous and satisfies

$$\gamma'(s) = \sum_{j=1}^{m} \omega_j(s) X_j(\gamma(s)) + \lambda(s) Y(\gamma(s)), \qquad \text{for a.e. } s \in [0,T], \tag{1.2}$$

and for some suitable piecewise constant real functions $\omega_1, \dots \omega_m, \lambda$, with $\lambda \ge 0$. We say that γ *connects* (x,t) *to* (ξ,τ), for $t > \tau$, if $\gamma(0) = (x,t)$ and $\gamma(T) = (\xi,\tau)$. We refer the reader to Section 2 for the definition of a homogeneous Lie group $\mathbb{L} = (\mathbb{R}^{N+1}, \circ, d_\lambda)$. In this paper we impose the following restrictions on the vector fields $X_1,\dots,X_m$ and Y:

(H.1) there exists a homogeneous Lie group $\mathbb{L} = (\mathbb{R}^{N+1}, \circ, d_\lambda)$ such that

- (i) $X_1,\dots,X_m, Y$ are left translation invariant on $\mathbb{L}$,
- (ii) $X_1,\dots,X_m$ are d_λ-homogeneous of degree one and Y is d_λ-homogeneous of degree two,

(H.2) for every $(x,t), (\xi,\tau) \in \mathbb{R}^{N+1}$ with $t > \tau$, there exists a $\mathscr{L}$-admissible path $\gamma : [0,T] \to \mathbb{R}^{N+1}$ connecting (x,t) to (ξ,τ).

In order to simplify our exposition we also require the technical assumption $X_0(0) = 0$. We note that operators in (1.1), satisfying assumptions (H.1) and (H.2), have previously been studied by Kogoj and Lanconelli in [11]. Furthermore, we recall that (H.1) and (H.2) yield the well known Hörmander condition [10]

$$\text{rank Lie}\{X_1,\dots,X_m,Y\}(z) = N+1, \qquad \text{for every } z \in \mathbb{R}^{N+1}, \tag{1.3}$$

where $\mathrm{Lie}\{X_1, \dots, X_m, Y\}$ denotes the Lie algebra generated by the vector fields. In particular, (H.1) and (H.2) implies that $\mathscr{L}$ is a hypoelliptic operator, *i.e.*, any distributional solution u to $\mathscr{L}u = 0$ is actually a smooth classical solution. We refer to [11, Proposition 10.1] for the proof of the fact that (H.1) and (H.2) imply (1.3).

This paper is devoted to Harnack inequalities for non-negative solutions to the equation $\mathscr{L}u = 0$ on arbitrary open subsets Ω of $\mathbb{R}^{N+1}$. In particular, for any given $(x_0, t_0) \in \Omega$, we intend to characterize the compact sets $K \subseteq \Omega$ for which the following Harnack inequality holds: *there exists* $C_K > 0$ *such that*

$$\sup_K u \le C_K\, u(x_0, t_0) \tag{1.4}$$

for every non-negative solution u *of* $\mathscr{L}u = 0$ *in* Ω. In order to formulate our main result, we consider any open subset Ω of $\mathbb{R}^{N+1}$, any $(x_0, t_0) \in \Omega$, and we denote by $\mathscr{A}_{(x_0,t_0)} = \mathscr{A}_{(x_0,t_0)}(\Omega)$ the following set

$$\begin{aligned} \mathscr{A}_{(x_0,t_0)} = \big\{ (x,t) \in \Omega \mid \ & \text{there exists an } \mathscr{L}\text{-admissible path } \gamma : [0,T] \to \Omega \\ & \text{such that } \gamma(0) = (x_0,t_0), \gamma(T) = (x,t) \big\}. \end{aligned} \tag{1.5}$$

We will refer to the set $\mathscr{A}_{(x_0,t_0)}$ as the *attainable set* and we let $\mathrm{int}\big(\mathscr{A}_{(x_0,t_0)}\big)$ denote the interior of $\mathscr{A}_{(x_0,t_0)}$. We can now formulate the main result proved in this paper.

Theorem 1.1 *Let* Ω *be an open subset of* $\mathbb{R}^{N+1}$ *and let* $(x_0, t_0) \in \Omega$. *For every compact set* $K \subseteq \mathrm{int}\big(\mathscr{A}_{(x_0,t_0)}\big)$, *there exists a positive constant* $C_K = C_K(\Omega, (x_0, t_0))$, *only dependent on* Ω, (x_0, t_0), K *and on the operator* $\mathscr{L}$, *such that*

$$\sup_K u \le C_K\, u(x_0, t_0),$$

for every non-negative solution u *of* $\mathscr{L}u = 0$ *in* Ω.

We recall that $\mathscr{A}_{(x_0,t_0)}$ is related to the Bony's minimum principle [5] which can be stated as follows. *Let* u *be a solution of* $\mathscr{L}u = 0$. *If* u *attains its minimum at* $(x_0, t_0) \in \Omega$, *then this minimum value is attained at every point of* $\mathscr{A}_{(x_0,t_0)}$.

We recall that operators $\mathscr{L}$ satisfying (H.1)-(H.2) have been studied in [7] in the setting of the Potential Theory. We refer to the monographs [1] and [9]. We point out that Potential Theory provides us with a statement analogous to Theorem 1.1 where the set $\mathscr{A}_{(x_0,t_0)}$ is replaced by the "smallest absorbent set" $\Omega_{(x_0,t_0)}$, see formula (4.3) and Corollary 4.2 below. However, unlike $\mathscr{A}_{(x_0,t_0)}$, the geometric structure of the set $\Omega_{(x_0,t_0)}$ is not explicitly known. Of course, if the identity $\mathscr{A}_{(x_0,t_0)} = \Omega_{(x_0,t_0)}$ was true, then Theorem 1.1 would be a direct consequence of the Potential Theory. We recall that the identity $\mathscr{A}_{(x_0,t_0)} = \Omega_{(x_0,t_0)}$ has been proven for some classes of Hörmander operators, see Remarks 4.3 and 4.4 below. However, due to the presence of the drift term X_0 in the operator $\mathscr{L}$, the same result has not (yet) been proved for operators $\mathscr{L}$ satisfying (H.1)-(H.2). For the convenience of the reader, in Section 4 we recall some definitions from Potential Theory and we discuss, in more details, the relations between the sets $\mathscr{A}_{(x_0,t_0)}$ and $\Omega_{(x_0,t_0)}$.

The proof of Theorem 1.1 relies on an invariant Harnack inequality due to Kogoj and Lanconelli [11, Theorem 7.1]. In addition we use a method introduced in [15] and in [6], which allows us to construct *Harnack chains* along suitable $\mathscr{L}$-admissible paths. Specifically, we say that a finite set $\{z_0, z_1, \dots, z_k\} \subseteq \Omega$ is a *Harnack chain connecting* z_0 *to* z_k if there exist k positive constants $C_1, \dots, C_k$ such that

$$u(z_j) \leq C_j\, u(z_{j-1}), \qquad j = 1, \dots, k, \tag{1.6}$$

for every non-negative solution $u : \Omega \to \mathbb{R}$ to the equation $\mathscr{L}u = 0$. The conclusion of the proof of Theorem 1.1 is achieved by a straightforward compactness argument.

We remark that, among the operators satisfying (H.1)-(H.2), we find general families of well-known operators. We quote, for instance, Kolmogorov operators studied by Lanconelli and Polidoro in [12]. Moreover, our result also applies to heat kernels on Carnot groups. Even if the family of operators $\mathscr{L}$ considered in this paper is wide, our method seems to be applicable to a more general class of hypoelliptic operators $\mathscr{L}$, since we only rely on the *local* properties of homogeneous Lie groups. We refer to the recent paper [4] by Bonfiglioli and Uguzzoni and to its bibliography for problems and conclusions related to our main result.

This paper is organized as follows. In Section 2 we recall some known results concerning operators $\mathscr{L}$ satisfying hypotheses (H.1)-(H.2). In Section 3 we first formulate and prove two statements of Harnack inequalities which are useful for our purposes, then we prove Theorem 1.1. In Section 4 we relate Theorem 1.1 to the general result from Potential Theory mentioned above. Furthermore, we recall a result by Bonfiglioli, Lanconelli and Uguzzoni [2] where a geometric characterization of $\Omega_{(x_0,t_0)}$ is given for operators of the form $\sum_{j=1}^m X_j^2 - \partial_t$, and we explicitly describe the set $\mathscr{A}_{(x_0,t_0)}$ in the case of the Kolmogorov operator $\partial_{x_1}^2 + x_1\partial_{x_2} - \partial_t$, where the drift term is $X_0 = x_1\partial_{x_2}$.

2 Preliminaries

In this section we introduce some notations and recall some basic notions concerning homogeneous Lie groups. We refer to the monograph [3] for a detailed treatment of the subject. Let $\circ$ be a given group law on $\mathbb{R}^{N+1}$ and suppose that the map $(z, \zeta) \mapsto \zeta^{-1} \circ z$ is smooth. Then $\mathbb{L} = (\mathbb{R}^{N+1}, \circ)$ is called a *Lie group*. Moreover, $\mathbb{L}$ is said *homogeneous* if there exists a family of dilations $(d_\lambda)_{\lambda>0}$ on $\mathbb{L}$ and if $(d_\lambda)_{\lambda>0}$ defines an automorphism of the group, *i.e.*,

$$d_\lambda(z \circ \zeta) = (d_\lambda z) \circ (d_\lambda \zeta)\,, \quad \text{for all } z, \zeta \in \mathbb{R}^{N+1} \text{ and } \lambda > 0.$$

Furthermore, the dilation d_λ induces a direct sum decomposition on $\mathbb{R}^N$

$$\mathbb{R}^N = V_1 \oplus \cdots \oplus V_k, \tag{2.1}$$

as follows. If we denote $x = x^{(1)} + x^{(2)} + \cdots + x^{(k)}$ with $x^{(j)} \in V_j$, then $d_\lambda(x,t) = (D(\lambda)x, \lambda^2 t)$, where

$$D(\lambda)\left(x^{(1)} + x^{(2)} + \cdots + x^{(k)}\right) = \left(\lambda x^{(1)} + \lambda^2 x^{(2)} + \cdots + \lambda^k x^{(k)}\right). \tag{2.2}$$

The natural number

$$Q := \mathrm{dim}V_1 + 2\,\mathrm{dim}V_2 + \cdots + k\,\mathrm{dim}V_k + 2 \tag{2.3}$$

is usually called the *homogeneous dimension* of $\mathbb{L}$ with respect to d_λ. We next recall some useful facts about homogeneous Lie groups. As stated in the Introduction, hypotheses (H.1)-(H.2) yield the Hörmander condition (1.3). We note that (H.1) and (1.3) imply that $\mathrm{span}\big\{X_1(0),\dots,X_m(0)\big\} = V_1$. Hence it is not restrictive to assume $m = \mathrm{dim}V_1$ and $X_j(0) = \mathbf{e}_j$ for $j = 1,\dots,m$, where $\{\mathbf{e}_i\}_{1\le i\le N}$ denotes the canonical basis of $\mathbb{R}^N$. We set

$$|x|_{\mathbb{L}} = \left(\sum_{j=1}^{k}\sum_{i=1}^{m_j}\left(x_i^{(j)}\right)^{\frac{2k!}{j}}\right)^{\frac{1}{2k!}}, \qquad \|(x,t)\|_{\mathbb{L}} = \left(|x|_{\mathbb{L}}^{2k!} + |t|^{k!}\right)^{\frac{1}{2k!}},$$

and we observe that the above functions are homogeneous of degree 1, on $\mathbb{R}^N$ and $\mathbb{R}^{N+1}$ respectively, in the sense that

$$\left|\left(\lambda x^{(1)} + \cdots + \lambda^k x^{(k)}\right)\right|_{\mathbb{L}} = \lambda|x|_{\mathbb{L}}, \qquad \|d_\lambda(x,t)\|_{\mathbb{L}} = \lambda\|(x,t)\|_{\mathbb{L}},$$

for every $(x,t)\in\mathbb{R}^{N+1}$ and for any $\lambda > 0$. We define the *quasi-distance* d by setting

$$d(z,\zeta) := \|\zeta^{-1}\circ z\|_{\mathbb{L}}, \qquad z,\zeta\in\mathbb{R}^{N+1}, \tag{2.4}$$

and we introduce the associated ball

$$\mathscr{B}(z_0,r) := \{z\in\mathbb{R}^{N+1} \mid d(z,z_0) < r\}. \tag{2.5}$$

We recall that, for every compact set $K\subset\mathbb{R}^{N+1}$ there exist two positive constants c_K^- and c_K^+, such that

$$c_K^-|z-\zeta| \le d(z,\zeta) \le c_K^+|z-\zeta|^{\frac{1}{k}}, \qquad \text{for all } z,\zeta\in K. \tag{2.6}$$

Here $|\cdot|$ denotes the usual Euclidean distance. For a proof of (2.6) we refer to [14]. From the above inequalities it follows that the topology generated by the quasi-distance d is equivalent to the standard Euclidean topology. For any $z\in\mathbb{R}^{N+1}$ and $H\subset\mathbb{R}^{N+1}$, we define

$$d(z,H) := \inf\{d(z,\zeta) \mid \zeta\in H\},$$

and for any open set $\Omega\subset\mathbb{R}^{N+1}$ and for any $\varepsilon\in]0,1[$, we introduce the set

$$\Omega_\varepsilon = \{z\in\Omega \mid d(z,\partial\Omega)\ge\varepsilon\}. \tag{2.7}$$

3 Harnack inequalities and proof of Theorem 1.1

The purpose of this section is to prove Theorem 1.1, but to do this we first have to develop some results concerning Harnack inequalities. In particular, we first recall an invariant Harnack inequality due to Kogoj and Lanconelli [11], then we prove a version of this inequality which is useful for our purposes. After that, we give a non-local Harnack inequality stated in terms of the $\mathscr{L}$-admissible path.

We first introduce some definitions. Let $Q^- = B \times [-1, 0]$, where $B = \{x \in \mathbb{R}^N : |x|_{\mathbb{L}} \leq 1\}$, and let $S = \left\{(x,t) \in Q^- : \frac{1}{4} \leq -t \leq \frac{3}{4}\right\}$. We define, for any positive R and $(x_0, t_0) \in \mathbb{R}^{N+1}$,

$$Q_R^-(x_0,t_0) = (x_0,t_0) \circ d_R(Q^-), \;\; S_R(x_0,t_0) = (x_0,t_0) \circ d_R(S).$$

We explicitly note the following fact that will be useful in the sequel. There exists a constant $\mu \in]0,1[$ such that

$$Q_{\mu R}^-(x_0,t_0) \subseteq \mathscr{B}((x_0,t_0),R), \quad \text{for every } (x_0,t_0) \in \mathbb{R}^{N+1} \text{ and } R > 0. \tag{3.1}$$

With the above notation, the Harnack inequality proved in [11, Theorem 7.1] reads as follows. *There exist two positive constants θ and M, only depending on the operator $\mathscr{L}$, such that*

$$\sup_{S_{\theta R}(x_0,t_0)} u \leq M\, u(x_0,t_0) \tag{3.2}$$

for every non-negative solution u of $\mathscr{L} u = 0$ in the cylinder $Q_R^-(x_0,t_0)$. To proceed and to state our first result we need to introduce some additional notations. In particular, for $\alpha, \beta, \gamma, \delta \in \mathbb{R}$, with $0 < \alpha < \beta < \gamma < \delta^2$, we set

$$\widetilde{Q}_R^-(x_0,t_0) = \left\{(x,t) \in Q_{\delta R}^-(x_0,t_0) \mid t_0 - \gamma R^2 \leq t \leq t_0 - \beta R^2\right\},$$
$$\widetilde{Q}_R^+(x_0,t_0) = \left\{(x,t) \in Q_{\delta R}^-(x_0,t_0) \mid t_0 - \alpha R^2 \leq t \leq t_0\right\}.$$

We then have the following invariant Harnack inequality for non-negative solutions u of $\mathscr{L} u = 0$.

Theorem 3.1 *Let $\mathscr{L}$ be an operator of the form* (1.1) *and assume that (H.1)-(H.2) are fulfilled. Then there exist constants $M > 1$ and $\alpha, \beta, \gamma, \delta \in]0,1[$, $0 < \alpha < \beta < \gamma < \delta^2$, which only depend on the operator $\mathscr{L}$, such that*

$$\sup_{\widetilde{Q}_R^-(x_0,t_0)} u \leq M \inf_{\widetilde{Q}_R^+(x_0,t_0)} u,$$

for every non-negative solution u of $\mathscr{L} u = 0$ in $Q_R^-(x_0,t_0)$ and for any $R > 0$, $(x_0,t_0) \in \mathbb{R}^{N+1}$.

Proof. By invariance with respect to the dilations d_λ and with respect to left translations it is not restrictive to assume that $R = 1$ and $(x_0,t_0) = (0,0)$. Hence we consider a non-negative solution u to $\mathscr{L} u = 0$ in Q^-. Let $r \in]0,1[$ be a fixed constant, and set $(x_1,t_1) := \left(0, -\frac{\theta^2 r^2}{2}\right)$. We claim that there exist a small $\delta > 0$ such that

$$Q_r^-(x,t) \subseteq Q^-, \quad (x_1,t_1) \in S_{\theta r}(x,t) \quad \text{for any } (x,t) \in Q_\delta^-(0,0). \tag{3.3}$$

The above statement is a direct consequence of the continuity of the Lie group law "$\circ$". The first assertion is trivial. To prove the second one we first note that (x_1, t_1) is an interior point of $S_{\theta r}(0,0)$. Then, since $(x,t)^{-1} \circ (x_1, t_1) \to (x_1, t_1)$ as (x,t) tends to $(0,0)$, there exists a sufficiently small positive δ such that $(x_1, t_1) \in S_{\theta r}(x,t)$ for any $(x,t) \in Q_\delta^-(0,0)$. Thus, from (3.2) and (3.3) it follows that

$$u(x_1, t_1) \le M \inf_{Q_\delta^-(0,0)} u. \tag{3.4}$$

Moreover, if δ is small enough, we also have $Q_{\delta/\theta}^-(x_1, t_1) \subseteq Q^-$, thus, by using again (3.2), we find

$$\sup_{S_\delta(x_1,t_1)} u \le M\, u(x_1, t_1).$$

Since we assume $X_0(0) = 0$, we have $S_\delta(x_1, t_1) = \widetilde{Q}_1^-(0,0)$ for some suitable $\beta, \gamma \in]0,1[$. This concludes the proof of Theorem 3.1. □

Our second result is a non-local Harnack inequality which is stated in terms of the $\mathscr{L}$-admissible paths defined in (1.2). We recall that the $\mathscr{L}$-admissible paths with $\lambda \equiv 1$ are the trajectories that naturally contain the *Harnack chains* (1.6) for the operators $\mathscr{L}$ satisfying assumptions (H.1)-(H.2). These trajectories have previously been used in the papers [16] and [6] to prove Gaussian lower bounds for the fundamental solution. In this paper we improve upon Proposition 3.4 in [16] and Proposition 1.1 in [6] as we allow Ω to be an arbitrary open set of $\mathbb{R}^{N+1}$. An useful tool in the proof is Lemma 2.2 in [6]. Here we give an equivalent statement of it in terms of our notation. *Let $\gamma : [0,T] \to \mathbb{R}^{N+1}$ be an $\mathscr{L}$-admissible path with $\lambda \equiv 1$ and let a, b be two constants such that $0 \le a < b \le T$. Then there exists a positive constant h, only dependent on $\mathscr{L}$, such that*

$$\int_a^b |\omega(s)|^2 ds \le h \quad \Rightarrow \quad \gamma(b) \in \widetilde{Q}_r^-(\gamma(a)), \ \text{with } r = \sqrt{\frac{b-a}{\beta}}. \tag{3.5}$$

In the above statement, as well as in the sequel, we may allow $\omega_1, \dots, \omega_m \in L^2([0,T])$ in the definition of $\mathscr{L}$-admissible paths (1.2). We recall the definition (2.7) of Ω_ε.

Theorem 3.2 *Let $\mathscr{L}$ be an operator of the form* (1.1) *and assume that (H.1)-(H.2) are fulfilled. Let Ω be an open subset of $\mathbb{R}^{N+1}$, and let $\varepsilon \in]0,1]$ be small enough to ensure that $\Omega_\varepsilon \neq \emptyset$. Let γ an $\mathscr{L}$-admissible path as in* (1.2) *contained in Ω_ε, and assume that $\omega_1, \dots \omega_m \in L^2([0,T])$, and that λ is measurable with $\inf_{[0,T]} \lambda > 0$. Then there exists a constant $C(\gamma, \varepsilon) > 0$ which also depends on the operator $\mathscr{L}$, such that*

$$u(\xi, \tau) \le C(\gamma, \varepsilon)\, u(x,t), \qquad (x,t) = \gamma(0), \ (\xi, \tau) = \gamma(T),$$

for every non-negative solution u of $\mathscr{L} u = 0$ in Ω. Moreover,

$$C(\gamma, \varepsilon) = \exp\left(c_0 + c_1 \frac{t - \tau}{\varepsilon^2} + c_2 \int_0^T \frac{\omega_1^2(s) + \dots + \omega_m^2(s)}{\lambda(s)} ds \right)$$

for positive constants c_0, c_1, c_2 only depending on the operator $\mathscr{L}$.

Remark 3.3 *The Harnack constant in the above theorem blows up as* $\lambda \to 0$. *On the other hand, in the proof of Theorem 1.1, we show that any* $\mathscr{L}$*-admissible path* (1.2) *can be approximated by a suitable* $\mathscr{L}$*-admissible path* γ_ε *with* $\inf \lambda_\varepsilon \geq \varepsilon$.

Proof of Theorem 3.2. To prove Theorem 3.2 we proceed along the lines of the proof of Proposition 1.1 in [6]. In particular, we aim to use (3.5) to construct a Harnack chain connecting (x, t) to (ξ, τ). In order to use (3.5), we first consider the case $\lambda \equiv 1$, so that $T = t - \tau$. Then, at the end of the proof, we will remove this additional assumption. We next show that, assuming $\lambda \equiv 1$, there exists a finite sequence $\sigma_0, \sigma_1, \dots, \sigma_k \in [0, t-\tau]$, with $0 = \sigma_0 < \sigma_1 < \cdots < \sigma_k = t - \tau$, such that

$$u(\gamma(\sigma_j)) \leq M\, u(\gamma(\sigma_{j-1})), \quad j = 1, \dots, k,$$

where $M > 1$ is the constant in Theorem 3.1. Hence

$$u(\gamma(t-\tau)) \leq M^k u(\gamma(0)). \tag{3.6}$$

To prove the above claim we first show that there exist positive numbers $r_0, r_1, \dots, r_{k-1}$, such that

$$Q^-_{r_j}(\gamma(\sigma_j)) \subset \Omega, \quad \gamma(\sigma_{j+1}) \in \widetilde{Q}^-_{r_j}(\gamma(\sigma_j)) \qquad j = 0, 1, \dots, k-1. \tag{3.7}$$

Then, we can apply Theorem 3.1 k times to complete the proof. However, to conclude the proof of Theorem 3.2 we also have to establish a suitable bound for k.

We next prove (3.7). We first note that $\gamma(\sigma) \in \Omega_\varepsilon$ for every $\sigma \in [0, t-\tau]$, then by (3.1) there exists $\mu \in]0,1[$ such that $Q^-_{\mu\varepsilon}(\gamma(\sigma)) \subset \mathscr{B}((\gamma(\sigma)), \varepsilon) \subset \Omega$. Thus, in order to have $Q^-_r(\gamma(\sigma)) \subset \Omega$, it is sufficient to choose $r \in]0, \mu\varepsilon]$. We next select $\sigma_0, \dots, \sigma_k \in [0, t-\tau]$ recursively. Suppose that $\sigma_0, \dots, \sigma_j$ and $r_0, \dots, r_{j-1}$ have been chosen and satisfy (3.7). In order to choose $r_j \in]0, \mu\varepsilon]$ and $\sigma_{j+1} \in]\sigma_j, t-\tau]$ such that (3.7) holds, we rely on (3.5) with $a = \sigma_j$, and $b = \sigma_{j+1}$. To this aim, we set

$$\sigma_{j+1} = \min\left\{\sigma_j + \beta\,(\mu\varepsilon)^2, \inf\left\{\sigma \in]\sigma_j, t-\tau] : \int_{\sigma_j}^{\sigma} \frac{|\omega(s)|^2}{h}\, ds > 1\right\}\right\}. \tag{3.8}$$

Note that, as the L^2 norm of ω is assumed to be finite, there exists a integer $j =: k-1$ such that the integral in (3.8) does not exceed 1. In this case we agree to set $\sigma_k = t - \tau$. Based on the definition in (3.8) we see that σ_{j+1} satisfies the restrictions

$$\max\left\{\frac{\sigma_{j+1} - \sigma_j}{\beta\,(\mu\varepsilon)^2}, \int_{\sigma_j}^{\sigma_{j+1}} \frac{|\omega(s)|^2}{h}\, ds\right\} \leq 1, \qquad 0 \leq \sigma_j < \sigma_{j+1} \leq t - \tau, \tag{3.9}$$

and we see that if we let $r_j = \sqrt{\frac{\sigma_{j+1}-\sigma_j}{\beta}}$, then $r_j \leq \mu\varepsilon$. Hence if we initialize the recursion by setting $\sigma_0 = 0$ then the sequences $\{\sigma_j\}_{j=0}^k$ and $\{r_j\}_{j=0}^{k-1}$ are well-defined and satisfy (3.7). It now only remains to estimate k. However, from the definition in (3.8) it first follows that

$$1 < \int_{\sigma_{j-1}}^{\sigma_j} \left(\frac{|\omega(s)|^2}{h} + \frac{1}{\beta\,(\mu\varepsilon)^2} \right) ds \leq 2, \qquad j = 1, \dots, k-1,$$
$$0 < \int_{\sigma_{k-1}}^{t-\tau} \left(\frac{|\omega(s)|^2}{h} + \frac{1}{\beta\,(\mu\varepsilon)^2} \right) ds \leq 2,$$

and then, by summation, that

$$k - 1 < \int_0^{t-\tau} \left(\frac{|\omega(s)|^2}{h} + \frac{1}{\beta\,(\mu\varepsilon)^2} \right) ds \leq 2k.$$

In particular, the inequalities in the last display imply that

$$k \leq 1 + \frac{t-\tau}{c\,\varepsilon^2} + \frac{1}{h} \int_0^{t-\tau} |\omega(s)|^2 ds. \tag{3.10}$$

The proof of Theorem 3.2, in the case $\lambda \equiv 1$, is a direct consequence of (3.6) and (3.10), if we set

$$c_0 = \log(M), \qquad c_1 = \frac{\log(M)}{c}, \qquad c_2 = \frac{\log(M)}{h}. \tag{3.11}$$

We next remove the assumption $\lambda \equiv 1$. In that order, we consider any measurable function $\lambda : [0,T] \to \mathbb{R}$ such that $\inf_{[0,T]} \lambda > 0$ and we introduce the function $\varphi : [0,T] \to [0, t-\tau]$ through the relation

$$\varphi(s) = \int_0^s \lambda(\rho) d\rho, \;\; s \in [0,T].$$

We define the function $\widetilde{\gamma}(s) := \gamma(\varphi^{-1}(s))$. This function satisfies

$$\widetilde{\gamma} : [0, t-\tau] \to \Omega, \quad \widetilde{\gamma}(0) = (x,t), \quad \widetilde{\gamma}(t-\tau) = (\xi, \tau)$$
$$\widetilde{\gamma}'(s) = \sum_{j=1}^m \frac{\omega_j(\varphi^{-1}(s))}{\lambda(\varphi^{-1}(s))} X_j(\widetilde{\gamma}_\varepsilon(s)) + Y(\widetilde{\gamma}(s)), \quad \text{for a.e. } s \in [0, t-\tau].$$

We then apply the first part of the proof to $\widetilde{\gamma}$ and we note that

$$\int_0^{t-\tau} \left(\frac{\omega_1(\varphi^{-1}(s))}{\lambda(\varphi^{-1}(s))} \right)^2 + \dots + \left(\frac{\omega_m(\varphi^{-1}(s))}{\lambda(\varphi^{-1}(s))} \right)^2 ds = \int_0^T \frac{\omega_1^2(\rho) + \dots + \omega_m^2(\rho)}{\lambda(\rho)} d\rho.$$

This accomplishes the proof of Theorem 3.2 □

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let K be a compact subset of $\text{int}\big(\mathscr{A}_{(x_0,t_0)}\big)$. Consider any $(x,t) \in K$. Since $(x,t) \in \text{int}\big(\mathscr{A}_{(x_0,t_0)}\big)$, it follows that

$$Q_r^-(\bar{x},\bar{t}) \subset \mathscr{A}_{(x_0,t_0)}, \qquad (\bar{x},\bar{t}) = (x,t) \circ \left(0, -r^2 \frac{\beta+\gamma}{2}\right)^{-1},$$

for all positive sufficiently small r. In the last display β, γ are defined as in Theorem 3.1. Applying Theorem 3.1 we see that

$$\sup_{Q_r(x,t)} u \le M \inf_{\widetilde{Q}_r^+(\bar{x},\bar{t})} u, \qquad Q_r(x,t) := \widetilde{Q}_r^-(\bar{x},\bar{t}). \tag{3.12}$$

Note that $Q_r(x,t)$ is an open neighborhood of (x,t). We next show that there exists a Harnack chain starting from (x_0,t_0) with end point in $\widetilde{Q}_r^+(\bar{x},\bar{t})$. To see this we first note that, by the very definition of $\mathscr{A}_{(x_0,t_0)}$, there exist $T > 0$ and a $\mathscr{L}$-admissible curve $\gamma : [0,T] \to \Omega$, defined by $\omega_1, ..., \omega_m, \lambda$, connecting (x_0,t_0) to $(\bar{x},\bar{t})$. Moreover, for every positive ε, we denote by γ_ε the solution to

$$\begin{aligned}
&\gamma_\varepsilon : [0,T] \to \mathbb{R}^{N+1}, \qquad \gamma_\varepsilon(0) = (x_0,t_0),\\
&\gamma_\varepsilon'(s) = \sum_{j=1}^m \omega_j(s) X_j(\gamma_\varepsilon(s)) + (\lambda(s)+\varepsilon) Y(\gamma_\varepsilon(s)), \quad \text{for a.e. } s \in [0,T].
\end{aligned}$$

In particular, since γ_ε converges uniformly to γ as $\varepsilon \to 0$, and $\gamma([0,T])$ is a compact subset of Ω, it is possible to choose ε such that $\gamma_\varepsilon([0,T])$ is a compact subset of Ω. Note that if we let $\gamma_\varepsilon(T) = (x_\varepsilon, t_\varepsilon)$, then $t_\varepsilon = \bar{t} - \varepsilon T$. Hence $(x_\varepsilon, t_\varepsilon) \in \widetilde{Q}_r^+(\bar{x},\bar{t})$, provided ε is suitably small. Since $\inf_{[0,T]}(\lambda(s)+\varepsilon) \ge \varepsilon$, we can apply Theorem 3.2 and we find that there exists a positive constant $C(\gamma,\varepsilon)$ such that

$$u(x_\varepsilon, t_\varepsilon) \le C(\gamma,\varepsilon)\, u(x_0,t_0). \tag{3.13}$$

Then, using (3.12) and (3.13) we conclude that there exists an open neighborhood $Q_r(x,t)$ of (x,t), and a positive constant $C(\gamma,\varepsilon)$ only depending on (x,t), such that

$$\sup_{Q_r(x,t)} u \le M\, C(\gamma,\varepsilon)\, u(x_0,t_0).$$

Theorem 1.1 then follows from a standard covering argument. □

4 Potential Theory

In this section, we first recall some known result concerning the Potential Theory for the operator $\mathscr{L}$ and we give the definition of the *absorbent set*. We then discuss, in detail, the relation between the absorbent sets and the attainable sets (1.5).

In the following we let Ω be any open subset of $\mathbb{R}^{N+1}$. If $u : \Omega \to \mathbb{R}$ is a smooth function satisfying $\mathscr{L}u = 0$ in Ω, then we say that u is *$\mathscr{L}$-harmonic* in Ω. We denote by $\mathscr{H}(\Omega)$ the linear space of functions which are $\mathscr{L}$-harmonic in Ω. Given a bounded open set $V \subset \mathbb{R}^{N+1}$, we say that V is *$\mathscr{L}$-regular* if for any $\varphi \in C(\partial V, \mathbb{R})$ there exists a unique function $H_\varphi^V \in \mathscr{H}(V)$ such that $\lim_{z \to z_0} H_\varphi^V(z) = \varphi(z_0)$ for every $z_0 \in \partial V$. If this function exists, then $H_\varphi^V \geq 0$ whenever $\varphi \geq 0$, as the classical Picone's maximum principle holds for $\mathscr{L}$ [11, Proposition 2.1]. Furthermore, if V is $\mathscr{L}$-regular, then, for every fixed $z \in V$, the map $\varphi \mapsto H_\varphi^V(z)$ defines a linear positive functional on $C(\partial V, \mathbb{R})$. Hence, Riesz representation theorem implies that there exists a Radon measure μ_z^V, supported in ∂V, such that

$$H_\varphi^V(z) = \int_{\partial V} \varphi(\zeta)\, \mathrm{d}\mu_z^V(\zeta), \qquad \text{for every } \varphi \in C(\partial V, \mathbb{R}). \tag{4.1}$$

We will refer to μ_z^V as the *$\mathscr{L}$-harmonic measure* defined with respect to V and z.

We recall that a lower semi-continuous function $u : \Omega \to]-\infty, \infty]$ is said to be *$\mathscr{L}$-superharmonic* in Ω if $u < \infty$ in a dense subset of Ω and if

$$u(z) \geq \int_{\partial V} u(\zeta)\, \mathrm{d}\mu_z^V(\zeta),$$

for every open $\mathscr{L}$-regular set $V \subset \overline{V} \subset \Omega$ and for every $z \in V$. We denote by $\overline{\mathscr{S}}(\Omega)$ the set of $\mathscr{L}$-superharmonic functions in Ω, and by $\overline{\mathscr{S}}^+(\Omega)$ the set of the functions in $\overline{\mathscr{S}}(\Omega)$ which are non-negative. A function $v : \Omega \to [-\infty, \infty[$ is said to be *$\mathscr{L}$-subharmonic* in Ω if $-v \in \overline{\mathscr{S}}(\Omega)$ and we write $\underline{\mathscr{S}}(\Omega) := -\overline{\mathscr{S}}(\Omega)$. As the collection of $\mathscr{L}$-regular sets is a basis for the Euclidean topology, it follows that $\overline{\mathscr{S}}(\Omega) \cap \underline{\mathscr{S}}(\Omega) = \mathscr{H}(\Omega)$.

With the terminology of the Potential Theory [1, 9], the map $\mathbb{R}^{N+1} \supseteq \Omega \mapsto \mathscr{H}(\Omega)$ is a *harmonic sheaf* and $(\mathbb{R}^{N+1}, \mathscr{H})$ is a *harmonic space.* Since the constant functions are $\mathscr{L}$-harmonic, the second statement is a consequence of the following properties:

- the $\mathscr{L}$-regular sets form a basis for the Euclidean topology [5, Corollaire 5.2];
- $\mathscr{H}$ satisfies *the Doob convergence property*, *i.e.*, the pointwise limit of any increasing sequence of $\mathscr{L}$-harmonic functions, on any open set, is $\mathscr{L}$-harmonic whenever the pointwise limit is finite in a dense set [11, Proposition 7.4];
- the family $\overline{\mathscr{S}}(\mathbb{R}^{N+1})$ *separates the points of* $\mathbb{R}^{N+1}$, *i.e.*, for every $z, \zeta \in \mathbb{R}^{N+1}$, $z \neq \zeta$, there exists $u \in \overline{\mathscr{S}}(\mathbb{R}^{N+1})$ such that $u(z) \neq u(\zeta)$ [7, Proposition 7.1].

We remark that $(\mathbb{R}^{N+1}, \mathscr{H})$ enjoys of the stronger property that the family $\overline{\mathscr{S}}^+(\mathbb{R}^{N+1}) \cap C(\mathbb{R}^{N+1})$ separates points of $\mathbb{R}^{N+1}$ [7, Proposition 7.1]. For this reason, $(\mathbb{R}^{N+1}, \mathscr{H})$ is said to be a *$\mathfrak{B}$-harmonic space.*

We summarize the above facts in the following remark for further reference.

Remark 4.1 *Let $\mathscr{L}$ be an operator of the form* (1.1) *and assume that (H.1)-(H.2) are fulfilled. Let Ω be an open subset of $\mathbb{R}^{N+1}$. Then, the linear space of the $\mathscr{L}$-harmonic functions in Ω, $\mathscr{H}(\Omega)$, is a* harmonic sheaf *and $(\mathbb{R}^{N+1}, \mathscr{H})$ is a $\mathfrak{B}$-*harmonic space.

It is noteworthy that *Wiener resolutivity theorem* holds in $\mathfrak{B}$-harmonic spaces. To use the Wiener theorem, as well as other general results from Potential Theory [1, 9], we introduce some additional notations. We recall that if $\Omega \subset \mathbb{R}^{N+1}$ is a bounded open set, then an extended real function $f : \partial\Omega \to [-\infty, \infty]$ is called *resolutive* if

$$\inf \overline{\mathscr{U}}_f^\Omega = \sup \underline{\mathscr{U}}_f^\Omega =: H_f^\Omega \in \mathscr{H}(\Omega),$$

where

$$\overline{\mathscr{U}}_f^\Omega := \big\{u \in \overline{\mathscr{S}}(\Omega) : \inf_\Omega u > -\infty \text{ and } \liminf_{z\to\zeta} u(z) \geq f(\zeta),\ \forall\, \zeta \in \partial\Omega\big\},$$
$$\underline{\mathscr{U}}_f^\Omega := \big\{u \in \underline{\mathscr{S}}(\Omega) : \sup_\Omega u < \infty \text{ and } \limsup_{z\to\zeta} u(z) \leq f(\zeta),\ \forall\, \zeta \in \partial\Omega\big\}.$$

We say that H_f^Ω is the *generalized solution in the sense of Perron-Wiener-Brelot* to the problem

$$\begin{cases} u \in \mathscr{H}(\Omega), \\ u = f \quad \text{on } \partial\Omega. \end{cases}$$

By Remark 4.1, Wiener resolutivity theorem implies that every $f \in C(\partial\Omega, \mathbb{R})$ is resolutive. The map $C(\partial\Omega, \mathbb{R}) \ni f \mapsto H_f^\Omega(z)$ defines a positive functional for every $z \in \Omega$. Again, there exists a Radon measure μ_z^Ω on $\partial\Omega$ such that

$$H_f^\Omega(z) = \int_{\partial\Omega} f(\zeta)\, \mathrm{d}\mu_z^\Omega(\zeta). \tag{4.2}$$

We call μ_z^Ω the *$\mathscr{L}$-harmonic measure* relative to Ω and z, and when Ω is $\mathscr{L}$-regular this definition coincides with the one in (4.1). We recall that $f : \partial\Omega \to [-\infty, \infty]$ is resolutive if and only if $f \in L^1(\partial\Omega, \mu_z^\Omega)$ for every $z \in \Omega$, and in this case $H_f^\Omega(z) = \int_{\partial\Omega} f\, \mathrm{d}\mu_z^\Omega$. Finally, a point $\zeta \in \partial\Omega$ is called *$\mathscr{L}$-regular* for Ω if

$$\lim_{\Omega\ni z\to\zeta} H_f^\Omega(z) = f(\zeta), \qquad \text{for every } f \in C(\partial\Omega, \mathbb{R}).$$

Obviously, Ω is $\mathscr{L}$-regular if and only if every $\zeta \in \partial\Omega$ is $\mathscr{L}$-regular.

Let $\Omega \subset \mathbb{R}^{N+1}$ be an open set. A closed subset F of Ω is called an *absorbent set* if for any $z \in F$ and any neighborhood U of z, there exists a $\mathscr{L}$-regular neighborhood V of z contained in U such that $\mu_z^V(\Omega \setminus F) = 0$. For any given $(x_0, t_0) \in \Omega$ we set

$$\mathscr{F}_{(x_0,t_0)} = \{F \subseteq \Omega : F \ni (x_0, t_0), F \text{ is an absorbent set}\}$$

and we let

$$\Omega_{(x_0,t_0)} = \bigcap_{F\in\mathscr{F}_{(x_0,t_0)}} F \tag{4.3}$$

denote the *smallest* absorbent set containing (x_0, t_0). The following general result, analogous to our Theorem 1.1, holds. *Let $(\mathbb{R}^{N+1}, \mathscr{H})$ be a $\mathfrak{B}$-harmonic space, let Ω*

be an open subset of $\mathbb{R}^{N+1}$ *and let* $(x_0, t_0) \in \Omega$*. Then, for every compact set* $K \subset \operatorname{int}(\Omega_{(x_0,t_0)})$ *there exists a positive constant* C_K *such that*

$$\sup_K u \le C_K\, u(x_0, t_0),$$

for every non-negative function $u \in \mathscr{H}(\Omega)$*.* We refer to Theorem 1.4.4 in [1] and Proposition 6.1.5 in [9]. As a consequence of the above inequality, $\Omega_{(x_0,t_0)}$ is a propagation set in the sense that if $u(x_0, t_0) = \min_\Omega u$, then $u \equiv u(x_0, t_0)$ in $\Omega_{(x_0,t_0)}$. Using Remark 4.1 we see that the stated results apply to our operator $\mathscr{L}$. In particular, we have

Corollary 4.2 *Let* $\mathscr{L}$ *be an operator of the form* (1.1) *and assume that (H.1)-(H.2) are fulfilled, let* Ω *be an open subset of* $\mathbb{R}^{N+1}$ *and let* $(x_0, t_0) \in \Omega$*. For every compact set* $K \subset \operatorname{int}(\Omega_{(x_0,t_0)})$ *there exists a positive constant* C_K *such that*

$$\sup_K u \le C_K\, u(x_0, t_0),$$

for every non-negative solution u *of* $\mathscr{L}u = 0$ *in* Ω*.*

As mentioned in the introduction in general it is not easy to give a geometric characterization of $\Omega_{(x_0,t_0)}$ since it is defined in terms of the carrier set of the $\mathscr{L}$-harmonic measure μ_z^V. Nevertheless, characterizations of the set $\Omega_{(x_0,t_0)}$ have been given for some classes of operators.

Remark 4.3 *Stationary Hörmander operators in the form of sum of squares in* $\mathbb{R}^N$ *give rise to elliptic harmonic spaces, we refer to [9] for the definition. It is known that in this case the relevant absorbent set* Ω_{x_0} *agrees with the connected component of* Ω *containing* x_0*, which coincides with the attainable set* $\mathscr{A}_{x_0}$*. In particular,* $\Omega_{x_0} = \mathscr{A}_{x_0}$*.*

Remark 4.4 *Consider operators of the form* $\mathscr{L} = \sum_{j=1}^m X_j^2 - \partial_t$ *having no drift term* X_0*. In this case, Bonfiglioli, Lanconelli and Uguzzoni have proved that* $\Omega_{(x_0,t_0)} = \Omega \cap \{t \le t_0\}$*, where* $\Omega = B \times]t_1, t_2[$ *is a cylinder and* B *is a regular domain with respect to the stationary operator* $\sum_{j=1}^m X_j^2$*, see formula (4.3) in [2]. Thus, also in this case, we have* $\Omega_{(x_0,t_0)} = \mathscr{A}_{(x_0,t_0)}$*.*

However, the presence of the drift term considerably changes the geometric structure of $\mathscr{A}_{(x_0,t_0)}$ and, seemingly, the one of $\Omega_{(x_0,t_0)}$. To clarify this fact, in the following example we consider the simplest degenerate Kolmogorov operator in the variables $(x_1, x_2, t) \in \mathbb{R}^3$,

$$\mathscr{L}u = X^2 u + Yu = 0, \qquad X = \partial_{x_1}, \quad \text{and} \quad Y = x_1 \partial_{x_2} - \partial_t. \tag{4.4}$$

The fact that the Kolmogorov operator $\mathscr{L}$ satisfies the hypotheses (H.1)-(H.2) is discussed in several paper, see for instance [11], [6] and [8]. Consider the domain

$$\Omega =]-R, R[\times]-1,1[\times]-1,1], \tag{4.5}$$

where R is a given positive constant. In this case

$$\mathscr{A}_{(0,0,0)} = \big\{(x_1, x_2, t) \in \Omega : |x_2| \le -tR\big\}. \tag{4.6}$$

We also recall that the points of the sets

$$\big\{(x_1, x_2, t) \in \partial\Omega : t = -1\big\}, \qquad \big\{(x_1, x_2, t) \in \partial\Omega : |x_1| = 1\big\},$$
$$\big\{(x_1, -1, t) \in \partial\Omega : x_1 \le 0\big\}, \qquad \big\{(x_1, 1, t) \in \partial\Omega : x_1 \ge 0\big\}$$

are $\mathscr{L}$-regular for Ω [13, Example 6.5]. The following result shows that the strong minimum principle cannot hold in a set bigger than $\mathscr{A}_{(0,0,0)}$.

Proposition 4.5 *Let Ω be defined as in* (4.5)*. Then there exists a solution $u \ge 0$ of* (4.4) *in Ω such that $u \equiv 0$ in $\mathscr{A}_{(0,0,0)}$ and such that $u > 0$ in $\Omega \setminus \mathscr{A}_{(0,0,0)}$.*

Proof. Let φ be any function in $C(\partial\Omega)$, such that $\varphi \equiv 0$ in $\partial\Omega \cap \overline{\mathscr{A}_{(0,0,0)}}$ and $\varphi > 0$ in $\partial\Omega \setminus \overline{\mathscr{A}_{(0,0,0)}}$. Then the Perron-Wiener-Brelot solution $u := H_\varphi^\Omega$ of the following Cauchy-Dirichlet problem

$$\begin{cases} X^2 u + Y u = 0 & \text{in } \Omega \\ u = \varphi & \text{in } \partial\Omega \end{cases}$$

is non-negative. We next prove that $u > 0$ in $\Omega \setminus \mathscr{A}_{(0,0,0)}$. By contradiction, we suppose that there exists $(x_1, x_2, t) \in \Omega \setminus \mathscr{A}_{(0,0,0)}$ such that $u(x_1, x_2, t) = 0$. Then (x_1, x_2, t) is a minimum for u, so that from Bony's minimum principle it follows that $u(R, x_2, t) = \varphi(R, x_2, t) = 0$. This contradicts the assumption on φ. Suppose now that there exists $(x_1, x_2, t) \in \mathscr{A}_{(0,0,0)}$ such that $u(x_1, x_2, t) > 0$. Since every point of the set $\partial\Omega \cap \overline{\mathscr{A}_{(0,0,0)}}$ is $\mathscr{L}$-regular, u is continuous in $\overline{\mathscr{A}_{(0,0,0)}}$. Hence there exists a $(\bar{x}_1, \bar{x}_2, \bar{t}) \in \overline{\mathscr{A}_{(0,0,0)}}$ such that $u(\bar{x}_1, \bar{x}_2, \bar{t}) = \max_{\overline{\mathscr{A}_{(0,0,0)}}} u > 0$. By Bony's minimum principle we then have $u(R, \bar{x}_2, \bar{t}) = \varphi(R, \bar{x}_2, \bar{t}) > 0$, and this contradicts our assumption on φ. ☐

Remark 4.6 *The above example shows that a Harnack inequality cannot hold in a set bigger than $\mathscr{A}_{(0,0,0)}$. Indeed, the example shows that it is impossible to find a positive constant C such that $u(x_1, x_2, t) \le C\, u(0,0,0)$ whenever $(x_1, x_2, t) \notin \mathscr{A}_{(0,0,0)}$. Hence, as a consequence of Corollary 4.2, we have that* $\operatorname{int}\big(\Omega_{(0,0,0)}\big) \subseteq \mathscr{A}_{(0,0,0)}$.

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