NECESSARY CONDITIONS FOR DISCONTINUITIES OF MULTIDIMENSIONAL SIZE FUNCTIONS

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Abstract. Some new results about multidimensional Topological Persistence are presented, proving that the discontinuity points of a k-dimensional size function are necessarily related to the pseudocritical or special values of the associated measuring function.

Introduction

Topological Persistence is devoted to the study of stable properties of sublevel sets of topological spaces and, in the course of its development, has revealed itself to be a suitable framework when dealing with applications in the field of Shape Analysis and Comparison. Since the beginning of the 1990s research on this subject has been carried out under the name of Size Theory, studying the concept of *size function*, a mathematical tool able to describe the qualitative properties of a shape in a quantitative way. More precisely, the main idea is to model a shape by a topological space $\mathcal{M}$ endowed with a continuous function φ, called *measuring function*. Such a function is chosen according to applications and can be seen as a descriptor of the features considered relevant for shape characterization. Under these assumptions, the size function $\ell_{(\mathcal{M},\varphi)}$ associated with the pair $(\mathcal{M},\varphi)$ is a descriptor of the topological attributes that persist in the sublevel sets of $\mathcal{M}$ induced by the variation of φ. According to this approach, the problem of comparing two shapes can be reduced to the simpler comparison of the related size functions. Since their introduction, these shape descriptors have been widely studied and applied in quite a lot of concrete applications concerning Shape Comparison and Pattern Recognition (cf., e.g., [4, 8, 15, 34, 35, 36]). From a more theoretical point of view, the notion of size function plays an essential role since it is strongly related to the one of *natural pseudodistance*. This is another key tool of Size Theory, defining a (dis)similarity measure between compact and locally connected topological spaces endowed with measuring functions (see [3] for historical references and [16, 18, 19] for a detailed review about the concept of natural pseudodistance). Indeed, size functions provide easily computable lower bounds for the natural pseudodistance (cf. [12, 13, 17]).

Approximately ten years after the introduction of Size Theory, Persistent Homology re-proposed similar ideas from the homological point of view (cf. [22]; for a survey on this topic see [21]). In this context, the notion of size function coincides with the dimension of the 0-th persistent homology group, i.e. the 0-th rank invariant [7].

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We refer the interested reader to Appendix A for more information about the relationship existing between Size Theory and Persistent Homology.

The study of Topological Persistence is capturing more and more attention in the mathematical community, with particular reference to the multidimensional setting (see [21, 29]). When dealing with size functions, the term *multidimensional* means that the measuring functions are vector-valued, and has no reference to the dimension of the topological space under study. However, while the basic properties of a size function ℓ are now clear when it is associated with a measuring function φ taking values in $\mathbb{R}$, very little is known when φ takes values in $\mathbb{R}^k$. More precisely, some questions about the structure of size functions associated with $\mathbb{R}^k$-valued measuring functions need further investigation, with particular reference to the localization of their discontinuities. Indeed, this last research line is essential in the development of efficient algorithms allowing us to apply Topological Persistence to concrete problems in the multidimensional context.

In this paper we start to fill this gap by proving a new result on the discontinuities of the so-called multidimensional size functions, showing that they can be located only at points with at least one *pseudocritical* or *special* coordinate (Theorem 2.11 and Theorem 2.13). This is proved by using an approximation technique and the theoretical machinery developed in [2], improving the comprehension of multidimensional Topological Persistence and laying the basis for its computation.

This paper is organized in two sections. In Section 1 the basic results about multidimensional size functions are recalled, while in Section 2 our main theorems are proved.

1. Preliminary Results on Size Theory

The main idea in Size Theory is to study a given shape by performing a geometrical/topological exploration of a suitable topological space $\mathcal{M}$, with respect to some properties expressed by an $\mathbb{R}^k$-valued continuous function $\vec{\varphi} = (\varphi_1, \dots, \varphi_k)$ defined on $\mathcal{M}$. Following this approach, Size Theory introduces the concept of *size function* as a stable and compact descriptor of the topological changes occurring in the lower level sets $\{P \in \mathcal{M} : \varphi_i(P) \le t_i, i = 1, \dots, k\}$ as $\vec{t} = (t_1, \dots, t_k)$ varies in $\mathbb{R}^k$.

In this section we recall some basic definitions and results about size functions, confining ourselves to those that will be useful in the rest of this paper. For a deeper investigation on these topics, the reader is referred to [2, 3, 28]. For further details about Topological Persistence in the multidimensional setting, see [7, 28].

In proving our new results we need to assume that $\mathcal{M}$ is a closed C^1 Riemannian manifold. However, we prefer to report here the basic concepts of Size Theory in their classical formulation, i.e. by supposing that $\mathcal{M}$ is a non-empty compact and locally connected Hausdorff space. We shall come back to the case of a C^1 Riemannian manifold later.

In the context of Size Theory, any pair $(\mathcal{M}, \vec{\varphi})$, where $\vec{\varphi} = (\varphi_1, \dots, \varphi_k) : \mathcal{M} \to \mathbb{R}^k$ is a continuous function, is called a *size pair*. The function $\vec{\varphi}$ is said to be a *k-dimensional measuring function*. The relations $\preceq$ and $\prec$ are defined in $\mathbb{R}^k$ as follows: for $\vec{x} = (x_1, \dots, x_k)$ and $\vec{y} = (y_1, \dots, y_k)$, we write $\vec{x} \preceq \vec{y}$ (resp. $\vec{x} \prec \vec{y}$) if and only if $x_i \le y_i$ (resp. $x_i < y_i$) for every index $i = 1, \dots, k$. Furthermore, $\mathbb{R}^k$ is equipped with the usual max-norm: $\|(x_1, x_2, \dots, x_k)\|_\infty = \max_{1 \le i \le k} |x_i|$. Now we are ready to introduce the concept of size function for a size pair $(\mathcal{M}, \vec{\varphi})$. We shall denote the open set $\{(\vec{x},\vec{y}) \in \mathbb{R}^k \times \mathbb{R}^k : \vec{x} \prec \vec{y}\}$ by Δ^+, while $\bar{\Delta}^+$ will be the closure of Δ^+. For every k-tuple $\vec{x} = (x_1,\dots,x_k) \in \mathbb{R}^k$, the set $\mathcal{M}\langle\vec{\varphi} \preceq \vec{x}\,\rangle$ will be defined as $\{P \in \mathcal{M} : \varphi_i(P) \le x_i,\ i = 1,\dots,k\}$.

Definition 1.1. For every k-tuple $\vec{y} = (y_1,\dots,y_k) \in \mathbb{R}^k$, we shall say that two points $P, Q \in \mathcal{M}$ are *$\langle\vec{\varphi} \preceq \vec{y}\rangle$-connected* if and only if a connected subset of $\mathcal{M}\langle\vec{\varphi} \preceq \vec{y}\,\rangle$ exists, containing P and Q.

Definition 1.2. We shall call the *(k-dimensional) size function* associated with the size pair $(\mathcal{M},\vec{\varphi})$ the function $\ell_{(\mathcal{M},\vec{\varphi})} : \Delta^+ \to \mathbb{N}$, defined by setting $\ell_{(\mathcal{M},\vec{\varphi})}(\vec{x},\vec{y})$ equal to the number of equivalence classes in which the set $\mathcal{M}\langle\vec{\varphi} \preceq \vec{x}\,\rangle$ is divided by the $\langle\vec{\varphi} \preceq \vec{y}\rangle$-connectedness relation.

Remark 1.3. In other words, $\ell_{(\mathcal{M},\vec{\varphi})}(\vec{x},\vec{y})$ is equal to the number of connected components in $\mathcal{M}\langle\vec{\varphi} \preceq \vec{y}\,\rangle$ containing at least one point of $\mathcal{M}\langle\vec{\varphi} \preceq \vec{x}\,\rangle$. The finiteness of this number is a consequence of the compactness and local connectedness of $\mathcal{M}$ (cf. [26]).

In the following, we shall refer to the case of measuring functions taking value in $\mathbb{R}^k$ by using the term "k-dimensional". Before going on, we introduce the following notations: when $\vec{y} \in \mathbb{R}^k$ is fixed, the symbol $\ell_{(\mathcal{M},\vec{\varphi})}(\cdot,\vec{y})$ will be used to denote the function that takes each k-tuple $\vec{x} \prec \vec{y}$ to the value $\ell_{(\mathcal{M},\vec{\varphi})}(\vec{x},\vec{y})$. An analogous convention will hold for the symbol $\ell_{(\mathcal{M},\vec{\varphi})}(\vec{x},\cdot)$.

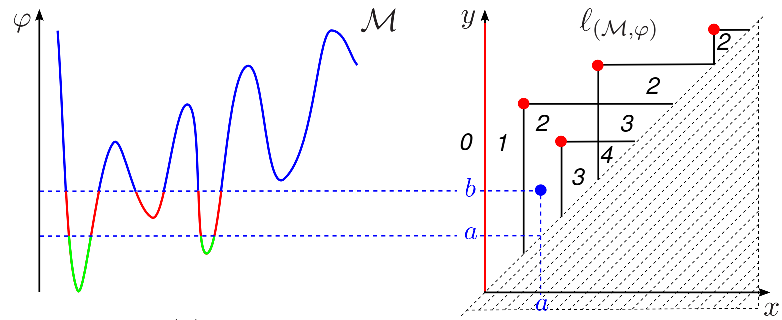
Remark 1.4. From Remark 1.3 it can be immediately deduced that for every fixed $\vec{y} \in \mathbb{R}^k$ the function $\ell_{(\mathcal{M},\varphi)}(\cdot,\vec{y})$ is non–decreasing with respect to $\preceq$, while for every fixed $\vec{x} \in \mathbb{R}^k$ the function $\ell_{(\mathcal{M},\varphi)}(\vec{x},\cdot)$ is non–increasing.

1.1. **The particular case $k = 1$.** In this section we will discuss the specific framework of measuring functions taking values in $\mathbb{R}$, namely the 1-dimensional case. Indeed, Size Theory has been extensively developed in this setting (cf. [3]), showing that each 1-dimensional size function admits a compact representation as a formal series of points and lines of $\mathbb{R}^2$ (cf. [27]). Due to this representation, a suitable *matching distance* between 1-dimensional size functions can be easily introduced, proving that these descriptors are stable with respect to such a distance [11, 13]. Moreover, the role of 1-dimensional size functions is crucial in the approach to the k-dimensional case proposed in [2].

Following the notations used in the literature about the case $k = 1$, the symbols $\vec{\varphi}$, $\vec{x}$, $\vec{y}$, $\preceq$, $\prec$ will be replaced respectively by φ, x, y, $\le$, $<$.

When dealing with a (1-dimensional) measuring function $\varphi : \mathcal{M} \to \mathbb{R}$, the size function $\ell_{(\mathcal{M},\varphi)}$ associated with $(\mathcal{M},\varphi)$ gives information about the pairs $(\mathcal{M}\langle\varphi \le x\rangle, \mathcal{M}\langle\varphi \le y\rangle)$, where $\mathcal{M}\langle\varphi \le t\rangle$ is defined by setting $\mathcal{M}\langle\varphi \le t\rangle = \{P \in \mathcal{M} : \varphi(P) \le t\}$ for $t \in \mathbb{R}$.

Figure 1 shows an example of a size pair and the associated 1-dimensional size function. On the left (Figure 1(a)) one can find the considered size pair $(\mathcal{M},\varphi)$, where $\mathcal{M}$ is the curve depicted by a solid line, and φ is the ordinate function. On the right (Figure 1(b)) the associated 1-dimensional size function $\ell_{(\mathcal{M},\varphi)}$ is given. As can be seen, the domain $\Delta^+ = \{(x,y) \in \mathbb{R}^2 : x < y\}$ is divided into bounded and unbounded regions, in each of which the 1-dimensional size function takes a constant value. The displayed numbers coincide with the values of $\ell_{(\mathcal{M},\varphi)}$ in each region. For example, let us now compute the value of $\ell_{(\mathcal{M},\varphi)}$ at the point (a,b). By applying Remark 1.3 in the case $k = 1$, it is sufficient to count how many of the



(a) (b)

FIGURE 1. (a) The topological spaces $\mathcal{M}$ and the measuring function φ. (b) The related size function $\ell_{(\mathcal{M},\varphi)}$.

three connected components in the sublevel $\mathcal{M}\langle\varphi \leq b\rangle$ contain at least one point of $\mathcal{M}\langle\varphi \leq a\rangle$. It can be easily verified that $\ell_{(\mathcal{M},\varphi)}(a,b) = 2$.

Following the 1-dimensional framework, the problem of comparing two size pairs can be easily translated into the simpler one of comparing the related 1-dimensional size functions. In [13], the *matching distance* d_{match} has been formally proven to be the most suitable distance between these descriptors. The definition of d_{match} is based on the observation that 1-dimensional size functions can be compactly described by a formal series of points and lines lying on the real plane, called respectively *proper cornerpoints* and *cornerpoints at infinity* (or *cornerlines*) and defined as follows:

Definition 1.5. For every point $P = (x, y)$ with $x < y$, consider the number $\mu(P)$ defined as the minimum, over all the positive real numbers ε with $x+\varepsilon < y-\varepsilon$, of

$$\ell_{(\mathcal{M},\varphi)}(x+\varepsilon, y-\varepsilon) - \ell_{(\mathcal{M},\varphi)}(x-\varepsilon, y-\varepsilon) - \ell_{(\mathcal{M},\varphi)}(x+\varepsilon, y+\varepsilon) + \ell_{(\mathcal{M},\varphi)}(x-\varepsilon, y+\varepsilon).$$

When this finite number, called *multiplicity of* P, is strictly positive, the point P will be called a *proper cornerpoint* for $\ell_{(\mathcal{M},\varphi)}$.

Definition 1.6. For every line r with equation $x = a$, consider the number $\mu(r)$ defined as the minimum, over all the positive real numbers ε with $a + \varepsilon < 1/\varepsilon$, of

$$\ell_{(\mathcal{M},\varphi)}(a+\varepsilon, 1/\varepsilon) - \ell_{(\mathcal{M},\varphi)}(a-\varepsilon, 1/\varepsilon).$$

When this finite number, called *multiplicity of* r, is strictly positive, the line r will be called a *cornerpoint at infinity* (or *cornerline*) for $\ell_{(\mathcal{M},\varphi)}$.

The fundamental role of proper cornerpoints and cornerpoints at infinity is explicitly shown in the following Representation Theorem, claiming that their multiplicities completely and univocally determine the values of 1-dimensional size functions.

For the sake of simplicity, each line of equation $x = a$ will be identified to a point at infinity with coordinates (a, ∞).

Theorem 1.7 (Representation Theorem)**.** *For every $\bar{x} < \bar{y} < \infty$, it holds that*

$$\ell_{(\mathcal{M},\varphi)}(\bar{x}, \bar{y}) = \sum_{\substack{x \leq \bar{x} \\ \bar{y} < y \leq \infty}} \mu((x, y)).$$

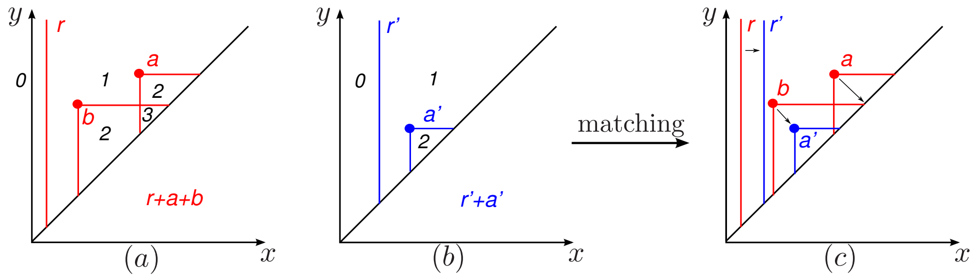


FIGURE 2. (a) Size function corresponding to the formal series $r + a + b$. (b) Size function corresponding to the formal series $r' + a'$. (c) The matching between the two formal series, realizing the matching distance between the two size functions.

Remark 1.8. In plain words, the Representation Theorem 1.7 claims that the value $\ell_{(\mathcal{M},\varphi)}(\bar{x},\bar{y})$ equals the number of cornerpoints lying above and on the left of $(\bar{x},\bar{y})$. By means of this theorem we are able to compactly represent 1-dimensional size functions as formal series of cornerpoints and cornerlines (An example is given by Figure 2(a) and Figure 2(b)).

As a first and simple consequence of the Representation Theorem 1.7, we have the following result, that will be useful in Section 2 (cf. [27]):

Corollary 1.9. *Each discontinuity point* $(\bar{x},\bar{y})$ *for* $\ell_{(\mathcal{M},\varphi)}$ *is such that either* $\bar{x}$ *is a discontinuity point for* $\ell_{(\mathcal{M},\varphi)}(\cdot,\bar{y})$*, or* $\bar{y}$ *is a discontinuity point for* $\ell_{(\mathcal{M},\varphi)}(\bar{x},\cdot)$*, or both these conditions hold.*

We are now able to introduce the matching distance d_{match}. Before going on, we observe that the Representation Theorem 1.7 allows us to reduce the problem of comparing 1-dimensional size functions into the comparison of the related multisets of cornerpoints. Indeed, the matching distance d_{match} can be seen as a measure of the cost of transporting the cornerpoints of a 1-dimensional size function into the cornerpoints of another one, with respect to a functional δ depending on the L_∞-distance between two matched cornerpoints and on their L_∞-distance from the diagonal $\{(x,y)\in\mathbb{R}^2 : x = y\}$. An example of matching between two formal series is given by Figure 2(c).

Let us now define more formally the matching distance d_{match}. Assume that two 1-dimensional size functions ℓ_1, ℓ_2 are given. Consider the multiset C_1 (respectively C_2) of cornerpoints for ℓ_1 (resp. ℓ_2), counted with their multiplicities and augmented by adding the points of the diagonal $\{(x,y)\in\mathbb{R}^2 : x = y\}$ counted with infinite multiplicity. If we denote by $\bar{\Delta}^*$ the set $\bar{\Delta}^+$ extended by the points at infinity of the kind (a,∞), i.e. $\bar{\Delta}^* = \bar{\Delta}^+ \cup \{(a,\infty) : a \in \mathbb{R}\}$, the matching distance $d_{match}(\ell_1,\ell_2)$ is then defined as

$$d_{match}(\ell_1,\ell_2) = \min_{\sigma} \max_{P\in C_1} \delta(P,\sigma(P)),$$

where σ varies among all the bijections between C_1 and C_2 and

$$\delta((x,y),(x',y')) = \min\left\{\max\left\{|x-x'|,|y-y'|\right\}, \max\left\{\frac{y-x}{2},\frac{y'-x'}{2}\right\}\right\},$$

for every (x, y), $(x', y') \in \bar{\Delta}^*$ and with the convention about ∞ that $\infty - y = y - \infty = \infty$ when $y \neq \infty$, $\infty - \infty = 0$, $\frac{\infty}{2} = \infty$, $|\infty| = \infty$, $\min\{c, \infty\} = c$ and $\max\{c, \infty\} = \infty$.

In plain words, the pseudometric δ measures the pseudodistance between two points (x, y) and (x', y') as the minimum between the cost of moving one point onto the other and the cost of moving both points onto the diagonal, with respect to the max-norm and under the assumption that any two points of the diagonal have vanishing pseudodistance (we recall that a pseudodistance d is just a distance missing the condition $d(X, Y) = 0 \Rightarrow X = Y$, i.e. two distinct elements may have vanishing distance with respect to d).

An application of the matching distance is given by Figure 2(c). As can be seen by this example, different 1-dimensional size functions may in general have a different number of cornerpoints. Therefore d_{match} allows a proper cornerpoint to be matched to a point of the diagonal: this matching can be interpreted as the destruction of a proper cornerpoint. Moreover, we stress that the matching distance is stable with respect to perturbations of the measuring functions, as the following Matching Stability Theorem states:

Theorem 1.10 (Matching Stability Theorem)**.** *If $(\mathcal{M}, \varphi)$, $(\mathcal{M}, \psi)$ are two size pairs with $\max_{P \in \mathcal{M}} |\varphi(P) - \psi(P)| \leq \varepsilon$, then it holds that $d_{match}(\ell_{(\mathcal{M},\varphi)}, \ell_{(\mathcal{M},\psi)}) \leq \varepsilon$.*

For a proof of the previous theorem and more details about the matching distance the reader is referred to [12, 13] (see also [10] for the analogue of the matching distance in Persistent Homology and its stability).

1.1.1. *Coordinates of cornerpoints and discontinuity points.* Following the related literature (see also [14] for the case of measuring functions with a finite number of critical homological values), it can be easily deduced that, if finite, both the coordinates of a cornerpoint for a 1-dimensional size function $\ell_{(\mathcal{M},\varphi)}$ are critical values of the measuring function φ, under the assumption that φ is C^1. However, to the best of our knowledge, this result has never been explicitly proved until now. Therefore, for the sake of completeness we formalize here this statement, that will be used in Section 2:

Theorem 1.11. *Let $\mathcal{M}$ be a closed C^1 Riemannian manifold, and let $\varphi : \mathcal{M} \to \mathbb{R}$ be a C^1 measuring function. Then if $(\bar{x}, \bar{y})$ is a proper cornerpoint for $\ell_{(\mathcal{M},\varphi)}$, it follows that both $\bar{x}$ and $\bar{y}$ are critical values of φ. If $(\bar{x}, \infty)$ is a cornerpoint at infinity for $\ell_{(\mathcal{M},\varphi)}$, it follows that $\bar{x}$ is a critical value of φ.*

Proof. We confine ourselves to prove the former statement, since the proof of the latter is analogous.

First of all, let us remark that there exists a closed C^∞ Riemannian manifold $\widetilde{\mathcal{M}}$ that is C^1-diffeomorphic to $\mathcal{M}$ through a C^1-diffeomorphism $h : \widetilde{\mathcal{M}} \to \mathcal{M}$ (cf. [30, Thm. 2.9]). Set $\tilde{\varphi} = \varphi \circ h$. Obviously, the size functions associated with the size pairs $(\widetilde{\mathcal{M}}, \tilde{\varphi})$ and $(\mathcal{M}, \varphi)$ coincide. Therefore, $(\bar{x}, \bar{y})$ is also a cornerpoint for $\ell_{(\widetilde{\mathcal{M}},\tilde{\varphi})}$.

We observe that the claim of our theorem holds for a closed C^∞ Riemannian manifold endowed with a Morse measuring function (see [25, Thm. 2.2]). Now, for every real value $\varepsilon > 0$ it is possible to find a Morse measuring function $\varphi_\varepsilon : \widetilde{\mathcal{M}} \to \mathbb{R}$ such that $\max_{Q \in \widetilde{\mathcal{M}}} |\tilde{\varphi}(Q) - \varphi_\varepsilon(Q)| \leq \varepsilon$ and $\max_{Q \in \widetilde{\mathcal{M}}} \|\nabla\tilde{\varphi}(Q) - \nabla\varphi_\varepsilon(Q)\| \leq \varepsilon$: We can obtain φ_ε by considering first the smooth measuring function given by the convolution of $\tilde{\varphi}$ and an opportune "regularizing" function, and then a Morse measuring function φ_ε approximating in $C^1(\widetilde{\mathcal{M}}, \mathbb{R})$ the previous measuring function (cf. [32, Corollary 6.8]). Therefore, from the Matching Stability Theorem 1.10 it follows that for every $\varepsilon > 0$ we can find a cornerpoint $(\bar{x}_\varepsilon, \bar{y}_\varepsilon)$ for the size function $\ell_{(\widetilde{\mathcal{M}},\varphi_\varepsilon)}$ with $\|(\bar{x}, \bar{y}) - (\bar{x}_\varepsilon, \bar{y}_\varepsilon)\|_\infty \le \varepsilon$ and $\bar{x}_\varepsilon, \bar{y}_\varepsilon$ as critical values for φ_ε. Passing to the limit for $\varepsilon \to 0$ we obtain that both $\bar{x}$ and $\bar{y}$ are critical values for $\tilde{\varphi}$. The claim follows by observing that, since $\tilde{\varphi}$ and φ have the same critical values, both $\bar{x}$ and $\bar{y}$ are also critical values for φ. □

From the Representation Theorem 1.7 and Theorem 1.11 we can obtain the following corollary, refining Corollary 1.9 in the C^1 case (we skip the easy proof):

Corollary 1.12. *Let $\mathcal{M}$ be a closed C^1 Riemannian manifold, and let $\varphi : \mathcal{M} \to \mathbb{R}$ be a C^1 measuring function. Let also $(\bar{x}, \bar{y})$ be a discontinuity point for $\ell_{(\mathcal{M},\varphi)}$. Then at least one of the following statements holds:*

(i): *$\bar{x}$ is a discontinuity point for $\ell_{(\mathcal{M},\varphi)}(\cdot, \bar{y})$ and $\bar{x}$ is a critical value for φ;*
(ii): *$\bar{y}$ is a discontinuity point for $\ell_{(\mathcal{M},\varphi)}(\bar{x}, \cdot)$ and $\bar{y}$ is a critical value for φ.*

The generalization of Corollary 1.12 in the k-dimensional setting is not so simple and requires some new ideas which are given in Section 2, which also provides our main results.

1.2. **Reduction to the 1-dimensional case.** We are now ready to review the approach to multidimensional Size Theory proposed in [2]. In that work, the authors prove that the case $k > 1$ can be reduced to the 1-dimensional framework by a change of variable and the use of a suitable foliation. In particular, they show that there exists a parameterized family of half-planes in $\mathbb{R}^k \times \mathbb{R}^k$ such that the restriction of a k-dimensional size function $\ell_{(\mathcal{M},\vec{\varphi})}$ to each of these half-planes can be seen as a particular 1-dimensional size function. The motivations at the basis of this approach move from the fact that the concepts of proper cornerpoint and cornerpoint at infinity, defined for 1-dimensional size functions, appear not easily generalizable to an arbitrary dimension (namely the case $k > 1$). As a consequence, at a first glance it does not seem possible to obtain the multidimensional analogue of the matching distance d_{match} and therefore it is not clear how to generalize the Matching Stability Theorem 1.10. On the other hand, all these problems can be bypassed by means of the results we recall in the rest of this subsection.

Definition 1.13. For every unit vector $\vec{l} = (l_1, \ldots, l_k)$ of $\mathbb{R}^k$ such that $l_i > 0$ for $i = 1, \ldots, k$, and for every vector $\vec{b} = (b_1, \ldots, b_k)$ of $\mathbb{R}^k$ such that $\sum_{i=1}^k b_i = 0$, we shall say that the pair $(\vec{l}, \vec{b})$ is *admissible*. We shall denote the set of all admissible pairs in $\mathbb{R}^k \times \mathbb{R}^k$ by Adm_k. Given an admissible pair $(\vec{l}, \vec{b})$, we define the half-plane $\pi_{(\vec{l},\vec{b})}$ of $\mathbb{R}^k \times \mathbb{R}^k$ by the following parametric equations:

$$\begin{cases} \vec{x} = s\vec{l} + \vec{b} \\ \vec{y} = t\vec{l} + \vec{b} \end{cases}$$

for $s, t \in \mathbb{R}$, with $s < t$.

The following proposition implies that the collection of half-planes given in Definition 1.13 is actually a foliation of Δ^+.

Proposition 1.14. *For every* $(\vec{x}, \vec{y}) \in \Delta^+$ *there exists one and only one admissible pair* $(\vec{l}, \vec{b})$ *such that* $(\vec{x}, \vec{y}) \in \pi_{(\vec{l},\vec{b})}$.

Now we can show the reduction to the 1-dimensional case.

Theorem 1.15 (Reduction Theorem)**.** *Let* $(\vec{l}, \vec{b})$ *be an admissible pair, and* $F^{\vec{\varphi}}_{(\vec{l},\vec{b})} : \mathcal{M} \to \mathbb{R}$ *be defined by setting*

$$F^{\vec{\varphi}}_{(\vec{l},\vec{b})}(P) = \max_{i=1,\dots,k} \left\{ \frac{\varphi_i(P) - b_i}{l_i} \right\} .$$

Then, for every $(\vec{x}, \vec{y}) = (s\vec{l} + \vec{b}, t\vec{l} + \vec{b}) \in \pi_{(\vec{l},\vec{b})}$ *the following equality holds:*

$$\ell_{(\mathcal{M},\vec{\varphi})}(\vec{x}, \vec{y}) = \ell_{(\mathcal{M},F^{\vec{\varphi}}_{(\vec{l},\vec{b})})}(s, t) .$$

In the following, we shall use the symbol $F^{\vec{\varphi}}_{(\vec{l},\vec{b})}$ in the sense of the Reduction Theorem 1.15.

Remark 1.16. In plain words, the Reduction Theorem 1.15 states that each multidimensional size function corresponds to a 1-dimensional size function on each half-plane of the given foliation. It follows that each multidimensional size function can be represented as a parameterized family of formal series of points and lines, following the description introduced in Subsection 1.1 for the case $k = 1$. Indeed, it is possible to associate a formal series $\sigma_{(\vec{l},\vec{b})}$ with each admissible pair $(\vec{l}, \vec{b})$, with $\sigma_{(\vec{l},\vec{b})}$ describing the 1-dimensional size function $\ell_{(\mathcal{M},F^{\vec{\varphi}}_{(\vec{l},\vec{b})})}$. Therefore, on each half-plane $\pi_{(\vec{l},\vec{b})}$, the matching distance d_{match} and the Matching Stability Theorem 1.10 can be applied. Moreover, the family $\left\{\sigma_{(\vec{l},\vec{b})} : (\vec{l}, \vec{b}) \in Adm_k\right\}$ turns out to be a complete descriptor for $\ell_{(\mathcal{M},\vec{\varphi})}$, since two multidimensional size functions coincide if and only if the corresponding parameterized families of formal series coincide.

Before proceeding, we now introduce an example showing how the Reduction Theorem 1.15 works.

Example 1.17. In $\mathbb{R}^3$ consider the set $\mathcal{Q} = [-1, 1] \times [-1, 1] \times [-1, 1]$ and the unit sphere S^2 of equation $x^2 + y^2 + z^2 = 1$. Let also $\vec{\Phi} = (\Phi_1, \Phi_2) : \mathbb{R}^3 \to \mathbb{R}^2$ be the continuous function, defined as $\vec{\Phi}(x, y, z) = (|x|, |z|)$. In this setting, consider the size pairs $(\mathcal{M}, \vec{\varphi})$ and $(\mathcal{N}, \vec{\psi})$ where $\mathcal{M} = \partial\mathcal{Q}$, $\mathcal{N} = S^2$, and $\vec{\varphi}$ and $\vec{\psi}$ are respectively the restrictions of $\vec{\Phi}$ to $\mathcal{M}$ and $\mathcal{N}$. In order to compare the size functions $\ell_{(\mathcal{M},\vec{\varphi})}$ and $\ell_{(\mathcal{N},\vec{\psi})}$, we are interested in studying the foliation in half-planes $\pi_{(\vec{l},\vec{b})}$, where $\vec{l} = (\cos\theta, \sin\theta)$ with $\theta \in (0, \frac{\pi}{2})$, and $\vec{b} = (a, -a)$ with $a \in \mathbb{R}$. Any such half-plane is represented by

$$\begin{cases} x_1 = s\cos\theta + a \\ x_2 = s\sin\theta - a \\ y_1 = t\cos\theta + a \\ y_2 = t\sin\theta - a \end{cases} ,$$

with $s, t \in \mathbb{R}$, $s < t$. Figure 3 shows the size functions $\ell_{(\mathcal{M},F^{\vec{\varphi}}_{(\vec{l},\vec{b})})}$ and $\ell_{(\mathcal{N},F^{\vec{\psi}}_{(\vec{l},\vec{b})})}$, for $\theta = \frac{\pi}{4}$ and $a = 0$, i.e. $\vec{l} = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ and $\vec{b} = (0, 0)$. With this choice, we obtain that $F^{\vec{\varphi}}_{(\vec{l},\vec{b})} = \sqrt{2}\max\{\varphi_1, \varphi_2\} = \sqrt{2}\max\{|x|, |z|\}$ and $F^{\vec{\psi}}_{(\vec{l},\vec{b})} = \sqrt{2}\max\{\psi_1, \psi_2\} =$

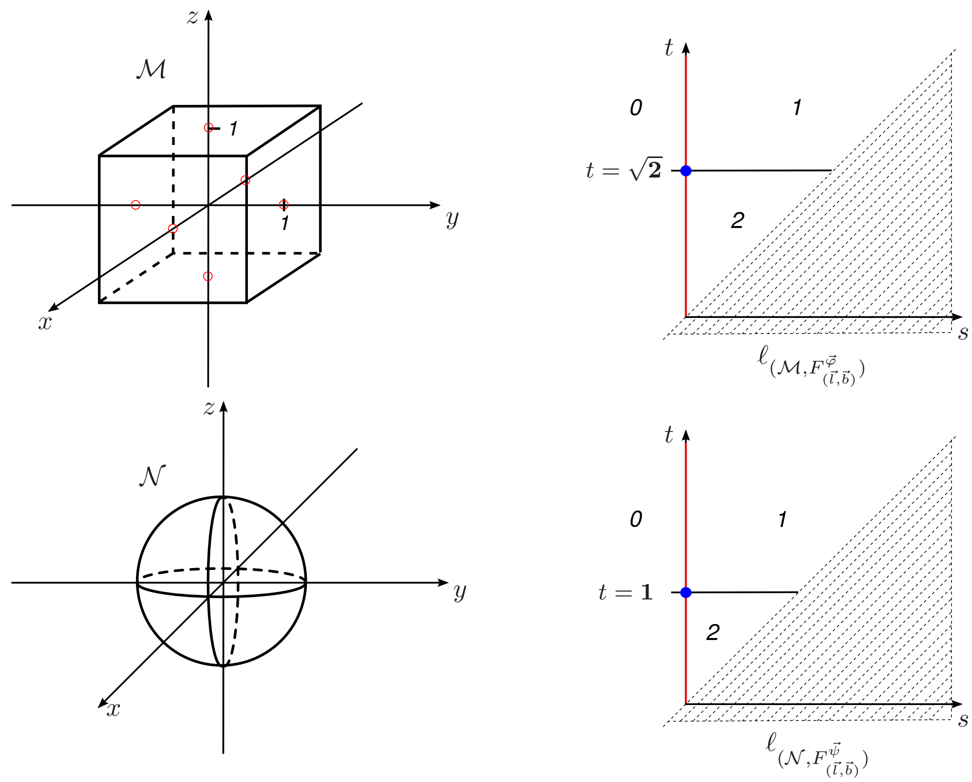


FIGURE 3. The topological spaces $\mathcal{M}$ and $\mathcal{N}$ and the size functions $\ell_{(\mathcal{M},F^{\vec{\varphi}}_{(\vec{l},\vec{b})})}, \ell_{(\mathcal{N},F^{\vec{\psi}}_{(\vec{l},\vec{b})})}$ associated with the half-plane $\pi_{(\vec{l},\vec{b})}$, for $\vec{l} = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ and $\vec{b} = (0, 0)$.

$\sqrt{2}\max\{|x|, |z|\}$. Therefore, Theorem 1.15 implies that, for every $(x_1, x_2, y_1, y_2) \in \pi_{(\vec{l},\vec{b})}$, we have

$$\ell_{(\mathcal{M},\vec{\varphi})}(x_1, x_2, y_1, y_2) = \ell_{(\mathcal{M},\vec{\varphi})}\left(\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}}, \frac{t}{\sqrt{2}}, \frac{t}{\sqrt{2}}\right) = \ell_{(\mathcal{M},F^{\vec{\varphi}}_{(\vec{l},\vec{b})})}(s, t)$$
$$\ell_{(\mathcal{N},\vec{\psi})}(x_1, x_2, y_1, y_2) = \ell_{(\mathcal{N},\vec{\psi})}\left(\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}}, \frac{t}{\sqrt{2}}, \frac{t}{\sqrt{2}}\right) = \ell_{(\mathcal{N},F^{\vec{\psi}}_{(\vec{l},\vec{b})})}(s, t) .$$

The matching distance $d_{match}(\ell_{(\mathcal{M},F^{\vec{\varphi}}_{(\vec{l},\vec{b})})}, \ell_{(\mathcal{N},F^{\vec{\psi}}_{(\vec{l},\vec{b})})})$ is equal to $\sqrt{2} - 1$, i.e. the cost of moving the point of coordinates $(0, \sqrt{2})$ onto the point of coordinates $(0, 1)$, computed with respect to the max-norm. The points $(0, \sqrt{2})$ and $(0, 1)$ are representative of the characteristic triangles of the size functions $\ell_{(\mathcal{M},F^{\vec{\varphi}}_{(\vec{l},\vec{b})})}$ and $\ell_{(\mathcal{N},F^{\vec{\psi}}_{(\vec{l},\vec{b})})}$, respectively. Note that the matching distance computed for $\vec{l} = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ and $\vec{b} = (0, 0)$ induces a pseudodistance. This means that, even by considering just one half-plane of the foliation, it is possible to effectively compare multidimensional size functions. We conclude by observing that $\ell_{(\mathcal{M},\varphi_1)} \equiv \ell_{(\mathcal{N},\psi_1)}$ and $\ell_{(\mathcal{M},\varphi_2)} \equiv \ell_{(\mathcal{N},\psi_2)}$.

In other words, the multidimensional size functions, with respect to $\vec{\varphi}, \vec{\psi}$, are able to discriminate the cube and the sphere, while both the 1-dimensional size functions, with respect to φ_1, φ_2 and ψ_1, ψ_2, cannot do that. This higher discriminatory power of multidimensional size functions gives a further motivation for their definition and use.

The next result proves the stability of d_{match} with respect to the choice of the half-planes of the foliation. Indeed, the next proposition states that small enough changes in $(\vec{l}, \vec{b})$ with respect to the max-norm induce small changes of $\ell_{(\mathcal{M}, F^{\vec{\varphi}}_{(\vec{l},\vec{b})})}$ with respect to the matching distance.

Proposition 1.18. *If $(\mathcal{M}, \vec{\varphi})$ is a size pair, $(\vec{l}, \vec{b}) \in Adm_k$ and ε is a real number with $0 < \varepsilon < \min_{i=1,\dots,k} l_i$, then for every admissible pair $(\vec{l}', \vec{b}')$ with $\|(\vec{l}, \vec{b}) - (\vec{l}', \vec{b})\|_\infty \le \varepsilon$, it holds that*

$$d_{match}(\ell_{(\mathcal{M}, F^{\vec{\varphi}}_{(\vec{l},\vec{b})})}, \ell_{(\mathcal{M}, F^{\vec{\varphi}}_{(\vec{l}',\vec{b}')})}) \le \varepsilon \cdot \frac{\max_{P \in \mathcal{M}} \|\vec{\varphi}(P)\|_\infty + \|\vec{l}\|_\infty + \|\vec{b}\|_\infty}{\min_{i=1,\dots,k}\{l_i(l_i - \varepsilon)\}}.$$

Remark 1.19. Analogously, it is possible to prove (cf. [2, Prop. 2]) that d_{match} is stable with respect to the chosen measuring function, i.e. that small enough changes in $\vec{\varphi}$ with respect to the max-norm induce small changes of $\ell_{(\mathcal{M}, F^{\vec{\varphi}}_{(\vec{l},\vec{b})})}$ with respect to the matching distance.

Proposition 1.18 and Remark 1.19 guarantee the stability of this approach.

2. Main Results

In this section we are going to prove some new results about the discontinuities of multidimensional size functions. In order to do that, we will confine ourselves to the case of a size pair $(\mathcal{M}, \vec{\varphi})$, where $\mathcal{M}$ is a closed C^1 Riemannian m-manifold.

From now to Theorem 2.11 we shall assume that an admissible pair $(\vec{l}, \vec{b}) \in Adm_k$ is fixed, considering the 1-dimensional size function $\ell_{(\mathcal{M},F)}$, where $F(Q) = \max_{i=1,\dots,k} \frac{\varphi_i(Q) - b_i}{l_i}$. We shall say that F and $\ell_{(\mathcal{M},F)}$ are the (1-dimensional) measuring function and the size function corresponding to the half-plane $\pi_{(\vec{l},\vec{b})}$, respectively.

The main results of this section are stated in Theorem 2.11 and Theorem 2.13, showing a necessary condition for a point $(\vec{x}, \vec{y}) \in \Delta^+$ to be a discontinuity point for the size function $\ell_{(\mathcal{M},\vec{\varphi})}$, under the assumption that $\vec{\varphi}$ is C^1 and C^0, respectively. For the sake of clarity, we will now provide a sketch of the arguments that will lead us to the proof of our main results.

Theorem 2.11 is a generalization in the k-dimensional setting of Corollary 1.12, stating that each discontinuity point for a 1-dimensional size function $\ell_{(\mathcal{M},\varphi)}$, related to a C^1 measuring function φ, is such that at least one of its coordinates is a critical value for φ. We recall that Corollary 1.12 directly descends from the Representation Theorem 1.7 and from Theorem 1.11, according to which each finite coordinate of a cornerpoint for $\ell_{(\mathcal{M},\varphi)}$ has to be a critical value for φ. Our first goal is to prove that a modified version of this last statement holds for the 1-dimensional size function $\ell_{(\mathcal{M},F)}$ corresponding to the half-plane $\pi_{(\vec{l},\vec{b})}$. The reason for such an adaptation is that the 1-dimensional measuring function F is not C^1 (even in case $\vec{\varphi}$ is C^1), and therefore we need to generalize the concepts of critical point and critical value by introducing the definitions of $(\vec{l},\vec{b})$-*pseudocritical point* and $(\vec{l},\vec{b})$-*pseudocritical value* for a C^1 function (Definition 2.1). These notions, together with an approximation in $C^0(\mathcal{M},\mathbb{R})$ of the function F by C^1 functions, are used to prove that, if $\vec{\varphi}\in C^1(\mathcal{M},\mathbb{R}^k)$, each finite coordinate of a cornerpoint for $\ell_{(\mathcal{M},F)}$ has to be an $(\vec{l},\vec{b})$-pseudocritical value for $\vec{\varphi}$ (Theorem 2.3). Next, we show (Proposition 2.4) that a correspondence exists between the discontinuity points of $\ell_{(M,F)}$ and the ones of $\ell_{(M,\vec{\varphi})}$. Theorem 2.3 and Proposition 2.4 lead us to the relation (Theorem 2.7) between the discontinuity points for $\ell_{(\mathcal{M},\vec{\varphi})}$, lying on the half-plane $\pi_{(\vec{l},\vec{b})}$, and the $(\vec{l},\vec{b})$-pseudocritical values for $\vec{\varphi}$. This last result is refined in Theorem 2.11 under the assumption that $\vec{\varphi}$ is C^1, providing a necessary condition for discontinuities of $\ell_{(\mathcal{M},\vec{\varphi})}$ that does not depend on the half-planes of the foliation. This can be done by introducing the concepts of *pseudocritical point* and *pseudocritical value* for an $\mathbb{R}^k$-valued C^1 function (Definition 2.8), and considering a suitable projection $\rho:\mathbb{R}^k\to\mathbb{R}^h$. The necessary condition given in Theorem 2.11 is finally generalized to the case of continuous measuring functions (Theorem 2.13), once more by means of an approximation technique, and the notions of *special point* and *special value*.

Before going on, we need the following definition:

Definition 2.1. Assume that $\vec{\varphi}\in C^1(\mathcal{M},\mathbb{R}^k)$. For every $Q\in\mathcal{M}$, set $I_Q=\left\{i\in\{1,\dots,k\}:\frac{\varphi_i(Q)-b_i}{l_i}=F(Q)\right\}$. We shall say that Q is an $(\vec{l},\vec{b})$-*pseudocritical point for* $\vec{\varphi}$ if the convex hull of the gradients $\nabla\varphi_i(Q)$, $i\in I_Q$, contains the null vector, i.e. for every $i\in I_Q$ there exists a real value λ_i such that $\sum_{i\in I_Q}\lambda_i\nabla\varphi_i(Q)=\mathbf{0}$, with $0\le\lambda_i\le 1$ and $\sum_{i\in I_Q}\lambda_i=1$. If Q is an $(\vec{l},\vec{b})$-pseudocritical point for $\vec{\varphi}$, the value $F(Q)$ will be called an $(\vec{l},\vec{b})$-*pseudocritical value for* $\vec{\varphi}$.

Remark 2.2. The concept of $(\vec{l},\vec{b})$-pseudocritical point is strongly connected, via the function F introduced in Definition 2.1, with the notion of generalized gradient introduced by F. H. Clarke [9]. For a point $Q\in\mathcal{M}$, the condition of being $(\vec{l},\vec{b})$-pseudocritical for $\vec{\varphi}$ corresponds to the one of being "critical" for the generalized gradient of F [9, Prop. 2.3.12]. However, in this context we prefer to adopt a terminology highlighting the dependence on the considered half-plane.

We can now state our first result.

Theorem 2.3. *Assume that $\vec{\varphi}\in C^1(\mathcal{M},\mathbb{R}^k)$. If (σ,τ) is a proper cornerpoint of $\ell_{(\mathcal{M},F)}$, then both σ and τ are $(\vec{l},\vec{b})$-pseudocritical values for $\vec{\varphi}$. If (σ,∞) is a cornerpoint at infinity of $\ell_{(\mathcal{M},F)}$, then σ is an $(\vec{l},\vec{b})$-pseudocritical value for $\vec{\varphi}$.*

Proof. We confine ourselves to proving the former statement, since the proof of the latter is analogous. The idea is to show that our thesis holds for a C^1 function approximating the measuring function $F:\mathcal{M}\to\mathbb{R}$ in $C^0(\mathcal{M},\mathbb{R})$, and verify that this property passes to the limit. Let us now set $\Phi_i(Q)=\frac{\varphi_i(Q)-b_i}{l_i}$ and choose $c\in\mathbb{R}$ such that $\min_{Q\in\mathcal{M}}\Phi_i(Q)>-c$, for every $i=1,\dots,k$. Consider the function sequence (F_p), $p\in\mathbb{N}^+=\mathbb{N}\setminus\{0\}$, where $F_p:\mathcal{M}\to\mathbb{R}$ and $F_p(Q)=\left(\sum_{i=1}^k(\Phi_i(Q)+c)^p\right)^{\frac{1}{p}}-c$: Such a sequence converges uniformly to the function F. Indeed, for every $Q\in\mathcal{M}$

and for every index p we have that

$$\begin{aligned}
|F(Q)-F_p(Q)| &= \left|\max_i \Phi_i(Q) - \left(\left(\sum_{i=1}^k (\Phi_i(Q)+c)^p\right)^{\frac{1}{p}} - c\right)\right| = \\
&= \left|\max_i\{\Phi_i(Q)+c\} - \left(\sum_{i=1}^k (\Phi_i(Q)+c)^p\right)^{\frac{1}{p}}\right| = \\
&= \left(\sum_{i=1}^k (\Phi_i(Q)+c)^p\right)^{\frac{1}{p}} - \max_i\{\Phi_i(Q)+c\} \leq \\
&\leq \max_i\{\Phi_i(Q)+c\}\cdot(k^{\frac{1}{p}}-1).
\end{aligned}$$

Let us now consider a proper cornerpoint $\bar{C}$ of the size function $\ell_{(\mathcal{M},F)}$. By the Matching Stability Theorem 1.10 it follows that it is possible to find a large enough p and a proper cornerpoint C_p of the 1-dimensional size function $\ell_{(\mathcal{M},F_p)}$ (associated with the size pair $(\mathcal{M},F_p)$) such that C_p is arbitrarily close to $\bar{C}$. Since C_p is a proper cornerpoint of $\ell_{(\mathcal{M},F_p)}$, it follows from Theorem 1.11 that its coordinates are critical values of the C^1 function F_p. By focusing our attention on the abscissa of C_p (analogous considerations hold for the ordinate of C_p) it follows that there exists $Q_p \in \mathcal{M}$ with $x(C_p) = F_p(Q_p)$ and (in respect to local coordinates $x_1,\ldots,x_m$ of the m-manifold $\mathcal{M}$)

$$\begin{aligned}
0 &= \frac{\partial F_p}{\partial x_1}(Q_p) = \left(\sum_{i=1}^k (\Phi_i(Q_p)+c)^p\right)^{\frac{1-p}{p}} \cdot \left(\sum_{i=1}^k (\Phi_i(Q_p)+c)^{p-1}\cdot\frac{\partial\Phi_i}{\partial x_1}(Q_p)\right) \\
&\vdots \\
0 &= \frac{\partial F_p}{\partial x_m}(Q_p) = \left(\sum_{i=1}^k (\Phi_i(Q_p)+c)^p\right)^{\frac{1-p}{p}} \cdot \left(\sum_{i=1}^k (\Phi_i(Q_p)+c)^{p-1}\cdot\frac{\partial\Phi_i}{\partial x_m}(Q_p)\right) .
\end{aligned}$$

Hence we have

$$\begin{aligned}
&\sum_{i=1}^k \quad (\Phi_i(Q_p)+c)^{p-1}\cdot\frac{\partial\Phi_i}{\partial x_1}(Q_p) = 0 \\
&\vdots \\
&\sum_{i=1}^k \quad (\Phi_i(Q_p)+c)^{p-1}\cdot\frac{\partial\Phi_i}{\partial x_m}(Q_p) = 0 .
\end{aligned}$$

Therefore, by setting

$$\boldsymbol{v}_p = (v_p^1,\ldots,v_p^k) = \left((\Phi_1(Q_p)+c)^{p-1},\ldots,(\Phi_k(Q_p)+c)^{p-1}\right),$$

we can write ${}^t\!J(Q_p)\cdot{}^t\boldsymbol{v}_p = \mathbf{0}$, where $J(Q_p)$ is the Jacobian matrix of $\vec{\Phi} = (\Phi_1,\ldots,\Phi_k)$ computed at the point Q_p. By the compactness of $\mathcal{M}$, we can assume (possibly by extracting a subsequence) that (Q_p) converges to a point $\bar{Q}$. Let us define $\boldsymbol{u}_p = \frac{\boldsymbol{v}_p}{\|\boldsymbol{v}_p\|_\infty}$. By compactness (recall that $\|\boldsymbol{u}_p\|_\infty = 1$) we can also assume (possibly by considering a subsequence) that the sequence $(\boldsymbol{u}_p)$ converges to a vector $\bar{\boldsymbol{u}} = (\bar{u}^1, \ldots, \bar{u}^k)$, where $\bar{u}^i = \lim_{p\to\infty} \frac{v_p^i}{\|\boldsymbol{v}_p\|_\infty}$ and $\|\bar{\boldsymbol{u}}\|_\infty = 1$. Obviously ${}^t\!J(Q_p) \cdot {}^t\boldsymbol{u}_p = \mathbf{0}$ and hence we have

$$ {}^t\!J(\bar{Q}) \cdot {}^t\bar{\boldsymbol{u}} = \mathbf{0}. \tag{2.1}$$

Since for every index p and for every $i = 1, \ldots, k$ the relation $0 < u_p^i \leq 1$ holds, for each $i = 1, \ldots, k$ the condition $0 \leq \bar{u}^i = \lim_{p\to\infty} u_p^i \leq 1$ is satisfied. Let us now recall that $F(\bar{Q}) = \max_i \Phi_i(\bar{Q})$, by definition, and consider the set $I_{\bar{Q}} = \{i \in \{1, \ldots, k\} : \Phi_i(\bar{Q}) = F(\bar{Q})\} = \{i_1, \ldots, i_h\}$. For every $r \notin I_{\bar{Q}}$ the component $\bar{u}^r$ is equal to 0, since $0 \leq u_p^r = \left(\frac{\Phi_r(Q_p)+c}{\max_i\{\Phi_r(Q_p)+c\}}\right)^{p-1}$ and $\lim_{p\to\infty} \frac{\Phi_r(Q_p)+c}{\max_i\{\Phi_r(Q_p)+c\}} = \frac{\Phi_r(\bar{Q})+c}{F(\bar{Q})+c}$, which is strictly less than 1 for $\Phi_r(\bar{Q}) < F(\bar{Q})$. Hence we have $\bar{\boldsymbol{u}} = \bar{u}^{i_1} \cdot \boldsymbol{e}_{i_1} + \cdots + \bar{u}^{i_h} \cdot \boldsymbol{e}_{i_h}$, where $\boldsymbol{e}_i$ is the i^{th} vector of the standard basis of $\mathbb{R}^k$. Thus, from equality (2.1) we have $\sum_{j=1}^h \bar{u}^{i_j} \cdot \frac{\partial \Phi_{i_j}}{\partial x_1}(\bar{Q}) = 0, \ldots, \sum_{j=1}^h \bar{u}^{i_j} \cdot \frac{\partial \Phi_{i_j}}{\partial x_m}(\bar{Q}) = 0$, that is $\sum_{j=1}^h \frac{\bar{u}^{i_j}}{l_{i_j}} \cdot \frac{\partial \varphi_{i_j}}{\partial x_1}(\bar{Q}) = 0, \ldots, \sum_{j=1}^h \frac{\bar{u}^{i_j}}{l_{i_j}} \cdot \frac{\partial \varphi_{i_j}}{\partial x_m}(\bar{Q}) = 0$, since $\Phi_i = \frac{\varphi - b_i}{l_i}$. Hence, $\sum_{j=1}^h \frac{\bar{u}^{i_j}}{l_{i_j}} \nabla\varphi_{i_j}(\bar{Q}) = \mathbf{0}$. By recalling that $\bar{u}^{i_j} \geq 0$, $l_{i_j} > 0$ and $\bar{\boldsymbol{u}}$ is a non–vanishing vector, it follows immediately that $\sum_{j=1}^h \frac{\bar{u}^{i_j}}{l_{i_j}} > 0$ and therefore the convex hull of the gradients $\nabla\varphi_{i_1}(\bar{Q}), \ldots, \nabla\varphi_{i_h}(\bar{Q})$ contains the null vector. Thus, $\bar{Q}$ is an $(\vec{l}, \vec{b})$-pseudocritical point for $\vec{\varphi}$ and hence $F(\bar{Q})$ is an $(\vec{l}, \vec{b})$-pseudocritical value for $\vec{\varphi}$. Moreover, from the uniform convergence of the sequence (F_p) to F and from the continuity of the function F, we have (recall that $\bar{C} = \lim_{p\to\infty} C_p$)

$$x(\bar{C}) = \lim_{p\to\infty} x(C_p) = \lim_{p\to\infty} F_p(Q_p) = F(\bar{Q}).$$

In other words, the abscissa $x(\bar{C})$ of a proper cornerpoint of $\ell_{(\mathcal{M},F)}$ is the image of an $(\vec{l}, \vec{b})$-pseudocritical point $\bar{Q}$ through F, i.e. an $(\vec{l}, \vec{b})$-pseudocritical value for $\vec{\varphi}$. An analogous reasoning holds for the ordinate $y(\bar{C})$ of a proper cornerpoint. □

Our next result shows that each discontinuity of $\ell_{(\mathcal{M},\vec{\varphi})}$ corresponds to a discontinuity of the 1-dimensional size function associated with a suitable half-plane of the foliation.

Proposition 2.4. *A point $(\vec{x}, \vec{y}) = (s \cdot \vec{l} + \vec{b}, t \cdot \vec{l} + \vec{b}) \in \pi_{(\vec{l},\vec{b})}$ is a discontinuity point for $\ell_{(\mathcal{M},\vec{\varphi})}$ if and only if (s, t) is a discontinuity point for $\ell_{(\mathcal{M},F)}$.*

Proof. Obviously, if (s, t) is a discontinuity point for $\ell_{(\mathcal{M},F)}$, then $(\vec{x}, \vec{y}) = (s \cdot \vec{l} + \vec{b}, t \cdot \vec{l} + \vec{b}) \in \pi_{(\vec{l},\vec{b})}$ is a discontinuity point for $\ell_{(\mathcal{M},\vec{\varphi})}$, because of the Reduction Theorem 1.15. In order to prove the inverse implication, we shall verify the contrapositive statement, i.e. if (s, t) is not a discontinuity point for $\ell_{(\mathcal{M},F)}$, then $(s \cdot \vec{l} + \vec{b}, t \cdot \vec{l} + \vec{b})$ is not a discontinuity point for $\ell_{(\mathcal{M},\vec{\varphi})}$. Indeed, if (s, t) is not a discontinuity point for $\ell_{(\mathcal{M},F)}$, then $\ell_{(\mathcal{M},F)}$ is locally constant at (s, t) (recall that each size function is natural–valued). Therefore it will be possible to choose a real number $\eta > 0$ such that

$$\ell_{(\mathcal{M},F)}(s - \eta, t + \eta) = \ell_{(\mathcal{M},F)}(s + \eta, t - \eta). \tag{2.2}$$

Before proceeding in our proof, we need the following result:

Lemma 2.5. *Let $(\mathcal{M},\psi)$, $(\mathcal{M},\psi')$ be two size pairs, with $\psi,\psi':\mathcal{M}\to\mathbb{R}$. If $d_{match}\left(\ell_{(\mathcal{M},\psi)},\ell_{(\mathcal{M},\psi')}\right)\le 2\varepsilon$, then it holds that*

$$\ell_{(\mathcal{M},\psi)}(s-\varepsilon,t+\varepsilon)\le \ell_{(\mathcal{M},\psi')}(s+\varepsilon,t-\varepsilon),$$

for every (s,t) with $s+\varepsilon<t-\varepsilon$.

Proof of Lemma 2.5. Let Δ^* be the set given by $\Delta^+\cup\{(a,\infty):a\in\mathbb{R}\}$. For every (s,t) with $s<t$, let us define the set $L_{(s,t)}=\{(\sigma,\tau)\in\Delta^*:\sigma\le s,\tau>t\}$. By the Representation Theorem 1.7 we have that $\ell_{(\mathcal{M},\psi)}(s-\varepsilon,t+\varepsilon)$ equals the number of proper cornerpoints and cornerpoints at infinity for $\ell_{(\mathcal{M},\psi)}$ belonging to the set $L_{(s-\varepsilon,t+\varepsilon)}$. Since $d_{match}\left(\ell_{(\mathcal{M},\psi)},\ell_{(\mathcal{M},\psi')}\right)\le 2\varepsilon$, the number of proper cornerpoints and cornerpoints at infinity for $\ell_{(\mathcal{M},\psi')}$ in the set $L_{(s+\varepsilon,t-\varepsilon)}$ is not less than $\ell_{(\mathcal{M},\psi)}(s-\varepsilon,t+\varepsilon)$. The reason is that the change from ψ to ψ' does not move the cornerpoints more than 2ε, with respect to the max-norm, because of the Matching Stability Theorem 1.10. By applying the Representation Theorem 1.7 once again to $\ell_{(\mathcal{M},\psi')}$, we get our thesis. □

Let us go back to the proof of Proposition 2.4. By Proposition 1.18, we can then consider a real value $\varepsilon=\varepsilon(\eta)$ with $0<\varepsilon<\min_{i=1,\dots,k} l_i$ such that for every admissible pair $(\vec{l}',\vec{b}')$ with $\left\|(\vec{l},\vec{b})-(\vec{l}',\vec{b}')\right\|_\infty\le\varepsilon$, the relation $d_{match}(\ell_{(\mathcal{M},F)},\ell_{(\mathcal{M},F')})\le\frac{\eta}{2}$ holds, where $\ell_{(\mathcal{M},F')}$ is the 1-dimensional size function corresponding to the half-plane $\pi_{(\vec{l}',\vec{b}')}$. By applying Lemma 2.5 twice and the monotonicity of $\ell_{(\mathcal{M},F')}$ in each variable (cf. Remark 1.4), we get the inequalities

$$\begin{aligned}\ell_{(\mathcal{M},F)}(s-\eta,t+\eta)\;&\le \ell_{(\mathcal{M},F')}(s-\frac{\eta}{2},t+\frac{\eta}{2})\\ &\le \ell_{(\mathcal{M},F')}(s+\frac{\eta}{2},t-\frac{\eta}{2})\le \ell_{(\mathcal{M},F)}(s+\eta,t-\eta).\end{aligned}\tag{2.3}$$

Because of equality (2.2) we have that the inequalities (2.3) imply

$$\begin{aligned}\ell_{(\mathcal{M},F)}(s-\eta,t+\eta)\;&= \ell_{(\mathcal{M},F')}(s-\frac{\eta}{2},t+\frac{\eta}{2})\\ &= \ell_{(\mathcal{M},F')}(s+\frac{\eta}{2},t-\frac{\eta}{2})= \ell_{(\mathcal{M},F)}(s+\eta,t-\eta).\end{aligned}\tag{2.4}$$

Therefore, once again because of the monotonicity of $\ell_{(\mathcal{M},F')}$ in each variable, for every (s',t') with $\|(s,t)-(s',t')\|_\infty\le\frac{\eta}{2}$ and for every $(\vec{l}',\vec{b}')$ with $\|(\vec{l},\vec{b})-(\vec{l}',\vec{b}')\|_\infty\le\varepsilon$ the equality $\ell_{(\mathcal{M},F')}(s',t')=\ell_{(\mathcal{M},F)}(s,t)$ holds. By applying the Reduction Theorem 1.15 we get $\ell_{(\mathcal{M},\vec{\varphi})}(s'\cdot\vec{l}'+\vec{b}',t'\cdot\vec{l}'+\vec{b}')=\ell_{(\mathcal{M},\vec{\varphi})}(s\cdot\vec{l}+\vec{b},t\cdot\vec{l}+\vec{b})$. In other words, $\ell_{(\mathcal{M},\vec{\varphi})}$ is locally constant at the point $(\vec{x},\vec{y})$, and hence $(\vec{x},\vec{y})$ is not a discontinuity point for $\ell_{(\mathcal{M},\vec{\varphi})}$. □

Remark 2.6. Let us observe that Proposition 2.4 holds under weaker hypotheses, i.e. in the case that $\mathcal{M}$ is a non-empty, compact and locally connected Hausdorff space. However, for the sake of simplicity, we prefer here to confine ourselves to the setting assumed at the beginning of the present section.

The following theorem associates the discontinuities of a multidimensional size function to the $(\vec{l},\vec{b})$-pseudocritical values of $\vec{\varphi}$.

Theorem 2.7. *Let $(\vec{x},\vec{y})\in\Delta^+$ with $(\vec{x},\vec{y})=(s\cdot\vec{l}+\vec{b},t\cdot\vec{l}+\vec{b})\in\pi_{(\vec{l},\vec{b})}$. If $(\vec{x},\vec{y})$ is a discontinuity point for $\ell_{(\mathcal{M},\vec{\varphi})}$ then at least one of the following statements holds:*

(i): *s is a discontinuity point for $\ell_{(\mathcal{M},F)}(\cdot,t)$;*

(ii): *t is a discontinuity point for $\ell_{(\mathcal{M},F)}(s,\cdot)$.*

Moreover, (i) and (ii) are equivalent to

(i'): $\vec{x}$ is a discontinuity point for $\ell_{(\mathcal{M},\vec{\varphi})}(\cdot,\vec{y})$;

(ii'): $\vec{y}$ is a discontinuity point for $\ell_{(\mathcal{M},\vec{\varphi})}(\vec{x},\cdot)$,

respectively. If $\vec{\varphi}\in C^1(\mathcal{M},\mathbb{R}^k)$, statement (i) implies that s is an $(\vec{l},\vec{b})$-pseudocritical value for $\vec{\varphi}$, and statement (ii) implies that t is an $(\vec{l},\vec{b})$-pseudocritical value for $\vec{\varphi}$.

Proof. By Proposition 2.4 we have that (s,t) is a discontinuity point for $\ell_{(\mathcal{M},F)}$, and from Corollary 1.9 it follows that either s is a discontinuity point for $\ell_{(\mathcal{M},F)}(\cdot,t)$ or t is a discontinuity point for $\ell_{(\mathcal{M},F)}(s,\cdot)$, or both these conditions hold, thus proving the first part of the theorem.

Let us now suppose that s is a discontinuity point for $\ell_{(\mathcal{M},F)}(\cdot,t)$. Since the function $\ell_{(\mathcal{M},F)}(\cdot,t)$ is monotonic, then for every real value $\varepsilon>0$ we have that $\ell_{(\mathcal{M},F)}(s-\varepsilon,t)\neq\ell_{(\mathcal{M},F)}(s+\varepsilon,t)$. Moreover, the following equalities hold because of the Reduction Theorem 1.15:

$$\begin{aligned}\ell_{(M,F)}(s-\varepsilon,t) &= \ell_{(M,\vec{\varphi})}((s-\varepsilon)\cdot\vec{l}+\vec{b},t\cdot\vec{l}+\vec{b})=\ell_{(M,\vec{\varphi})}(\vec{x}-\varepsilon\cdot\vec{l},\vec{y})\\ \ell_{(M,F)}(s+\varepsilon,t) &= \ell_{(M,\vec{\varphi})}((s+\varepsilon)\cdot\vec{l}+\vec{b},t\cdot\vec{l}+\vec{b})=\ell_{(M,\vec{\varphi})}(\vec{x}+\varepsilon\cdot\vec{l},\vec{y}).\end{aligned}\tag{2.5}$$

By setting $\vec{\varepsilon}=\varepsilon\cdot\vec{l}$, we get $\ell_{(M,\vec{\varphi})}(\vec{x}-\vec{\varepsilon},\vec{y})\neq\ell_{(M,\vec{\varphi})}(\vec{x}+\vec{\varepsilon},\vec{y})$. Therefore $\vec{x}$ is a discontinuity point for $\ell_{(M,\vec{\varphi})}(\cdot,\vec{y})$, thus proving that $(i)\Rightarrow(i')$.

Let us now prove that $(i')\Rightarrow(i)$. If $\vec{x}$ is a discontinuity point for $\ell_{(M,\vec{\varphi})}(\cdot,\vec{y})$, from the monotonicity in the variable $\vec{x}$ (cf. Remark 1.4) it follows that $\ell_{(M,\vec{\varphi})}(\vec{x}-\varepsilon\cdot\vec{l},\vec{y})\neq\ell_{(M,\vec{\varphi})}(\vec{x}+\varepsilon\cdot\vec{l},\vec{y})$ for every $\varepsilon>0$. Therefore, because of the equalities (2.5) we get $\ell_{(\mathcal{M},F)}(s-\varepsilon,t)\neq\ell_{(\mathcal{M},F)}(s+\varepsilon,t)$, proving that $(i')\Rightarrow(i)$. Analogously, we can show that $(ii)\Leftrightarrow(ii')$.

Furthermore, if s is a discontinuity point for $\ell_{(\mathcal{M},F)}(\cdot,t)$, from the Representation Theorem 1.7 it follows that s is the abscissa of a cornerpoint (possibly at infinity). Hence, if $\vec{\varphi}\in C^1(\mathcal{M},\mathbb{R}^k)$ then by Theorem 2.3 we have that s is an $(\vec{l},\vec{b})$-pseudocritical value for $\vec{\varphi}$.

In a similar way, we can examine the case that t is a discontinuity point for $\ell_{(\mathcal{M},F)}(s,\cdot)$, and get the final statement. □

Before giving the first of our main results, we need the following definition.

Definition 2.8. Let $\vec{\xi}:\mathcal{M}\to\mathbb{R}^h$, and suppose that $\vec{\xi}$ is C^1 at a point $Q\in\mathcal{M}$. The point Q is said to be a *pseudocritical point for* $\vec{\xi}$ if the convex hull of the gradients $\nabla\xi_i(Q)$, $i=1,\dots,h$, contains the null vector, i.e. there exist $\lambda_1,\dots,\lambda_h\in\mathbb{R}$ such that $\sum_{i=i}^h\lambda_i\cdot\nabla\xi_i(Q)=\mathbf{0}$, with $0\le\lambda_i\le1$ and $\sum_{i=1}^h\lambda_i=1$. If Q is a pseudocritical point of $\vec{\xi}$, then $\vec{\xi}(Q)$ will be called a *pseudocritical value for* $\vec{\xi}$.

Remark 2.9. Definition 2.8 corresponds to the Fritz John necessary condition for optimality in Nonlinear Programming [1]. We shall use the term “pseudocritical” just for the sake of conciseness. For further references see [33]. The concept of pseudocritical point is strongly related also to the one of Jacobi Set (cf. [20]).

The next example makes Definition 2.8 clearer.

Example 2.10. Let us compute the pseudocritical points and values for the measuring function $\vec{\xi}=(\xi_1,\xi_2):\mathcal{M}\to\mathbb{R}^2$, where $\mathcal{M}$ is the surface coinciding with the unit sphere $S^2 \subset \mathbb{R}^3$, and $\vec{\xi}$ is obtained as the restriction to $\mathcal{M}$ of the function $\vec{\Xi} = (\Xi_1, \Xi_2) : \mathbb{R}^3 \to \mathbb{R}^2$, with $\vec{\Xi}(x, y, z) = (x, z)$ (see Figure 4). According to Definition 2.8, it follows that a point $Q \in \mathcal{M}$ is pseudocritical for $\vec{\xi}$ if and only if either $\nabla\xi_1(Q) = \mathbf{0}$, or $\nabla\xi_2(Q) = \mathbf{0}$, or these two gradient vectors are parallel with opposite verse. Referring to our example, $\nabla\xi_1(Q)$ and $\nabla\xi_2(Q)$ are the projections of $\nabla\Xi_1(Q) = (1, 0, 0)$ and $\nabla\Xi_2(Q) = (0, 0, 1)$ onto the tangent space of $\mathcal{M}$ at Q, respectively. Therefore, it can be easily verified that the pseudocritical points of $\mathcal{M}$ for the function $\vec{\xi}$ are given by the set $\{(\cos\alpha, 0, \sin\alpha),\ 0 \le \alpha \le \frac{\pi}{2} \ \vee\ \pi \le \alpha \le \frac{3}{2}\pi\}$. Hence, the corresponding pseudocritical values are the elements of the set $\{(\cos\alpha, \sin\alpha),\ 0 \le \alpha \le \frac{\pi}{2} \ \vee\ \pi \le \alpha \le \frac{3}{2}\pi\}$.

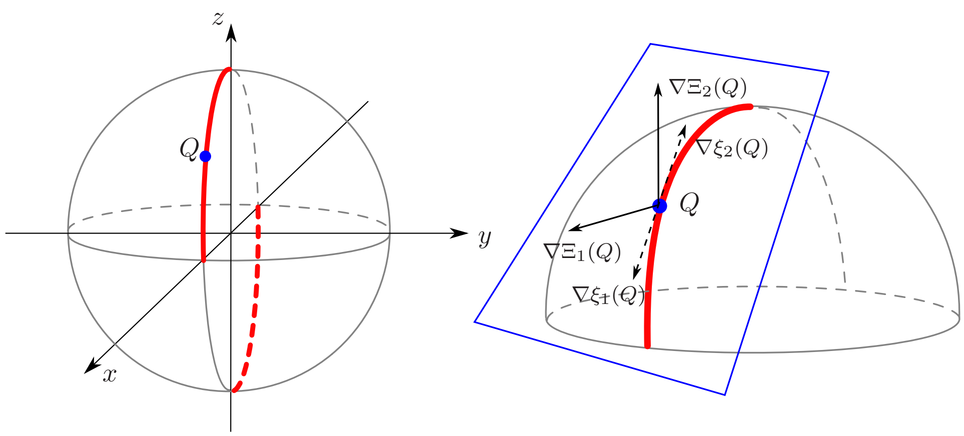


FIGURE 4. (a) The sphere $S^2 \subseteq \mathbb{R}^3$ endowed with the measuring function $\vec{\xi} = (\xi_1, \xi_2) : S^2 \to \mathbb{R}^2$, defined as $\vec{\xi}(x, y, z) = (x, z)$ for each $(x, y, z) \in S^2$. The pseudocritical points of $\vec{\xi}$ are depicted in bold red. (b) The point Q is a pseudocritical point for $\vec{\xi}$, since the vectors $\nabla\xi_1(Q)$ and $\nabla\xi_2(Q)$ are parallel with opposite verse.

In the following, we shall say that $\rho : \mathbb{R}^k \to \mathbb{R}^h$ is a *projection* if there exist h indices $i_1, \ldots, i_h$ such that $\rho((x_1, \ldots, x_k)) = (x_{i_1}, \ldots, x_{i_h})$, for every $\vec{x} = (x_1, \ldots, x_k) \in \mathbb{R}^k$. In other words, such a function ρ is used to delete some components of a vector $\vec{x} \in \mathbb{R}^k$.

We are now ready to give the first main result of this paper.

Theorem 2.11. *Assume that $\vec{\varphi} \in C^1(\mathcal{M}, \mathbb{R}^k)$. Let $(\vec{x}, \vec{y}) \in \Delta^+$ be a discontinuity point for $\ell_{(\mathcal{M},\vec{\varphi})}$. Then at least one of the following statements holds:*

(i): $\vec{x}$ is a discontinuity point for $\ell_{(\mathcal{M},\vec{\varphi})}(\cdot, \vec{y})$;

(ii): $\vec{y}$ is a discontinuity point for $\ell_{(\mathcal{M},\vec{\varphi})}(\vec{x}, \cdot)$.

Moreover, if (i) holds, then a projection ρ exists such that $\rho(\vec{x})$ is a pseudocritical value for $\rho \circ \vec{\varphi}$. If (ii) holds, then a projection ρ exists such that $\rho(\vec{y})$ is a pseudocritical value for $\rho \circ \vec{\varphi}$.

Proof. Because of Proposition 1.14, an admissible pair $(\vec{l}, \vec{b})$ exists, such that $(\vec{x}, \vec{y}) = (s \cdot \vec{l} + \vec{b}, t \cdot \vec{l} + \vec{b})$ for a suitable pair (s, t). Statements (i) and (ii) are guaranteed by Theorem 2.7, assuring that either $\vec{x}$ is a discontinuity point for $\ell_{(\mathcal{M},\vec{\varphi})}(\cdot, \vec{y})$ and

s is an $(\vec{l},\vec{b})$-pseudocritical value for $\vec{\varphi}$, or $\vec{y}$ is a discontinuity point for $\ell_{(\mathcal{M},\vec{\varphi})}(\vec{x},\cdot)$ and t is an $(\vec{l},\vec{b})$-pseudocritical value for $\vec{\varphi}$, or both these conditions hold.

Let us now confine ourselves to assume that $\vec{x}$ is a discontinuity point for $\ell_{(\mathcal{M},\vec{\varphi})}(\cdot,\vec{y})$ and s is an $(\vec{l},\vec{b})$-pseudocritical value for $\vec{\varphi}$. We shall prove that a projection ρ exists such that $\rho(\vec{x})$ is a pseudocritical value for $\rho\circ\vec{\varphi}$. The proof in the case that $\vec{y}$ is a discontinuity point for $\ell_{(\mathcal{M},\vec{\varphi})}(\vec{x},\cdot)$ and t is an $(\vec{l},\vec{b})$-pseudocritical value for $\vec{\varphi}$ proceeds in quite a similar way. Since s is an $(\vec{l},\vec{b})$-pseudocritical value for $\vec{\varphi}$, by Definition 2.1 there exist a point $Q\in\mathcal{M}$ and some indices $i_1,\ldots,i_h$ with $1\le h\le k$, such that $s=F(Q)=\frac{\varphi_{i_1}(Q)-b_{i_1}}{l_{i_1}}=\cdots=\frac{\varphi_{i_h}(Q)-b_{i_h}}{l_{i_h}}$ and $\sum_{j=1}^{h}\lambda_j\cdot\nabla\vec{\varphi}_{i_j}(Q)=\mathbf{0}$, with $0\le\lambda_j\le 1$ for $j=1,\ldots,h$, and $\sum_{j=1}^{h}\lambda_j=1$. Let us now consider the projection $\rho:\mathbb{R}^k\to\mathbb{R}^h$ defined by setting $\rho(\vec{x})=(x_{i_1},\ldots,x_{i_h})$. Since $(\vec{x},\vec{y})=(x_1,\ldots,x_k,y_1,\ldots,y_k)=(s\cdot l_1+b_1,\ldots,s\cdot l_k+b_k,t\cdot l_1+b_1,\ldots,t\cdot l_k+b_k)$, we observe that $x_{i_j}=\left(\frac{\varphi_{i_j}(Q)-b_{i_j}}{l_{i_j}}\right)\cdot l_{i_j}+b_{i_j}=\varphi_{i_j}(Q)$, for every $j=1,\ldots,h$. Therefore it follows that $\rho(\vec{x})$ is a pseudocritical value for $\rho\circ\vec{\varphi}$. □

Remark 2.12. We stress that Theorem 2.11 improves the result obtained in Theorem 2.7, providing a necessary condition for discontinuities of multidimensional size functions that does not depend on the foliation of the domain Δ^+.

2.1. **Refining Theorem 2.11 to less regular measuring functions.** In this section we generalize Theorem 2.11 to the case of continuous measuring functions. In what follows, we shall call a *special point for a continuous function* $\vec{\xi}:\mathcal{M}\to\mathbb{R}^h$ any point $Q\in\mathcal{M}$ where $\vec{\xi}$ is not C^1. If Q is a special point for $\vec{\xi}$, the value $\vec{\xi}(Q)$ will be called a *special value for* $\vec{\xi}$.

Theorem 2.13. *Let $(\vec{x},\vec{y})\in\Delta^+$ be a discontinuity point for $\ell_{(\mathcal{M},\vec{\varphi})}$. Then at least one of the following statements holds:*

- (i)**:** *$\vec{x}$ is a discontinuity point for $\ell_{(\mathcal{M},\vec{\varphi})}(\cdot,\vec{y})$;*
- (ii)**:** *$\vec{y}$ is a discontinuity point for $\ell_{(\mathcal{M},\vec{\varphi})}(\vec{x},\cdot)$.*

Moreover, if (i) holds, then a projection ρ exists such that $\rho(\vec{x})$ is either a special value or a pseudocritical value for $\rho\circ\vec{\varphi}$. If (ii) holds, then a projection ρ exists such that $\rho(\vec{y})$ is either a special value or a pseudocritical value for $\rho\circ\vec{\varphi}$.

Proof. Because of Proposition 1.14, an admissible pair $(\vec{l},\vec{b})$ exists, such that $(\vec{x},\vec{y})=(s\cdot\vec{l}+\vec{b},t\cdot\vec{l}+\vec{b})$ for a suitable pair (s,t). Statements (i) and (ii) are guaranteed by Theorem 2.7, assuring that either $\vec{x}$ is a discontinuity point for $\ell_{(\mathcal{M},\vec{\varphi})}(\cdot,\vec{y})$ and s is a discontinuity point for $\ell_{(\mathcal{M},F)}(\cdot,t)$, or $\vec{y}$ is a discontinuity point for $\ell_{(\mathcal{M},\vec{\varphi})}(\vec{x},\cdot)$ and t is a discontinuity point for $\ell_{(\mathcal{M},F)}(s,\cdot)$, or both these conditions hold.

Let us now assume that $\vec{x}$ is a discontinuity point for $\ell_{(\mathcal{M},\vec{\varphi})}(\cdot,\vec{y})$ and s is a discontinuity point for $\ell_{(\mathcal{M},F)}(\cdot,t)$. We shall prove that a projection ρ exists such that $\rho(\vec{x})$ is either a special value or a pseudocritical value for $\rho\circ\vec{\varphi}$.

Call $\mathbb{S}_j$ the set of special points of $\varphi_j:\mathcal{M}\to\mathbb{R}$, for $j=1,\ldots,k$. For every $i\in\mathbb{N}^+=\mathbb{N}\setminus\{0\}$ and $j=1,\ldots,k$, consider the compact set $K_j^i=\{Q\in\mathcal{M}: d(Q,\mathbb{S}_j)\ge\frac{1}{i}\}$, and take a C^1 function $\varphi_j^i:\mathcal{M}\to\mathbb{R}$ such that

(1) $\max_{Q\in\mathcal{M}}|\varphi_j(Q)-\varphi_j^i(Q)|\le\frac{1}{i}$;
(2) $\max_{Q\in K_j^i}\|\nabla\varphi_j(Q)-\nabla\varphi_j^i(Q)\|\le\frac{1}{i}$.

This can be done by considering the convolution of each component φ_j, $j = 1, \dots, k$, with a suitable "regularizing" function.

From now on, for the sake of conciseness we shall use the symbols F and F^i to denote the functions $F^{\vec{\varphi}}_{(\vec{l}\,\vec{b})} = \max_{j=1,\dots,k}\left\{\frac{\varphi_j - b_j}{l_j}\right\}$ and $F^{\vec{\varphi}^i}_{(\vec{l}\,\vec{b})} = \max_{j=1,\dots,k}\left\{\frac{\varphi^i_j - b_j}{l_j}\right\}$, respectively. For every $i \in \mathbb{N}^+$, we also set $\vec{\varphi}^i = (\varphi^i_1, \dots, \varphi^i_k)$.

Since s is a discontinuity point for $\ell_{(\mathcal{M},F)}(\cdot, t)$, by the Representation Theorem 1.7 it follows that a cornerpoint of $\ell_{(\mathcal{M},F)}$ (proper or at infinity) of coordinates $(s, \bar{t})$ exists, with $\bar{t} > t$. Moreover, by condition (1) we have that the sequence (F^i) uniformly converges to F. Therefore, the Matching Stability Theorem 1.10 implies that a sequence $((s^i, \bar{t}^i))$ exists, such that $(s^i, \bar{t}^i)$ is a cornerpoint for $\ell_{(\mathcal{M},F^i)}$ and $((s^i, \bar{t}^i))$ converges to $(s, \bar{t})$. For every large enough index i, once more by the Representation Theorem 1.7, s^i is then a discontinuity point for $\ell_{(\mathcal{M},F^i)}(\cdot, t)$, and hence by Theorem 2.7 we have that $\vec{x}^{\,i} = s^i \cdot \vec{l} + \vec{b}$ is a discontinuity point for $\ell_{(\mathcal{M},\vec{\varphi}^i)}(\cdot, \vec{y})$. From Theorem 2.11 it follows that a projection ρ^i exists, such that $\rho^i(\vec{x}^{\,i})$ is a pseudocritical value for $\rho^i \circ \vec{\varphi}^i$. Possibly by considering a subsequence, we can suppose that all the ρ^i equal a projection ρ. Moreover, we can consider a sequence (Q^i) such that $Q^i \in \mathcal{M}$, $\rho \circ \vec{\varphi}^i(Q^i) = \rho(\vec{x}^{\,i})$ and Q^i is a pseudo-critical point for $\rho \circ \vec{\varphi}^i$. Furthermore, by the compactness of $\mathcal{M}$, possibly by extracting a subsequence we can assume (Q^i) converging to a point $Q \in \mathcal{M}$. From the continuity of $\vec{\varphi}$ and from the uniform convergence of $(\vec{\varphi}^i)$ to $\vec{\varphi}$, we can deduce

$$(3)\quad \rho \circ \vec{\varphi}(Q) = \lim_{i\to\infty} \rho \circ \vec{\varphi}(Q^i) = \lim_{i\to\infty} \rho \circ \vec{\varphi}^i(Q^i) = \lim_{i\to\infty} \rho(\vec{x}^{\,i}) = \rho(\vec{x}).$$

If $\rho(\vec{x})$ is a special value for $\rho \circ \vec{\varphi}$ then our claim is proved. If $\rho(\vec{x}) = (x_{j_1}, \dots, x_{j_h})$ is not a special value for $\rho \circ \vec{\varphi}$ then $Q \notin \mathbb{S}_{j_1} \cup \ldots \cup \mathbb{S}_{j_h}$. Hence, for any large enough index i, it follows that $Q, Q^i \in K^i_{j_1} \cap \ldots \cap K^i_{j_h}$. By recalling that each point Q^i is a pseudocritical point for $\rho \circ \vec{\varphi}^i$, and by observing that the property of being a pseudocritical point passes to the limit, we get that $\rho(\vec{x})$ is a pseudocritical value for $\rho \circ \vec{\varphi}$. In other words, we have just proved that if $\vec{x}$ is a discontinuity point for $\ell_{(\mathcal{M},\vec{\varphi})}(\cdot, \vec{y})$, then a projection ρ exists such that $\rho(\vec{x})$ is either a special value or a pseudocritical value for $\rho \circ \vec{\varphi}$.

Analogously, it is possible to prove that if $\vec{y}$ is a discontinuity point for $\ell_{(\mathcal{M},\vec{\varphi})}(\vec{x}, \cdot)$, then a projection ρ exists such that $\rho(\vec{y})$ is either a special value or a pseudocritical value for $\rho \circ \vec{\varphi}$. □

2.2. **Consequences of our results.** The results proved in this paper imply several relevant consequences. First of all they contribute to clarifying the structure of multidimensional size functions. In order to explain this point let us consider the case of a compact smooth surface $\mathcal{S}$ endowed with a smooth function $\vec{\varphi} : \mathcal{S} \to \mathbb{R}^2$. It is immediate to verify that all pseudocritical points belong to the Jacobi set of $\vec{\varphi}$, that is the set where the gradients $\nabla\varphi_1$ and $\nabla\varphi_2$ are parallel. This implies (cf. [20]) that in the generic case the pseudocritical points belong to a 1-submanifold $\mathcal{J}$ of $\mathcal{S}$ (in local coordinates such a manifold is determined by the vanishing of the Jacobian of $\vec{\varphi}$). Now, Theorem 2.13 allows us to claim that all discontinuity points (x_1, x_2, y_1, y_2) of the size function $\ell_{(\mathcal{M},\vec{\varphi})}$ belong either to $\mathcal{J} \times \mathbb{R}^2$ or to $\mathbb{R}^2 \times \mathcal{J}$. For the computation of $\mathcal{J}$ we refer to [20].

In the light of this new information, we can imagine the possibility of constructing new algorithms to efficiently compute multidimensional size functions. Let us consider the connected components in which the domain of $\ell_{(\mathcal{M},\vec{\varphi})}$ is divided by the set $(\mathcal{J} \times \mathbb{R}^2) \cup (\mathbb{R}^2 \times \mathcal{J})$. Since size functions are locally constant at each point of continuity (we recall that they are natural-valued), we immediately obtain that $\ell_{(\mathcal{M},\vec{\varphi})}$ is constant at each of those connected components. It follows that the computation of $\ell_{(\mathcal{M},\vec{\varphi})}$ just requires the computation of its value at only one point for each connected component. These observations open the way to new and more efficient methods of computation for multidimensional size functions.

Our results also make new pseudodistances between size functions computable in an easier way. Indeed, let us consider two size pairs $(\mathcal{M},\vec{\varphi})$, $(\mathcal{N},\vec{\psi})$ and the value δ_H giving the Hausdorff distance between the sets where $\ell_{(\mathcal{M},\vec{\varphi})}$ and $\ell_{(\mathcal{N},\vec{\psi})}$ are discontinuous. It is trivial to check that the function d_D defined by setting $d_D\left(\ell_{(\mathcal{M},\vec{\varphi})},\ell_{(\mathcal{N},\vec{\psi})}\right)=\delta_H$ is a pseudodistance between multidimensional size functions. Helping us to localize the discontinuities of multidimensional size functions, Theorem 2.13 makes the computation of d_D easier.

Conclusions and future work

In this paper we have proved that a discontinuity point for a multidimensional size function has at least one special or pseudocritical coordinate, under the hypothesis that the considered measuring function is (at least) continuous. This result is a first step in the development of Size Theory for $\mathbb{R}^k$-valued measuring functions. Indeed, the localization of the unique points where k-dimensional size functions can be discontinuous allows us to better understand Topological Persistence and opens the way to the formulation of effective algorithms for its computation. On the other hand, it is worth noting that our framework could be applicable also to the study of discontinuities in persistent algebraic topology, including persistent homology groups and size homotopy groups. However, some difficulties could derive from the present lack of the analogue of Theorem 1.10 for those structures, i.e. a stability result in the case of *continuous* (possibly non-tame [10]) measuring functions. These last research lines appear to be promising, both from the theoretical and the applicative point of view.

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This paper is dedicated to Martina and Riccardo.

A. Appendix

A.1. **Relationship between Size Theory and Persistent Homology.** Size Theory and Persistent Homology are deeply connected theories. We shall recall some similarities and differences between them in this appendix. For more details we refer to the survey papers [3] and [21].

Size Theory was born at the beginning on the 1990s (cf. [23, 24]) as a mathematical approach to shape comparison. The main idea is to describe shape as a pseudometric (the *natural pseudodistance*) between topological spaces endowed with real-valued functions, called *measuring functions*. The measuring functions are used as descriptors of the properties with respect to which the topological spaces are compared. For example, if we are interested in the comparison of two objects A and B with respect to their bumps and hollows, it can be natural to consider two subsets of $\mathbb{R}^3$ representing their bodies, endowed with two functions associating with each point its distance from the center of mass of the body it belongs to. On the other hand, if we are interested in the colorings of A and B, we can consider two surfaces endowed with functions representing the color taken at each point. The natural pseudodistance between two pairs *(topological space, measuring function)*, called *size pairs*, is the infimum of the change of the measuring function under the action of all possible homeomorphisms from one topological space to the other. Size functions and size homotopy groups (their algebraic-topological equivalent; cf. [28]) appeared as mathematical tools useful for computing lower bounds for the natural pseudodistance, introducing *ante litteram* the study of Topological Persistence.

Persistent Homology was born approximately ten years later, at the beginning of the 00s, as a mathematical approach to studying the homology of topological spaces known just by a sampling. In this case, the attention was focused on the radius r of the spheres centered at the sample points, whose union approximates the topological space. The problem of choosing the value of r led to the concept of persistence, emphasizing the topological properties stable under the change of r. In other words, the main goal was topological simplification, in order to get the relevant topological information concerning the object under study. The value r, playing the role of the measuring function in Size Theory, has been subsequently extended to more general functions.

Despite their different origins and goals, Size Theory and Persistent Homology have developed similar structures and concepts, under different names. In order to help readers who are not familiar with both these theories, this section compares some of their key concepts, explaining their reciprocal links. These connections and relationships are summarized in Table 1.

As we have already said previously, the objects under study in Size Theory are the pairs *(topological space, measuring function)*, called size pairs. The main results of this paper, stated in Theorem 2.11 and Theorem 2.13, are given under the assumption that the topological space is a closed C^1 Riemannian manifold, while the measuring function is supposed to be at least continuous. In Persistent Homology the object of study is usually a simplicial complex K, endowed with a filtration, i.e. a nested sequence of subcomplexes that starts with the empty complex and ends with the complete complex K. The filtration is usually obtained by a real-valued function defined at the vertices of K and extended to the simplexes. Each level K_c in the filtration is obtained by taking just the simplexes having vertexes at which the function takes a value less than (or equal to) a parametrical value c.

As a matter of fact, Size Theory is more focused on continuous data (topological spaces or manifolds, endowed with continuous or C^k functions), while Persistent Homology usually studies discrete structures (simplicial complexes endowed with piecewise linear functions) or structures satisfying some finiteness hypotheses (topological spaces endowed with tame functions). As a consequence, the results obtained in the two theories are often expressed and proved in similar but different mathematical settings. For example, while the fact that the persistent homology groups are finitely generated is just a trivial consequence of the assumed hypotheses, the finiteness of size functions requires a (simple but not trivial) proof. Analogously, while the localization of discontinuities for the rank of the 0-th persistent homology group (i.e. the 0-th rank invariant) is usually trivial in the 1-dimensional setting, this does not hold for the discontinuities of a size function. This is actually what happens in this paper, where the measuring functions are not required to be tame

Size Theory	**references**	**Persistent Homology**	**references**
size pair	[23, 25]	*filtration of a complex*	[10, 22]
natural pseudodistance between size pairs	[16, 18]	———	———
measuring function	[23, 25]	*filtrating function*	[21, 22]
(multidimensional) size function	[2, 23, 24]	0*th rank invariant*	[7]
size homotopy group	[28]	———	———
size functor	[5]	———	———
———	———	*persistent* k*-th homology group*	[21, 22]
multiplicity of cornerpoints	[27, 31]	*multiplicity of points in persistence diagrams*	[21, 22]
formal series of cornerpoints and cornerlines	[27, 31]	*persistence diagrams*	[21, 22]
multidimensional Size Theory	[2, 28]	*multidimensional Persistent Homology*	[6, 7]

Table 1. Approximative correspondence between some concepts in Size Theory and Persistent Homology. For each concept bibliographic references are reported. A line denotes a missing correspondence.

(cf. [10] for a formal definition of tame function). Obviously, this creates many technical difficulties, even in the case of C^1 measuring functions, since they are allowed to have an infinite number of critical values. This kind of problem does not usually appear in literature regarding Persistent Homology.

Size functions are the most usual tool in Size Theory, while persistent homology groups constitute the main object of research in Persistent Homology. Size functions are simply the rank of persistent 0-homology groups. On the other hand, the relationship between persistent homology groups (introduced in [22]) and size homotopy groups (introduced in [28]) is the same that links homology groups and homotopy groups. For example, the first persistent homology group is the Abelianization of the first size homotopy group.

Both size functions and persistent homology groups are often represented by sets of points with multiplicities. The representation for size functions is called *formal series of cornerpoints (proper and at infinity)* and was introduced in [31]. The correspondent representation for persistent homology groups is named *persistence diagram* and was introduced in [22]. The formulas defining the multiplicities of the considered points are quite analogous. However, because of the hypotheses usually assumed in Persistent Homology, persistence diagrams are finite collections of points, while the formal series used in Size Theory can contain an infinite number of cornerpoints. The k-Triangle Lemma in [22] is essentially equivalent to the

Representation Theorem recalled in this paper and proved in [27] (under slightly different hypotheses).

Formal series representing size functions and persistence diagrams representing the ranks of persistent homology groups can be compared by using some matching distances (cf. [27, 31] for size functions and [10] for persistence diagrams). The matching distance used in this paper has been studied in [11, 13] for size functions and in [10] for persistent homology groups.

The study of multidimensional measuring functions has started in [28] for Size Theory and in [7] for Persistent Homology.

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