Hilden Braid Groups

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Abstract

Let H_g be a genus g handlebody and $\mathrm{MCG}_{2n}(\mathrm{T}_g)$ be the group of the isotopy classes of orientation preserving homeomorphisms of $\mathrm{T}_g = \partial \mathrm{H}_g$, fixing a given set of $2n$ points. In this paper we study two particular subgroups of $\mathrm{MCG}_{2n}(\mathrm{T}_g)$ which generalize Hilden groups defined by Hilden in [16]. As well as Hilden groups are related to plat closures of braids, these generalizations are related to Heegaard splittings of manifolds and to bridge decompositions of links. Connections between these subgroups and motion groups of links in closed 3-manifolds are also provided.

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1 Introduction

In [16] Hilden introduced and found generators for two particular subgroups of the mapping class group of the sphere with $2n$ punctures. Roughly speaking these groups consist of (the isotopy classes of) homeomorphisms of the puctured sphere which admit an extension to the 3-ball fixing n arcs embedded in the 3-ball and bounded by the punctures. The interest in these groups was motivated by the theory of links in $\mathbb{S}^3$ (or in $\mathbb{R}^3$). In [5], Hilden's generators were used in order to find a finite number of explicit equivalence moves relating two braids having the same plat closure. More recently, many authors investigated different groups related to Hilden's ones (see [1, 6, 26, 27]) and in particular motion groups (introduced in [12]). The second author introduced in [10] a higher genus generalization of Hilden's groups. These groups are subgroups of punctured mapping class groups of closed surfaces and are related to the study of link theory in a closed 3-manifold.

In this paper we define and study a different higher genus generalization of Hilden's groups: the Hilden braid groups. These groups can be seen as subgroups of the ones studied in [10] and they can be thought as a generalization of Hilden's groups in the "braid direction". Analogously to the genus zero case, our interest in these groups is mainly motivated by the theory of links in a closed 3-manifold. With respect to groups introduced in [10], our groups seem to be more useful in studying links in a fixed manifold.

The paper is organized as follows: Section 2 is devoted to the definition of Hilden braid groups and pure Hilden braid groups; a set of generators for these groups will be provided in Section 3 (Theorems 2 and 3). As we will prove in Section 4, by fixing a Heegaard decomposition of a given 3-manifold, it is possible to define a plat-like closure for $2n$-string braids on the Heegaard surface. In this setting, Hilden braid groups play a role similar to Hilden groups in the plat closure of classical braids. Moreover, as in the genus zero case, Hilden braid groups are connected with motion groups of links in closed 3-manifolds; this relation is established in Section 5 (Theorem 7).

2 Hilden groups: topological generalizations

Referring to Figure 1, let H_g be an oriented handlebody of genus $g \geq 0$ and $\partial \mathrm{H}_g = \mathrm{T}_g$. We recall that a system of n pairwise disjoint arcs $\mathcal{A}_n = \{A_1, \ldots, A_n\}$ properly embedded in H_g is called *trivial* or *boundary parallel* if there exist n disks (the grey ones in Figure 1) $D_1, \ldots, D_n$, called *trivializing disks*, embedded in H_g such that $A_i \cap D_i = A_i \cap \partial D_i = A_i$, $\partial D_i - A_i \subset \partial \mathrm{H}_g$ and $A_i \cap D_j = \emptyset$, for $i, j = 1, \ldots, n$ and $i \neq j$.

By means of the trivializing disks D_i we can "project" each arc A_i into the arc $a_i = \partial D_i - \mathrm{int}(A_i)$ embedded in T_g and with the property that $a_i \cap a_j = \emptyset$, if $i \neq j$. We denote with $P_{i,1}, P_{i,2}$ the endpoints of the arc A_i (which clearly coincide with the endpoints of a_i), for $i = 1, \ldots, n$.

Let $\mathrm{MCG}_{2n}(\mathrm{T}_g)$ (resp. $\mathrm{MCG}_n(\mathrm{H}_g)$) be the group of the isotopy classes of orientation preserving homeomorphisms of T_g (resp. H_g) fixing the set $\mathcal{P}_{2n} = \{P_{i,1}, P_{i,2} \,|\, i = 1, \ldots, n\}$ (resp. $A_1 \cup \cdots \cup A_n$).

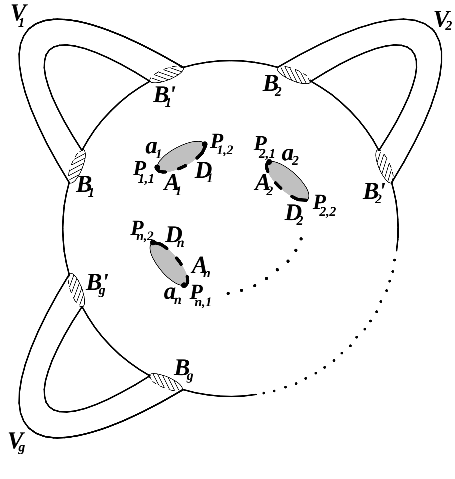

Figure 1: The model for a genus g handlebody and a trivial system of arcs.

The *Hilden mapping class group* $\mathcal{E}^g_{2n}$ is the subgroup of $\mathrm{MCG}_{2n}(\mathrm{T}_g)$ defined as the image of the injective group homomorphism $\mathrm{MCG}_n(\mathrm{H}_g) \longrightarrow \mathrm{MCG}_{2n}(\mathrm{T}_g)$ induced by restriction to the boundary. In other words, $\mathcal{E}^g_{2n}$ consists of the isotopy classes of homeomorphisms that admit an extension to H_g fixing $A_1 \cup \cdots \cup A_n$. Moreover, if $\mathrm{PMCG}_{2n}(\mathrm{T}_g)$ denotes the subgroup of $\mathrm{MCG}_{2n}(\mathrm{T}_g)$ consisting of the isotopy classes of the homeomorphisms of T_g fixing the punctures pointwise, we set $\bar{\mathcal{E}}^g_{2n} = \mathrm{PMCG}_{2n}(\mathrm{T}_g) \cap \mathcal{E}^g_{2n}$ and call it the *pure Hilden mapping class group*. As recalled before, the groups $\mathcal{E}^0_{2n}$ and $\bar{\mathcal{E}}^0_{2n}$ were first introduced and studied by Hilden in [16], where the author found a finite set of generators, while in [10] has been provided a set of generators for all Hilden (pure) mapping class groups.

Now we are ready to define Hilden braid groups. Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{MCG}_n(\mathrm{H}_g) & \xrightarrow{\cong} & \mathcal{E}^g_{2n} \subset \mathrm{MCG}_{2n}(\mathrm{T}_g) \\ \Big\downarrow \bar{\Omega}_{g,n} & & \Big\downarrow \Omega_{g,n} \\ \mathrm{MCG}(\mathrm{H}_g) & \xrightarrow{\cong} & \mathcal{E}^g_0 \subset \mathrm{MCG}(\mathrm{T}_g). \end{array}$$

where the vertical rows are forgetfull homomorphisms. The *n-th Hilden braid group* of the surface T_g is the group $\mathrm{Hil}^g_n := \mathcal{E}^g_{2n} \cap \ker \Omega_{g,n} \cong \ker \bar{\Omega}_{g,n}$. The *n-th Hilden pure braid group* $P\mathrm{Hil}^g_n$ of the surface T_g is the pure part of Hil^g_n, that is $\mathrm{Hil}^g_n \cap \mathrm{PMCG}_{2n}(\mathrm{T}_g)$. Notice that, since $\mathrm{MCG}(\mathbb{S}^2) = \mathrm{PMCG}(\mathbb{S}^2) = 1$,then $\mathcal{E}^0_{2n} = \mathrm{Hil}^0_n$ and $\bar{\mathcal{E}}^0_{2n} = P\mathrm{Hil}^0_n$. Moreover, in [3], it is shown that $\ker(\Omega_{g,n})$ is isomorphic to the quotient of the braid group $\mathrm{B}_{2n}(\mathrm{T}_g)$ by its center, which is trivial if $g \geq 2$. So, if $g \geq 2$ we can see Hil^g_n as a subgroup of the braid group of the surface T_g.

In [26, 27], Tawn found a finite presentation for two groups that he called the Hilden group $\mathbf{H}_{2n}$ and pure Hilden group $\mathbf{PH}_{2n}$. The definition proposed by Tawn is slightly different from ours; indeed, $\mathbf{H}_{2n}$ and $\mathbf{PH}_{2n}$ are subgroups of, respectively, the braid group B_{2n} and the pure braid group P_{2n}, instead of, respectively, the mapping class group $\mathrm{MCG}_{2n}(\mathbb{S}^2)$ and the pure mapping class group $\mathrm{PMCG}_{2n}(\mathbb{S}^2)$. Nevertheless, from these presentations it is not difficult to obtain a presentation for Hil^0_n and $P\mathrm{Hil}^0_n$ as sketched in the following. Consider the inclusion of the punctured $2n$ disk into the $2n$ punctured sphere: it induces surjective maps from $\mathrm{B}_{2n} = \mathrm{MCG}_{2n}(D^2)$ to $\mathrm{MCG}_{2n}(\mathbb{S}^2)$ and from $\mathrm{P}_{2n} = \mathrm{PMCG}_{2n}(D^2)$ to $\mathrm{PMCG}_{2n}(\mathbb{S}^2)$. The kernels of these maps coincide with the subgroup normally generated by the center of $\mathrm{MCG}_{2n}(D^2)$ and the element $\sigma_1 \dots \sigma^2_{2n-1} \dots \sigma_1$, where σ_i denotes the usual generator of the braid group B_{2n}. One can easily show that such elements belong to $\mathbf{PH}_{2n}$ and that these maps restrict to surjective homomorphisms $\mathbf{H}_{2n} \longrightarrow \mathrm{Hil}^0_n$ and $\mathbf{PH}_{2n} \longrightarrow P\mathrm{Hil}^0_n$. Therefore a finite presentation of Hil^0_n (respectively of $P\mathrm{Hil}^0_n$) is given by the same set of generators of $\mathbf{H}_{2n}$ (respectively $\mathbf{PH}_{2n}$) and the same set of relations of $\mathbf{H}_{2n}$ (respectively $\mathbf{PH}_{2n}$) plus the relation $W_1 = 1$ and $W_2 = 1$, where W_1 and W_2 are, respectively, the generator of the center of $\mathrm{MCG}_{2n}(D^2)$ and the element $\sigma_1 \dots \sigma^2_{2n-1} \dots \sigma_1$ both written as words in the generators of $\mathbf{H}_{2n}$ (respectively $\mathbf{PH}_{2n}$). For further details on the genus zero case see [10], while in this paper we will mainly focus on the positive genus cases.

3 Generators of Hil^g_n

In this section we find a set of generators for Hil^g_n and $P\mathrm{Hil}^g_n$. We start by fixing some notations.

Referring to Figure 1, for each $k = 1, \dots, g$, we denote with V_k the k-th 1-handle (i.e. a solid cylinder) obtained by cutting H_g along the two (isotopic) meridian disks B_k and B'_k. Moreover, we set $b_k = \partial B_k$ and $b'_k = \partial B'_k$ and call them meridian curves. For each $i = 1, \dots, n$ the disk D_i denotes the trivializing disk for the i-th arc A_i and $a_i = \partial D_i \setminus \mathrm{int}(A_i)$. The endpoints of both a_i and A_i are denoted with $P_{i,1}, P_{i,2}$ and we set $\mathcal{P}_{2n} = \{P_{i,1}, P_{i,2} \mid i = 1, \dots, n\}$. We denote with D a disk embedded in T_g containing all the arcs a_i and not intersecting any meridian curve b_k or b'_k. Finally δ_i denotes a disk in D containing a_i and such that $\delta_i \cap a_j = \emptyset$ for $i \neq j$ and $j = 1, \ldots, n$.

Let us describe certain families of homeomorphisms of T_g fixing setwise $\mathcal{P}_{2n}$ and whose isotopy classes belong to Hil_n^g. We will keep the same notation for a homeomorphism and its isotopy class.

Intervals For $i = 1, \ldots, n$, we denote with ι_i the homeomorphism of T_g that exchanges the endpoints of a_i inside δ_i and that is the identity outside δ_i. The interval ι_i is also called braid twist (see for instance [21]).

Elementary exchanges of arcs For $i = 1, \ldots, n-1$, let N_i be a tubular neighborhood of $\delta_i \cup \beta_i \cup \delta_{i+1}$ where β_i is a band connecting δ_i and δ_{i+1}, lying inside D and not intersecting any arc a_j, for $j = 1, \ldots, n$. We denote with λ_i the homeomorphism of T_g that exchanges a_i and a_{i+1}, mapping $P_{i,j}$ to $P_{i+1,j}$ inside N_i, for $j = 1, 2$, and that is the identity outside N_i.

Elementary twists We denote with s_i the Dehn twist along the curve $d_i = \partial\delta_i$. Notice that $s_i = \iota_i^2$ in $\mathrm{MCG}_{2n}(\mathrm{T}_g)$.

Slides of arcs Let C be an oriented simple closed curve curve in $\mathrm{T}_g \setminus \mathcal{P}_{2n}$ intersecting a_i transversally in one point. Consider an embedded closed annulus $A(C)$ in T_g whose core is C, containing a_i in its interior part and such that $A(C) \cap \mathcal{P}_{2n} = \{P_{i,1}, P_{i,2}\}$. We denote with C_1 and C_2 the boundary curves of $A(C)$ with the convention that C_1 is the one on the left of C according to its orientation, (see Figure 2). The slide $S_{i,C}$ of the arc a_i along the curve C is the multi-twist $T_{C_1}^{-1} T_{C_2} s_i^{\varepsilon}$, where $\varepsilon = 1$ if travelling along C we see $P_{i,1}$ on the right and $\varepsilon = -1$ otherwise. Such an element fixes a_i and determines on T_g the same deformation caused by "sliding" the arc a_i along the curve C according to its orientation. We denote the set of all the arc slides by $\mathcal{S}_n^g$.

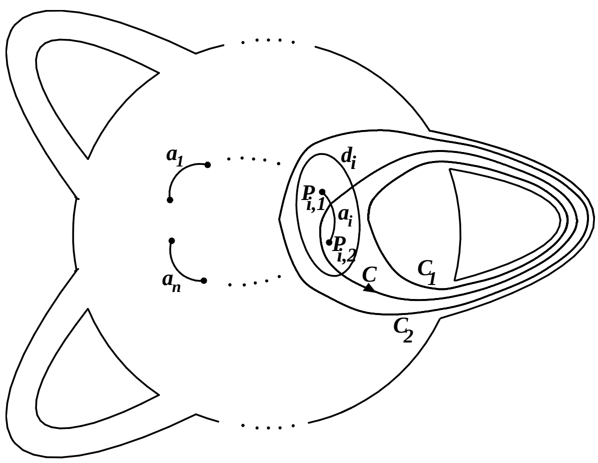

Figure 2: The slide $S_{i,C} = T_{C_1}^{-1} T_{C_2} s_i$ of the arc a_i along the curve C.

Admissible slides of meridian disks Let $\mathrm{T}_g(i)$ be the genus $g-1$ surface obtained by cutting out from T_g the boundary of the i-th handle, and capping the resulting holes with the two meridian disks B_i and B'_i as in Figure 3. A simple closed curve C on $\mathrm{T}_g(i)$ will be called an admissible curve for the meridian disk B_i if it does not intersect $B'_i \cup \mathcal{P}_{2n}$, it intersects B_i in a simple arc and is homotopic to the trivial loop in $\mathrm{T}_g(i) \setminus B'_i$ rel Q, where Q is any point of $B_i \cap C$. By exchanging the roles of B_i and B'_i we obtain the definition of admissible curve for the meridian disk B'_i.

Let C be an admissible oriented curve for the meridian disk B_i. Let $A(C)$ be an embedded closed annulus in $\mathrm{T}_g(i) \setminus (B'_i \cup \mathcal{P}_{2n})$ whose core is C and containing B_i in its interior part. We denote with C_1 and C_2 the boundary curves of $A(C)$ with the convention that C_1 is the one on the left of C according to its orientation, (see Figure 3). Notice that one between C_1 and C_2 is homotopic to b_i in $\mathrm{T}_g(i) \setminus B'_i$, while the other is trivial in $\mathrm{T}_g(i) \setminus B'_i$. An admissible slide $M_{i,C}$ of the meridian disk B_i along the curve C is the multi-twist $T_{C_1}^{-1} T_{C_2} T_{b_i}^{\varepsilon}$, where $\varepsilon = 1$ if C_1 is homotopic to b_i and $\varepsilon = -1$ otherwise. Since this homeomorphism fixes both the meridian disks B_i and B'_i, it could be extended, via the identity on the boundary of i-th handle, to a homeomorphism of T_g, and determines on T_g the same deformation caused by "sliding" the disk B_i along the curve C according to its orientation. In an analogous way we define $M'_{i,C}$ an admissible slide of the meridian disk B'_i along an admissible oriented curve C for B'_i. We denote the set of all admissible meridian slides with $\mathcal{M}_n^g$.

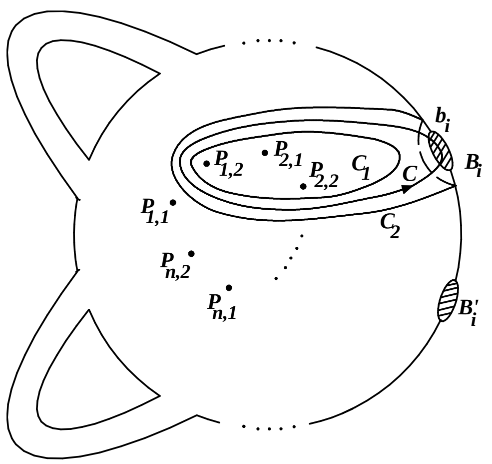

Figure 3: The slide $M_{i,C} = T_{C_1}^{-1} T_{C_2} T_{b_i}^{-1}$ of the meridian disk B_i along the curve C.

Remark 1 In [10] one can find explicit extensions of all above homeomorphisms to the couple $(\mathrm{H}_g, \mathcal{A}_n)$, that is they all belong to $\mathcal{E}_{2n}^g$. Moreover it is straightforward to see that all the above elements belong also to the kernel of $\Omega_{g,n}$ and so to Hil_n^g.

It is possible to define the slide of a meridian disk B_i (resp. B'_i) along a generic simple closed curve on $\mathrm{T}_g(i) \setminus B'_i \cup \mathcal{P}_{2n}$ (resp. $\mathrm{T}_g(i) \setminus B_i \cup \mathcal{P}_{2n}$). Such a meridian slide still belongs to $\mathcal{E}^g_{2n}$; however, it is easy to see that a slide of a meridian disk belong to the kernel of $\Omega_{g,n}$ (and so to Hil^g_n) if and only if the sliding curve is admissible.

Let $(Z_2)^n \rtimes S_n$ be the signed permutation group and let $p : \mathrm{MCG}_{2n}(\mathrm{T}_g) \longrightarrow S_{2n}$ be the map which associates to any element of $\mathrm{MCG}_{2n}(\mathrm{T}_g)$ the permutation induced on the punctures. The next proposition shows that Hil^g_n is generated by ι_1, λ_i, for $i = 1, \ldots, n-1$ and a set of generators for $P\mathrm{Hil}^g_n$.

Proposition 1 *The exact sequence*

$$1 \longrightarrow \mathrm{PMCG}_{2n}(\mathrm{T}_g) \longrightarrow \mathrm{MCG}_{2n}(\mathrm{T}_g) \longrightarrow S_{2n} \longrightarrow 1$$

restricts to an exact sequence

$$1 \longrightarrow P\mathrm{Hil}^g_n \longrightarrow \mathrm{Hil}^g_n \longrightarrow (Z_2)^n \rtimes S_n \longrightarrow 1.$$

Proof. The signed permutation group can be considered as the subgroup of S_{2n} generated by the transposition $(1\ 2)$ and the permutations $(2i{-}1\ 2i{+}1)(2i\ 2i{+}2)$, for $i = 1, \ldots n-1$. Let $\sigma \in \mathrm{Hil}^g_n$: since the extension of σ induces a permutation of the arcs $A_1, \ldots, A_n$, then $p(\sigma) \in (Z_2)^n \rtimes S_n$. Moreover if $p(\sigma) = 1$ then an extension of it fixes the arcs pointwise, so $\sigma \in P\mathrm{Hil}^g_n$. ∎

We say that an element $\sigma \in P\mathrm{Hil}^g_n$ is an *arcs-stabilizer* if σ is the identity on a_i, for each $i = 1, \ldots, n$. The set of all arcs-stabilizer elements of $P\mathrm{Hil}^g_n$ determines a subgroup of $P\mathrm{Hil}^g_n$ that we call the *arcs-stabilizer subgroup.*

The subgroup $FP_n(\mathrm{T}_g)$ of $\ker(\Omega_{g,n}) \cap \mathrm{PMCG}_{2n}(\mathrm{T}_g)$, consisting of the elements fixing the arcs $a_1, \ldots, a_n$ is called in [2] *n-th framed pure braid group of* Σ_g: this group is a (non trivial) generalization of pure framed braid groups considered in [20, 23] and several equivalent definitions have been provided in [2].

Proposition 2 *Let $FP_n(\mathrm{T}_g)$ be the n-th framed braid group of T_g defined above. The arcs-stabilizer subgroup of $P\mathrm{Hil}^g_n$ coincides with $FP_n(\mathrm{T}_g)$. In particular the arcs-stabilizer subgroup of $P\mathrm{Hil}^g_n$ is generated by the elementary twists s_i and the slides $m_{i,j}$ and $l_{i,j}$ of the arc a_i along the curves $\mu_{i,j}$ and $\lambda_{i,j}$ depicted in Figure 4, for $i = 1, \ldots, n$ and $j = 1, \ldots, g$. As a consequence, the slide $t_{i,k}$ of the arc a_i along the curve $\tau_{i,k}$ depicted in Figure 5, is the composition of the above elementary twists and slides, for $1 \leq i < k \leq n$.*

Proof. By the above definitions, it is enough to show that $FP_n(\mathrm{T}_g)$ is a subgroup of $P\mathrm{Hil}^g_n$. In [2] is shown that $FP_n(\mathrm{T}_g)$ is generated by the elementary twists s_i and the slides slides $m_{i,j}$ and $l_{i,j}$ of the arc a_i along the curves $\mu_{i,j}$ and $\lambda_{i,j}$ depicted in Figure 4, for $i = 1, \ldots, n$ and $j = 1, \ldots, g$. Therefore $FP_n(\mathrm{T}_g)$ is a subgroup of $P\mathrm{Hil}^g_n$. The multitwists $t_{i,k}$, for $1 \leq i < k \leq n$, belong to $FP_n(\mathrm{T}_g)$: in particular they can be obtained by the above set of generators using lantern relations (see [2]). ∎

Now we will prove that an infinite set of generators for $P\mathrm{Hil}^g_n$ is given by the elementary twists, all the arc slides and all the admissible meridian slides.

Theorem 2 *The group $P\mathrm{Hil}^g_n$ is generated by $\mathcal{M}^g_n \cup \mathcal{S}^g_n \cup \{s_1, \ldots, s_n\}$.*

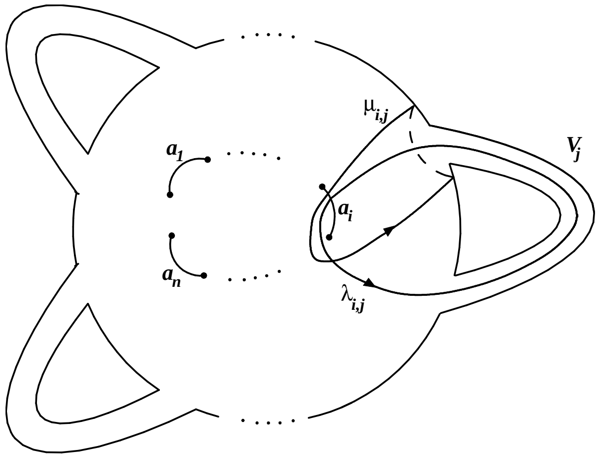

Figure 4: The arc slides $m_{i,j}$ and $l_{i,j}$.

Proof. Let $\mathcal{G}_n^g$ be the subgroup of $P\mathrm{Hil}_n^g$ generated by $\mathcal{M}_n^g \cup \mathcal{S}_n^g$. By Proposition 2 it is enough to show that for any $\sigma \in P\mathrm{Hil}_n^g$ there exists an element $h \in \mathcal{G}_n^g$ such that $h\sigma$ is an arcs-stabilizer. In order to do so, the first step will be to find an element $h_n \in \mathcal{G}_n^g$ such that

(i) $h_n\sigma(a_n) = a_n$;

(ii) for each $i = 1, \dots n$ such that $\sigma(a_n) \cap a_i = \emptyset$ we have $h_n(a_i) = a_i$.

The element h_n will be defined as the composition of arc slides and admissible meridian slides along opportunely chosen curves.

We denote with $\mathcal{D}$ the union of all the disk D_i for $i = 1, \dots, n$ and let $\mathcal{I} = \sigma(D_n) \cap \mathcal{D}$. Up to isotopy, we can assume that $\mathcal{I}$ consists of a finite number of arcs. Clearly, each arc l in $\mathcal{I}$ is a component of $\sigma(D_n) \cap D_k$ for a unique k; moreover, since $\sigma(A_n) \cap A_k = A_n \cap A_k$, if $k \neq n$ then $\sigma(A_n) \cap A_k = \emptyset$, so the endpoints of l belong to $\sigma(a_n) \cap a_k$. If, instead, $k = n$ we can assume that at least one endpoint of l belongs to $\sigma(a_n) \cap a_n$, since if both the endpoints lie in A_n then, by composing with a homeomorphism isotopic to the identity, the intersection l can be removed.

By an innermost argument, it is possible to choose $l_0 \in \mathcal{I}$ with $l_0 \subset \sigma(D_n) \cap D_k$ such that l_0 determines a disk both in $\sigma(D_n)$ and in D_k, whose union is a disk $\bar{D}$, properly embedded in H_g. Moreover $\bar{D}$ intersects $\sigma(D_n) \cap \mathcal{D}$ only in l_0 if $k \neq n$, while, if $k = n$ and one of the endpoints of l_0 lies in A_n, then the intersection of $\bar{D}$ with $\sigma(D_n) \cap \mathcal{D}$ is an arc which is the union of l_0 with a subarc of A_n going from $l_0 \cap A_n$ to one of the punctures $\partial A_n = \{P_{n,1}, P_{n,2}\}$. In any case, the boundary of $\bar{D}$ is the union of two simple arcs m_1 and m_2 on H_g with $m_1 \subset \sigma(a_n)$ and $m_2 \subset a_k$ (see Figure 6).

We set $K_1 = \{k \mid \bar{D} \cap V_k \neq \emptyset\}$ and $K_2 = \{1, \dots g\} \setminus K_1$. Notice that if $k \in K_1$, then for each arc $\alpha \in \bar{D} \cap B_k$, there exists a corresponding arc $\alpha' \in \bar{D} \cap B_k'$ such that the union of the disks bounded by α and α' on B_k, B_k' and $\bar{D}$ is a properly embedded disk in the handle V_k bounding a ball (see Figure 6). Indeed, if this is not the case, the intersection α could be removed composing with an element

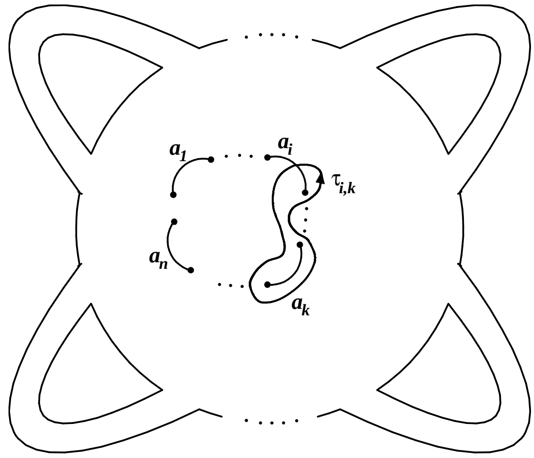

Figure 5: The arc slide $t_{i,k}$.

isotopic to the identity. If $\mathrm{H}_g(K_2)$ denotes the handlebody (of genus $g - |(K_2)|$) obtained from H_g by removing all the handles V_k with $k \in K_2$, then $\bar{D}$ separates $\mathrm{H}_g(K_2)$ into two connected components Δ_1 and Δ_2. Let Δ be the connected component of $\mathrm{H}_g(K_2) \setminus \bar{D}$ that does not contain A_k, which is the dotted zone in Figure 6.

Each disk D_i with $i \neq k$ and each meridian disk B_k or B'_k, with $k \in K_2$ is either contained or disjoint from Δ. Let $I_1 = \{i \mid A_i \subset \Delta\}$, $I_2 = \{i \mid i \in K_2, B_i \subset \Delta\}$ and $I_3 = \{i \mid i \in K_2, B'_i \subset \Delta\}$. For each $i \in I_1 \cup I_2 \cup I_3$ we choose a simple closed oriented curve C_i on $\mathrm{H}_g(K_2)$ such that

> (*) both $C_i \cap h(a_n) = C_i \cap m_1$ and $C_i \cap a_k = C_i \cap m_2$ consist of a single point; travelling along C_i the intersection with a_k comes before the one with $\sigma(a_n)$; if $i \in I_1$ then C_i intersects a_i in a single point, while if $i \in I_2$ (resp. I_3) then C_i is an admissible curve for the meridian disk B_i (resp. B'_i); C_i does not intersects all the others arcs a_j and all the other meridian disks B_j, B'_j, with $j \in K_2$.

To see that such a C_i exists, consider an arc a that starts from a_i, B_i or B'_i travels inside $\partial\Delta$ to m_1 without intersecting all the other arcs a_j and all the other meridian disks B_j, B'_j, goes along m_1 to a_k and finally goes along a_k to an endpoint of a_k (that is $P_{k,1}$ or $P_{k,2}$) without travelling along m_2. Then C_i can be chosen as the boundary of a small tubular neighboorhood of a. In Figure 6 it is depicted the case $I_1 = \{i\}$, $I_1 = \{i'\}$ and $I_3 = \emptyset$.

If we set $f_0 = \prod_{i \in I_1} S_{i,C_i} \prod_{i \in I_2} M_{i,C_i} \prod_{i \in I_3} M'_{i,C_i}$ then $f_0(\Delta)$ does not contain any arc and any meridian disk. This means that $f_0(\Delta)$ bounds a ball in $\mathrm{H}_g \setminus \mathcal{A}_n$ and so, up to composing with an element isotopic to the identity, we can remove the intersection l_0. Moreover, $f_0\sigma(D_n) \cap \mathcal{D} = \mathcal{I} \setminus l_0$.

By repeating the above procedure a finite number of times, we get an element $f = f_k f_{k-1} \cdots f_0 \in \mathcal{G}^g_n$ such that $f\sigma(D_n) \cap \mathcal{D} = A_n$. Then $f\sigma(D_n) \cup D_n$ is a properly embedded disk $\widetilde{D}$ in H_g. If we denote with $\mathrm{H}_g(K_2)$ the handlebody obtained from H_g by removing all the handles that have no intersection with $\widetilde{D}$

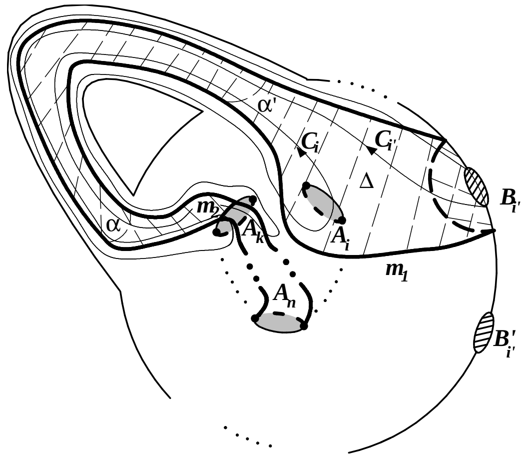

Figure 6: Reducing to arcs-stabilizer elements.

(that is $K_2 = \{k \mid V_k \cap \widetilde{D} = \emptyset\}$), then $\widetilde{D}$ separates $\mathrm{H}_g(K_2)$ into two connected components Δ_1 and Δ_2. As above, each disk D_i with $i \neq n$ and each meridian disk B_i or B'_i, corresponding to the removed handles, is either contained or disjoint from Δ_k, for $k = 1, 2$. If there exists $k = 1, 2$ such that Δ_k does not contain any disk D_i, B_i or B'_i then, up to composing with an element isotopic to the identity, we have $f\sigma(D_n) = D_n$ and so $h_n = f$. If this is not the case, we choose one of the two connected components, for example Δ_1, and, as before, we set $I_1 = \{i \mid i \in K_2, A_i \subset \Delta_1\}$, $I_2 = \{i \mid i \in K_2, B_i \subset \Delta_1\}$ and $I_3 = \{i \mid i \in K_2, B'_i \subset \Delta_1\}$ and for each $i \in I_1 \cup I_2 \cup I_3$ we choose a simple oriented closed curve C_i on $\mathrm{H}_g(K_2)$ satisfying (*). Then by taking $h_n = \prod_{i \in I_1} S_{i,C_i} \prod_{i \in I_2} M_{i,C_i} \prod_{i \in I_3} M'_{i,C_i} f$ we get $h_n\sigma(a_n) = a_n$. Moreover h_n satisfies (ii) since it is the compositions of slides (of arcs or of meridian disks) along curves that by (*) intersects only the arcs a_i such that $a_i \cap \sigma(a_n) \neq \emptyset$, and so fixes all the other arcs.

We can repeat the same procedure on a_{n-1}, that is we can find $h_{n-1} \in \mathcal{G}_n^g$ with $h_{n-1}h_n\sigma(a_{n-1}) = a_{n-1}$ and such that for each $i = 1, \ldots, n$ with $h_n\sigma(a_{n-1}) \cap a_i = \emptyset$ we have $h_{n-1}(a_i) = a_i$. This implies that $h_{n-1}h_n\sigma(a_n) = a_n$, that is $h_{n-1}h_n\sigma$ fixes the last two arcs. Proceeding in this way, we construct, for each $i = 1, \ldots, n$, an element $h_i \in \mathcal{G}_n^g$ such that $h_i h_{i+1} \cdots h_n \sigma(a_j) = a_j$ for each $j \geq i$. So $h = h_1 h_2 \cdots h_{n-1} h_n$ is the required element of $\mathcal{G}_n^g$, that is $h\sigma$ is arcs-stabilizer. ∎

In order to find a finite set of generators it would be enough to show that the subgroup of $P\mathrm{Hil}_n^g$ generated by $\mathcal{S}_n^g$ and the one generated by the $\mathcal{M}_n^g$ are finitely generated. The next two propositions show that the first subgroup is finitely generated, and the second one is finitely generated when $g = 1$.

Proposition 3 *The subgroup of $P\mathrm{Hil}_n(\mathrm{T}_g)$ generated by $\mathcal{S}_n^g$ is finitely generated by*

(1) the elementary twist s_i, for $i = 1, \ldots, n$;

(2) *the slides $m_{i,j}$ and $l_{i,j}$ of the arc a_i along the curves $\mu_{i,j}$ and $\lambda_{i,j}$ depicted in Figure 4, for $i = 1, \dots, n$ and $j = 1, \dots, g$;*

(3) *the slides $s_{i,k}$, of the arc a_i along the curve $\sigma_{i,k}$ depicted in Figure 7, for $1 \leq i \neq k \leq n$.*

Proof. By the definition of slide if $C \simeq C_1 \cdots C_r$ in $\pi_1(\mathrm{T}_g \setminus \{P_{j,k} \mid j = 1, \dots, n,\ j \neq i,\ k = 1,2\}, *)$ then $S_{i,C} = S_{i,C_r} \cdots S_{i,C_1}$ in $\mathrm{MCG}_{2n}(\mathrm{T}_g)$, where $* \in a_i$ is any fixed point. Then all the slides of the i-th arc are generated by the slides of the i-th arc along a set of generators for $\pi_1(\mathrm{T}_g \setminus \{P_{j,k} \mid j = 1, \dots, n,\ j \neq i,\ k = 1,2\}, *)$. The set of slides $m_{i,j},\ l_{i,j}$ for $j = 1, \dots, g$ and $s_{i,k}, s'_{i,k}$, for $k = 1, \dots, n$, $i \neq k$ is therefore a set of generators for all the slides of the i-th arc, where $s'_{i,k}$ is the slide of the arc a_i along the curve $\sigma'_{i,k}$ depicted in Figure 7. Applying a lantern relation one obtains that $s_{i,k}s'_{i,k} = t_{i,k}s_i^{-1}$. The statement therefore, follows from the fact that, by Proposition 2, the slide $t_{i,k}$ can be written as composition of the slides $m_{j,r}$, $l_{j,r}$ and elementary twists. ∎

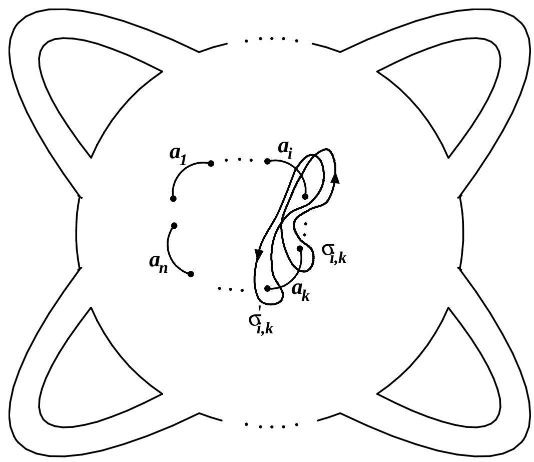

Figure 7: The arc slides $s_{i,k}$ and $s'_{i,k}$.

Proposition 4 *The subgroup of $P\mathrm{Hil}_n(\mathrm{T}_1)$ generated by $\mathcal{M}_n^1$ is finitely generated.*

Proof. Let $\mathbb{S}^2$ be the sphere obtained by cutting out from T_1 the boundary of the handle, and capping the resulting holes with the two meridian disks B_1 and B'_1 as in the definiton of meridian slides. Any simple closed curve C on $\mathbb{S}^2$ which does not intersect $B'_1 \cup \mathcal{P}_{2n}$ and intersects B_1 in a simple arc, is an admissible curve for the meridian disk B_1. Therefore, if we write C as a product $C_1 \cdots C_r$ of a finite set of generators of $\pi_1(\mathbb{S}^2 \setminus (\mathcal{P}_{2n} \cup B'_1), *)$, where $* \in B_1$ is any fixed point, we have $M_{i,C} = M_{i,C_r} \cdots M_{i,C_1}$. An analogous statement holds for $M'_{i,C}$. ∎

¿From these propositions it follows that $P\mathrm{Hil}_n^1$ is finitely generated. The problem of whether $P\mathrm{Hil}_n^g$ is finitely generated or not when $g > 1$ remains open. We end this section by giving the generators for Hil_n^g.

Theorem 3 *The group* Hil_n^g *is generated by*

(1) ι_1 *and* λ_j*,* $j = 1, \dots n-1$*;*

(2) $m_{1,k}$*,* $l_{1,k}$*,* $s_{1,r}$ *with* $k = 1, \dots, g$ *and* $r = 2, \dots n$*;*

(3) the elements of $\mathcal{M}_n^g$*.*

Proof. The statement follows from Theorem 2, Proposition 3 and the remark that $s_1 = \iota_1^2$ and that arc slides of the arc a_i can be reduced to arc slides of the arc a_1 by using (compositions of) elementary exchanges of arcs. ∎

4 Generalized plat closure

One of the main motivations to study topological generalizations of Hilden groups comes from link theory in 3-manifolds. In this section we describe a representation of links in 3-manifolds via braids on closed surfaces: this approach generalizes the concept of plat closure and explains the role played by Hil_n^g in this representation. We start by recalling the definition of (g, n)-links.

Let L be a link in a 3-manifold M. We say that L is a (g, n)*-link* if there exists a genus g Heegaard surface S for M such that

(i) L intersects S transversally and

(ii) the intersection of L with both of handlebodies into which M is divided by S, is a trivial system of n arcs.

Such a decomposition for L is called (g, n)*-decomposition* or n*-bridge decomposition of genus* g. The minimum n such that L admits a (g, n)-decomposition is called *genus* g *bridge number* of L.

Clearly if $g = 0$ we get the usual notion of bridge decomposition and bridge number of links in the 3-sphere (or in $\mathbb{R}^3$). Given two links $L \subset M$ and $L' \subset M'$ we say that L and L' are *equivalent* if there exists an orientation preserving homeomorphism $f : M \longrightarrow M'$ such that $f(L) = L'$ and we write $L \cong L'$.

The notion of (g, n)-decompositions was used in [10] to develop an algebraic representation of $\mathcal{L}_{g,n}$, the set of equivalence classes of (g, n)-links, as follows. Let $(\mathrm{H}_g, \mathcal{A}_n)$ be as in Figure 1 and let $(\bar{\mathrm{H}}_g, \bar{\mathcal{A}}_n)$ be a homeomorphic copy of $(\mathrm{H}_g, \mathcal{A}_n)$. Fix an orientation reversing homeomorphism $\tau : \mathrm{H}_g \longrightarrow \bar{\mathrm{H}}_g$ such that $\tau(A_i) = \bar{A}_i$, for each $i = 1, \dots, n$. Then the following application is well defined and surjective

$$\Theta_{g,n} : \mathrm{MCG}_{2n}(\mathrm{T}_g) \longrightarrow \mathcal{L}_{g,n} \qquad \Theta_{g,n}(\psi) = L_\psi \tag{1}$$

where L_ψ is the (g, n)-link in the 3-manifold M_ψ defined by

$$(M_\psi, L_\psi) = (\mathrm{H}_g, \mathcal{A}_n) \cup_{\tau\psi} (\bar{\mathrm{H}}_g, \bar{\mathcal{A}}_n).$$

This means that it is possible to describe each link admitting a (g, n)-decomposition in a certain 3-manifold by a element of $\mathrm{MCG}_{2n}(\mathrm{T}_g)$. This element is not unique, since we have the following result.

Proposition 5 ([10]) *If ψ and ψ' belong to the same double coset of $\mathcal{E}^g_{2n}$ in* $\mathrm{MCG}_{2n}(\mathrm{T}_g)$ *then $L_\psi \cong L_{\psi'}$.*

Therefore, in order to describe all (g,n)-links via (1) it is enough to consider double coset classes of $\mathcal{E}^g_{2n}$ in $\mathrm{MCG}_{2n}(\mathrm{T}_g)$. This representation has revealed to be a useful tool for studying links in 3-manifolds, see [7, 8, 11, 19]. However, if we represent links using (1), we have to deal with links that lie in different manifolds. If we want to fix the ambient manifold, then the following remark holds.

Remark 4 If $\psi_1, \psi_2 \in \mathrm{MCG}_{2n}(\mathrm{T}_g)$ are such that $\Omega_{g,n}(\psi_1) = \Omega_{g,n}(\psi_2)$ then L_{ψ_1} and L_{ψ_2} belong to the same ambient manifold.

So, in order to fix the ambient manifold, we want to modify representation (1) by separating the part that determines the manifold from the part that determines the link.

Referring to Figure 1, let D be a disk embedded in T_g containing all the arcs a_i and not intersecting any meridian curve b_k or b'_k, for $i = 1, \ldots, n$ and $k = 1, \ldots, g$. Let $\mathcal{T}^g_n$ be the subgroup of $\mathrm{MCG}_{2n}(\mathrm{T}_g)$ generated by Dehn twist along curves that do not intersect the disk D. We have the following proposition.

Proposition 6 *For each $\psi \in \mathcal{T}^g_n$ the link L_ψ is a n-components trivial link in M_ψ.*

Proof. Since, the action of ψ on the punctures is trivial, for each $i = 1, \ldots, n$, the arc A_i is glued, via $\tau\psi$, to $\bar{A}_i$, giving rise to a connected component of L_ψ. Moreover for each $i = 1, \ldots, n$ we have $\psi(a_i) = a_i$, so if we set $\tau(D_i) = \bar{D}_i \subset \bar{\mathrm{H}}_g$, the i-th component $A_i \cup_{\tau\psi} \bar{A}_i$ of L_ψ bounds in M_ψ the embedded disk $D_i \cup_{\tau\psi} \bar{D}_i$. ∎

Let $\mathrm{T}_{g,1}$ be the compact surface obtained by removing the interior part of the disk D. The natural inclusion of $\mathrm{T}_{g,1}$ into T_g with $2n$ marked points induces an injective map $\mathrm{MCG}(\mathrm{T}_{g,1}) \longrightarrow \mathrm{MCG}_{2n}(\mathrm{T}_g)$ (see [25]) and the mapping class group $\mathrm{MCG}(\mathrm{T}_{g,1})$ turns out to be isomorphic to the group $\mathcal{T}^g_n$. On the other hand, we have also the following exact sequence:

$$1 \longrightarrow \pi_1(U\mathrm{T}_{g,1}) \longrightarrow \mathcal{T}^g_n \longrightarrow \mathrm{MCG}(\mathrm{T}_g) \longrightarrow 1$$

where $U\mathrm{T}_{g,1}$ is the unit tangent bundle of $\mathrm{T}_{g,1}$ (see [2]). As a consequence each element of $\mathrm{MCG}(\mathrm{T}_g)$ admits a lifting as an element of $\mathcal{T}^g_n$, so we can realize any genus g Heegaard decomposition of a 3-manifold M using an element of $\mathcal{T}^g_n$, for any $n > 0$. Now we are ready to define the *generalized plat closure*. Let M be a fixed manifold, and choose an element $\psi \in \mathcal{T}^g_n$ such that $M = M_\psi$. We define a map

$$\Theta^\psi_{g,n} : \ker(\Omega_{g,n}) \longrightarrow \{(g,n) - \text{links in } M_\psi\} \tag{2}$$

given by $\Theta^\psi_{g,n}(\sigma) = \Theta_{g,n}(\psi\sigma)$. We set $\hat{\sigma}^\psi = \Theta^\psi_{g,n}(\sigma)$.

Remark 5 As recalled before, $\ker(\Omega_{g,n})$ is isomorphic to the braid group $\mathrm{B}_{2n}(\mathrm{T}_g)$, quotiented by its center, which is trivial if $g \geq 2$. This means that we can interpretate $\Theta^\psi_{g,n}$ as a generalization of the notion of plat closure for classical braids, as shown schematically in Figure 8. Indeed, for $g = 0$ and $\psi = \mathrm{id}$ we obtain the classical plat closure. This is the only representation in the case of the 3-sphere (i. e. with $g = 0$), since $\mathcal{T}_n^0$ is trivial. On the contrary, the generalized plat closure in a 3-manifold different from $\mathbb{S}^3$ depends on the choice of the element $\psi \in \mathcal{T}_n^g$, and, topologically, this corresponds to the choice of a Heegaard surface of genus g for M.

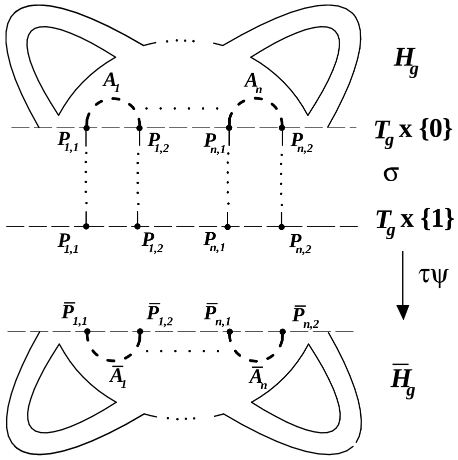

Figure 8: A generalized plat closure

In this setting a natural question arises.

Question 1 *Is it possible to determine when two element $\sigma_1 \in \ker(\Omega_{g,n_1})$ and $\sigma_2 \in \ker(\Omega_{g,n_2})$ determine equivalent links via* (2)*?*

A partial answer is given by the following statement, that is a straightforward corollary of Proposition 5.

Corollary 6 *Let $\psi \in \mathcal{T}_n^g$. Denote with $\mathrm{Hil}_n^g(\psi) = \psi^{-1}\mathrm{Hil}_n^g\psi$.*

1) if σ_1 and σ_2 belong to the same left coset of Hil_n^g in $\ker(\Omega_{g,n})$ then $\hat{\sigma_1}^{\psi}$ and $\hat{\sigma_2}^{\psi}$ are equivalent links in the manifold M_ψ.

2) if σ_1 and σ_2 belong to the same right coset of $\mathrm{Hil}_n^g(\psi)$ in $\ker(\Omega_{g,n})$ then $\hat{\sigma_1}^{\psi}$ and $\hat{\sigma_2}^{\psi}$ are equivalent links in the manifold M_ψ.

Proof. To prove the equivalence it is enough to exhibit two orientation preserving homeomorphisms $f : (\mathrm{H}_g, \mathcal{A}_n) \longrightarrow (\mathrm{H}_g, \mathcal{A}_n)$ and $\bar{f} : (\bar{H}_g, \bar{\mathcal{A}}_n) \longrightarrow (\bar{H}_g, \bar{\mathcal{A}}_n)$ making the following diagram commute

$$\begin{array}{ccc} (\partial \mathrm{H}_g, \partial \mathcal{A}_n) & \xrightarrow{\tau\psi\sigma_1} & (\partial \bar{\mathrm{H}}_g, \partial \bar{\mathcal{A}}_n) \\ \big\downarrow f_{|\partial} & & \bar{f}_{|\partial} \big\downarrow \\ (\partial \mathrm{H}_g, \partial \mathcal{A}_n) & \xrightarrow{\tau\psi\sigma_2} & (\partial \bar{\mathrm{H}}_g, \partial \bar{\mathcal{A}}_n) \end{array}$$

In the first case there exists $\varepsilon \in \mathrm{Hil}_n^g$ such that $\sigma_2 = \sigma_1\varepsilon$ so we can choose $f = \varepsilon^{-1}$ and $\bar{f} = \mathrm{id}$. In the second case there exists $\varepsilon \in \mathrm{Hil}_n^g$ such that $\sigma_2 = \psi^{-1}\varepsilon\psi\sigma_1$ so we can choose $f = \mathrm{id}$ and $\bar{f} = \tau\varepsilon\tau^{-1}$. ∎

In the case of classical plat closure Question 1 was solved in [5], where it is shown that two braids determine the same plat closure if and only if they are related by a finite sequence of moves corresponding to generators of Hil_n^0 and a stabilization move (see also [24]).

Another non trivial question concerns the surjectivity of the map (2). The following proposition deals with this problem.

Proposition 7 *Let M be a 3-manifold with a finite number of equivalence classes of genus g Heegaard splittings*[1]*. Then there exist $\psi_1, \cdots, \psi_k \in \mathcal{T}_n^g$ such that for each (g,n)-link $L \subset M$ we have $L \cong \hat{\sigma}^{\psi_i}$ with $\sigma \in \ker(\Omega_{g,n})$ and $i \in \{1, \ldots, k\}$.*

Proof. By result of [4], it is possible to choose elements $\psi_1, \cdots, \psi_k \in \mathrm{MCG}(\mathrm{T}_g)$ such that, each $\psi \in \mathrm{MCG}(\mathrm{T}_g)$ that induces an Heegaard decomposition $H_g \cup_{\psi\tau} \bar{H}_g$ of M, belongs to the same double coset class of ψ_i in $\mathrm{MCG}(\mathrm{T}_g)$ modulo $\mathcal{E}_0^g$, for a certain $i \in \{1, \ldots, k\}$. Now let $\psi_1, \cdots, \psi_k \in \mathcal{T}_n^g$ such that $\Omega_{g,n}(\psi_i) = \psi_i$, for $i = 1, \ldots, k$. Since L is a (g,n)-link in M, there exists $\psi \in \mathrm{MCG}_{2n}(\mathrm{T}_g)$ such that $L = \theta_{g,n}(\psi)$ and $\mathrm{H}_g \cup_{\Omega_{g,n}(\psi)} \bar{\mathrm{H}}_g$ is a genus g Heegaard splitting for M. So there exist $\bar{\varepsilon_1}, \bar{\varepsilon_2} \in \mathcal{E}_0^g$ and $i \in \{1, \ldots k\}$ such that $\Omega_{g,n}(\psi_i) = \bar{\varepsilon_1}\Omega_{g,n}(\psi)\bar{\varepsilon_2}$. Since $\Omega_{g,n}$ restricts to a surjective homomorphism $\mathcal{E}_{2n}^g \longrightarrow \mathcal{E}_0^g$, then there exists $\varepsilon_i \in \mathcal{E}_{2n}^g$ such that $\Omega_{g,n}(\varepsilon_i) = \bar{\varepsilon_i}$, for $i = 1, 2$. If we set $\sigma = \psi_i^{-1}\varepsilon_1\psi\varepsilon_2$, then $\sigma \in \ker(\Omega_{g,n})$ and $L \cong \hat{\sigma}^{\psi_i}$. ∎

The Waldhausen conjecture, which has been proved in [17, 18, 22], tells us that every manifold admits a finite number of homeomorphism classes of irreducible genus g Heegaard splittings. So, for example, Proposition 7 holds whenever g is the Heegaard genus of M. In [9] it is analyzed the case $g = n = 1$.

5 The Hilden map and the motion groups

In this section we describe the connections between Hilden braid groups and the so-called *motion groups*. We start by recalling few definitions (see [12]).

A *motion* of a compact submanifold N in a manifold M is a path f_t in $Homeo_c(M)$ such that $f_0 = \mathrm{id}$ and $f_1(N) = N$, where $\mathrm{Homeo}_c(M)$ denotes the group of homeomorphisms of M with compact support. A motion is called *stationary* if $f_t(N) = N$ for all $t \in [0,1]$. The *motion group* $\mathcal{M}(M,N)$ of N in M is the group of equivalence classes of motion of N in M where two motions f_t, g_t are equivalent if $(g^{-1}f)_t$ is homotopic relative to endpoints to a stationary motion.

Notice that the motion group of k points in M is the braid group $\mathrm{B}_k(M)$. Moreover, since each motion is equivalent to a motion that fixes a point $* \in M - N$, it is possible to define a homomorphism

$$\mathcal{M}(M,N) \longrightarrow \mathrm{Aut}(\pi_1(M - N, *)) \tag{3}$$

[1]Two Heegaard splittings of a manifold M are said equivalent if there exists an homeomorphism $f : M \longrightarrow M$ that send the one splitting surface into the other.

sending an element represented by the motion f_t into the automorphism induced on $\pi_1(M-N,*)$ by f_1.

We are mainly interested in the case of links in 3-manifolds. In [12] a finite set of generators for the motion groups $\mathcal{M}(\mathbb{S}^3, L_n)$ of the n-component trivial link in $\mathbb{S}^3$ is given, while a presentation can be found in [1]. Moreover in [13] a presentation for the motion group of all torus links in $\mathbb{S}^3$ is obtained. On the contrary, there are not known examples of computations of motion groups of links in 3-manifolds different from $\mathbb{S}^3$.

In [16] was described how to construct examples of motions of a link L in $\mathbb{S}^3$ presented as the plat closure of a braid $\sigma \in \mathrm{B}_{2n}(\mathbb{S}^2)$ using the elements of $\mathrm{Hil}_n^0 \cap \mathrm{Hil}_n^0(\sigma)$. In the following theorem we extend this result to links in 3-manifolds via Hilden braid groups of a surface.

THEOREM 7 *Let $\psi \in \mathcal{T}_n^g$ and let $\hat{\sigma}^\psi$ be a link in $M_\psi = \mathrm{H}_g \cup_{\tau\psi} \bar{\mathrm{H}}_g$, where $\sigma \in \ker(\Omega_{g,n})$. There exists a group homomorphism, that we call the Hilden map, $\mathcal{H}_{\psi\sigma} : \mathrm{Hil}_n^g \cap \mathrm{Hil}_n^g(\psi\sigma) \longrightarrow \mathcal{M}(M_\psi, \hat{\sigma}^\psi)$, where $\mathrm{Hil}_n^g(\psi\sigma) = (\psi\sigma)^{-1}\mathrm{Hil}_n^g\psi\sigma$.*

Proof. Let ε be a representative of an element in $\mathrm{Hil}_n^g \cap \mathrm{Hil}_n^g(\psi\sigma)$. By definition $\mathrm{Hil}_n^g \subset \ker\Omega_{g,n}$, so there exists an isotopy $g : I \times \mathrm{T}_g \longrightarrow T_g$ such that $g(0,\cdot) = g_0 = \mathrm{id}$ and $g(1,\cdot) = g_1 = \varepsilon$. Then $\psi\sigma g(\psi\sigma)^{-1}$ is an isotopy between the identity and $\psi\sigma\varepsilon(\psi\sigma)^{-1}$. Moreover by hypothesis the isotopy class of $\psi\sigma\varepsilon(\psi\sigma)^{-1}$ belongs to Hil_n^g and so extends to H_g. Since the rows of the commutative diagram

$$\begin{array}{ccc} \mathrm{MCG}_n(\mathrm{H}_g) & \longrightarrow & \mathcal{E}_{2n}^g \\ \big\downarrow & & \big\downarrow{\scriptstyle \Omega_{g,n}} \\ \mathrm{MCG}(\mathrm{H}_g) & \longrightarrow & \mathcal{E}_0^g. \end{array} \tag{4}$$

are isomorphisms, there exist two isotopies $f : I \times \mathrm{H}_g \longrightarrow \mathrm{H}_g$ and $\bar{f} : I \times \mathrm{H}_g \longrightarrow \mathrm{H}_g$ between the identity and an extension of, respectively, ε and $\psi\sigma\varepsilon(\psi\sigma)^{-1}$. We claim that it is possible to choose f and $\bar{f}$ such that they extend, respectively, g and $\psi\sigma g(\psi\sigma)^{-1}$. Indeed if $g = 0$ we can use the Alexander trick to extend the isotopy from the boundary sphere to the 3-ball. If $g > 0$ first we extend the isotopy on a system of meridian discs for H_g not intersecting the system of arcs and then we reduce to the previous case by cutting along them.

We define $\mathcal{H}_{\psi\sigma}([\varepsilon]) = [F]$ where $F : I \times M_\psi \longrightarrow M_\psi$ is defined by

$$F(t,x) = F_t(x) = \begin{cases} f(t,x) & \text{if } x \in \mathrm{H}_g \\ \bar{f}(t,x) & \text{if } x \in \bar{H}_g \end{cases}$$

The commutativity of the following diagram ensures that F_t is a well-defined homeomorphism of M_ψ

$$\begin{array}{ccc} \partial\mathrm{H}_g & \xrightarrow{\tau\psi\sigma} & \partial\bar{\mathrm{H}}_g \\ {\scriptstyle g_t}\big\downarrow & & \big\downarrow{\scriptstyle \tau\psi\sigma g_t(\tau\psi\sigma)^{-1}} \\ \partial\mathrm{H}_g & \xrightarrow{\tau\psi\sigma} & \partial\bar{\mathrm{H}}_g. \end{array} \tag{5}$$

It is immediate to check that $F_0 = \mathrm{id}$ and $F_1(\hat{\sigma}^\psi) = \hat{\sigma}^\psi$ and so F is a motion of $\hat{\sigma}^\psi$ in M_ψ. Moreover the definition of $\mathcal{H}_{\psi\sigma}$ does not depend on the homeomorphism choosen as a representative of the element in $\mathrm{Hil}_n^g \cap \mathrm{Hil}_n^g(\psi\sigma)$: indeed, if ε' is another representative, there exists an isotopy between ε and ε' fixing $A_1 \cup \cdots \cup A_n$ and so the corresponding motions are equivalent.

In order to prove both that the definition does not depend on the choice of the isotopies and that $\mathcal{H}_{\psi\sigma}$ is a group homomorphisms we distinguish three cases. For the case of $g = 0$ we refer to [1, 6, 16]. If $g > 1$, the statement follows from the fact that $\pi_1(\mathrm{Homeo}(\mathrm{T}_g), \mathrm{id}) = 1$, where $\mathrm{Homeo}(\mathrm{T}_g)$ is the group of orientation preserving homeomorphisms of T_g (see [15]). If $g = 1$, then $\pi_1(\mathrm{Homeo}(\mathrm{T}_g), \mathrm{id}) = \mathbb{Z}$, see [14]. Nevertheless, since $\mathrm{MCG}(\mathrm{T}_1) \cong \mathrm{MCG}_1(\mathrm{T}_1)$, we can suppose that all the isotopies that we take into consideration fix a point P; so the statement follows from $\pi_1(\mathrm{Homeo}(\mathrm{T}_g, P), \mathrm{id}) = 1$ (see [14]).

∎

In order to use the Hilden map to get informations on motion groups, it is natural to ask if $\mathcal{H}_{\psi\sigma}$ is surjective and/or injective. Clearly the answer will depend on $\psi\sigma$, that is on the ambient manifold and on the considered link. Before giving a (partial) answer in the case of $\mathbb{S}^3$, let us recall the main result of [12].

THEOREM 8 ([12]) *The homomorphism* $\mathcal{M}(\mathbb{S}^3, L_n) \longrightarrow \mathrm{Aut}(\pi_1(\mathbb{S}^3 - L_n, *)) \cong \mathbb{F}_n$ *is injective and* $\mathcal{M}(\mathbb{S}^3, L_n)$ *is generated by:*

R_i: *turn the i-th circle over, corresponding to the automorphism of* $\mathbb{F}_n$

$$\rho_i : \begin{cases} x_i \longrightarrow x_i^{-1} \\ x_k \longrightarrow x_k \text{ if } k \neq i \end{cases}$$

T_j: *interchange the j-th and* $(j+1)$*-th circles, corresponding to the automorphism of* $\mathbb{F}_n$

$$\tau_j : \begin{cases} x_j \longrightarrow x_{j+1} \\ x_{j+1} \longrightarrow x_j \\ x_h \longrightarrow x_h \text{ if } h \neq j, j+1 \end{cases}$$

A_{ik}: *pull the i-th circle throught the k-th circle, corresponding to the automorphism of* $\mathbb{F}_n$

$$\alpha_{ik} : \begin{cases} x_i \longrightarrow x_k x_i x_k^{-1} \\ x_h \longrightarrow x_h \text{ if } h \neq i \end{cases}$$

where $j = 1, \ldots, n-1$, $i, k = 1, \ldots n$ *and* $i \neq k$.

COROLLARY 9 *Let* $\psi \in \mathcal{T}_n^g$ *be an element such that* $M_\psi = \mathbb{S}^3$. *For example, choose* $\sigma = \mathrm{id}$ *if* $g = 0$ *and* $\sigma = T_{\alpha_1} \cdots T_{\alpha_g}$, *where* $\alpha_1, \ldots \alpha_g$ *denote the curves depicted in Figure 9 if* $g > 1$. *The homomorphism* $\mathcal{H}_\psi : \mathrm{Hil}_n^g \cap \mathrm{Hil}_n^g(\psi) \longrightarrow \mathcal{M}(\mathbb{S}^3, L_n)$ *is surjective. Moreover, it is injective if and only if* $(g, n) = (0, 1)$.

Proof. First of all notice that each element of $\mathcal{T}_n^g$ commutes with the following elements of Hil_n^g: ι_i, λ_j, $s_{i,k}$, for $j = 1, \ldots, n-1$, $i, k = 1, \ldots n$ and $i \neq k$. So all these elements belong to $\mathrm{Hil}_n^g \cap \mathrm{Hil}_n^g(\psi)$. Moreover, $\mathcal{H}_\psi(\iota_i) = R_i$, $\mathcal{H}_\psi(\lambda_j) = T_j$ and $\mathcal{H}_\psi(s_{i,k}) = A_{ik}$ so the surjectivity of $\mathcal{H}_\psi$ follows by Theorem 8. The same holds for $\mathcal{M}(\mathbb{S}^3, L_1)$, since, by Theorem 8, it is isomorphic to the subgroup of $\mathrm{Aut}(\mathbb{F}_2)$ generated by ρ_1 which has clearly order two. On the contrary, if

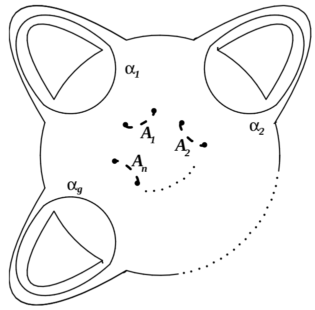

Figure 9: An example of Hilden map.

$(g, n) \neq (0, 1)$, then ι_1 has infinite order in $\mathrm{MCG}_{2n}(\mathrm{T}_g)$ while ρ_1, and so R_1, is an element of order two. ∎

Using results from [9] and [13] it would be possible to analyze the case of the torus links in $\mathbb{S}^3$. Moreover, the Hilden map could be used in order to get informations on motion groups of links that belong to 3-manifolds different from $\mathbb{S}^3$.

References

[1] P. Bellingeri and A. Cattabriga, *The Hilden group and its generalizations*, in progress.

[2] P. Bellingeri and S. Gervais, *Surface framed braids*, preprint (2009).

[3] J. Birman, *Braids, links and mapping class groups,* Ann. of Math. Studies **82** Princeton University Press (Princeton, 1974).

[4] J. Birman, *On the equivalence of Heegaard splitting of closed, orientable 3-manifold,* In *Knots, Groups and 3-manifolds*, Ann. of Math. Studies **84** (1975), 137-164.

[5] J. Birman, *On the stable equivalence of plat representation of knots and links,* Canad. J. Math. **28** (1976), 264–290.

[6] T. Brendle, A. Hatcher *Configuration Spaces of Rings and Wickets,* preprint (2008), ArXiv math.GT0805.4353.

[7] A. Cattabriga, *The Alexander polynomial of (1,1)-knots,* J. Knot Theory Ramifications **15** (2006), 1119-1129.

[8] A. Cattabriga and M. Mulazzani, *All strongly-cyclic branched coverings of (1,1)-knots are Dunwoody manifolds*, J. Lond. Math. Soc., (2004), **70**, 512-528.

[9] A. Cattabriga and M. Mulazzani, $(1,1)$*-knots via the mapping class group of the twice punctured torus*, Adv. Geom. **4** (2004), 263–277.

[10] A. Cattabriga and M. Mulazzani, *Extending homeomorphisms from punctured surfaces to handlebodies*, Topol. Appl. **155** (2008), 610–621.

[11] P. Cristofori, M. Mulazzani and A. Vesnin: *Strongly-cyclic branched coverings of knots via (g,1)-decompositions*, Acta Mathematica Hungarica **116** (2007), 163-176

[12] D. Goldsmith, *The theory of motion groups*, Michigan Math. J. **28** (1981), 3-17.

[13] D. Goldsmith, *Motions of links in the 3-sphere*, Math. Scand. **50** (1982), 167-205.

[14] M.-E. Hamstrom, *The space of homeomorphisms on a torus*, Illinois J. Math. **9** (1965), 59-65.

[15] M.-E. Hamstrom, *Homotopy groups of the space of homeomorphisms on a 2-manifold*, Illinois J. Math. **10** (1966), 563-573.

[16] H. M. Hilden, *Generators for two groups related to the braid groups*, *Pacific J.Math.* **59** (1975), 475-486.

[17] K. Johannson, *Heegaard surfaces in Haken 3 manifolds*, Bull. Amer. Math. Soc. **23** (1990), 91-98.

[18] K. Johannson, *Topology and combinatorics of 3-manifolds*, Lecture Notes in Mathematics **1599**, Springer-Verlag (Berlin,1995).

[19] Y. Koda, *Strongly-cyclic branched coverings and Alexander polynomial of knots in rational homology spheres*, Math. Proc. Cambridge Philos. Soc. **142** (2007), 259–268.

[20] H. Ko and L. Smolinsky, *The Framed Braid Group and 3-manifolds*, Proc. Amer. Math. Soc. **115** (1992) 541–551.

[21] C. Labruère and L. Paris, *Presentations for the punctured mapping class groups in terms of Artin groups*, Alg. Geom. Topol. **1** (2001) 73-114.

[22] T. Li, *Heegaard surfaces and measured laminations. II. Non-Haken 3-manifolds*, J. Amer. Math. Soc. **19** (2006), 625–657

[23] D. Margalit and J. McCammond, *Geometric presentations for pure braid group*, J. Knot Theory and Ramif **18** (2009) 1–20.

[24] J. M. Montesinos, *Minimal plat representation of knots and links is knot unique*, Canad. J. Math. **28** (1976), 161–167.

[25] L. Paris and D. Rolfsen, *Geometric subgroups of mapping class groups*, J. Reine Angew. Math. **521** (2000), 47-83

[26] S. Tawn, *A presentation for Hilden's subgroup of the braid group*, Math. Res. Lett. **15** (2008), 1277-1293.

[27] S. Tawn, *A presentation for the pure Hilden group*, preprint (2008), ArXiv 0902.4840.

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