Obstacle problem for Arithmetic Asian options

Laura Monti [a], Andrea Pascucci [a]

[a] *Dipartimento di Matematica, Università di Bologna, Piazza di Porta S. Donato 5, 40126 Bologna (Italy)*

Received *****; accepted after revision +++++

Presented by

Abstract

We prove existence, regularity and a Feynman-Kač representation formula of the strong solution to the free boundary problem arising in the financial problem of the pricing of the American Asian option with arithmetic average.

Résumé

Problème de l'obstacle pour l'option américain asiatique à moyenne arithmétique. On démontre l'existence, la régularité et une formule de représentation de Feynman-Kač de la solution forte d'un problème avec frontière libre. Ce type de problème on le retrouve en finance pour évaluer le prix d'une option asiatique à moyenne arithmétique de style américain.

1. Introduction

According to the classical financial theory (see, for instance, [12]) the study of Asian options of American style leads to free boundary problems for degenerate parabolic PDEs. More precisely, let us assume that, in the standard setting of local volatility models, the dynamics of the underlying asset is driven by the SDE

$$dS_t = \mu(t, S_t)S_t dt + \sigma(t, S_t)S_t dW_t, \tag{1}$$

and consider the process $dA_t = f(S_t)dt$, where $f(S) = S$ and $f(S) = \log S$ occur respectively in the study of the Arithmetic average and Geometric average Asian options. Then the price of the related Amerasian option with payoff function φ is the solution of the obstacle problem with final condition

$$\begin{cases} \max\{Lu, \varphi - u\} = 0, &]0, T[\times\mathbb{R}^2_+, \\ u(T, s, a) = \varphi(T, s, a), & s, a > 0, \end{cases} \tag{2}$$

Email addresses: monti@unibo.it (Laura Monti), pascucci@dm.unibo.it (Andrea Pascucci).

where

$$Lu = \frac{\sigma^2 s^2}{2}\,\partial_{ss}u + rs\partial_s u + f(s)\partial_a u + \partial_t u - ru, \qquad s,a>0, \tag{3}$$

is the Kolmogorov operator of (S_t, A_t) and r is the risk free rate. Recently this problem has also been considered in the study of pension plans in [7] and stock loans in [2].

Typical Arithmetic average payoffs are of the form

$$\begin{aligned} \varphi(t,s,a) &= \left(\frac{a}{t} - K\right)^+ \qquad \text{(fixed strike)}, \\ \varphi(t,s,a) &= \left(\frac{a}{t} - s\right)^+ \qquad \text{(floating strike)}. \end{aligned} \tag{4}$$

A direct computation shows that in these cases a super-solution[1] to (2) with $f(s) = s$ is given by

$$\bar{u}(t,s,a) = \frac{\alpha}{t}\left(1 + e^{-\beta t}\sqrt{s^2 + a^2}\right) \tag{5}$$

for α, β are positive constants, with β suitably large. On the other hand it is well-known that generally (2) does not admit a smooth solution in the classical sense.

Recently the Geometric Asian option has been studied under the following hypotheses:

(H1) σ is bounded, locally Hölder continuous and such that $\sigma \geq \sigma_0$ for some positive constant σ_0;

(H2) φ is locally Lipschitz continuous on $]0,T[\times\mathbb{R}^2_+$ and the distributional derivative $\partial_{ss}\varphi$ is locally lower bounded (to fix ideas, this includes $\varphi(s) = (s-K)^+$ and excludes $\varphi(s) = -(s-K)^+$).

In [4], [11] it is proved that problem (2), with $f(s) = \log s$, has a strong solution u in the Sobolev space

$$\mathcal{S}^p_{\text{loc}} = \{u \in L^p \mid \partial_s u, \partial_{ss}u, (f(s)\partial_a + \partial_t)u \in L^p_{\text{loc}}\}, \qquad p \geq 1. \tag{6}$$

Moreover uniqueness has been proved via Feynman-Kač representation. However, as we shall see below, the Geometric and Arithmetic cases are structurally quite different.

The aim of this note is to give an outlined proof of the following

Theorem 1.1 *Consider problem* (2) *with* $f(s) = s$, *under the assumptions (H1) and (H2). Then we have:*

i) if there exists a super-solution $\bar{u}$ *then there also exists a strong solution* $u \in \mathcal{S}^p_{\text{loc}} \cap C\left(]0,T]\times\mathbb{R}^2_+\right)$ *for any* $p \geq 1$, *such that* $u \leq \bar{u}$;

ii) if u *is a strong solution to* (2) *such that*

$$|u(t,s,a)| \leq \frac{C}{t}(1 + s^q + a^q), \qquad s,a>0,\ t\in]0,T], \tag{7}$$

for some positive constants C, q, *then*

$$u(t,s,a) = \sup_{\tau\in\mathcal{T}_{t,T}} E\left[e^{-r\tau}\varphi(\tau, S^{t,s,a}_\tau, A^{t,s,a}_\tau)\right], \qquad t,s,a>0, \tag{8}$$

where $\mathcal{T}_{t,T} = \{\tau \in \mathcal{T} \mid \tau \in [t,T]$ *a.s.*$\}$ *and* $\mathcal{T}$ *is the set of all stopping times with respect to the Brownian filtration. In particular there exists at most one strong solution of* (2) *verifying the growth condition* (7).

For simplicity here we only consider the case of constant σ. Then by a transformation (cf. formula (4.4) in [1]), operator L in (3) with $f(s) = s$ can be reduced in the canonical form

$$L_A = x_1^2\partial_{x_1x_1} + x_1\partial_{x_2} + \partial_t, \qquad x = (x_1,x_2)\in\mathbb{R}^2_+. \tag{9}$$

Before proceeding with the proof, we make some preliminary comments.

[1] $\bar{u}$ is a super-solution of (2) if

$$\begin{cases} \max\{L\bar{u}, \varphi - \bar{u}\} \leq 0, &]0,T[\times\mathbb{R}^2_+, \\ \bar{u}(T,s,a) \geq \varphi(T,s,a), & s,a>0. \end{cases}$$

Remark 1. In the Geometric case $f(s) = \log s$, by a logarithmical change of variables the pricing operator L takes the form

$$L_G = \partial_{x_1x_1} + x_1\partial_{x_2} + \partial_t, \qquad x = (x_1, x_2) \in \mathbb{R}^2. \tag{10}$$

It is known (cf. [9]) that L_G has remarkable invariance properties with respect to a homogeneous Lie group structure: precisely, L_G is invariant with respect to the left translation in the law $(t', x') \circ (t, x) = (t' + t, x_1' + x_1, x_2' + x_2 + tx_1')$ and homogeneous of degree two with respect to the dilations $\delta_l(t, x) = (l^2t, lx_1, l^3x_2)$ in the sense that, setting $z = (t, x)$,

$$L_G(u(z' \circ z)) = (L_G u)(z' \circ z), \qquad L_G(u(\delta_l(z))) = l^2(L_G u)(\delta_l(z)), \qquad z, z' \in \mathbb{R}^3,\ l > 0.$$

Moreover L_G has a fundamental solution $\Gamma_G(t, x; T, X)$ (here (t, x) and (T, X) represent respectively the starting and ending points of the underlying stochastic process) of Gaussian type whose explicit expression is known explicitly:

$$\Gamma_G(t, x; T, X) = \Gamma_G\left((T, X)^{-1} \circ (t, x); 0, 0\right)$$

where $(T, X)^{-1} = (-T, -X_1, -X_2 + TX_1)$ and

$$\Gamma_G(t, x; 0, 0) = \frac{\sqrt{3}}{2\pi t^2} \exp\left(\frac{x_1^2}{t} + \frac{3x_1(x_2 - tx_1)}{t^2} + \frac{3(x_2 - tx_1)^2}{t^3}\right), \qquad t < 0,\ x_1, x_2 \in \mathbb{R}.$$

On the contrary, the Arithmetic operator L_A does not admit a homogeneous structure: nevertheless in this case we are able to find an interesting invariance property with respect to the "translation" operator

$$\ell_{(t',x')}(t, x) = (t' + t, x_1'x_1, x_2' + x_1'x_2), \qquad t, t' \in \mathbb{R},\ x, x' \in \mathbb{R}_+^2; \tag{11}$$

precisely we have

$$L_A(u(\ell_{(t',x')})) = (L_A u)\left(\ell_{(t',x')}\right). \tag{12}$$

We remark that the fundamental Γ_A of L_A is not known explicitly and one of the key point in the proof consists in showing suitable summability properties of Γ_A near the pole. Note also that, due to the invariance property (12), the fundamental solution Γ_A verifies

$$\Gamma_A(t, x; T, X) = \frac{1}{X_1^2}\Gamma_A\left(t - T, \frac{x_1}{X_1}, \frac{x_2 - X_2}{X_1}; 0, 1, 0\right), \qquad x_1, x_2, X_1, X_2 > 0,\ x_2 > X_2,\ t < T.$$

Remark 2. Given a domain $\emptyset$ and $\alpha \in]0, 1[$, we denote by $C_G^\alpha(\emptyset)$ and $C_G^{1,\alpha}(\emptyset)$ the intrinsic Hölder spaces defined by the norms

$$\|u\|_{C_G^\alpha(\emptyset)} = \sup_{\emptyset} |u| + \sup_{\substack{z, z_0 \in \emptyset \\ z \neq z_0}} \frac{|u(z) - u(z_0)|}{\|z_0^{-1} \circ z\|_G^\alpha}, \qquad \|u\|_{C_G^{1,\alpha}(\emptyset)} = \|u\|_{C_G^\alpha(\emptyset)} + \|\partial_{x_1} u\|_{C_G^\alpha(\emptyset)}$$

where $\|\cdot\|_G$ is a δ_l-homogeneous norm in $\mathbb{R}^3$. Due to the embedding Theorem 2.1 in [4], the strong solution u of Theorem 1.1 is locally in $C_G^{1,\alpha}$ for any $\alpha \in]0, 1[$. In particular this ensures the validity of the standard "smooth pasting" condition in the x_1 variable (corresponding to the asset price). We also recall that the (optimal) $\mathcal{S}^\infty_{\text{loc}}$-regularity of the solution has been recently proved in [5].

Remark 3. In the standard Black & Scholes setting and for particular homogeneous payoffs, it is possible to reduce the *spatial* dimension of the pricing problem (from two to one). In the case of European Asian options, this was first suggested in [8] and [13] respectively for the floating and fixed strike payoffs. In the American case, the dimensional reduction is only possible for the floating strike payoff while the fixed strike Amerasian option necessarily involves a 2-dimensional degenerate PDE.

Proof of Theorem 1.1. It is not restrictive to assume that φ is continuous on $[0, T] \times \mathbb{R}_+^2$ or equivalently we may study the problem on $[\varepsilon, T] \times \mathbb{R}_+^2$ for a fixed, but arbitrary, positive ε. We divide the proof in some steps.

Step 1. Let $D_r(x)$ denote the Euclidean ball centered at $x \in \mathbb{R}^2$, with radius r. We construct a sequence of "lentil shaped" domains $O_n = D_n\left(n + \frac{1}{n}, 0\right) \cap D_n\left(0, n + \frac{1}{n}\right)$ covering $\mathbb{R}^2_+$. For any $n \in \mathbb{N}$, the cylinder $H_n =]0, T[\times O_n$ is a L_A*-regular domain* in the sense that there exists a barrier function at any point of the parabolic boundary $\partial_P H_n := \partial H_n \setminus (\{0\} \times O_n)$ and therefore the obstacle problem on H_n has a strong solution (cf. Theorem 3.1 in [4]). Indeed on any compact subset H of $\mathbb{R} \times \mathbb{R}^2_+$, we have $L_A \equiv L_H$ where

$$L_H = a_H(t,x)\partial_{x_1x_1} + x_1\partial_{x_2} + \partial_t \tag{13}$$

and a_H is a some smooth function such that $0 < \underline{a}_H \leq a_H \leq \bar{a}_H$ on $]0, T[\times\mathbb{R}^2_+$, with $\underline{a}_H, \bar{a}_H$ suitable positive constants. Note that L_H is a perturbation of the Geometric operator L_G: as such, by Theorem 1.4 in [3], it has a fundamental solution Γ_H that is bounded from above and below by Gaussian functions. Then by Theorem 3.1 in [4] we have: for any $n \in \mathbb{N}$ and $g \in C(H_n \cup \partial_P H_n)$, $g \geq \varphi$, there exists a strong solution[2] $u_n \in \mathcal{S}^p_{\text{loc}}(H_n) \cap C(H_n \cup \partial_P H_n)$ to problem

$$\begin{cases} \max\{L_A u, \varphi - u\} = 0 & \text{in } H_n, \\ u|_{\partial_P H_n} = g. \end{cases} \tag{14}$$

Moreover, for every $p \geq 1$ and $H \subset\subset H_n$ there exists a positive constant C, only dependent on $H, H_n, p, \|\varphi\|_{L^\infty(H_n)}, \|g\|_{L^\infty(H_n)}$ such that $\|u_n\|_{\mathcal{S}^p(H)} \leq C$.

Step 2. To prove part *i)*, we consider a sequence of cut-off functions $\chi_n \in C^\infty_0(\mathbb{R}^2_+)$ such that $\chi_n = 1$ on O_{n-1}, $\chi_n = 0$ on $\mathbb{R}^2_+ \setminus O_n$ and $0 \leq \chi_n \leq 1$. We set $g_n(t,x,y) = \chi_n(x,y)\varphi(t,x,y) + (1 - \chi_n(x,y))\bar{u}(t,x,y)$ and denote by u_n the strong solution to (14) with $g = g_n$. By the comparison principle we have $\varphi \leq u_n \leq u_{n+1} \leq \bar{u}$ and therefore, by the a priori estimate in $\mathcal{S}^p$, for every $p \geq 1$ and $H \subset\subset H_n$ we have $\|u_n\|_{\mathcal{S}^p(H)} \leq C$ for some constant C dependent on H but not on n. Then we can pass to the limit as $n \to \infty$, on compacts of $]0, T[\times\mathbb{R}^2_+$, to get a solution of $\max\{Lu, \varphi - u\} = 0$. A standard argument based on barrier functions shows that $u(t,x)$ is continuous up to $t = T$ and attains the final datum. This concludes the proof of part *i)*.

Step 3. To prove part *ii)*, we first construct the fundamental solution Γ_A of L_A as the limit of an increasing sequence of Green functions for L_A on the cylinders H_n, $n \in \mathbb{N}$: to this end we combine some classical PDE technique (cf. Chapter 15 in [6]) with the recent interior and boundary Schauder estimates for degenerate Kolmogorov operators proved in [10]. We also show that Γ_A is the transition density of the underlying stochastic process.

Next we observe that, for fixed $(t,x) \in]0, T[\times\mathbb{R}^2_+$ and $n \in \mathbb{N}$ suitably large so that $x \in O_n$, by the maximum principle we have

$$\Gamma_A(t,x;\cdot,\cdot) \leq \Gamma_{H_n}(t,x;\cdot,\cdot) + \max_{[t,T]\times\partial O_n} \Gamma_A(t,x;\cdot,\cdot) \qquad \text{in }]t,T] \times O_n.$$

Consequently we infer (cf. formula (4.8) in [11]) that $\Gamma_A(t,x;\cdot,\cdot) \in L^p(H_n)$ for some $p > 1$. This local summability property of Γ_A can be combined with the standard maximal estimate

$$E\left[\sup_{t\leq\tau\leq T} |X^{t,x}_\tau|^q\right] < \infty, \qquad q \geq 1,$$

valid for the solution X of a SDE whose coefficients have at most linear growth: then, adapting the arguments used in the proof of Theorem 4.3 in [11], we can show formula (8). This concludes the proof of part *ii)*. □

[2] Here $\mathcal{S}^p_{\text{loc}}$ is the Sobolev space defined in (6) with $f(s) = s$ and $s = x_1$, $a = x_2$.

References

[1] E. Barucci, S. Polidoro, and V. Vespri, *Some results on partial differential equations and Asian options*, Math. Models Methods Appl. Sci., 11 (2001), pp. 475–497.

[2] M. Dai and Z. Q. Xu, *Optimal redeeming strategy of stock loans*, arXiv:0906.0702v1, (2009).

[3] M. Di Francesco and A. Pascucci, *On a class of degenerate parabolic equations of Kolmogorov type*, AMRX Appl. Math. Res. Express, (2005), pp. 77–116.

[4] M. Di Francesco, A. Pascucci, and S. Polidoro, *The obstacle problem for a class of hypoelliptic ultraparabolic equations*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., 464 (2008), pp. 155–176.

[5] M. Frentz, K. Nyström, A. Pascucci, and S. Polidoro, *Optimal regularity in the obstacle problem for Kolmogorov operators related to American Asian options*, to appear in Math. Ann., (2009).

[6] A. Friedman, *Stochastic differential equations and applications. Vol. 2.*, Probability and Mathematical Statistics. Vol. 28. Academic Press, 1976.

[7] A. Friedman and W. Shen, *A variational inequality approach to financial valuation of retirement benefits based on salary*, Finance Stoch., 6 (2002), pp. 273–302.

[8] J. E. Ingersoll, *Theory of Financial Decision Making*, Blackwell, Oxford, 1987.

[9] E. Lanconelli and S. Polidoro, *On a class of hypoelliptic evolution operators*, Rend. Sem. Mat. Univ. Politec. Torino, 52 (1994), pp. 29–63. Partial differential equations, II (Turin, 1993).

[10] K. Nyström, A. Pascucci, and S. Polidoro, *Regularity near the initial state in the obstacle problem for a class of hypoelliptic ultraparabolic operators*, submitted, (2009).

[11] A. Pascucci, *Free boundary and optimal stopping problems for American Asian options*, Finance Stoch., 12 (2008), pp. 21–41.

[12] G. Peskir and A. Shiryaev, *Optimal stopping and free-boundary problems*, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 2006.

[13] L. C. G. Rogers and Z. Shi, *The value of an Asian option*, J. Appl. Probab., 32 (1995), pp. 1077–1088.