DECOMPOSITION OF HOMOGENEOUS POLYNOMIALS OVER AN ALGEBRAICALLY CLOSED FIELD

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ABSTRACT: *Let F be a homogeneous polynomial of degree d in $m+1$ variables defined over an algebraically closed field of characteristic zero and suppose that F belongs to the s-th secant varieties of the standard Veronese variety $X_{m,d} \subset \mathbb{P}^{\binom{m+d}{d}-1}$ but that its minimal decomposition as sum of d-th powers of linear forms $M_1, \ldots, M_r$ is $F = M_1^d + \cdots + M_r^d$ with $r > s$. We show that if $s+r \leq 2d+1$ then such a decomposition of F can be split in two parts: one of them is uniquely determined by linear forms that can be written using only two variables, the other part is algorithmically computable. We also show that the 0-dimensional scheme $\mathcal{Z}$ of degree s that is contained in $X_{m,d}$ and such that $F \in \langle \mathcal{Z} \rangle$ is uniquely determined by F itself.*

Introduction

The decomposition of a homogeneous polynomial that combines a minimum number of addenda and that involves a minimum number of variables is a problem arising form classical algebraic geometry (see e.g. [1], [14]), computational complexity (see e.g. [20], [4]) and signal processing (see e.g. [11], [6], [33]).

The so called Big Waring problem (coming from a question in number theory stated by E. Waring in 1770, see [34]) asked which is the minimum positive integer s such that the generic polynomial of degree d in $m+1$ variables can be written as a sum of s d-th powers of linear forms. That problem was solved for polynomials over an algebraically closed field of characteristic zero by J. Alexander and A. Hirshowitz in 1995 by computing the dimensions of all s-th secant varieties to Veronese varieties (see [1] for the original proof and [5] for a recent proof).
In fact the Veronese variety $X_{m,d} \subset \mathbb{P}^{\binom{m+d}{d}-1}$ parameterizes those polynomials of degree d in $m+1$ variables that can be written as a d-th power of a linear form (see (4)). The s-th secant variety $\sigma_s(X_{m,d}) \subset \mathbb{P}^{\binom{m+d}{d}-1}$ of the Veronese variety $X_{m,d}$ is the Zariski closure of the set $\sigma_s^0(X_{m,d}) \subset \mathbb{P}^{\binom{m+d}{d}-1}$ that parameterizes homogeneous polynomials of degree d in $m+1$ variables that can be written as a sum of at most s d-th powers of linear forms (see Definition 3 and Notation 3 respectively).

If we call the symmetric rank of a homogeneous polynomial F the minimum positive integer s such that F can be written as a sum of s d-th powers of linear forms (cfr. Remark 1), we can explicit a question that is crucial in many applications:

"Which is the symmetric rank of a given homogeneous polynomial F?"

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Applications are interested in the cases of polynomials defined both over an algebraically closed field of characteristic zero and over the real numbers (see [27], [24], [23], [12]). In this paper we will restrict our attention on the cases of polynomials defined over an algebraically closed field K of characteristic 0.

By the definition of secant varieties of Veronese varieties (cfr. Definition 3) we see that, even if we were able to compute their equations (it is not needless to underline here that the knowledge of such equations is still an open problem, see [15], [7], [18], [19]), in general they will not be sufficient in order to compute the symmetric rank of a homogeneous polynomial, because $\sigma_s(X_{m,d})$ is the Zariski closure of $\sigma_s^0(X_{m,d})$ and $\sigma_s(X_{m,d}) \setminus \sigma_s^0(X_{m,d})$ is, in general, not empty and, when this case occurs, it contains polynomials whose symmetric rank is bigger than s (see Remark 2).
If $m = 1$ there is the very well known Sylvester's algorithm (due firstly to J. J. Sylvester himself in 1886, see [31], then reformulated in 2001 by G. Comas and M. Seiguer, see [9], and more recently different versions of the same appeared in [10], in [3] and in [2]) that, given a homogeneous polynomial of degree d in 2 variables, turns out its symmetric rank. If $m \geq 2$ the generalizations of the Sylvester's algorithm work effectively for small values of m and theoretically for all m's (cfr. [10], [3] and in [2]).

The notion of symmetric rank of a homogeneous polynomial is derived from the language of tensors. In fact the vector space $K[x_0, \dots, x_m]_d$ of homogeneous polynomials of degree d in $m+1$ variables over an algebraically closed field K of characteristic 0 is isomorphic to S^dV^* where V is an $(m+1)$-dimensional vector space over K. Now S^dV^* is the linear subset of symmetric tensors of $V^{\otimes d}$. Then there is a 1:1 correspondence between homogeneous polynomials of degree d in $m+1$ variables and S^dV^*. Therefore we can describe the Veronese variety both as $X_{m,d} \subset \mathbb{P}(K[x_0, \dots, x_m]_d)$ parameterizing the projective classes of those polynomials that can be written as the d-th power of linear forms, and as $X_{m,d} \subset S^dV^*$ that parameterizes the projective classes of the symmetric tensors of the type $v^{\otimes d} \in S^dV^*$ with $v \in V^*$ (see (3)). Hence the symmetric rank for symmetric tensors is nothing else than the minimum positive integer s such that a symmetric tensor $T \in S^dV^*$ can be written as $T = v_1^{\otimes d} + \cdots + v_s^{\otimes d}$ (see Definition 4).

Assume now that we are in one of the cases in which it is possible to compute the symmetric rank either of a homogeneous polynomial or of a symmetric tensors (because of the above identification we will use those two notions indifferently). Suppose therefore to be able to find $M_1, \dots, M_s \in K[x_0, \dots, x_m]_1$ such that a given $F \in K[x_0, \dots, x_m]_d$ can be written as $F = M_1^d + \cdots + M_s^d$. Is that decomposition unique? If it is not unique, is at least possible to write a canonical decomposition in such a way that some of the addenda are unique and the others can be algorithmically computable? For some references on the uniqueness questions see [22], [32], [26], [25], [16], [17], [30].
Moreover, is it possible to find such a canonical decomposition of $F \in K[x_0, \dots, x_m]_d$ in such a way that the appearing addenda use the minimum number of variables as possible? More precisely:

> "Is it possible to find a linear change of coordinates such that $F \in K[x_0, \dots, x_m]_d$ can be written as
>
> $$F = L_1^d + \cdots + L_q^d + M_1^d + \cdots + M_t^d$$
>
> with $L_1, \dots, L_q \in K[x_0, x_1]_1$ and $M_1, \dots, M_t \in K[x_0, \dots, x_m]_1$? Under which conditions? Is such a decomposition unique? Is it algorithmically computable?"

On the canonical decomposition of homogeneous polynomials see [26], [25], [32], [21], [11], [14], [28], [29] In this paper, after a section of some preliminaries and of the geometric construction that will be necessary for the sequel (Section 1), we will state in Section 2 the main theorem of the paper, that is the following:

Theorem 1. *Let $P \in \mathbb{P}^N$ and let $\mathrm{sbr}(P)$ be the symmetric border rank of P, i.e. $\mathrm{sbr}(P)$ is the minimum positive integer s such that $P \in \sigma_s(X_{m,d})$. Suppose that:*

$$\mathrm{sbr}(P) < \mathrm{sr}(P)$$

and

$$\mathrm{sbr}(P) + \mathrm{sr}(P) \leq 2d + 1.$$

Let $\mathcal{S} \subset X_{m,d}$ be a 0-dimensional reduced sub-scheme that realizes the symmetric rank of P, and let $\mathcal{Z} \subset X_{m,d}$ be a 0-dimensional non-reduced sub-scheme such that $P \in \langle \mathcal{Z} \rangle$, $\deg \mathcal{Z} \leq \mathrm{sbr}(P)$ and $P \notin \langle \mathcal{Z}' \rangle$ for any 0-dimensional non-reduced sub-scheme $\mathcal{Z}' \subset X_{m,d}$ with $\deg(\mathcal{Z}') < \deg(\mathcal{Z})$. Let also $C_d \subset X_{m,d}$ be the unique rational normal curve that intersects $\mathcal{S} \cup \mathcal{Z}$ in degree at least $d+2$.

Then, for all points $P \in \mathbb{P}^N$ as above we have that:

$$\mathcal{S} = \mathcal{S}_1 \sqcup \mathcal{S}_2, \tag{1}$$

$$\mathcal{Z} = \mathcal{Z}_1 \sqcup \mathcal{S}_2, \tag{2}$$

where $\mathcal{S}_1 = \mathcal{S} \cap C_d$, $\mathcal{Z}_1 = \mathcal{Z} \cap C_d$ and $\mathcal{S}_2 = (\mathcal{S} \cap \mathcal{Z}) \setminus \mathcal{S}_1$.

Moreover in Proposition 2 we prove the uniqueness of $\mathcal{Z}$, C_d and $\mathcal{S}_2$ and then we conclude that the scheme $\mathcal{S}$ can be algorithmically computed.

Theorem 1 and Proposition 2 can be rephrased in terms of homogeneous polynomials as follows:

Corollary 2. *Let $F \in K[x_0, \ldots, x_m]_d$ be a homogeneous polynomial of degree d in $m+1$ variables such that $[F] \in \sigma_s(X_{m,d}) \setminus \sigma_s^0(X_{m,d})$ and $s + \mathrm{sr}(F) \leq 2d+1$. Then, after a linear change of coordinates, there exist $L_1, \ldots, L_q \in K[x_0, x_1]_1$ and $M_1, \ldots, M_t \in K[x_0, \ldots, x_m]_1$ such that*

$$F = L_1^d + \cdots + L_q^d + M_1^d + \cdots + M_t^d.$$

Moreover $[L_1], \ldots, [L_q] \in \mathbb{P}(K[x_0, x_1]_1)$ are uniquely determined by $[F] \in \mathbb{P}(K[x_0, \ldots, x_m]_d)$ and $M_1, \ldots, M_t \in K[x_0, \ldots, x_m]_1$ are algorithmically computable via the Sylvester algorithm (see [31], [9], [10], [3], [2]*).*

In Remark 2 and in Remark 3 we will show how to check the symmetric rank s of a homogeneous polynomial of degree d and the uniqueness of the decomposition in the cases $s \leq d$ and $s = d+1$ respectively.

Finally in Section 3 we will improve Theorem 1 in the case of homogeneous polynomials of degree d in 3 variables. Actually we will prove the following:

Theorem 2. *Fix an integer $d \geq 7$. Fix $P \in \mathbb{P}^{\binom{d+2}{2}-1}$ such that $\mathrm{sr}(P) + \mathrm{sbr}(P) \leq 3d-1$ and $\mathrm{sr}(P) \neq \mathrm{sbr}(P)$. Fix $Z, S \subset \mathbb{P}^2$ such that $\nu_d(Z) = \mathcal{Z}$ computes $\mathrm{sbr}(P)$ and $\nu_d(S) = \mathcal{S}$ computes $\mathrm{sr}(X)$. Assume that Z and S are not as described in Theorem 1, i.e. assume that there is no line $L \subset \mathbb{P}^2$ such that $Z = (Z \cap L) \sqcup S_2$ and $S = (S \cap L) \sqcup S_2$. Then there are a smooth conic $E \subset \mathbb{P}^2$, a set $S_2 \subset \mathbb{P}^2 \setminus E$, two schemes $Z_1, S_1 \subset E$ and a point $P_1 \in \langle \nu_d(E) \rangle$ such that $S = S_1 \sqcup S_2$, $Z = Z_1 \sqcup S_2$, $P \in \langle \{P_1\} \cup S_2 \rangle$, Z_1 computes $br_{\nu_d(E)}(P_1)$ and S_1 computes $r_{\nu_d(E)}(P_1)$ (for the definitions of $br_{\nu_d(E)}(P_1)$ and of $r_{\nu_d(E)}(P_1)$ see 8).*

The converse holds in the following sense. Fix a smooth conic $E \subset \mathbb{P}^2$, $S_2 \subset \mathbb{P}^2 \setminus E$ such that $2 \cdot \sharp(S_2) \leq d-3$, and any $S_1 \subset E$, $Z_1 \subset E$ such that

S_1 is reduced and the pair (Z_1, S_1) computes $(br_{\nu_d(E)}(P_1), r_{\nu_d(E)}(P_1))$ for some $P_1 \in \langle \nu_d(E) \rangle$. Set $S := S_1 \sqcup S_2$, $Z := Z_1 \sqcup S_2$ and take any $P \in \langle \{P_1\} \cup S_2 \rangle$ not in the linear span of a proper subset $\{P_1\} \cup S_2$. Then Z computes the symmetric border rank of P, while S computes the symmetric rank of P.

1. Preliminaries and Construction

As already observed in the introduction, we can give the notion of Veronese variety in two equivalent ways. We start with the one given by symmetric tensors and we suddenly relate it to the one given by homogeneous polynomials.

Notation 1. For all this paper, V will be a vector space of dimension $m+1$ defined over an algebraically closed field K of characteristic 0.
With S^dV we indicate the linear subspace of $V^{\otimes d}$ made by symmetric tensors of order d.

Definition 1. Let $T \in S^dV$ be a symmetric tensor of order d. The symmetric rank $\mathrm{sr}(T)$ of T is the minimum integer r such that there exist $v_1, \dots, v_r \in V$ such that $T = \sum_{i=1}^r v_i^{\otimes d}$.

A way of defining Veronese variety via symmetric tensors is the following.

Definition 2. The Veronese variety $X_{m,d} \subset \mathbb{P}(S^dV)$ is the variety parameterizing projective classes of symmetric tensors $T \in S^dV$ such that $\mathrm{sr}(T) = 1$.

With this definition, the Veronese variety can be obviously viewed as the image of the following map:

$$
\begin{array}{rccc}
\nu_{m,d} : & \mathbb{P}(V) & \to & \mathbb{P}(S^dV) \\
 & [v] & \mapsto & [v^{\otimes d}].
\end{array}
\tag{3}
$$

Now, consider V^* the dual space of V as the vector space $K[x_0, \dots, x_m]_1$ of homogeneous linear forms in $m+1$ variables defined over the field K. Then, in the map (3), we can replace the space V with the space $K[x_0, \dots, x_m]_1$ and give the analogous definition of Veronese variety in terms of homogeneous polynomials of degree d (in fact S^dV^* will be the space of homogeneous polynomials $K[x_1, \dots, x_m]_d$ in $m+1$ variables of degree d over K). With this notation the Veronese variety can be interpreted also as the image of the following map:

$$
\begin{array}{rccc}
\nu_{m,d} : & \mathbb{P}(K[x_0, \dots, x_m]_1) & \to & \mathbb{P}(K[x_0, \dots, x_m]_d) \\
 & [L] & \mapsto & [L^d].
\end{array}
\tag{4}
$$

Therefore the Veronese variety $X_{m,d}$ can be also viewed as the variety that parameterizes homogeneous polynomials of degree d in $m+1$ variables over K that can be written as d-th powers of linear forms.

Obviously, the map $\nu_{m,d}$ of (3) and (4) is nothing else than the embedding of a $\mathbb{P}^m$ into $\mathbb{P}^{\binom{m+d}{d}-1}$ given by the sections of the sheaf $\mathcal{O}(d)$ (the image of such embedding is in fact the classical way of defining Veronese variety).

Notation 2. We will always refer to the projective space $\mathbb{P}^{\binom{m+d}{d}-1} = \mathbb{P}(S^dV) = \langle \nu_{m,d}(\mathbb{P}(V)) \rangle \simeq \langle \nu_{m,d}(\mathbb{P}(V^*)) \rangle = \mathbb{P}(S^dV^*) = \mathbb{P}(K[x_0, \dots, x_m]_d)$ with $\mathbb{P}^N$.

This correspondence allows to speak of symmetric rank of a homogeneous polynomial $F \in K[x_0, \dots, x_m]_d$ as the minimum number r of linear forms $L_1, \dots, L_r \in K[x_0, \dots, x_m]_1$ such that $F = \sum_{i=1}^r L_i^d$.
Both in the case of a symmetric tensor $T \in S^dV$, with $\dim(V) = m+1$, and in the case of

a homogeneous polynomial $F \in K[x_0, \dots, x_m]_d$, the notion of symmetric rank can be resumed in the minimum number of points belonging to the Veronese variety $X_{m,d} \subset \mathbb{P}^N$ whose span contains $[T] \in \mathbb{P}(S^dV)$ or $[F] \in \mathbb{P}(K[x_0, \dots, x_m]_d)$. We will be more precise in the following remark.

Remark 1. A symmetric tensor $T \in S^dV$, with $\dim(V) = m+1$ (or a homogeneous polynomial $F \in K[x_0, \dots, x_m]_d$) has symmetric rank r if and only if the two following conditions are both satisfied:

- there exists a set $\mathcal{S}$ of r distinct points $\mathcal{S} = \{P_1, \dots, P_r\} \subset X_{m,d}$ such that $[T] \in \langle \mathcal{S} \rangle$ (or $[F] \in \langle \mathcal{S} \rangle$) and $\dim(\langle \mathcal{S} \rangle) = r - 1$,
- for any set of points $\mathcal{S}' \subset X_{m,d}$ such that $\sharp(\mathcal{S}') < r$ we have that $[T] \notin \langle \mathcal{S}' \rangle$ (or $[F] \notin \langle \mathcal{S}' \rangle$).

This remark allows us to use the notion of symmetric rank for symmetric tensors, for homogeneous polynomials and, more generally, for points in $P \in \mathbb{P}^N$ where $\mathbb{P}^N$ is as in Notation 2, i.e. $\mathbb{P}^N \simeq \mathbb{P}(K[x_0, \dots, x_m]_d) \simeq \mathbb{P}(S^dV)$. Hence, with an abuse of notation, we will say that a point $P \in \mathbb{P}^N$ has symmetric rank r if there exisits a symmetric tensor $T \in S^dV$ such that $[T] = P$ has $\mathrm{sr}(T) = r$, i.e. if r is the minimum number of different points $P_1 \dots, P_r \in X_{m,d}$ such that $P \in \langle P_1, \dots, P_r \rangle$.

Notation 3. We indicate with $\sigma_r^0(X_{m,d}) \subset \mathbb{P}(S^dV)$ the set of points $P \in \mathbb{P}(S^dV)$ whose symmetric rank is at most r:

$$\sigma_r^0(X_{m,d}) := \bigcup_{P_1, \dots P_r \in X_{m,d}} \langle P_1, \dots, P_r \rangle.$$

Definition 3. Let $X \subset \mathbb{P}^n$ be a projective variety. The s-th secant variety $\sigma_s(X)$ of X is defined as follows:

$$\sigma_s(X) := \overline{\bigcup_{P_1, \dots P_s \in X} \langle P_1, \dots, P_s \rangle}.$$

Definition 4. Let $T \in S^dV$ be a symmetric tensor. The minimum integer s such that $[T] \in \sigma_s(X_{m,d}) \setminus \sigma_{s-1}(X_{m,d})$ is called the symmetric border rank of T and we write $\mathrm{sbr}(T) = s$ (we will often use the same definition for the projective class $[T]$, and more generally for any point $P \in \mathbb{P}(S^dV) = \mathbb{P}^N$).

Remark 2. Let now $P \in \sigma_s(X_{m,d}) \setminus \sigma_s^0(X_{m,d}) \subsetneq \mathbb{P}^N$. In [2] (see Proposition 2.8), it is proved that there exists a non-reduced sub-scheme $\mathcal{Z} \subset X_{m,d}$ of degree s such that the projective dimension of $\langle \mathcal{Z} \rangle \subset \mathbb{P}^N$ is $s-1$ and $P \in \langle \mathcal{Z} \rangle$. By definition of symmetric border rank we also have that $P \notin \langle \mathcal{Z}' \rangle$ for any other 0-dimensional non-reduced sub-scheme $\mathcal{Z}' \subset X_{m,d}$ with $\deg(\mathcal{Z}') < s$.

Moreover, since $P \in \sigma_s(X_{m,d}) \setminus \sigma_s^0(X_{m,d})$, then $\mathrm{sr}(P) > \mathrm{sbr}(P)$.

Notation 4. Let $P \in \mathbb{P}^N$ be a point such that $\mathrm{sbr}(P) = s < \mathrm{sr}(P) = r$. We fix here the notation that we will use all along the paper for the schemes that realize the symmetric border rank and the symmetric rank of P respectively.

- We will indicate with $\mathcal{Z} \subset X_{m,d}$ a non-reduced 0-dimensional sub-scheme of degree at most s such that $P \in \langle \mathcal{Z} \rangle$ and $P \notin \langle \mathcal{Z}' \rangle$ for a 0-dimensional non-reduced sub-scheme $\mathcal{Z}' \subset X_{m,d}$ such that $\deg(\mathcal{Z}') < \deg(\mathcal{Z})$ (i.e. $\mathcal{Z}$ as in Remark 2).
- We indicate also with $\mathcal{S} \subset X_{m,d}$ a reduced 0-dimensional sub-scheme of degree r computing the symmetric rank of $P \in \mathbb{P}^N$ (i.e. $\mathcal{S}$ as in Remark 1).

From now on we will always consider a point $P \in \mathbb{P}^N = \mathbb{P}(S^d V)$, as in Notation 2, such that $\mathrm{sbr}(P) = s < \mathrm{sr}(P) = r$ and

$$\mathrm{sbr}(P) < \mathrm{sr}(P), \tag{5}$$

$$\mathrm{sbr}(P) + \mathrm{sr}(P) \leq 2d + 1. \tag{6}$$

Notation 5. Let now $Z, S \subset \mathbb{P}(V) = \mathbb{P}^m$ be the pre-images via the Veronese map $\nu_{m,d}$ of $\mathcal{Z}, \mathcal{S} \subset \mathbb{P}^N$ respectively as in Notation 4. I.e. $Z, S \subset \mathbb{P}^m$ are two 0-dimensional sub-schemes such that

$$\nu_{m,d}(Z) = \mathcal{Z} \text{ and } \nu_{m,d}(S) = \mathcal{S} \tag{7}$$

with $\mathcal{Z}$ and $\mathcal{S}$ realizing the symmetric border rank and the symmetric rank of P respectively.

Remark 3. Obviously we have that:

- $\deg(S) = \deg(\mathcal{S}) = \mathrm{sr}(P)$,
- $\deg(Z) = \deg(\mathcal{Z}) = \mathrm{sbr}(P)$,
- $\dim(\langle S \rangle) = \dim(\langle \mathcal{S} \rangle) = \mathrm{sr}(P) - 1$,
- $\dim(\langle Z \rangle) = \dim(\langle \mathcal{Z} \rangle) \leq \mathrm{sbr}(P) - 1$,
- $\sharp(\mathrm{Supp}(S)) = \sharp(\mathrm{Supp}(\mathcal{S}))$,
- $\sharp(\mathrm{Supp}(Z)) = \sharp(\mathrm{Supp}(\mathcal{Z}))$.

Notation 6. Now define a 0-dimensional scheme $W \subset \mathbb{P}^m$ as the union of Z and S as above in Notation 4:

$$W := Z \cup S. \tag{8}$$

Then define also:

$$\mathcal{W} := \nu_{m,d}(W) \tag{9}$$

in such a way that $\mathcal{W} \subset \mathbb{P}^N$ is a scheme obtained by the union of $\mathcal{S} \subset \mathbb{P}^N$ that realizes the symmetric rank of P and $\mathcal{Z} \subset \mathbb{P}^N$ that realizes its symmetric border rank.

Remark 4. The hypothesis (6) on the relation between the symmetric rank and the symmetric border rank of $P \in \mathbb{P}^N$ assures that $\deg(W) \leq 2d + 1$.

This Remark allows us to apply the Lemma 4.6 proved in [2] that assures the existence of a line $L \subset \mathbb{P}^m$ that intersects the scheme $W \subset \mathbb{P}^m$ defined in (8) with multiplicity at least $d + 2$. We prove now the uniqueness of such a line L.

Lemma 1. *Fix an integer $d \geq 1$. Let $W \subset \mathbb{P}^m$ with $m \geq 2$, be a zero-dimensional scheme of degree $\deg(W) \leq 2d + 1$ and such that $h^1(\mathcal{I}_W(d)) > 0$. Then there is a unique line $L \subset \mathbb{P}^m$ such that $\deg(L \cap W) \geq d + 2$ and*

$$\deg(W \cap L) = d + 1 + h^1(\mathcal{I}_W(d)).$$

Proof. For the existence of the line $L \subset \mathbb{P}^m$ see [2], Lemma 4.6. We prove here the uniqueness. Since $\deg(W) \leq 2d + 1$ and since the scheme-theoretic intersection of two different lines has length at most one, the uniqueness of the line L will follow once we will have proved the formula $\deg(W \cap L) = d + 1 + h^1(\mathcal{I}_W(d))$ of the statement. We will prove it by induction on m.

First assume $m = 2$. In this case L is a Cartier divisor of $\mathbb{P}^m$, hence the residual scheme $\mathrm{Res}_L(W)$ of W with respect to L has degree $\deg(\mathrm{Res}_L(W)) = \deg(W) - \deg(W \cap L)$. Look at the exact sequence that defines the residual scheme $\mathrm{Res}_L(W)$:

$$0 \to \mathcal{I}_{\mathrm{Res}_{L(W)}}(d-1) \to \mathcal{I}_W(d) \to \mathcal{I}_{W \cap L, L}(d) \to 0. \tag{10}$$

Since $\dim(\mathrm{Res}_L(W)) \leq \dim(W) \leq 0$ and $d-1 \geq -2$, we have $h^2(\mathcal{I}_{\mathrm{Res}_L(W)}(d-2)) = 0$. Since $\deg(W \cap L) \geq d+1$, we have $h^0(L, \mathcal{I}_{W\cap L}(d)) = 0$. Since $\deg(\mathrm{Res}_L(W)) = \deg(W) - \deg(W \cap L) \leq d$, we have $h^1(\mathcal{I}_{\mathrm{Res}_L(W)}(d-1)) = 0$ (e.g. use [2]), Thus the cohomology exact sequence of (10) gives $h^1(\mathcal{I}_W(d)) = \deg(W \cap L) - d - 1$, proving the lemma for $m = 2$.

Now assume $m \geq 3$ and that the result is true for $\mathbb{P}^{m-1}$.
Take a general hyperplane $H \subset \mathbb{P}^m$ containing L and set $W' := W \cap L$. The inductive assumption gives $h^1(H, \mathcal{I}_{W'}(d)) = \deg(W' \cap L) - d - 1$. Since $\deg(\mathrm{Res}_H(W)) \leq d-1$, as above we get $h^1(\mathcal{I}_{\mathrm{Res}_H(W)}(d-1)) = 0$. Consider now the analogue exact sequence to (10) with H instead of L:

$$0 \to \mathcal{I}_{\mathrm{Res}_H(W)}(d-1) \to \mathcal{I}_W(d) \to \mathcal{I}_{W\cap H,H}(d) \to 0.$$

Then, since $W \cap L = W' \cap L$, we get, as above, that $h^1(\mathcal{I}_W(d)) = \deg(W \cap L) - d - 1$. $\square$

Now, let $W, L \subset \mathbb{P}^m$ be as in Lemma 1, then if we indicate with $C_d \subset X_{m,d}$ the image of L via the Veronese map $\nu_{m,d}$, i.e.

$$C_d := \nu_{m,d}(L), \tag{11}$$

then we can translate Lemma 1 in terms of an unique rational normal curve $C_d \subset X_{m,d}$ that intersects $\mathcal{W}$ in degree at least $d+2$.

Proposition 1. *Let $P \in \mathbb{P}^N = \mathbb{P}^{\binom{m+d}{d}-1}$ be such that* $\mathrm{sbr}(P) < \mathrm{sr}(P)$ *and* $\mathrm{sbr}(P) + \mathrm{sr}(P) \leq 2d+1$. *Let $\mathcal{S}, \mathcal{Z} \subset X_{m,d}$ be as in Notation 4. Let also $\mathcal{W} \subset X_{m,d}$ be as in (9), i.e. the 0-dimensional sub-scheme obtained as the union of $\mathcal{S} \subset \mathbb{P}^N$ that realizes the symmetric rank of P and $\mathcal{Z} \subset \mathbb{P}^N$. Then there exists a unique rational normal curve $C_d \subset X_{m,d}$ of degree d such that the degree of the schematic intersection $C_d \cap \mathcal{W}$ is at least $d+2$ and*

$$\deg(C_d \cap \mathcal{W}) = d + 1 + h^1(\mathcal{I}_{\mathcal{W}}(1)).$$

Proof. Let $Z, S \subset \mathbb{P}^m$ be 0-dimensional schemes such that $\nu_{m,d}(Z) = \mathcal{Z}$ and $\nu_{m,d}(S) = \mathcal{S}$ as in (7), and let $W = S \cup Z$, i.e. $\nu_{m,d}(W) = \mathcal{W}$. Therefore, by the hypothesis on $\mathrm{sbr}(P)$ and $\mathrm{sr}(P)$, we have that $\deg(W) \leq 2d+1$, hence we can apply Lemma 1 to the scheme $W \subset \mathbb{P}^m$ in order to get the existence of an unique line L that intersects W with multiplicity at least $d+2$ and such that $\deg(L \cap W) = d + 1 + h^1(\mathcal{I}_W(d))$. Now the unique rational normal curve $C_d \subset X_{m,d}$ of the statement of this proposition is nothing but $\nu_d(L) \subset X_{m,d}$. Since obviously $h^1(\mathcal{I}_W(d)) = h^1(\mathcal{I}_{\mathcal{W}}(1))$ we get the statement. $\square$

We state here the following remark because it is an obvious consequence of Lemma 1 but we will use it in the next section.

Remark 5. Let $W, L \subset \mathbb{P}^m$ be as in Lemma 1. Observe that, since obviously $\mathrm{lenght}(W \cap L) = \mathrm{lenght}((W \cap L) \cap L)$, we have, as a consequence of Lemma 1, that

$$h^1(\mathcal{I}_W(d)) = h^1(\mathcal{I}_{W\cap L}(d))$$

and also that

$$h^0(\mathcal{I}_W(d)) = h^0(\mathcal{I}_{L\cap W}(d)) + \deg(W) - \deg(W \cap L).$$

Both those equalities can be expressed in terms of $C_d = \nu_{m,d}(L)$ and $\mathcal{W} = \nu_{m,d}(W)$:

$$h^1(\mathcal{I}_{\mathcal{W}}(1)) = h^1(\mathcal{I}_{\mathcal{W}\cap C_d}(1))$$

and

$$h^0(\mathcal{I}_{\mathcal{W}}(1)) = h^0(\mathcal{I}_{\mathcal{W}\cap C_d}(1)) + \deg(\mathcal{W}) - \deg(\mathcal{W} \cap C_d).$$

What we will do in the sequel of the paper is to study the intersection of the schemes $\mathcal{Z}, \mathcal{S} \subset X_{m,d}$ given in Notation 4 with the unique rational normal curve $C_d \subset X_{m,d}$ of the previous proposition. Hence we need to introduce the notation that we will use in order to give names to all the parts in which it is possible to split the schemes $\mathcal{Z}$ and $\mathcal{S}$ with respect to the curve C_d. The notation that we are going to introduce it will be coherent and functional to the main result of the paper that is Theorem 1 which describes the structure of the schemes that realize the symmetric rank and the symmetric border rank of a point $P \in \mathbb{P}^N$ satisfying the relation $\mathrm{sbr}(P) + \mathrm{sr}(P) \leq 2d+1$ given in (6).

Notation 7. Let $P \in \mathbb{P}^m$ be such that $\mathrm{sbr}(P) < \mathrm{sr}(P)$ and $\mathrm{sbr}(P) + \mathrm{sr}(P) \leq 2d+1$. Let $Z, S, W \subset \mathbb{P}^m$ as in (4) and (8) respectively. Let also $L \subset \mathbb{P}^m$ be the line of Lemma 1 that intersects W in degree at least $d+2$ and $C_d = \nu_{m,d}(L)$ as in Proposition 1. Then we call:

- $Z_1 := Z \cap L \subset \mathbb{P}^m$,
- $S_1 := S \cap L \subset \mathbb{P}^m$,
- $S_2 := (S \cap Z) \setminus S_1 \subset \mathbb{P}^m$,
- $S_3 := S \setminus (S_1 \cup S_2) \subset \mathbb{P}^m$

and

- $\mathcal{Z}_1 := \nu_{m,d}(Z_1) = \mathcal{Z} \cap C_d \subset X_{m,d}$,
- $\mathcal{S}_i := \nu_{m,d}(S_i) \subset X_{m,d}$ for $i = 1, 2, 3$.

In this way we write the reduced sub-scheme $\mathcal{S} \subset X_{m,d}$ that realizes the symmetric rank of P as the union of three disjoint sub-schemes:

- $\mathcal{S}_1$ is the intersection of $\mathcal{S}$ with C_d,
- $\mathcal{S}_2$ is the part that $\mathcal{S}$ and $\mathcal{Z}$ have in common out of $\mathcal{S}_1$,
- $\mathcal{S}_3$ is the remaining part of $\mathcal{S}$.

This notation will be functional to the proof of the main theorem of this paper (Theorem 1) where we will show the structure of the schemes $\mathcal{Z}$ and $\mathcal{S}$ that realizes the symmetric border rank and the symmetric rank of a point $P \in \mathbb{P}^N$ such that $\mathrm{sbr}(P) < \mathrm{sr}(P)$ and $\mathrm{sbr}(P) + \mathrm{sr}(P) \leq 2d+1$. In fact we will prove that, once one has fixed the point $P \in \mathbb{P}^N$ such that $\mathrm{sbr}(P) < \mathrm{sr}(P)$ and $\mathrm{sbr}(P) + \mathrm{sr}(P) \leq 2d+1$, then if $\mathcal{Z}, \mathcal{S} \subset X_{m,d}$ are two 0-dimensional sub-schemes as in Notation 4 that realize the symmetric rank and the symmetric border rank of P respectively, then the scheme $\mathcal{S}$ is actually equal to $\mathcal{S}_1 \cup \mathcal{S}_2$ (hence $\mathcal{S}_3 = \emptyset$ that means that the scheme $\mathcal{S}$ is made only by its part on C_d and the part that it has in common with $\mathcal{Z}$) and the scheme $\mathcal{Z}$ is made by its intersection with $C_d \subset X_{m,d}$ and its intersection with $\mathcal{S}$, i.e. $\mathcal{Z} = \mathcal{Z}_1 \cup \mathcal{S}_2$.

2. Results

In this section we give the main result of the paper: Theorem 1. The following two lemmas are functional to the main theorem that will follow.

Lemma 2. *Let $A \subset \mathbb{P}^m$, $m \geq 2$, be a 0-dimensional scheme. Let $L \subset \mathbb{P}^m$ be a line. Set $A_1 := L \cap A$, $\mathcal{A} := \nu_{m,d}(A)$, $\mathcal{A}_1 := \nu_{m,d}(A_1)$ and $C_d := \nu_{m,d}(L)$. Assume $\deg(A) - \deg(A_1) \leq d-1$. Then:*

$$h^0(\mathcal{I}_{A \cup L}(d)) = \binom{m+d}{m} - d - 1 - \deg(A) + \deg(A_1), \tag{12}$$

$$\dim\langle C_d \cup \mathcal{A}\rangle = d + \deg(A) - \deg(A_1) \tag{13}$$

and

$$\langle \mathcal{A}\rangle \cap \langle C_d\rangle = \langle \mathcal{A}_1\rangle. \tag{14}$$

Proof. We first prove equation (12) adapting the proof of Lemma 1.
First assume $m = 2$. Let $\mathrm{Res}_L(A)$ be the residual scheme of A with respect to the line L. We have $\deg(\mathrm{Res}_L(A)) = \deg(A) - \deg(A_1)$. Obviously $h^0(\mathbb{P}^2, \mathcal{I}_{A\cup L}(d)) = h^0(\mathbb{P}^2, \mathcal{I}_{\mathrm{Res}_L(A)}(d-1))$. Since $\deg(A) - \deg(A_1) \leq d-1$, we have $h^1(\mathbb{P}^2, \mathcal{I}_{\mathrm{Res}_L(A)}(d-1)) = 0$. Hence $h^0(\mathbb{P}^2, \mathcal{I}_{\mathrm{Res}_L(A)}(d-1)) = \binom{d+1}{2} - \deg(A) + \deg(A_1)$. Since $\binom{d+2}{2} - \binom{d+1}{2} = d+1$, we get (12) when $m = 2$.
In the case $m > 2$ the proof works by induction on m as in the proof of Lemma 1.

For the equation (13) we simply have to observe that $h^0(\mathcal{I}_{\mathcal{A}\cup C_d}(1)) = h^0(\mathcal{I}_{A\cup L}(d))$ and to apply the equation (12) to $h^0(\mathcal{I}_{\mathcal{A}\cup C_d}(1))$.

For the equation (14) we need to apply equation (13) to the Grassmann formula $\dim(\langle \mathcal{A}\rangle \cap \langle C_d\rangle) = \dim(\langle C_d\rangle) + \dim(\langle \mathcal{A}\rangle) - \dim(\langle C_d \cup \mathcal{A}\rangle)$ and get the result. □

Lemma 3. *In the same setting of Lemma 2, consider a scheme $\mathcal{A}_2 \subset X_{m,d}$ disjoint from $\mathcal{A}$ and such that $\deg(\mathcal{A}) - \deg(\mathcal{A}_1) + \sharp(\mathcal{A}_2) \leq d-1$. Then $\langle \mathcal{A}\cup\mathcal{A}_2\rangle \cap \langle C_d \cup \mathcal{A}_2\rangle = \langle \mathcal{A}_1 \cup \mathcal{A}_2\rangle$.*

Proof. By applying Lemma 2 to the scheme $A\cup\mathcal{A}_2$ instead of A, we get from (12) that $\dim(\langle C_d \cup \mathcal{A}\cup\mathcal{A}_2\rangle) = d + \deg(A\cup\mathcal{A}_2) - \deg(A_1)$. Then by Grassmann formula we get that $\dim(\langle C_d \cup \mathcal{A}_2\rangle \cap \langle \mathcal{A}\cup\mathcal{A}_2\rangle) = \dim(\langle C_d\cup\mathcal{A}_2\rangle) + \deg(A_1) - d$. Now $\dim(\langle C_d \cup \mathcal{A}_2\rangle) = d + \sharp(\mathcal{A}_2)$ then $\dim(\langle C_d\cup\mathcal{A}_2\rangle \cap \langle \mathcal{A}\cup\mathcal{A}_2\rangle) = \deg(A_1) + \sharp(\mathcal{A}_2)$ that is equal to $\dim(\langle \mathcal{A}_1\cup\mathcal{A}_2\rangle)$. Then $\langle \mathcal{A}\cup\mathcal{A}_2\rangle \cap \langle C_d\cup\mathcal{A}_2\rangle = \langle \mathcal{A}_1\cup\mathcal{A}_2\rangle$. □

We can now prove the main theorem of this paper. We will need all the construction given in the previous section.

Theorem 1. *Let $P \in \mathbb{P}^N$ be such that:*

$$\mathrm{sbr}(P) < \mathrm{sr}(P)$$

and

$$\mathrm{sbr}(P) + \mathrm{sr}(P) \leq 2d+1.$$

Let $\mathcal{S}, \mathcal{Z} \subset X_{m,d}$ be as in Notation 4, i.e. $\mathcal{S} \subset X_{m,d}$ is a 0-dimensional reduced sub-scheme that realizes the symmetric rank of P, and $\mathcal{Z} \subset X_{m,d}$ is a 0-dimensional non-reduced sub-scheme such that $P \in \langle \mathcal{Z}\rangle$, $\deg \mathcal{Z} \leq \mathrm{sbr}(P)$ and $P \notin \langle \mathcal{Z}'\rangle$ for any 0-dimensional non-reduced sub-scheme $\mathcal{Z}' \subset X_{m,d}$ with $\deg(\mathcal{Z}') < \deg(\mathcal{Z})$. Let also $C_d \subset X_{m,d}$ be the unique rational normal curve that intersects $\mathcal{S}\cup\mathcal{Z}$ in degree at least $d+2$ (as proved in Proposition 1).

Then, for all points $P \in \mathbb{P}^N$ as above we have that:

$$\mathcal{S} = \mathcal{S}_1 \sqcup \mathcal{S}_2, \tag{15}$$

$$\mathcal{Z} = \mathcal{Z}_1 \sqcup \mathcal{S}_2, \tag{16}$$

where $\mathcal{S}_1 = \mathcal{S}\cap C_d$, $\mathcal{Z}_1 = \mathcal{Z}\cap C_d$ and $\mathcal{S}_2 = (\mathcal{S}\cap\mathcal{Z}) \setminus \mathcal{S}_1$ (as in Notation 7).

Proof. The existence of the scheme $\mathcal{Z} \subset X_{m,d}$ is assured by Proposition 2.8 in [2]. Set $\mathcal{W} := \mathcal{Z}\cup\mathcal{S}$.

(a) Here we check that the scheme $\mathcal{W} \subset X_{m,d}$ is linearly dependent in $\mathbb{P}^N$, i.e. we check that:

$$h^1(\mathcal{I}_{\mathcal{W}}(1)) > 0. \tag{17}$$

Since $\mathrm{sbr}(P) < \mathrm{sr}(P)$ by hypothesis, there exists a point $Q \in \mathcal{S}$ such that $Q \notin \mathcal{Z}_{red}$. Clearly $h^0(\mathcal{I}_{\mathcal{W}\setminus\{Q\}}(1)) - 1 \leq h^0(\mathcal{I}_{\mathcal{W}}(1)) \leq h^0(\mathcal{I}_{\mathcal{W}\setminus\{Q\}}(1))$ and $h^1(\mathcal{I}_{\mathcal{W}}(1)) = h^1(\mathcal{I}_{\mathcal{W}\setminus\{Q\}}(1)) + 1 + (h^0(\mathcal{I}_{\mathcal{W}}(1)) - h^0(\mathcal{I}_{\mathcal{W}\setminus\{Q\}}(1)))$. Thus to prove (17) it is sufficient to prove that

$$h^0(\mathcal{I}_{\mathcal{W}}(1)) = h^0(\mathcal{I}_{\mathcal{W}\setminus\{Q\}}(1)). \tag{18}$$

Since $P \in \langle \mathcal{Z} \rangle$ and $\mathcal{Z} \subset \mathcal{W} \setminus \{Q\}$ we have that $f(P) = 0$ for any $f \in H^0(\mathcal{I}_{\mathcal{W}\setminus\{Q\}}(1))$. Now, since $\mathcal{S}$ is by definition a set of points of $X_{m,d}$ computing the symmetric rank of P, we have that $P \in \langle \mathcal{S} \rangle$ and $P \notin \langle \mathcal{S} \setminus \{Q\} \rangle$, then $Q \in \langle \mathcal{S} \cup \{P\} \rangle$. Thus $f(Q) = 0$ for any $f \in H^0(\mathcal{I}_{\mathcal{W}\setminus\{Q\}}(1))$, because $f(P) = 0$ as shown above, $\mathcal{S}\setminus\{Q\} \subset \mathcal{W}\setminus\{Q\}$ and $f(\mathcal{A}) = 0$ for all $\mathcal{A} \in \mathcal{S}\setminus\{Q\}$. We have then proved that $H^0(\mathcal{I}_{\mathcal{W}\setminus\{Q\}}(1)) \subseteq H^0(\mathcal{I}_{\mathcal{W}}(1))$. Since the reverse inclusion is obvious, we have that $H^0(\mathcal{I}_{\mathcal{W}\setminus\{Q\}}(1)) = H^0(\mathcal{I}_{\mathcal{W}}(1))$ that proves (18). Hence $\mathcal{W} \subset X_{m,d}$ is linearly dependent.

(b) Step (a) gives $\dim(\langle \mathcal{W} \rangle) \leq \mathrm{length}(\mathcal{W}) - 2$. Since $\mathrm{length}(\mathcal{W}) \leq \mathrm{length}(\mathcal{Z}) + \mathrm{length}(\mathcal{S}) = \mathrm{sbr}(P) + \mathrm{sr}(P) \leq 2d + 1$, then, by Lemma 4.6 in [2], there is a line $L \subset \mathbb{P}^m$ whose image $C_d = \nu_{m,d}(L)$ in $X_{m,d}$ contains a sub-scheme of $\mathcal{W}$ with length at least $d+2$.
Set $\mathcal{S}_3 := \mathcal{S} \setminus (\mathcal{S}_1 \cup \mathcal{S}_2)$ where $\mathcal{S}_1, \mathcal{S}_2 \subset \mathcal{S}$ are defined in the statement, set also $\mathcal{W}' := \mathcal{W} \setminus \mathcal{S}_3$ and notice that it is well-defined, because each point of $\mathcal{S}_3$ is a connected component of the scheme $\mathcal{W}$.

(c) Here we prove $\mathcal{S}_3 = \emptyset$.

Assume that this is not the case and that $\sharp(\mathcal{S}_3) > 0$. Observe that Remark 5 gives that $h^1(\mathcal{I}_{\mathcal{W}\cap C_d}(1)) = h^1(\mathcal{I}_{\mathcal{W}}(1))$ and that $h^0(\mathcal{I}_{\mathcal{W}}(1)) = h^0(\mathcal{I}_{C_d \cap \mathcal{W}}(1)) - \deg(\mathcal{W}) + \deg(\mathcal{W} \cap C_d)$. Hence we get

$$\dim(\langle \mathcal{W} \rangle) = \dim(\langle \mathcal{W}' \rangle) + \sharp(\mathcal{S}_3).$$

Now, by definition, we have that $\mathcal{S} \cap \mathcal{W}' = \mathcal{S}_1 \cup \mathcal{S}_2$, $\mathcal{W} = \mathcal{W}' \sqcup \mathcal{S}_3$ and $\mathcal{Z} \cup \mathcal{S}_1 \cup \mathcal{S}_2 = \mathcal{W}'$, then obviously $\dim(\langle \mathcal{W}' \rangle \cap \langle \mathcal{S} \rangle) = \dim(\langle \mathcal{W}' \rangle) + \dim(\langle \mathcal{S} \rangle) - \dim(\langle \mathcal{W}' \cup \mathcal{S} \rangle) = \dim(\langle \mathcal{S} \rangle) - \sharp(\mathcal{S}_3)$. Since $\mathcal{Z} \subseteq \mathcal{W}'$, we have $P \in \langle \mathcal{W}' \rangle \cap \langle \mathcal{S} \rangle$ that has dimension $\dim(\mathcal{S}) - \sharp(\mathcal{S}_3)$ as just proved. Notice that $\dim(\langle \mathcal{S}_1 \cup \mathcal{S}_2 \rangle) = \dim(\langle \mathcal{S} \rangle) - \sharp(\mathcal{S}_3)$, hence $\dim(\langle \mathcal{S}_1 \cup \mathcal{S}_2 \rangle) = \dim(\langle \mathcal{W}' \rangle \cap \langle \mathcal{S} \rangle)$; since $\langle \mathcal{S}_1 \cup \mathcal{S}_2 \rangle \subset \langle \mathcal{W}' \rangle \cap \langle \mathcal{S} \rangle$ we get that $\langle \mathcal{S}_1 \cup \mathcal{S}_2 \rangle = \langle \mathcal{W}' \rangle \cap \langle \mathcal{S} \rangle$. Since $P \in \langle \mathcal{Z} \rangle \cap \langle \mathcal{S} \rangle \subset \langle \mathcal{W}' \rangle \cap \langle \mathcal{S} \rangle = \langle \mathcal{S}_1 \cup \mathcal{S}_2 \rangle$, we get that $P \in \langle \mathcal{S}_1 \cup \mathcal{S}_2 \rangle$. Since we supposed that $\mathcal{S} \subset X_{m,d}$ is a set computing symmetric rank of P, it is absurd that P belongs to the span a proper subset of $\mathcal{S}$, then necessarily $\sharp(\mathcal{S}_3) = 0$, that is equivalent to the fact that $\mathcal{S}_3 = \emptyset$.

(d) Here we check that $\mathcal{Z} = \mathcal{Z}_1 \cup \mathcal{S}_2$ and that $\mathcal{Z}_1 \cap \mathcal{S}_2 = \emptyset$; moreover we show that the equality $\mathcal{Z} = \mathcal{Z}_1 \cup \mathcal{S}_2$ implies the theorem.

We apply Lemma 3 with $\mathcal{A} = \mathcal{Z}\setminus\mathcal{S}_2$ and $\mathcal{A}_2 = \mathcal{S}_2$, then we get that $\langle \mathcal{Z} \rangle \cap \langle C_d \cup \mathcal{S}_2 \rangle = \langle \mathcal{Z}_1 \cup \mathcal{S}_2 \rangle$. Since obviously $P \in \langle \mathcal{Z} \rangle \cap \langle C_d \cup \mathcal{S}_2 \rangle$, by the minimality of $\mathcal{Z}$ we have that $\mathcal{Z} = \mathcal{Z}_1 \cup \mathcal{S}_2$. □

Remark 6. Observe that if $\mathcal{S}, \mathcal{Z} \subset X_{m,d}$ are as in (15) and (16), then all the points $P \in \langle \mathcal{S} \rangle \cap \langle \mathcal{Z} \rangle \setminus \langle \mathcal{S}_2 \rangle$ are such that $\mathrm{sbr}(P) < \mathrm{sr}(P)$ and $\mathrm{sbr}(P) + \mathrm{sr}(P) \leq 2d + 1$.

We can be more precise one the uniqueness of the construction given above. Actually it happens that the scheme $Z \subset \mathbb{P}^m$ (and by consequence the scheme $\mathcal{Z} \subset \mathbb{P}^N$) can be uniquely described by $P \in \mathbb{P}^N$, while the scheme $S \subset \mathbb{P}^m$ (and hence the scheme $\mathcal{S} \subset \mathbb{P}^N$) can only be algorithmically computed from $P \in \mathbb{P}^m$, but not in an unique way, at least for the part $S_1 = S\setminus Z$ (the same holds for $\mathcal{S}_1 = \mathcal{S} \setminus \mathcal{Z}$).

Proposition 2. *Fix $P \in \mathbb{P}^N$ such that $\mathrm{sbr}(P) + \mathrm{sr}(P) \leq 2d+1$ and $\mathrm{sbr}(P) < \mathrm{sr}(P)$ and consider all the construction given above:*

- *$Z, S \subset \mathbb{P}^m$ as in (7) (i.e. Z is a minimal non-reduced 0-dimensional sub-scheme such that $\deg(Z) \leq \mathrm{sbr}(P)$ and $S \subset \mathbb{P}^m$ is a reduced 0-dimensional scheme whose image via $\nu_{m,d}$ realizes the symmetric rank of P), and $\mathcal{Z} = \nu_{m,d}(Z), \mathcal{S} = \nu_{m,d}(S) \subset X_{m,d}$;*
- *$L \subset \mathbb{P}^m$ the line that intersects $Z \cup S$ in degree at least $d+2$ (as in Lemma 1) and $C_d = \nu_{m,d}(L)$;*

- *$Z_1, S_1 \subset \mathbb{P}^m$ be the schematic intersections of L with Z and S respectively (as in Notation 7) and $\mathcal{Z}_1 = \nu_{m,d}(Z_1), \mathcal{S}_1 = \nu_{m,d}(S_1) \subset X_{m,d}$;*
- *$S_2 = S \setminus S_1$ and $\mathcal{S}_2 = \nu_{m,d}(S_2) \subset X_{m,d}$.*

Let also $P_1 := \langle \mathcal{Z}_1 \rangle \cap \langle \mathcal{S}_1 \rangle \in \mathbb{P}^N$.

Then the following sentences hold:

(1) *Z, L, $S_2 \subset \mathbb{P}^m$ and $P_1 \in \mathbb{P}^N$ are uniquely determined by $P \in \mathbb{P}^N$; moreover $\deg(Z) = \mathrm{sbr}(P)$;*
(2) *The line L is the unique line of $\mathbb{P}^m$ containing an unreduced connected component of Z;*
(3) *The scheme $\mathcal{Z}_1 \subset X_{m,d}$ computes symmetric border rank of $P_1 \in \langle C_d \rangle$ with respect to the rational normal curve C_d;*
(4) *The set $S_1 \subset \mathbb{P}^m$ is not uniquely determined by P, but it may be computed using Sylvester algorithm from P_1 (see* [31], [9], [10], [3] *and* [2]*). Hence S is algorithmically computable from P.*

Proof. We first show that it is sufficient to prove the uniqueness of $Z \subset \mathbb{P}^m$. Indeed, $L \subset \mathbb{P}^m$ is uniquely determined by Z, because it is the only line of $\mathbb{P}^m$ containing an unreduced connected component of Z. Thus $Z_1 := Z \cap L$ and $S_2 := Z \setminus Z_1$ are uniquely determined. The point $P_1 \in \langle \nu_{m,d}(L) \rangle$ is uniquely determined, because $\{P_1\} = \langle \{P\} \cup \mathcal{S}_2 \rangle \cap \langle C_d \rangle$.

Now we prove the uniqueness of Z. Assume the existence of $Z' \neq Z$ such that $\deg(Z') \leq \mathrm{sbr}(P)$, $P \in \langle \nu_{m,d}(Z') \rangle$, and $P \notin \langle \nu_{m,d}(Z'') \rangle$ for any $Z'' \subsetneqq Z'$.
Call $L' \subset \mathbb{P}^m$ the unique line that intersects $Z' \cup S$ in degree at least $d+2$ and set $S_2' := S \setminus (S \cap L')$. Call $P_1' := \langle \nu_{m,d}(Z' \cap L') \rangle \cap \langle \nu_{m,d}(S \cap L') \rangle$. Set $Z_1' := Z' \cap L''$.
We saw in the proof of Theorem 1 that both Z and Z' have degree $\mathrm{sbr}(P)$. Thus $\deg(Z \cup Z') \leq 2d$. Since $P \in \langle \nu_{m,d}(Z) \rangle \cap \langle \nu_{m,d}(Z') \rangle$, $Z' \neq Z$ and $P \notin \langle \nu_{m,d}(Z'') \rangle$ for any Z'' such that either $Z'' \subsetneqq Z$ or $Z'' \subsetneqq Z'$, the proof of Lemma 1 gives $h^1(\mathcal{I}_{Z \cup Z'}(d)) > 0$. Hence there is a line $R \subset \mathbb{P}^m$ such that $\deg(Z \cup B) \geq d + 2 \geq 5$. Since no line, except at most L (resp. L') contains a degree ≥ 3 subscheme of Z (resp. Z') we get that either $L = R$ or $L' = R$.

(a) Here we assume $L = L'$. Hence $S_2' = S \setminus (S \cap L') = S_2$. Notice that $\{P_1\} = \langle \{P\} \cup \mathcal{S}_2 \rangle$. Hence $P_1 = P_1'$ (under the assumption $L = L'$). Hence both $\nu_{m,d}(Z_1)$ and $\nu_{m,d}(Z_1')$ computes symmetric border rank of $P_1 \in \langle C_d \rangle$ with respect to the rational normal curve C_d. Observe that since $P_1 \in \langle \nu_{m,d}(S_1) \rangle \cap \langle \nu_{m,d}(Z_1) \rangle$ and since $\deg(Z_1) < \deg(S_1)$ we get that the symmetric border rank of P_1 with respect to the rational normal curve C_d is less or equal than $(d + 1)/2$. This implies that the scheme computing the symmetric border rank of P_1 with respect to C_d (just use that any zero-dimensional subscheme with degree at most $d + 1$ of the rational normal curve C_d is linearly independent). Hence $Z_1 = Z_1'$. Therefore $Z = Z_1' \sqcup S_2' = Z'$.

(b) Here we assume $L \neq L'$. Just to fix the notation we assume $R = L'$. Since $L \neq L'$, we have $\deg(Z \cap L') \leq 2$. Thus $\deg((Z \cup B) \cap R) \leq 2 + \deg(B \cap R) \leq 2 + \mathrm{sbr}(P_1') \leq d + 1$, where $\mathrm{sbr}(P_1')$ is with respect to C_d. But this last sentence is a contradiction, hence $L = L'$. □

Corollary 1. *In the same setting of Proposition 2, the line $L \subset \mathbb{P}^m$ is spanned by Z_1.*

Proof. This is an obvious consequence of the just proved fact that L is the unique line that contains an unreduced component of the 0-dimensional scheme Z. □

We can interpret Theorem 1 and Proposition 2 in terms of homogeneous polynomials.

Corollary 2. *Let $F \in K[x_0, \dots, x_m]_d$ be a homogeneous polynomial of degree d in $m+1$ variables such that $[F] \in \sigma_s(X_{m,d}) \setminus \sigma_s^0(X_{m,d})$ and $s + \mathrm{sr}(F) \leq 2d + 1$. Then, after a linear change of coordinates, there exist $L_1, \dots, L_q \in K[x_0, x_1]_1$ and $M_1, \dots, M_t \in K[x_0, \dots, x_m]_1$ such that*

$$F = L_1^d + \cdots + L_q^d + M_1^d + \cdots + M_t^d.$$

Moreover $[L_1], \dots, [L_q] \in \mathbb{P}(K[x_0, x_1]_1)$ *are uniquely determined by* $[F] \in \mathbb{P}(K[x_0, \dots, x_m]_d)$ *and* $M_1, \dots, M_t \in K[x_0, \dots, x_m]_1$ *are algorithmically computable via the Sylvester algorithm (see* [31], [9], [10], [3], [2]*).*

An analogous corollary can be stated for symmetric tensors if we assume the following identification. Let $W \subset V$ be a linear subspace of dimension 2 of a vector space V of dimension $m+1$. Now, since $S^dW^* \simeq K[x_0, x_1]_d \subset K[x_0, \dots, x_m]_d \simeq S^dV^*$ and since $(S^dV^*)^* \simeq S^dV$ for any finite dimensional vector space over an algebraically closed field of characteristic 0, we may assume that $S^dW \subset S^dV$ up to certain isomorphism.

Corollary 3. *Let $T \in S^dV$ be a symmetric tensor such that $[T] \in \sigma_s(X_{m,d}) \setminus \sigma_s^0(X_{m,d})$ and $s + \mathrm{sr}(T) \leq 2d+1$. Then there exist an unique subspace $W \subset V$ of dimension 2 and an unique non-reduced 0-dimensional scheme $Z \subset V$ of degree s such that $[T] \in \langle \nu_{m,d}(Z) \rangle$. Moreover*

$$T \in \langle T_1, \dots, T_q, Z_1, \dots, Z_t \rangle$$

with $q + t = \mathrm{sr}(T)$, $\{Z_1, \dots, Z_t\}$ is the reduced part of $\nu_{m,d}(Z)$ and $T_1, \dots, T_q \in S^dW$ are symmetric tensors of symmetric rank equal to 1. The tenosrs $T_1, \dots, T_q \in S^dW$ are uniquely determined, up to multiplication by constants, by $[T] \in \mathbb{P}(S^dV)$ and the tensors $Z_1, \dots, Z_t \in \nu_{m,d}(Z)$ can be algorithmically computed using one of the Sylvester's algorithms described in [31], [9], [10], [3], [2].

Remark 7. Take $P \in \mathbb{P}^N$. An obvious consequence of Theorem 1 is that $\mathrm{sr}(P) = \mathrm{sbr}(P)$ if $\mathrm{sr}(P) \leq d/2$. This observation is also a consequence of [2], Lemma 4.6, which was used in the proof of Theorem 1.

Remark 8. Fix $P \in \mathbb{P}^N$. If $\mathrm{sr}(P) \leq \lfloor (d+1)/2 \rfloor$, then [2], Lemma 4.6, gives $\mathrm{sbr}(P) = \mathrm{sr}(P)$ and that there is a unique 0-dimensional scheme $\mathcal{Z} \subset X_{m,d}$ such that $\mathrm{length}(\mathcal{Z}) = \mathrm{sbr}(P)$ and $P \in \langle \mathcal{Z} \rangle$. Since $\mathrm{sr}(P) = \mathrm{sbr}(P)$ and $\mathcal{Z}$ is unique and also reduced. It is the only set that computes $\mathrm{sr}(P)$. Hence $\mathrm{sbr}(A) = \mathrm{sr}(A)$ for every $A \in \mathbb{P}^n$ such that $\mathrm{sr}(A) \leq \lfloor (d+1)/2 \rfloor$.

In the proof of Theorem 1 we also pointed out the following statements.

Proposition 3. *Let $P \in \mathbb{P}^N$ be such that $\mathrm{sbr}(P) + \mathrm{sr}(P) \leq 2d+1$. Let z be the minimal length of a 0-dimensional sub-scheme $\mathcal{Z} \subset X_{m,d}$ such that $P \in \langle \mathcal{Z} \rangle$. Then $z = \mathrm{sbr}(P)$. Moreover, such a scheme $\mathcal{Z}$ is uniquely determined by P.*

Proposition 4. *Let $P \in \mathbb{P}^N$ be such that $\mathrm{sbr}(P) = \mathrm{sr}(P) \leq d$ and assume the existence of two 0-dimensional sub-schemes $\mathcal{Z}, \mathcal{S} \subset X_{m,d}$ with length $\mathrm{sbr}(P)$, $P \in \langle \mathcal{Z} \rangle \cap \langle \mathcal{S} \rangle$ and $\mathcal{S}$ reduced. Then $\mathcal{Z}, \mathcal{S}$ are as in the statement of Theorem 1, i.e. there are a rational normal curve C_d, and a subscheme $\mathcal{S}_2$ as in that statement such that $\mathcal{Z} = (\mathcal{Z} \cap C_d) \sqcup \mathcal{S}_2$ and $\mathcal{S} = (\mathcal{S} \cap C_d) \sqcup \mathcal{S}_2$.*

Remark 9. Fix a reduced set $\mathcal{S} \subset X_{m,d}$ such that $s := \sharp(\mathcal{S}) \leq d$. Fix any $P \in \langle \mathcal{S} \rangle$. Here we show (thanks to Theorem 1) how to check if $\mathrm{sr}_X(P) = s$ and $\mathcal{S}$ is the only subset of $X_{m,d}$ computing $\mathrm{sr}_X(P)$.
We will also check that, if this condition is satisfied, then $\mathrm{sbr}_X(P) = s$ and $\mathcal{S}$ is the only subset computing the symmetric border rank of P.
An obvious necessary condition is that $P \notin \langle \mathcal{S}' \rangle$ for all $\mathcal{S}' \subset \mathcal{S}$ such that $\sharp(\mathcal{S}') = s-1$. We assume that this condition is satisfied.
Let $S \subset \mathbb{P}^m$ be the only subset such that $\nu_{m,d}(S) = \mathcal{S}$. Assume the existence of a 0-dimensional scheme $A \subset \mathbb{P}^m$ such that $P \in \langle \nu_{m,d}(A) \rangle$, $\deg(A) \leq s$ and $A \neq S$. We apply the proof of Theorem 1 to these schemes A and S. However, here we want to translate all the conditions in conditions on S. A necessary condition (that is independent from P) is the existence of a line $L \subset \mathbb{P}^m$ such that $\sharp(L \cap S) \geq \lceil (d+1)/2 \rceil$. Assume that it exists. Then the set $S_2 := S \setminus (L \cap S)$

must be contained in any S and each point of S_2 must be a connected component of A, i.e. $S_2 \subset A$ and A is reduced at each point of S_2. We need to have $\sharp(A \cap L) + \deg(S \cap L) = d+2$ and there must be a unique $P_1 \in \langle \nu_{m,d}(A \cap L)\rangle \cap \langle \nu_{m,d}(S \cap L)\rangle$. Moreover, $P \in \langle \nu_{m,d}(S_2) \cup \{P_1\}\rangle$. Conversely, the construction given above the Theorem 1 gives how to get the point P form $A \cap L$, P_1 and S_2.

Remark 10. Fix a reduced set $\mathcal{S} \subset X_{m,d}$ such that $\sharp(\mathcal{S}) = d+1$. Fix any $P \in \langle \mathcal{S} \rangle$. Here we show (thanks to Theorem 1) how to check if $\mathrm{sr}(P) = \mathrm{sbr}(P) = d+1$ and if $\mathcal{S}$ is the only subset of $X_{m,d}$ computing the symmetric rank of P.
An obvious necessary condition is that $P \notin \langle \mathcal{S}' \rangle$ for all $\mathcal{S}' \subset \mathcal{S}$ such that $\sharp(\mathcal{S}') = d$. We assume that this condition is satisfied.
If $\mathrm{sbr}(P) \le d$, then there is a 0-dimensional scheme $\mathcal{Z} \subset X_{m,d}$ such that $\deg(\mathcal{Z}) \le d$ and $P \in \langle \mathcal{Z} \rangle$. Fix any such $\mathcal{Z}$. Let A (resp. B) be the only subscheme of $\mathbb{P}^m$ such that $\nu_{m,d}(A) = \mathcal{Z}$ (resp. $\nu_{m,d}(B) = \mathcal{S}$). The pair (A, B) must be as in the construction above of Theorem 1. In particular there is a line $L \subset \mathbb{P}^m$ such that $\sharp(L \cap B) \ge \lceil (d+1)/2 \rceil$. Assume that it exists. Then the set $F_2 := B \setminus L \cap B$ must be contained in any B and each point of F_2 must be a connected component of A, i.e. $F_2 \subset A$ and A is reduced at each point of F_2. We need to have $\sharp(A \cap L) + \deg(B \cap L) = d+2$ and there must be a unique $P_1 \in \langle \nu_{m,d}(A \cap L)\rangle \cap \langle \nu_{m,d}(B \cap L)$. Moreover, $P \in \langle \nu_{m,d}(F_2) \cup \{P_1\}\rangle$.

Question 1. Is it true that $\mathrm{sr}(P) \le d(\mathrm{sbr}(P) - 1)$ for all $P \in \mathbb{P}^N$ and that equality holds if and only if $P \in TX_{m,d} \setminus X_{m,d}$ where $TX_{m,d} \subset \mathbb{P}^N$ is the tangential variety of $X_{m,d}$?

3. The case of Veronese surface

In the case $m = 2$ we may use [13], Corollary 2, to go a little bit further on the sentence of Theorem 1.

We introduce, only for this section, the notion of "X-rank" and "X-border rank" of a point $P \in \mathbb{P}^N$ with respect to a projective, reduced variety $X \subset \mathbb{P}^N$ (actually we have implicitly used that concept also in the proof of Theorem 1 but it was useless to introduce that notion there).

Notation 8. Let $X \subset \mathbb{P}^N$ be a projective, reduced variety, and let $P \in \langle X \rangle$. Then we write

$$r_X(P) := \{\min s \in \mathbb{N} \mid P \in \langle P_1, \ldots, P_s \rangle, \text{ with } P_i \in X\}$$

and

$$br_X(P) := \{\min s \in \mathbb{N} \mid P \in \sigma_s(X)\}.$$

Remark 11. If $X \subset \mathbb{P}^N$ is the Veronese variety $X_{m,d}$, then $r_X(P) = \mathrm{sr}(P)$ and $br_X(P) = \mathrm{sbr}(P)$.

Remark 12. Obviously if $C \subset X \subset \mathbb{P}^N$ is a subvariety of a projective variety X, and if $P \in \langle C \rangle$ then $r_X(P) \le r_C(P)$.

Theorem 2. *Fix an integer $d \ge 7$. Fix $P \in \mathbb{P}^{\binom{d+2}{2}-1}$ such that $r_{X_{2,d}}(P) + br_{X_{2,d}}(P) \le 3d-1$ and $r_{X_{2,d}}(P) \ne br_{X_{2,d}}(P)$. Fix $Z, S \subset \mathbb{P}^2$ such that $\nu_d(Z) = \mathcal{Z}$ computes $br_{X_{2,d}}(P)$ and $\nu_d(S) = \mathcal{S}$ computes $r_{X_{2,d}}(X)$. Assume that Z and S are not as described in Theorem 1, i.e. assume that there is no line $L \subset \mathbb{P}^2$ such that $Z = (Z \cap L) \sqcup S_2$ and $S = (S \cap L) \sqcup S_2$. Then there are a smooth conic $E \subset \mathbb{P}^2$, a set $S_2 \subset \mathbb{P}^2 \setminus E$, two schemes $Z_1, S_1 \subset E$ and a point $P_1 \in \langle \nu_d(E) \rangle$ such that $S = S_1 \sqcup S_2$, $Z = Z_1 \sqcup S_2$, $P \in \langle \{P_1\} \cup S_2 \rangle$, Z_1 computes $br_{\nu_d(E)}(P_1)$ and S_1 computes $r_{\nu_d(E)}(P_1)$.*

The converse holds in the following sense. Fix a smooth conic $E \subset \mathbb{P}^2$, $S_2 \subset \mathbb{P}^2 \setminus E$ such that $2 \cdot \sharp(S_2) \le d-3$, and any $S_1 \subset E$, $Z_1 \subset E$ such that S_1 is reduced and the pair (Z_1, S_1)

computes $(br_{\nu_d(E)}(P_1), r_{\nu_d(E)}(P_1))$ *for some* $P_1 \in \langle \nu_d(E) \rangle$. *Set* $S := S_1 \sqcup S_2$, $Z := Z_1 \sqcup S_2$ *and take any* $P \in \langle \{P_1\} \cup S_2 \rangle$ *not in the linear span of a proper subset* $\{P_1\} \cup S_2$. *Then* Z *computes the symmetric border rank of* P, *while* S *computes the symmetric rank of* P.

Proof. Set $W := Z \cup S$. As in the proof of step (a) of Theorem 1 we get $h^1(\mathbb{P}^2, \mathcal{I}_W(d)) > 0$. Since we assumed that Z, S, P are not as in the statement of Theorem 1, its proof gives length$(W) \geq 2d+2$. By assumption and by the proof of Theorem 1 we have that for every line $D \subset \mathbb{P}^2$ the degree of $W \cap D$ is less or equal than $d+1$. Hence there is a smooth conic $E \subset \mathbb{P}^2$ such that $W \subset E$ ([13] Remark (i) at p. 116); here we use that $\deg(W) \leq \mathrm{sbr}(P) + \mathrm{sr}(P) < 3d$. Then the proof of Theorem 1 works verbatim.
Now we check the "converse " part. Notice that $\deg(Z_1) + \deg(S_1) = 2d+2$ (either [9] or [21], Theorem 4.1). Since $2 \cdot \sharp(S_2) \leq d-3$, we have $h^1(\mathbb{P}^2, \mathcal{I}_{S_2}(d-2)) = 0$. Thus $\dim(\langle S_2 \cup \nu_d(E) \rangle) = \dim(\langle \nu_d(E) \rangle) + \sharp(S_2)$. □

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