DECOMPOSITION OF HOMOGENEOUS POLYNOMIALS WITH LOW RANK

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ABSTRACT: *Let F be a homogeneous polynomial of degree d in $m+1$ variables defined over an algebraically closed field of characteristic zero and suppose that F belongs to the s-th secant varieties of the standard Veronese variety $X_{m,d} \subset \mathbb{P}^{\binom{m+d}{d}-1}$ but that its minimal decomposition as a sum of d-th powers of linear forms $M_1, \ldots, M_r$ is $F = M_1^d + \cdots + M_r^d$ with $r > s$. We show that if $s + r \leq 2d + 1$ then such a decomposition of F can be split in two parts: one of them is made by linear forms that can be written using only two variables, the other part is uniquely determined once one has fixed the first part. We also obtain a uniqueness theorem for the minimal decomposition of F if the rank is at most d and a mild condition is satisfied.*

Introduction

The decomposition of a homogeneous polynomial that combines a minimum number of addenda and that involves a minimum number of variables is a problem arising form classical algebraic geometry (see e.g. [1], [14]), computational complexity (see e.g. [19]) and signal processing (see e.g. [26]).

The so called Big Waring problem (coming from a question in number theory stated by E. Waring in 1770) asked which is the minimum positive integer s such that the generic polynomial of degree d in $m+1$ variables can be written as a sum of s d-th powers of linear forms. That problem was solved for polynomials over an algebraically closed field of characteristic zero by J. Alexander and A. Hirschowitz in 1995 by computing the dimensions of all s-th secant varieties to Veronese varieties (see [1] for the original proof and [4] for a recent proof).
In fact the Veronese variety $X_{m,d} \subset \mathbb{P}^N$, $N := \binom{m+d}{d} - 1$, parameterizes those polynomials of degree d in $m+1$ variables that can be written as a d-th power of a linear form. The s-th secant variety $\sigma_s(X_{m,d}) \subset \mathbb{P}^N$ of the Veronese variety $X_{m,d}$ is the Zariski closure of the set that parameterizes homogeneous polynomials of degree d in $m+1$ variables that can be written as a sum of at most s d-th powers of linear forms. The minimum s for which a point $P \in \mathbb{P}^N$ belongs to $\sigma_s(X_{m,d})$ is often called the symmetric border rank of P and we write it $\mathrm{sbr}(P)$, while the minimum integer r for which $P \in \langle P_1, \ldots, P_r \rangle$, with $P_1, \ldots, P_r$ distinct points of $X_{m,d}$, is called the symmetric rank of P and we write it as $\mathrm{sr}(P)$.

If $P \in \mathbb{P}^N \simeq \mathbb{P}(K[x_0, \ldots, x_m]_d)$ is the projective class of a homogeneous polynomial F of degree d in $m+1$ variables, the symmetric rank of F is the minimum positive integer r such that F can be written as a sum of r d-th powers of linear forms. The following question is crucial in many applications: "Which is the symmetric rank of a given homogeneous polynomial F?"

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Applications are interested in the cases of polynomials defined both over an algebraically closed field of characteristic zero and over the real numbers (see [23], [20], [13]). In this paper we will restrict our attention to the cases of polynomials defined over an algebraically closed field K of characteristic 0.

By the definition of secant varieties of Veronese varieties we see that, even if we were able to compute their equations (it is not needless to underline here that the knowledge of such equations is still an open problem, see [15], [6], [18]), in general they will not be sufficient in order to compute the symmetric rank of a homogeneous polynomial, because the symmetric rank of a homogeneous polynomial may be strictly bigger than its symmetric border rank.
If $m = 1$ there is the very well known Sylvester's algorithm (due firstly to J. J. Sylvester himself in 1886, then reformulated in 2001 by G. Comas and M. Seiguer, see [11], and more recently different versions of the same appeared in [12], [3], [2]) that, given a homogeneous polynomial of degree d in 2 variables, turns out its symmetric rank. If $m \geq 2$ the generalizations of the Sylvester's algorithm work effectively for small values of m and theoretically for all m's (cfr. [12], [3], [2]).

The notion of symmetric rank of a homogeneous polynomial is derived from the language of tensors. In fact the vector space $K[x_0, \dots, x_m]_d$ of homogeneous polynomials of degree d in $m+1$ variables over an algebraically closed field K of characteristic 0 is isomorphic to S^dV^* where V is an $(m+1)$-dimensional vector space over K, and V^* its dual space. Now S^dV^* is the linear subset of symmetric tensors of $V^{\otimes d}$. Then there is a 1:1 correspondence between homogeneous polynomials of degree d in $m+1$ variables and S^dV^*. Therefore we can describe the Veronese variety both as $X_{m,d} \subset \mathbb{P}(K[x_0, \dots, x_m]_d)$ parameterizing the projective classes of those polynomials that can be written as the d-th power of linear forms, and as $X_{m,d} \subset \mathbb{P}(S^dV^*)$ that parameterizes the projective classes of the symmetric tensors of the type $v^{\otimes d} \in S^dV^*$ with $v \in V^*$. Hence the symmetric rank for symmetric tensors is nothing else than the minimum positive integer s such that a symmetric tensor $T \in S^dV^*$ can be written as $T = v_1^{\otimes d} + \cdots + v_s^{\otimes d}$.

Assume now that we are in one of the cases in which it is possible to compute the symmetric rank either of a homogeneous polynomial or of a symmetric tensor. Suppose therefore to be able to find $M_1, \dots, M_r \in K[x_0, \dots, x_m]_1$ such that a given $F \in K[x_0, \dots, x_m]_d$ can be written as $F = M_1^d + \cdots + M_r^d$ (in the language of tensors: suppose to be able to find $v_1, \dots, v_r \in V^*$ s.t. $T \in S^dV^*$ can be written as $T = v_1^{\otimes d} + \cdots + v_r^{\otimes d}$). Is that decomposition unique? If it is not unique, is at least possible to write a canonical decomposition in such a way that some of the addenda are unique?

Moreover, is it possible to find such a canonical decomposition of $F \in K[x_0, \dots, x_m]_d$ in such a way that the appearing addenda use the minimum number of variables as possible?

In this paper we prove the following result.

Theorem 1. *Let $P \in \mathbb{P}^N$ and let $\mathrm{sbr}(P)$ be the symmetric border rank of P and $\mathrm{sr}(P)$ its symmetric rank. Suppose that:*

$$\mathrm{sbr}(P) < \mathrm{sr}(P) \text{ and}$$
$$\mathrm{sbr}(P) + \mathrm{sr}(P) \leq 2d + 1.$$

Let $\mathcal{S} \subset X_{m,d}$ be a 0-dimensional reduced subscheme that realizes the symmetric rank of P, and let $\mathcal{Z} \subset X_{m,d}$ be a 0-dimensional non-reduced subscheme such that $P \in \langle \mathcal{Z} \rangle$, $\deg \mathcal{Z} \leq \mathrm{sbr}(P)$ and $P \notin \langle \mathcal{Z}' \rangle$ for any 0-dimensional non-reduced subscheme $\mathcal{Z}' \subset X_{m,d}$ with $\deg(\mathcal{Z}') < \deg(\mathcal{Z})$. Let also $C_d \subset X_{m,d}$ be the unique rational normal curve that intersects $\mathcal{S} \cup \mathcal{Z}$ in degree at least $d+2$.

Then, for all points $P \in \mathbb{P}^N$ as above we have that:

$$\mathcal{S} = \mathcal{S}_1 \sqcup \mathcal{S}_2, \tag{1}$$

$$\mathcal{Z} = \mathcal{Z}_1 \sqcup \mathcal{S}_2, \tag{2}$$

where $\mathcal{S}_1 = \mathcal{S} \cap C_d$, $\mathcal{Z}_1 = \mathcal{Z} \cap C_d$ and $\mathcal{S}_2 = (\mathcal{S} \cap \mathcal{Z}) \setminus \mathcal{S}_1$.
Moreover the scheme $\mathcal{S}_2$ is unique.

For the existence of the scheme $\mathcal{Z}$ in the statement of Theorem 1, see Remark 1, which is just a quotation either of [5], Lemma 2.1.5 or of [2], Proposition 1. The assumption " $\mathrm{sbr}(P)+\mathrm{sr}(P) \leq 2d+1$ " in Theorem 1 is sharp (see Example 1).

If we interpret the point $P \in \mathbb{P}^N$ of Theorem 1 as the projective class either of a symmetric tensor $T \in S^dV^*$ or of a homogeneous polynomial $F \in K[x_0, \dots, x_m]_d$, we understand that the scheme $\mathcal{S} \subset X_{m,d}$ gives either the decomposition $T = v_1^{\otimes d} + \cdots + v_r^{\otimes d}$ or the decomposition $F = L_1^d + \cdots + L_r^d$ with r being the symmetric rank of P, $v_i \in V^*$ and $L_i \in K[x_0, \dots, x_m]_1$ for $i = 1, \dots, r$, if s is the symmetric border rank of P and if $s + r \leq 2d + 1$ and $r > s$. Now, under these conditions, the scheme $\mathcal{S}$ can be split in two parts as in (1) . The existence of the scheme $\mathcal{S}_1$ shows that there is a choice of a 2-dimensional vector space $W^* \subset V^*$ such that some of the v_i's appearing in the decomposition of $T \in S^dV^*$ belong to W^* (that is the same to say that some of the L_i's appearing in the decomposition of $F \in K[x_0, \dots, x_m]_d$ can be written using only two variables). Moreover there is a unique way to choose such a W^* if we require that $\mathbb{P}(W^*)$ intersects $\{[v_1], \dots, [v_r]\} \cup Z$ in degree at least $d + 2$, where $Z \subset \mathbb{P}(V^*)$ is the 0-dimensional sub-scheme whose image via the Veronese map ν_d spans a $\mathbb{P}^s$ that contains $[T]$. With this unique choice of W^* the other part of the decomposition of T (of F respectively) is uniquely determined.

Moreover in this paper we use Theorem 1 and a related lemma (see Lemma 3) to prove the following uniqueness statement, which is the second major result of this paper.

Theorem 2. *Assume $d \geq 5$. Fix a finite set $B \subset \mathbb{P}^m$ such that $\rho := \sharp(B) \leq d$ and no subset of it with cardinality $\lfloor (d+1)/2 \rfloor$ is collinear. Fix $P \in \langle \nu_d(B) \rangle$ such that $P \notin \langle E \rangle$ for any $E \subsetneq \nu_d(B)$. Then $\mathrm{sr}(P) = \mathrm{sbr}(P) = \rho$ and $\nu_d(B)$ is the only zero-dimensional scheme $\mathcal{Z} \subset X_{m,d}$ such that $\deg(\mathcal{Z}) \leq \rho$ and $P \in \langle \mathcal{Z} \rangle$.*

The uniqueness or non-uniqueness statement for the decomposition is an important problem. For non-symmetric tensors, see Kruskal's works and their extensions and simplifications ([17], [16], [25]). For symmetric tensors there is a uniqueness theorem for general points with prescribed non-maximal border rank k using the notion of $(k-1)$-weakly non-defectivity introduced by C. Ciliberto and L. Chiantini ([7], [10], Proposition 1.5). For uniqueness and non-uniqueness results when the border rank is maximal, see [24], [21], [22], [9]. Theorem 2 is an extension (with an additional assumption) of [5], Theorem 1.2.6. Without some additional assumptions [5], Theorem 1.2.6, cannot be extended (e.g. it is sharp when $m = 1$). We give an example showing that if $m = 2$, then Theorem 2 is sharp (see Example 2), even taking S in linearly general position. We do not know how to improve the upper bound in Theorem 2 in the case $m > 2$.

1. Preliminaries and Lemmas

Fix integers $m \geq 2$ and $d \geq 2$. Let $\nu_d : \mathbb{P}^m \to \mathbb{P}^N$, $N := \binom{m+d}{m} - 1$, the order d Veronese embedding of $\mathbb{P}^m$.

Definition 1. Let $X \subset \mathbb{P}^n$ be a projective variety. The s-th secant variety $\sigma_s(X)$ of X is defined as follows:

$$\sigma_s(X) := \overline{\bigcup_{P_1, \dots P_s \in X} \langle P_1, \dots, P_s \rangle}.$$

From now on we always take $n := N$ and $X := \nu_d(\mathbb{P}^m) = X_{m,d}$.

Definition 2. Fix any $P \in \mathbb{P}^N$. The minimum integer s such that $X \in \sigma_s(X) \setminus \sigma_{s-1}(X)$ is called the symmetric border rank of P and we write $\mathrm{sbr}(P) = s$.

Definition 3. Fix $P \in \mathbb{P}^N$. The minimum integer r such that $P \in \langle P_1, \dots, P_r\rangle$ for $P_1, \dots, P_r \in X_{m,d}$ is called the symmetric rank of P and we write $\mathrm{sr}(P) = r$.

Remark 1. Fix integers $m \geq 1$, $d \geq 2$ and $P \in \mathbb{P}^N$ such that $\mathrm{sbr}(P) \leq d+1$. By [5], Lemma 2.1.5, or [2], Proposition 1, there is a zero-dimensional scheme $E \subset X_{m,d}$ such that $\deg(E) \leq \mathrm{sbr}(P)$ and $P \in \langle E\rangle$. Moreover, $\mathrm{sbr}(P)$ is the minimal of the degrees of any such scheme E. Thus if $P \in \langle E\rangle$ and $\deg(E) \leq \mathrm{sbr}(P)$, then $\deg(E) = \mathrm{sbr}(P)$ and $P \notin \langle E'\rangle$ for any $E' \subsetneq E$.

Lemma 1. *Fix any $P \in \mathbb{P}^n$ and two zero-dimensional subschemes A, B of $\mathbb{P}^n$ such that $A \neq B$, $P \in \langle A\rangle$, $P \in \langle B\rangle$, $P \notin \langle A'\rangle$ for any $A' \subsetneq A$ and $P \notin \langle B'\rangle$ for any $B' \subsetneq B$. Then $h^1(\mathbb{P}^n, \mathcal{I}_{A\cup B}(1)) > 0$.*

Proof. Since A and B are zero-dimensional, we have the inequality $h^1(\mathbb{P}^n, \mathcal{I}_{A\cup B}(1)) \geq \max\{h^1(\mathbb{P}^n, \mathcal{I}_A(1)), h^1(\mathbb{P}^n, \mathcal{I}_B(1))\}$. Thus we may assume $h^1(\mathbb{P}^n, \mathcal{I}_A(1)) = h^1(\mathbb{P}^n, \mathcal{I}_B(1)) = 0$, i.e. $\dim(\langle A\rangle) = \deg(A) - 1$ and $\dim(\langle B\rangle) = \deg(B) - 1$. Set $D := A \cap B$ (scheme-theoretic intersection). Thus $\deg(A \cup B) = \deg(A) + \deg(B) - \deg(D)$. Since $D \subseteq A$ and A is linearly independent, we have $\dim(\langle D\rangle) = \deg(D)-1$. Since $h^1(\mathbb{P}^n, \mathcal{I}_{A\cup B}(1)) > 0$ if and only if $\dim(\langle A\cup B\rangle) \leq \deg(A \cup B) - 2$, we get $h^1(\mathbb{P}^n, \mathcal{I}_{A\cup B}(1)) > 0$ if and only if $\langle D\rangle \subsetneq \langle A\rangle \cap \langle B\rangle$. Since $A \neq B$, then $D \subsetneq A$. Hence $P \notin \langle D\rangle$. Since $P \in \langle A\rangle \cap \langle B\rangle$, we are done. $\square$

Lemma 2. *Fix an integer $d \geq 1$. Let $W \subset \mathbb{P}^m$ with $m \geq 2$, be a zero-dimensional scheme of degree $\deg(W) \leq 2d + 1$ and such that $h^1(\mathbb{P}^m, \mathcal{I}_W(d)) > 0$. Then there is a unique line $L \subset \mathbb{P}^m$ such that $\deg(L \cap W) \geq d + 2$ and*

$$\deg(W \cap L) = d + 1 + h^1(\mathbb{P}^m, \mathcal{I}_W(d)).$$

Proof. For the existence of the line $L \subset \mathbb{P}^m$ see [2], Lemma 2.
Since $\deg(W) \leq 2d + 1$ and since the scheme-theoretic intersection of two different lines has length at most one and $\deg(W) \leq 2d + 2$, there is no line $R \neq L$ such that $\deg(R \cap W) \geq d + 2$. Thus L is unique.
We prove the formula $\deg(W \cap L) = d + 1 + h^1(\mathcal{I}_W(d))$ by induction on m.

First assume $m = 2$. In this case L is a Cartier divisor of $\mathbb{P}^m$. Hence the residual scheme $\mathrm{Res}_L(W)$ of W with respect to L has degree $\deg(\mathrm{Res}_L(W)) = \deg(W) - \deg(W \cap L)$. Look at the exact sequence that defines the residual scheme $\mathrm{Res}_L(W)$:

$$0 \to \mathcal{I}_{\mathrm{Res}_L(W)}(d-1) \to \mathcal{I}_W(d) \to \mathcal{I}_{W\cap L,L}(d) \to 0. \tag{3}$$

Since $\dim(\mathrm{Res}_L(W)) \leq \dim(W) \leq 0$ and $d-1 \geq -2$, we have $h^2(\mathbb{P}^m, \mathcal{I}_{\mathrm{Res}_L(W)}(d-1)) = 0$. Since $\deg(W\cap L) \geq d+1$, we have $h^0(L, \mathcal{I}_{W\cap L}(d)) = 0$. Since $\deg(\mathrm{Res}_L(W)) = \deg(W)-\deg(W\cap L) \leq d$, we have $h^1(\mathbb{P}^m, \mathcal{I}_{\mathrm{Res}_L(W)}(d-1)) = 0$ (obvious and also a particular case of [2], Lemma 2). Thus the cohomology exact sequence of (3) gives $h^1(\mathbb{P}^m, \mathcal{I}_W(d)) = \deg(W \cap L) - d - 1$, proving the lemma for $m = 2$.

Now assume $m \geq 3$ and that the result is true for $\mathbb{P}^{m-1}$.
Take a general hyperplane $H \subset \mathbb{P}^m$ containing L and set $W' := W \cap L$. The inductive assumption gives $h^1(H, \mathcal{I}_{W'}(d)) = \deg(W' \cap L) - d - 1$. Since $\deg(\mathrm{Res}_H(W)) \leq d - 1$, as above we get $h^1(\mathbb{P}^m, \mathcal{I}_{\mathrm{Res}_H(W)}(d-1)) = 0$. Consider now the analogue exact sequence to (3) with H instead of L:

$$0 \to \mathcal{I}_{\mathrm{Res}_H(W)}(d-1) \to \mathcal{I}_W(d) \to \mathcal{I}_{W\cap H,H}(d) \to 0.$$

Then, since $W \cap L = W' \cap L$, we get, as above, that $h^1(\mathbb{P}^m, \mathcal{I}_W(d)) = \deg(W \cap L) - d - 1$. $\square$

2. The proofs

In this section we prove Theorems 1 and Theorem 2.

Proof of Theorem 1. The existence of the scheme $\mathcal{Z} \subset X$ is assured by Remark 1. Let S (resp. Z) be the only subset (resp. subscheme) of $\mathbb{P}^m$ such that $\mathcal{S} = \nu_d(S)$ (resp. $\mathcal{Z} = \nu_d(Z)$). We have $\sharp(S) = \mathrm{sr}(P)$ and $\deg(Z) = \mathrm{sbr}(P)$. Set $W := S \cup Z$ and $\mathcal{W} := \nu_d(W)$. We have $\deg(W) = \mathrm{sr}(P) + \mathrm{sbr}(P) \leq 2d+1$. Lemma 1 gives $h^1(\mathcal{I}_{\mathcal{W}}(1)) > 0$. Thus $\dim(\langle \mathcal{W} \rangle) \leq \deg(\mathcal{W}) - 2$. Since $\deg(\mathcal{W}) \leq \deg(\mathcal{Z}) + \deg(\mathcal{S}) = \mathrm{sbr}(P) + \mathrm{sr}(P) \leq 2d+1$, then, by Lemma 2 in [2], there is a line $L \subset \mathbb{P}^m$ whose image $C_d := \nu_d(L)$ in X contains a subscheme of $\mathcal{W}$ with length at least $d+2$. Since $C_d = \langle C_d \rangle \cap X$ (scheme-theoretic intersection), we have $\mathcal{W} \cap C_d = \nu_d(W \cap L)$, $\mathcal{Z} \cap C_d = \nu_d(Z \cap L)$ and $\mathcal{S} \cap C_d = \nu_d(S \cap L)$.

(a) Let $\mathcal{S}_1, \mathcal{S}_2 \subset \mathcal{S}$ be defined in the statement and set $\mathcal{S}_3 := \mathcal{S} \setminus (\mathcal{S}_1 \cup \mathcal{S}_2)$. Let $S_3 \subset \mathbb{P}^m$ be the only subset such that $\mathcal{S}_3 = \nu_d(S_3)$. Set $W' := W \setminus S_3$ and $\mathcal{W}' := \nu_d(W') = \mathcal{W} \setminus \mathcal{S}_3$. Notice that W' is well-defined, because each point of S_3 is a connected component of the scheme W. In this step we prove $S_3 = \emptyset$, i.e. $\mathcal{S}_3 = \emptyset$.
Assume that this is not the case and that $\sharp(\mathcal{S}_3) > 0$. Lemma 2 gives that $h^1(\mathbb{P}^N, \mathcal{I}_{W \cap L}(1)) = h^1(\mathbb{P}^N, \mathcal{I}_{\mathcal{W}}(1))$ and that $h^0(\mathcal{I}_{\mathcal{W}}(1)) = h^0(\mathcal{I}_{C_d \cap \mathcal{W}}(1)) - \deg(\mathcal{W}) + \deg(\mathcal{W} \cap C_d)$. Hence we get

$$\dim(\langle \mathcal{W} \rangle) = \dim(\langle \mathcal{W}' \rangle) + \sharp(\mathcal{S}_3).$$

Now, by definition, we have that $\mathcal{S} \cap \mathcal{W}' = \mathcal{S}_1 \cup \mathcal{S}_2$, $\mathcal{W} = \mathcal{W}' \sqcup \mathcal{S}_3$ and $\mathcal{Z} \cup \mathcal{S}_1 \cup \mathcal{S}_2 = \mathcal{W}'$. Grassmann's formula gives $\dim(\langle \mathcal{W}' \rangle \cap \langle \mathcal{S} \rangle) = \dim(\langle \mathcal{W}' \rangle) + \dim(\langle \mathcal{S} \rangle) - \dim(\langle \mathcal{W}' \cup \mathcal{S} \rangle) = \dim(\langle \mathcal{S} \rangle) - \sharp(\mathcal{S}_3)$. Since $\mathcal{Z} \subseteq \mathcal{W}'$, we have $P \in \langle \mathcal{W}' \rangle \cap \langle \mathcal{S} \rangle$ that has dimension $\dim(\mathcal{S}) - \sharp(\mathcal{S}_3)$ as just proved. Since $\mathcal{S}$ is linearly independent, we have $\dim(\langle \mathcal{S}_1 \cup \mathcal{S}_2 \rangle) = \dim(\langle \mathcal{S} \rangle) - \sharp(\mathcal{S}_3)$. Hence $\dim(\langle \mathcal{S}_1 \cup \mathcal{S}_2 \rangle) = \dim(\langle \mathcal{W}' \rangle \cap \langle \mathcal{S} \rangle)$; since $\langle \mathcal{S}_1 \cup \mathcal{S}_2 \rangle \subset \langle \mathcal{W}' \rangle \cap \langle \mathcal{S} \rangle$ we get that $\langle \mathcal{S}_1 \cup \mathcal{S}_2 \rangle = \langle \mathcal{W}' \rangle \cap \langle \mathcal{S} \rangle$. Since $P \in \langle \mathcal{Z} \rangle \cap \langle \mathcal{S} \rangle \subset \langle \mathcal{W}' \rangle \cap \langle \mathcal{S} \rangle = \langle \mathcal{S}_1 \cup \mathcal{S}_2 \rangle$, we get that $P \in \langle \mathcal{S}_1 \cup \mathcal{S}_2 \rangle$. Since we supposed that $\mathcal{S} \subset X$ is a set computing the symmetric rank of P, it is absurd that P belongs to the span of a proper subset of $\mathcal{S}$, then necessarily $\sharp(\mathcal{S}_3) = 0$, that is equivalent to the fact that $\mathcal{S}_3 = \emptyset$. Thus in this step we proved $\mathcal{S} = \mathcal{S}_1 \sqcup \mathcal{S}_2$.
In steps (b), (c) and (d) we will prove $\mathcal{Z} = (\mathcal{Z} \cap C_d) \sqcup \mathcal{S}_2$ in a very similar way (using Z instead of S). In each of these steps we take a subscheme $W_2 \subset W$ such that $S \subset W_2$, $W_2 \cap L = W \cap L$ and $W_2 \cup Z = W$. Then we play with Lemma 2. In steps (b) (resp. (c), resp. (d)) we call $W_2 = W''$ (resp. $W_2 = W_Q$, resp. $W_2 = W_1$).

(b) Let Z_4 be the union of the connected components of Z which does not intersect $L \cup S_2$. Here we prove $Z_4 = \emptyset$. Set $W'' := W \setminus Z_4$. The scheme W'' is well-defined, because Z_4 is a union of some of the connected components of W. Lemma 2 give $\dim(\nu_d(W)\rangle) = \dim(\nu_d(W''))\rangle) + \deg(Z_4)$. Since $W = W'' \cup Z$, Grassmann's formula gives $\dim(\langle \nu_d(W'' \cup Z) \rangle) = \dim(\langle \nu_d(W'') \rangle) + \dim(\langle \nu_d(Z) \rangle) - \dim(\langle \nu_d(W'') \rangle \cap \langle \nu_d(Z) \rangle)$. Thus $\dim(\langle \nu_d(Z) \rangle) = \dim(\langle \nu_d(W'') \rangle \cap \langle \nu_d(Z) \rangle) + \deg(Z_4)$. Since $\nu_d(Z)$ is linearly independent and $Z = (Z \cap W'') \sqcup Z_4$, we get $\dim(\langle \nu_d(Z) \rangle) = \dim(\langle \nu_d(Z \cap W'') \rangle) + \deg(Z_4)$. Thus $\dim(\langle \nu_d(W'') \rangle \cap \langle \nu_d(Z) \rangle) = \dim(\langle \nu_d(Z \cap W'') \rangle$. Since $\nu_d(W'' \cap Z) \subseteq \langle \nu_d(W'') \cap \nu_d(Z) \rangle$, $\deg(\langle \nu_d(W'') \rangle \cap \langle \nu_d(Z) \rangle) = \dim(\langle \nu_d(W'' \cap Z) \rangle) + 1$, and $\nu_d(W'')$ is linearly independent, then the linear space $\langle \nu_d(W'') \rangle \cap \langle \nu_d(Z) \rangle$ is spanned by $\nu_d(W'' \cap Z)$. Since $S \subseteq W''$ and $P \in \langle \nu_d(Z) \rangle \cap \langle \nu_d(S) \rangle$, we have $P \in \langle \nu_d(W'' \cap Z) \rangle$. Since Z computes the border rank of P, we get $W'' \cap Z = Z$, i.e. $Z_4 = \emptyset$.

(c) Here we prove that each point of S_2 is a connected component of Z. Fix $Q \in S_2$ and call Z_Q the connected component of Z such that $(Z_Q)_{red} = \{Q\}$. Set $Z[Q] := (Z \setminus Z_Q) \cup \{Q\}$ and $W_Q := (W \setminus Z_Q) \cup \{Q\}$. Since Z_Q is a connected component of W, the schemes $Z[Q]$ and W_Q are well-defined. Assume $Z_Q \neq \{Q\}$, i.e. $W_Q \neq W$, i.e. $Z[Q] \neq Z$. Since $W_Q \cap L = W \cap L$, Lemma 2 gives $\dim(\langle \nu_d(W)\rangle) - \dim(\langle \nu_d(W_Q)\rangle) = \deg(Z_Q) - 1 > 0$. Since $\nu_d(Z)$ is linearly independent, we have $\dim(\langle \nu_d(Z)\rangle) = \dim(\langle \nu_d(Z[Q]\rangle) + \deg(Z_Q) - 1$. The Grassmann's formula gives $\dim(\langle \nu_d(Z[Q])\rangle) = \dim(\langle \nu_d(W_Q)\rangle \cap \langle \nu_d(Z)\rangle)$. Since $\langle \nu_d(Z[Q])\rangle \subseteq \langle \nu_d(W_Q)\rangle \cap \langle \nu_d(Z)\rangle$ and $Z[Q]$ is linearly independent, we get $\langle \nu_d(Z[Q])\rangle = \langle \nu_d(W_Q)\rangle \cap \langle \nu_d(Z)\rangle$. Since $Q \in S_2 \subseteq S$, we have $S \subset W_Q$. Thus $P \in \langle W_Q\rangle$. Thus $P \in \langle \nu_d(Z)\rangle \cap \langle \nu_d(W_Q)\rangle = \langle \nu_d(Z[Q])\rangle$. Since $\mathcal{Z}$ computes the border rank of P, $Z[Q] \subseteq Z$ and $P \in \langle \nu_d(Z[Q])\rangle$, we get $Z[Q] = Z$. Thus each point of $\mathcal{S}_2$ is a connected component of $\mathcal{Z}$.

(d) To conclude that $Z = (Z \cap L) \sqcup S_2$ it is sufficient to prove that every connected component of Z whose support is a point of L is contained in L. Set $\eta := \deg(Z \cap L)$ and call μ the sum of the degrees of the connected components of Z whose support in contained in L.
Set $W_1 := (W \cap L) \cup S_2$. Notice that $\deg(W_1) = \deg(W) + \eta - \mu$. Lemma 2 gives $\dim(\langle \nu_d(W_1)\rangle) = \dim(\langle \nu_d(W)\rangle) + \eta - \mu$. Since $W = W_1 \cup Z$, Grassmann's formula gives $\dim(\langle \nu_d(W_1 \cup Z)\rangle) = \dim(\langle \nu_d(W_1)\rangle) + \dim(\langle \nu_d(Z)\rangle) - \dim(\langle \nu_d(W_1)\rangle \cap \langle \nu_d(Z)\rangle)$. Thus $\dim(\langle \nu_d(Z)\rangle) = \dim(\langle \nu_d(W_1)\rangle \cap \langle \nu_d(Z)\rangle) + \mu - \eta$. Notice that $Z \cap W_1 = (Z \cap L) \sqcup S_2$, i.e. $\deg(Z \cap W_1) = \deg(Z) - \eta + \mu$. Since $\nu_d(Z)$ is linearly independent, we get $\dim(\langle \nu_d(Z)\rangle) = \dim(\langle \nu_d(Z \cap W_1)\rangle) + \mu - \eta$. Thus $\dim(\langle \nu_d(W_1)\rangle \cap \langle \nu_d(Z)\rangle) = \dim(\langle \nu_d(Z \cap W_1)\rangle)$, i.e. $\langle \nu_d(W_1)\rangle \cap \langle \nu_d(Z)\rangle$ is spanned by $\nu_d(W_1 \cap Z)$. Since $S \subset W_1$ and $P \in \langle \nu_d(Z)\rangle \cap \langle \nu_d(S)\rangle$, we have $P \in \langle \nu_d(W_1 \cap Z)\rangle$. Since Z computes the symmetric border rank of P, we get $W_1 \cap Z = Z$, i.e. $\eta = \mu$. Together with steps (b) and (c) we get $Z = (Z \cap L) \sqcup S_2$. Thus from steps (b), (c) and (d) we get $\mathcal{Z} = (\mathcal{Z} \cap C_d) \sqcup \mathcal{S}_2$.

(e) Here we prove that there exists an unique rational normal curve C_d that intersects $\mathcal{S} \cup \mathcal{Z}$ in degree at least $d+2$. Notice that L and $C_d = \nu_d(L)$ are uniquely determined by the choice of a pair (Z, S) with $\nu_d(Z)$ computing $\mathrm{sbr}(P)$ and $\nu_d(S)$ computing $\mathrm{sr}(P)$. Fix another pair (Z', S') with $\nu_d(Z)$ computing $\mathrm{sbr}(P)$ and $\nu_d(S)$ computing $\mathrm{sr}(P)$. Let L' be the line associated to $Z' \cup S'$. Assume $L' \neq L$. First assume $S' = S$. The part of Theorem 1 proved before gives $Z = Z_1 \sqcup S_2$, $Z' = Z'_1 \sqcup S'_2$ and $S = S'_1 \sqcup S'_2$ with $Z_1 = Z \cap L$, $Z'_1 = Z' \cap L'$, $S_1 = S \cap L$ and $S'_1 = S_1 \cap L'$. Now $\mathrm{sbr}(P) = \deg(Z_1) + \sharp(S_2) = \deg(Z'_1) + \sharp(S'_2)$, $\mathrm{sr}(P) = \deg(S_1) + \sharp(S_2) = \deg(S'_1) + \sharp(S'_2)$, $\deg(S_1) > \deg(Z_1)$, $\deg(S_1) + \deg(Z_1) \geq d+2$ and $\deg(S'_1) + \deg(Z'_1) \geq d+2$. Since $L' \neq L$, at most one of the points of S_1 may be contained in L' and at most one of the points of S'_1 may be contained in L. Thus $\deg(S'_1) - 1 \leq \sharp(S_2)$ and $\deg(S_1) - 1 \leq \sharp(S'_2)$. Since $\deg(S_1) + \deg(Z_1) + 2(\sharp(S_2)) = \deg(S'_1) + \deg(Z'_1) + 2(\sharp(S'_2)) \leq 2d+1$, $\deg(S_1) + \deg(Z_1) \geq d+2$ and $\deg(S'_1) + \deg(Z'_1) \geq d+2$, we get $2(\sharp(S_2)) \leq d-1$ and $2(\sharp(S'_2)) \leq d-1$. Since $\deg(S_1) + \deg(Z_1) \geq d+2$ and $\deg(S_1) > \deg(Z_1)$, we have $\deg(S_1) \geq (d+3)/2$. Hence $\deg(S_1) - 1 \geq (d+1)/2 > (d-1)/2 \geq \sharp(S'_2)$, contradiction. Thus all pairs (Z', S) give the same line L. Now assume $S' \neq S$. Call L'' the line associated to the pair (Z, S'). The part of Theorem 1 proved in the previous steps gives that L is the only line containing an unreduced connected component of Z. Thus $L'' = L$. Since we proved that the lines associated to (Z', S') and (Z, S') are the same, we are done.

(f) Here we prove the uniqueness of $\mathcal{S}_2$. Take any pair (Z', S') with $\nu_d(Z)$ computing $\mathrm{sbr}(P)$ and $\nu_d(S)$ computing $\mathrm{sr}(P)$. By step (e) the same line L is associated to any pair (Z'', S'') as above. Hence the set $S''_2 := S'' \setminus (S'' \cap L)$ associated to the pair (Z, S') is the union of the connected components of Z not contained in L. Thus $S''_2 = S \setminus S \cap L = S_2$. We apply the part of Theorem 1 proved in steps (a), (b), (c) and (d) to the pair (Z', S). We get that $S \setminus S \cap L$ is the union of the connected components of Z' not contained in L. Applying the same part of Theorem 1 to the pair (Z', S') we get $S' \backslash S' \cap L = S \backslash S \cap L$, concluding the proof of the uniqueness of $\mathcal{S}_2$. $\square$

The following example shows that the assumption " $\mathrm{sbr}(P) + \mathrm{sr}(P) \le 2d + 1$ " in Theorem 1 is sharp.

Example 1. Fix integers $m \ge 2$ and $d \ge 4$. Let $C \subset \mathbb{P}^m$ be a smooth conic. Let $Z \subset C$ be any unreduced degree 3 subscheme. Set $\mathcal{Z} := \nu_d(Z)$. Since $d \ge 2$, $\mathcal{Z}$ is linearly independent. Since $\mathcal{Z}$ is curvilinear, it has only finitely many degree 2 subschemes. Thus the plane $\langle \mathcal{Z} \rangle$ contains only finitely many lines spanned by a degree 2 subscheme of $\mathcal{Z}$. Fix any $P \in \langle \mathcal{Z} \rangle$ not contained in one of these lines. Remark 1 gives $\mathrm{sbr}(P) = 3$. The proof of [2], Theorem 4, gives $\mathrm{sr}(P) = 2d - 1$ and the existence of a set $S \subset C$ such that $\sharp(S) = 2d - 1$, $S \cap Z = \emptyset$ and $\nu_d(S)$ computes $\mathrm{sr}(P)$. We have $\mathrm{sbr}(P) + \mathrm{sr}(P) = 2d + 2$.

Lemma 3. *Fix $P \in \mathbb{P}^N$ such that $\rho := \mathrm{sbr}(P) = \mathrm{sr}(P) \le d$. Let Ψ be the set of all zero-dimensional schemes $A \subset X$ such that $\deg(A) = \rho$ and $P \in \langle \nu_d(A) \rangle$. Assume $\sharp(\Psi) \ge 2$. Fix any $A \in \Psi$. Then there is a line $L \subset \mathbb{P}^m$ such that $\deg(L \cap A) \ge (d+2)/2$.*

Proof. Since $\mathrm{sr}(P) = \rho$ and $\sharp(\Psi) \ge 2$, there is $B \in \Psi$ such that $B \ne A$ and at least one among the schemes A and B is reduced. Since $\deg(A \cup B) \le 2d + 1$ and $h^1(\mathcal{I}_{A \cup B}(1)) > 0$ there is a line $L \subset \mathbb{P}^m$ such that $\deg((A \cup B) \cap L) \ge d + 2$. We may repeat verbatim the proof of Theorem 1, because it does not use the inequality $\deg(A) < \deg(B)$ We get $A = A_1 \sqcup A_2$ and $B = B_1 \sqcup A_2$ with A_2 reduced, $A_2 \cap L = \emptyset$ and $A_1 \cup B_1 \subset L$. Since $\deg(A) = \deg(B)$, we have $\deg(A_1) = \deg(B_1)$. Thus $\deg(A_1) \ge (d+2)/2$. $\square$

Proof of Theorem 2. Since $\mathrm{sbr}(P) \le \rho \le d$, the border rank is the minimal degree of a zero-dimensional scheme $W \subset X_{m,d}$ such that $P \in \langle W \rangle$ (Remark 1). Thus it is sufficient to prove the last assertion. Assume the existence of a zero-dimensional scheme $\mathcal{Z} \subset X_{m,d}$ such that $z := \deg(\mathcal{Z}) \le \rho$ and $P \in \langle \mathcal{Z} \rangle$. If $z = \rho$ we also assume $\mathcal{Z} \ne \nu_d(B)$. Taking z minimal, we may also assume $z = \mathrm{sbr}(P)$. Let $Z \subset \mathbb{P}^m$ be the only scheme such that $\nu_d(Z) = \mathcal{Z}$. If $z < \rho$ we apply a small part of the proof Theorem 1 to the pair $(\mathcal{Z}, \nu_d(B))$ (we just use or reprove that $\deg((Z \cup B) \cap L) \ge d + 2$ and that $\deg(B \cap L) = \deg(Z \cap L) + \rho - z \ge \deg(Z \cap L))$. We get a contradiction: indeed $B \cap L$ must have degree $\ge (d+1)/2$, contradiction. If $z = \rho$, then we use Lemma 3. $\square$

Example 2. Assume $m = 2$ and $d \ge 4$. Let $D \subset \mathbb{P}^2$ be a smooth conic. Fix sets $S, S' \subset D$ such that $\sharp(S) = \sharp(S') = d+1$ and $S \cap S' = \emptyset$. Since no 3 points of D are collinear, the sets S, S' and $S \cup S'$ are in linearly general position. Since $h^0(D, \mathcal{O}_D(d)) = 2d+1$ and D is projectively normal, we have $h^1(\mathcal{I}_S(d)) = h^1(\mathcal{I}_{S'}(d)) = 0$ and $h^1(\mathcal{I}_{S \cup S'}(d)) = 1$. Thus $\nu_d(S)$ and $\nu_d(S')$ are linearly independent and $\langle \nu_d(S) \rangle \cap \langle \nu_d(S') \rangle$ is a unique point. Call P this point. Obviously $\mathrm{sr}(P) \le d+1$. To get the example claimed in the introduction after the statement of Theorem 2, it is sufficient to prove that $\mathrm{sbr}(P) \ge d+1$. Assume $\mathrm{sbr}(P) \le d$ and take Z computing $\mathrm{sbr}(P)$. We may apply a small part of the proof of Theorem 1 to P, S, Z (even if a priori S may not compute $\mathrm{sr}(P)$). We get the existence of a line L such that $\deg(Z \cap L) < \sharp(S \cap L)$ and $\deg(Z \cap L) + \sharp(S \cap L) \ge d+2$. Since $d \ge 4$, we get $\sharp(S \cap L) \ge 3$, contradiction.

We do not have experimental evidence to raise the following question (see [2] for the cases with $\mathrm{sbr}(P) \le 3$).

Question 1. Is it true that $\mathrm{sr}(P) \le d(\mathrm{sbr}(P) - 1)$ for all $P \in \mathbb{P}^N$ and that equality holds if and only if $P \in TX_{m,d} \setminus X_{m,d}$ where $TX_{m,d} \subset \mathbb{P}^N$ is the tangential variety of the Veronese variety $X_{m,d}$?

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