Università degli Studi di Milano

FACOLTÀ DI SCIENZE MATEMATICHE, FISICHE E NATURALI
DIPARTIMENTO DI MATEMATICA
CORSO DI DOTTORATO DI RICERCA IN MATEMATICA
XVII CICLO

Varieties parameterizing forms and their secant varieties.

MAT/03

Tesi di Dottorato di:

Alessandra Bernardi

Relatore:

Chiar.mo Prof. **Alessandro Gimigliano**

Dipartimento di Matematica
Università degli Studi di Bologna

Coordinatore del Dottorato:

Chiar.mo Prof. **Antonio Lanteri**

Dipartimento di Matematica
Università degli Studi di Milano

Anno Accademico 2004/2005

Al mio nonno

To my grand-dad

Contents

Introduction

The problems studied in this thesis originate from classical problems in algebraic geometry (and commutative algebra). Actually, the story begins with a number theory question: in 1770 E. Waring in [**War**] stated (without proofs) that:

> "Every natural number is sum of at most 9 positive cubes."
> "Every natural number is sum of at most 19 biquadratics."

Moreover, he believed that:

> "for all integers $d \geq 2$ there exists a positive integer $g(d)$ such that each $n \in \mathbb{Z}^+$ can be written as $n = a_1^d + \cdots + a_{g(d)}^d$ with $a_i \geq 0$, $i = 1, \ldots, g(d)$."

Waring belief was showed to be true by Hilbert in 1909.

An analogous problem can be formulated for homogeneous polynomials of given degree d in $S := K[x_0, \ldots, x_n]$ where K is an algebraically closed field of characteristic zero:

> "Which is the least $g(d) \in \mathbb{Z}^+$ such that each degree d homogeneous polynomial f in S is the sum of at most $g(d)$ d-th power of linear forms?"

This problem is classically known as the "Little Waring problem".

We always indicate with S_d the degree d part of S.

Our work starts form the problem that is known as the "Big Waring problem":

> "Which is the least $G(d) \in \mathbb{Z}^+$ such that the *generic* form $f \in S_d$ is sum of at most $G(d)$ d-th powers of linear forms?"

This problem is completely solved by J. Alexander, A. Hirschowitz in [**AH**]. In order to see how the result in [**AH**] enter the problem, consider $S = K[x_1, \ldots, x_n]$ and $R = K[y_1, \ldots, y_n]$, two polynomial rings, and the action of R on S given by interpreting the y_j's as $\left(\frac{\partial}{\partial x_j}\right)$, the partial derivatives of the x_i's; this action is called "Apolarity".

If I is an homogeneous ideal of R, the "Inverse System" I^{-1} of I is the R-submodule of S containing all the elements of S annihilated by I (by the apolarity action).

When $X = \mathrm{Proj}(S/I(X))$ is a projective scheme then the Hilbert function $H(X,d)$ of X in degree d is $\dim(((I(X))^{-1})_d)$. So we can switch from the study of the Hilbert function of a scheme to the study of the inverse system of its ideal and vice-versa.

Via inverse systems, one can check (see [**Ge**]) that the least $G(d) \in \mathbb{Z}^+$ solving the Big Waring problem is also the minimum $G(d)$ such that the Hilbert function in degree d of the union of the first infinitesimal neighborhoods of $G(d)$ generic points in $\mathbb{P}^n$ is maximal, i.e. $G(d)\binom{n+d}{d}$. In [**AH**] the Hilbert function of these kind of schemes has been computed.

A connection with another classical problem in algebraic geometry is given by a secant varieties question.
Let $X \subset \mathbb{P}^N$ be a projective variety of dimension n; the "$(s-1)$-Secant variety of X" is

$$\mathrm{Sec}_{s-1}(X) := \overline{\bigcup_{P_1,\dots,P_s \in X} < P_1, \dots, P_s >}.$$

We recall that the "Veronese variety" of degree d and dimension n can be viewed as the image of the embedding

$$\begin{array}{rcl} \nu_d : \mathbb{P}(S_1) = (\mathbb{P}^n)^* & \hookrightarrow & \mathbb{P}(S_d) = \left(\mathbb{P}^{\binom{n+d}{d}-1}\right)^* \\ [L] & \overset{\nu_d}{\mapsto} & [L^d] \end{array},$$

hence it parameterizes the set of d-th powers of linear forms.

It is not difficult to prove that the $(s-1)$-secant variety to $\nu_d(\mathbb{P}^n)$ parameterizes the closure of the set of forms which can be written as sums of s d-th powers of linear forms. Therefore solving the Big Waring problem is equivalent to finding the least integer $G(d)$ such that $\mathrm{Sec}_{G(d)-1}(\nu_d(\mathbb{P}^n)) = \mathbb{P}^{\binom{n+d}{d}-1}$.

The study of the dimension of the secant variety of a projective variety X is actually a classical problem. In fact if $X \subset \mathbb{P}^N$ is a reduced irreducible variety of dimension n, there exists an expected dimension for $\mathrm{Sec}_{s-1}(X)$, i.e. $min\,\{ns+s-1, N\}$. When $\dim(\mathrm{Sec}_{s-1}(X)) < \mathrm{expdim}(\mathrm{Sec}_{s-1}(X))$ one says that $\mathrm{Sec}_{s-1}(X)$ is *"defective"* with defect $\delta_s(X) = \mathrm{expdim}(\mathrm{Sec}_{s-1}(X)) - \dim(\mathrm{Sec}_{s-1}(X))$.

In the thesis we will consider also a more general problem with respect to the Big Waring one:

> "Which is the least integer $G(d)$ such that the generic element of S_d can be written as $N_1 + \cdots + N_{G(d)}$ where each $N_i = M^{(i)}_{1,j(1)} \cdots M^{(i)}_{k,j(k)}$ and $M^{(i)}_{1,j(1)}, \dots, M^{(i)}_{k,j(k)}$ belong to $S_{j(1)}, \dots, S_{j(k)}$ respectively?"

The geometric translation of this algebraic problem is the following:

"Let ϕ be the map defined as follows:

$$\begin{array}{rcl} \phi : \mathbb{P}(S_{j(1)}) \times \cdots \times \mathbb{P}(S_{j(k)}) & \to & \mathbb{P}(S_d) \\ ([M_{1,j(1)}], \ldots, [M_{k,j(k)}]) & \mapsto & [M_{1,j(1)} \cdots M_{k,j(k)}] \end{array}$$

where $\sum_{l=1}^{k} j(l) = d$. We define now a variety X as the closure of the image of this map. The integer $G(d)$ we are looking for is the least integer $G(d)$ such that $\dim(\mathrm{Sec}_{G(d)-1}(X)) = \binom{n+d}{d} - 1$."

When the generic element of S_d can be written as $F = N_1 + \cdots + N_s$, where $N_1, \ldots, N_s$ are some specific kind of forms (e.g. powers of linear forms in the Waring problem), it is used to say that F is a "Canonical Form". In this thesis we are interested in developing three types of canonical forms and, more generally, in determining the dimension of the secant varieties associated to them. The third case is a generalization of this kind of problems to tensors (forms can be viewed as symmetric tensors).

1. Let $L_1, \ldots, L_s$ be linear forms of S and $F_1, \ldots, F_s \in S_k$,
$$F = L_1^{d-k} F_1 + \cdots + L_s^{d-k} F_s. \tag{1}$$

2. Let $L_i^{(j)}$ be linear forms of S for $i = 1, \ldots, d$ and $j = 1, \ldots, s$,
$$F = L_1^{(1)} \cdots L_d^{(1)} + \cdots + L_1^{(s)} \cdots L_d^{(s)}. \tag{2}$$

3. Let $V_1, \ldots, V_t$ vector spaces on K, a tensor $T \in V_1^* \otimes \cdots \otimes V_t^*$ is said to be "decomposable" if there exist vectors $v_i^* \in V_i^*$ such that $T = v_1^* \otimes \cdots \otimes v_t^*$. "Which is the minimum integer s such that the generic tensor T of $V_1^* \otimes \cdots \otimes V_t^*$ is sum of s decomposable tensors?"

First problem. Let $X \subset \mathbb{P}^r$ be a projective variety of dimension n. Let $O_{k,X,P}$ be the k-th osculating space to X at $P \in X$. Let $X_0 \subset X$ be the dense set of smooth points where $O_{k,X,P}$ has maximal dimension. The "k-th osculating variety to X" is defined as

$$O_{k,X} := \overline{\bigcup_{P \in X_0} O_{k,X,P}}.$$

We prove that the geometric problem associated to the canonicity of the form (1) is equivalent to the fact that $\mathrm{Sec}_{s-1}(O_{k,\nu_d(\mathbb{P}^n)})$ fills $\mathbb{P}^r$.

The first important result we use is Terracini's Lemma (see Lemma 2.6.1).

The method we use for the study of $\dim(\mathrm{Sec}_{s-1}(O_{k,\nu_d(\mathbb{P}^n)}))$ is the following:

1. let $W_i := T_{P_i}(O_{k,\nu_d(\mathbb{P}^n)})$ and $W =< W_1, \dots, W_s >$; try to compute directly $\dim(W)$; if this is not possible, then
2. compute the degree d part I_d of the inverse system I of $W \subset S_d$ and the degree d part $I_d^{(i)}$ of the inverse system $I^{(i)}$ of W_i. Now if $Z = \mathrm{Proj}(R/I)$ and $Z^{(i)} = \mathrm{Proj}(R/I^{(i)})$, so that $Z = Z^{(1)} \cup \dots \cup Z^{(s)}$, we can prove that the schemes $Z^{(i)}$ depend only on n, k and not on d and that the $Z^{(i)}$ are 0-dimensional projective schemes of length $\binom{k+n}{n} + n$, such that if $\wp_i$ are the ideals of points P_i which are the support of $Z^{(i)}$, then $\wp_i^{k+1} \supset I^{(i)} \supset \wp_i^{k+2}$;
3. compute, if it is possible, the Hilbert function of Z, then $\dim(\mathrm{Sec}_{s-1}(O_{k,\nu_d(\mathbb{P}^n)})) = H(Z, d) - 1$.
4. When it is not possible to compute directly $H(Z, d)$, one could use the following construction: consider

$$X_i = \mathrm{Proj}(S/\wp_i^{k+1}),\ Y_i = \mathrm{Proj}(S/\wp_i^{k+2}),\ X = X_1 \cup \dots \cup X_s,\ Y = Y_1 \cup \dots \cup Y_s, \quad (3)$$

   then $X \subset Z \subset Y$.

   We prove a crucial lemma (see Lemma 3.3.6) that allows us to move in many cases the problem from the study of $\dim(\mathrm{Sec}_{s-1}(O_{k,\nu_d(\mathbb{P}^n)}))$ to the study of X and Y. This is very interesting not only because it makes the defectiveness or the regularity of $\mathrm{Sec}_{s-1}(O_{k,\nu_d(\mathbb{P}^n)})$ be dependent on the regularity of the Hilbert functions of X and Y, but also because we didn't find any example where $\mathrm{Sec}_{s-1}(O_{k,\nu_d(\mathbb{P}^n)})$ is defective but X and Y are regular. We conjecture that this fact never happens.

In the case of $\mathbb{P}^2$, we are able to prove our conjecture for small values of s. Let X, Z be projective schemes defined as in (3), $n = 2$ and $3 \le s \le 9$; then: $H(Z, d) = \min\left\{H(X, d) + 2s, \binom{d+2}{2}\right\}$. The proof mainly uses "La méthode d'Horace" (e.g. see [**Hi**]) on a scheme Z' which is a specialization of Z. It consists in considering a curve $\mathcal{C}$ through $P_1, \dots, P_t$ with $t \le s$, and in studying the residual scheme $Res_{\mathcal{C}}(Z')$ whose representative ideal is $(I(Z') : I(\mathcal{C}))$; then, if $\mathcal{C}$ is a fixed component of multiplicity ν, $H(Z', d) = H(Res_{\mathcal{C}}(Z'), d - t\nu)$.

Studying this problem we find many varieties which are "very defective" (i.e. $\delta_s >> 0$), e.g. the secant varieties of $O_{4,\nu_5(\mathbb{P}^6)} \subset \mathbb{P}^{461}$. When $s = 2$ we have that $\mathrm{expdim}(\mathrm{Sec}_1(O_{4,\nu_5(\mathbb{P}^6)})) = 431$ but we get that the defect is $\delta_2 = 86$. When $s = 3, 4$ the defects are $\delta_3 = 44$ and $\delta_4 = 9$. Eventually, $\mathrm{Sec}_4(O_{4,\nu_5(\mathbb{P}^6)}) = \mathbb{P}^{461}$. So, even if we expect that $\mathrm{Sec}_2(O_{4,\nu_5(\mathbb{P}^6)})$ should fill up $\mathbb{P}^{461}$, even the 3-secant variety doesn't. In terms of forms we get that neither we can write a generic $f \in (K[x_0, \dots, x_6])_5$ as $f = L_1F_1 + L_2F_2 + L_3F_3$ with $L_i \in R_1$ and $F_i \in R_4$ (as we expect), nor as $f = L_1F_1 + \dots + L_4F_4$, but we need five addenda.

Second problem. We define the Split variety $\mathrm{Split}_d(\mathbb{P}^n)$ as the closure of the following map:

$$\begin{array}{rcl} \phi : \underbrace{\mathbb{P}(S_1) \times \cdots \times \mathbb{P}(S_1)}_{d} & \to & \mathbb{P}(S_d) \\ ([L_1], \ldots, [L_d]) & \mapsto & [L_1 \cdots L_d] \end{array}.$$

The problem of studying (2) is equivalent to studying the dimension of $\mathrm{Sec}_{s-1}(\mathrm{Split}_d(\mathbb{P}^n))$. Our first interest in this problem came from a conjecture stated in [**Eh**], for which we find a counterexample. Let $\mathbb{G}(k,n)$ be the Grassmannian of k-spaces in $\mathbb{P}^n$. Ehrenborg conjectures that the least positive integer s such that $\mathrm{Sec}_{s-1}(\mathbb{G}(n-1,n+d-1))$ fills up $\mathbb{P}^{\binom{n+d}{d}-1}$ is the same s such that $\mathrm{Sec}_{s-1}(\mathrm{Split}_d(\mathbb{P}^n)) = \mathbb{P}^{\binom{n+d}{d}-1}$. If this conjecture were true, we would be able to compute the dimension of $\mathrm{Sec}_{s-1}(\mathrm{Split}_d(\mathbb{P}^n))$ in many cases. It is a known result (see for example [**CGG3**]) that $\mathrm{Sec}_{3-1}(\mathbb{G}(3,6))$ is defective: one would expect that $\mathrm{Sec}_2(\mathbb{G}(3,6)) = \mathbb{P}^{34}$, but $\dim(\mathrm{Sec}_2(\mathbb{G}(3,6))) = 33$; only $\mathrm{Sec}_3(\mathbb{G}(3,6)) = \mathbb{P}^{34}$. Unfortunately this fact does not imply the same for $\mathrm{Sec}_{s-1}(\mathrm{Split}_4(\mathbb{P}^3))$: in fact $\mathrm{Sec}_2(\mathrm{Split}_4(\mathbb{P}^3)) = \mathbb{P}^{34}$. The only case where we are able to prove Ehremborg conjecture is when $d = 2$ for which $\mathrm{Split}_2(\mathbb{P}^n) = O_{1,\nu_2(\mathbb{P}^n)} = \mathrm{Sec}_1(\nu_2(\mathbb{P}^n))$ and many things are known (see [**CGG2**]).

For the study of $\mathrm{Sec}_{s-1}(\mathrm{Split}_d(\mathbb{P}^n))$ we follow two directions:

- First we prove, by using consequences of Terracini's Lemma (see Corollary 2.6.2 and Proposition 2.6.3), that if $d > 2$ and $n \geq 3(s-1)$, then

$$\dim(\mathrm{Sec}_{s-1}(\mathrm{Split}_d(\mathbb{P}^n))) = \mathrm{expdim}(\mathrm{Sec}_{s-1}(\mathrm{Split}_d(\mathbb{P}^n))).$$

- Second we study the intersection between $\mathbb{G}(n-1,n+d-1)$ and $\mathrm{Split}_d(\mathbb{P}^n)$.
  We can prove that $\mathbb{G}(n-1,n+1) \cap \mathrm{Split}_2(\mathbb{P}^n)$ is set-theoretically represented by the locus of the $(n-1)$-spaces of $\mathbb{P}^{n+1}$ that are $(n-1)$ secant to the rational normal curve $\nu_{n+1}(\mathbb{P}^1)$. We can partially generalize this result and prove that the locus $\{(n-1)-$ spaces that are $(n-1)-$secant to $\nu_{n+d-1}(\mathbb{P}^1)\}$ is contained in $\mathrm{Split}_d(\mathbb{P}^n) \cap \mathbb{G}(n-1,n+d-1)$. In the case of $d = 3$ we compute that also the reverse inclusion is true.

Third problem. The geometric problem associated at this last algebraic problem is the study of the dimension of the secant varieties to the Segre varieties. Let $\mathbb{P}^{n_i} = \mathbb{P}(V_i)$ for $i = 1, \ldots, k$, be the Segre variety which is defined as the image of the following map:

$$\begin{array}{rcl} \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k} & \to & \mathbb{P}^{(n_1+1)\cdots(n_k+1)-1} \\ ((x_0^{(1)}, \ldots, x_{n_1}^{(1)}), \ldots, (x_0^{(k)}, \ldots, x_{n_k}^{(k)})) & \mapsto & (\ldots, x_{i_1}^{(1)} \cdots x_{i_k}^{(k)}, \ldots) \end{array}.$$

The interest in the knowledge of the dimension of $\mathrm{Sec}_{s-1}(Seg(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}))$ is mostly motivated by its connections with questions in representation theory, coding theory, algebraic complexity theory and statistics (see [**BCS**]).

For this problem we do an exposition of results in [**CGG1**] and in [**LM1**].

In [**CGG1**] the authors solve some cases of this problem using Inverse Systems and "La méthode d'Horace".

Our exposition of [**LM1**] needs an introduction on Representation Theory; then we will present the firs part of that paper where the authors develop an algorithm to compute the decomposition of the degree d part of the ideal of $\mathrm{Sec}_{s-1}(Seg(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}))$. We conclude this section by enunciating the statements of the main result of [**LM1**] that is the proof of the Garcia, Stillman, Strumfeld conjecture (see [**GSS**]) on the generation of the ideal of the first secant variety to the first secant variety to the Segre variety in the case of three factors: $\mathrm{Sec}_1(Seg(\mathbb{P}^{n_1} \otimes \mathbb{P}^{n_2} \otimes \mathbb{P}^n_3))$.

I wish to thank firstly my advisor Prof. Alessandro Gimigliano for the many ideas and explanations he gave me and his good deal of patience. Secondly I want to thank all that gave me suggestions and ideas (and in some cases a place to sleep) for the work done in this thesis, in particular E. Arrondo, M.V. Catalisano, I. Dimitrov, A.V. Geramita, I. Huges, M. Idà, G. Kemper, J. M. Landsberg, A. Lanteri, M. Roth and B. Van Geemen. I would like also to thank all the people that, with their work, allowed me to participate to the many conferences and to spend various months of my PhD program abroad.

Chapter 1

Big Waring Problem

Let K be an algebraically closed field of characteristic zero. We will work on the projective space $\mathbb{P}^n = \mathbb{P}^n(K)$. The polynomial ring $S := K[x_0, \ldots, x_n]$ is a graduated ring and so we can write it as $K[x_0, \ldots, x_n] = \bigoplus_{d \geq 0} S_d$ where $S_d =< x_0^d, x_0^{d-1}x_1, \ldots, x_n^d >$ is the vector space of homogeneous forms of degree d. It is a well known fact that $\dim_K(S_d) = \begin{pmatrix} d+n \\ n \end{pmatrix}$. In a geometric language those vector spaces S_d are called *Complete Linear Systems of hypersurfaces of degree d in $\mathbb{P}^n$*. Sometimes we will write $\mathbb{P}(S_d)$ in order to mean the projectivization of S_d, therefore $\mathbb{P}(S_d)$ will be a $\mathbb{P}^{\binom{n+d}{d}-1}$ whose elements will be classes of forms of degree d: $[F] \in \mathbb{P}(S_d)$ with $F \in S_d$.

1.1 The Big Waring Problem

We want to introduce a number theory question presented by E. Waring in 1770 in [**War**]. He stated without any proof that "every positive integer is sum of at most 9 positive cubes", "every positive integer is sum of at most 19 fourth powers"... Waring believed that for every $d \in \mathbb{Z}^+$ there exists an integer $g(d)$ such that every $n \in \mathbb{N}$ may be written as

$$n = a_1^d + \cdots + a_{g(d)}^d.$$

In 1909 Hilbert proved that such a $g(d)$ exists for every $d \geq 2$ and he computed it.

An analogous problem can be formulated for homogeneous polynomials of S_d. It is the so called *Little Waring Problem*:

> "find the minimum $s \in \mathbb{Z}$ such that all forms $F \in S_d$ are sum of at most s d-th powers of linear forms."

The problem we are interested in is a slightly different form of the little Waring problem, it is called the *Big Waring Problem* and it is formulated as follows:

> "Which is the minimum $s \in \mathbb{Z}$ such that the *generic* form $F \in S_d$ is a sum of at most s d-th powers of linear forms?"
>
> $$F = L_1^d + \cdots + L_s^d$$

In order to know which elements of S_d can be written as sum of s d-th powers of linear forms, we study the image of the map

$$\phi : \underbrace{S_1 \times \cdots \times S_1}_{s} \longrightarrow S_d, \quad \phi(L_1, \ldots, L_s) = L_1^d + \cdots + L_s^d. \tag{1.1}$$

The Big Waring problem asks to find the smallest s such that $\overline{\mathrm{Im}(\phi_d)} = S_d$ (we just observe that if we require $\dim(\phi_d) = S_d$ we would solve the little Waring problem).

The map ϕ can be viewed as a polynomial map between affine spaces:

$$\phi : \mathbb{A}^{s(n+1)} \longrightarrow \mathbb{A}^{N=\binom{n+d}{n}}.$$

In order to know the dimension of the image of such a map we look at its differential

$$d\phi|_P : T_P(\mathbb{A}^{s(n+1)}) \longrightarrow \mathbb{A}^N.$$

Let $P = (L_1, \ldots, L_s) \in \mathbb{A}^{s(n+1)}$ and $v = (M_1, \ldots, M_s) \in T_P(\mathbb{A}^{s(n+1)}) \simeq \mathbb{A}^{s(n+1)}$ where $L_i, M_i \in S_1$ for $i = 1, \ldots, s$. Let us consider the following parameterizations $t \longmapsto (L_1 + M_1 t, L_2 + M_2 t, \ldots, L_s + M_s t)$ of a line $\mathcal{C}$ passing through P whose tangent vector at P is M. The image of $\mathcal{C}$ via ϕ is $\phi(L_1 + M_1 t, L_2 + M_2 t, \ldots, L_s + M_s t) = \sum_{i=1}^s (L_i + M_i t)^d$. The tangent vector to $\phi(\mathcal{C})$ in $\phi(P)$ is $\lim_{t \to 0} \frac{d}{dt} \left(\sum_{i=1}^s (L_i + M_i t)^d \right) = \lim_{t \to 0} \sum_{i=1}^s d(L_i + M_i t)^{d-1} M_i = \sum_{i=1}^s d L_i^{d-1} M_i$. Now, as $v = (M_1, \ldots, M_s)$ varies in $\mathbb{A}^{s(n+1)}$, the tangent vectors we get span $< L_1^{d-1} S_1, \ldots, L_s^{d-1} S_1 >$.

Hence we can say:

Proposition 1.1.1. *Let $L_1, \ldots, L_s$ be linear forms in $S = K[x_0, \ldots, x_n]$, where $L_i = a_{i_0} x_0 + \cdots + a_{i_n} x_n$ and*

$$\phi : \underbrace{S_1 \times \cdots \times S_1}_{s} \longrightarrow S_d, \quad \phi(L_1, \ldots, L_s) = L_1^d + \cdots + L_s^d;$$

then

$$\mathrm{rk}(d\phi)|_{(L_1, \ldots, L_s)} = \dim_K < L_1^{d-1} S_1, \ldots, L_s^{d-1} S_1 > .$$

It is very interesting to have a look at how the problem of determining this dimension has been solved, because the solution involves many algebraic and geometric tools.

1.2 Inverse Systems

1.2.1 Definition and observations

This section is an exposition of inverse systems techniques, and it follows [**Ge**].

Definition 1.2.1. *Let $S = K[x_1, \dots, x_n]$ and $R = K[y_1, \dots, y_n]$ be polynomial rings and consider the action of R on S (called Apolarity of R on S) defined as follows:*

$$y_i \circ x_j = \left(\frac{\partial}{\partial x_i}\right)(x_j) = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases} ;$$

i.e. we view the polynomials of R as "partial derivative operator" on S.

Now we can extend this action to the whole rings R, S by linearity and using properties of differentiation:

$$R_i \times S_j \longrightarrow S_{j-i}$$

$$r_i \times s_j := r_i \circ s_j$$

in particular

$$y^\alpha \circ x^\beta = \begin{cases} 0, & \text{if } \alpha \not\leq \beta; \\ \prod_{i=1}^n \frac{(b_i)!}{(b_i - a_i)!} x^{\beta - \alpha}, & \text{if } \alpha \leq \beta. \end{cases}$$

where $x^\beta := x_0^{b_1} \cdots x_n^{b_n}$ when $\beta = (b_1, \dots, b_n)$ and $b_i \geq 0$, and also $\alpha = (a_1, \dots, a_n) \leq \beta$ iff $a_i \leq b_i$ for all $i = 1, \dots, n$, that is equivalent to x^α divides x^β in S.

Remarks:

- The action of R on S makes S a (non finitely generated) R-module (but the converse is not true);
- the action of R on S lowers the degree;
- the apolarity action induces a non-singular K-bilinear pairing:

  $$R_j \times S_j \longrightarrow K \ \ \forall\, j = 0, 1, \dots$$

  that induces two bilinear maps; [1]

[1] Let $V \times W \longrightarrow K$ be a K-bilinear parity given by $v \times w \longrightarrow v \circ w$. It induces two K-bilinear maps:
$\phi : V \longrightarrow \mathrm{Hom}_K(W, K)$ such that $\phi(v) := \phi_v$ and $\phi_v(w) = v \circ w$ and $\chi : W \longrightarrow \mathrm{Hom}_K(V, K)$ such that $\chi(w) := \chi_w$ and $\chi_w(v) = v \circ w$.
$V \times W \longrightarrow K$ is not singular iff for all the bases $\{w_1, \dots, w_n\}$ of W the matrix $(b_{ij} = v_i \circ w_j)$ is invertible.

- Notice that if $\{y^A\}$ and $\{x^B\}$ are bases of R_j and S_j respectively, they are not exactly dual bases. The dual bases of R_j and S_j are: $\{y^{A_1}, \dots, y^{A_t}\}$ and $\{\frac{1}{c_1}x^{A_1}, \dots, \frac{1}{c_t}x^{A_t}\}$ for an appropriate choice of coefficients c_i. So $\{y_1, \dots, y_n\}$ in R_1 is a dual base of $\{x_1, \dots, x_n\}$, base of S_1, with respect to the apolarity action, but for $j > 1$ this is no longer true.

Definition 1.2.2. *Let I be a homogeneous ideal of R. The Inverse System I^{-1} of I is the R-submodule of S containing all the elements of S annihilated by I.*

Remarks:

- If $I = (F_1, \dots, F_t) \subset R$ and $G \in R$ then $G \in I^{-1} \Leftrightarrow F_1 \circ G = \cdots = F_t \circ G = 0$. Finding all such G's means finding all the polynomial solutions for the differential equations defined by the F_i's, so one can notice that determining I^{-1} is equivalent to solve (with polynomial solutions) a finite set of differential equations;

- I^{-1} is a graduated submodule of S but it is not necessarily multiplicatively closed and in general I^{-1} is not an ideal of S.

We need now a digression on the Hilbert function.

1.2.2 Hilbert Function and Inverse Systems

For this paragraph we refer to [**EH**].

Let $X \subset \mathbb{P}^n(K)$ be a closed subscheme whose representative homogeneous ideal is $I := I(X) \subset S$. Let $A = S/I$ be the homogeneous coordinate ring of X; A_d will be its degree d component.

Definition 1.2.3. *The Hilbert Function of the scheme X is:*

$$H(X, \cdot) : \mathbb{N} \to \mathbb{N};$$

$$H(X, d) = \dim_K(A_d).$$

We can easy observe that

$$H(X, d) = \dim_K(A_d) = \dim_K(S_d) - \dim_K(I_d).$$

Let us introduce the following theorem known as "Hilbert Theorem":

Theorem 1.2.4. *There exists an unique polynomial $P(X, d)$ in the variable d (the Hilbert polynomial) such that $H(X, d) = P(X, d)$ for all sufficiently large d.*

Remark: The degree of the Hilbert polynomial is the dimension of X:

$$\deg(P(X,d)) = \dim_K(X)$$

and so if $\dim_K(X) = 0$ then $P(X,d) = \text{constant} = \deg(X)$.

This observation will be useful in order to prove that the Hilbert function of a 0-dimensional scheme X is such that $H(X,d) = H(X,d_0)$ for certain d_0 and for any $d \geq d_0$.

Definition 1.2.5. *Let $X \subset \mathbb{P}^r$ be an n-dimensional projective scheme, and let $lc(P(X,d))$ be the leading coefficient of $P(X,d)$, then the degree of X is*

$$\deg(X) = n! \cdot lc(P(X,d)). \tag{1.2}$$

Remark: If X is a 0-dimensional scheme of degree δ then $P(X,d) = \delta$ (in general one has $H(X,d) \leq \delta$).

In our work the importance of inverse systems will be given by the following theorem, for a particular choice of the ideal I:

Theorem 1.2.6. *The dimension of the part of degree d of the inverse system of an ideal $I \subset R$ is the Hilbert function of R/I in degree d:*

$$\boxed{\dim_K(I^{-1})_d = \operatorname{codim}(I_d) = H(R/I,d)}\ . \tag{1.3}$$

Remark:

- $(I^{-1})_d \cong I_d^{\perp}$. [2]
- if I is a monomial ideal then $I_d^{\perp} =<$ monomials of R_d that are not in $I_d>$
- $(I \cap J)^{-1} = I^{-1} + J^{-1}$.

In order to discover which kind of ideals we need to consider to solve the big Waring problem via inverse system we need to introduce the study of zero-dimensional schemes.

[2] If $V \times W \longrightarrow K$ is a non degenerate bilinear form and V_1 is a subspace of W, then $V_1^{\perp}$ is a subspace of W and precisely: $V_1^{\perp} = \{w \in W / v \circ w = 0 \ \forall\ v \in V_1\} = \{w \in W / \chi_w(V_1) = 0\}$. Let $V \times W \longrightarrow K$ be non singular simmetry with $\dim_K(V) = \dim_K(V_1) = t$, then $\dim_K(V_1^{\perp}) = n - t$.

1.3 Reduced 0-dimensional schemes

Let $\overline{X} = \{P_1, \ldots, P_s\} \subset \mathbb{P}^n$ a set of s distinct points and $\wp_i \subset S = K[x_0, \ldots, x_n]$ be the prime ideals corresponding to the point P_i with $i = 1, \ldots, s$. With X we indicate the projective 0-dimensional scheme whose support is $\overline{X}$ and with representative ideal $I = \wp_1 \cap \cdots \cap \wp_s$. Moreover if we indicate with I_d and A_d the degree d part of an ideal I and a ring A respectively, then $I = \bigoplus_{d \geq 0} I_d$ and the coordinate ring of X is $A(X) := S/I = \bigoplus A(X)_d$.
So the Hilbert function of X is

$$H(X, d) = \dim_K(A(X)_d) = \dim_K(S_d) - \dim_K(I_d).$$

1.3.1 Hilbert function of reduced 0-dimensional schemes

We will consider an example of Hilbert function of simple points in the plane (sometime we say "simple points" instead of "reduced 0-dimensional schemes").

Example: Let $X = \{P_1, P_2, P_3, P_4\} \subset \mathbb{P}^2$. We already know that $P(X, d) = 4$.
If we want to study the Hilbert function of X in any degree d we have to distinguish three cases:

1. X contained in a line L having equation $l = 0$

   - The only line containing X is L, so $H(X, 1) = \dim_K(S_1) - \dim_K(I_1) = 3 - 1 = 2$.
   - If $q = 0$ is the equation of a plane conic containing X then, by Bezout Theorem, q must be identically zero on L, i.e. $q = 0$ gives the union of L with another line, and the equations of the conics containing X give the tridimensional space of the forms lm, $m \in S_1$. Then $H(X, 2) = \dim_K(S_2) - \dim_K(I_2) = 6 - 3 = 3$.
   - Suppose that $d \geq 3$.
     Let $X' \subset X$ be a subscheme of X with support on three points; consider the curves made by d lines, three of them passing through a different point of X' each. Such a curve is of degree d, it contains X' but it does not contain X. Moreover we can find such a curve for any $X' \subset X$ whose support is made by three points. This implies that the vanishing in 4 points imposes 4 independent conditions to the forms of degree d, so $H(X, d) = 4$ for all $d \geq 3$.

2. Only three points on a line.

   - There is not any linear form in $I(X) \Rightarrow H(X, 1) = 3$.

- With the same argument of the previous case, all plane quadrics containing three points on L must contain L; this means that all quadrics containing X are union of L and another line through the fourth point. Since the linear space corresponding to the lines through the fourth point is 2-dimensional, the space of the quadrics containing X is 2-dimensional, then $H(X, 2) = 6 - 2 = 4$.
- With the same argument of the previous case one can see that $H(X, d) = 4$ for all $d \geq 3$ (better: since $H(X, 2) = 4$ we can conclude that $H(X, d) = 4$ for all $d \geq 2$).

3. No three points on a line.

   - the scheme X does not lie on a line so $H(X, 1) = 3$.
   - For any three points of X one can find a quadric through them not containing the fourth point, so X imposes four independent conditions to quadrics and $H(X, d) = 4$ for all $d \geq 2$.

We treat now the general case of s simple points on $\mathbb{P}^n$.

Definition 1.3.1. *With $S_d(P_1, \ldots, P_s)$ we indicate the vector subspace of S_d whose elements are forms of degree d which are zero at the points $P_1, \ldots, P_s \in \mathbb{P}^n$.*

The goal is to write the dimension of $S_d(P_1, \ldots, P_s) \subset K[x_0, \ldots, x_n]_d$ in terms of Hilbert function.

Let $M_1, \ldots, M_N$, with $N = \binom{d+n}{n}$, be a monomial base of S_d: if $F \in S_d$ then $F = c_1 M_1 + \cdots + c_N M_N$ with $c_i \in K$ for all $i = 1, \ldots, N$.
Let $P_1, \ldots, P_s \in \mathbb{P}^n$; the degree d forms vanishing on those points are the solutions of the following linear system

$$\begin{cases} M_1(P_1)c_1 + \cdots + M_N(P_1)c_N = 0 \\ \vdots \\ M_1(P_s)c_1 + \cdots + M_N(P_s)c_N = 0 \end{cases} . \tag{1.4}$$

Let $\mathcal{M}_d$ be the matrix defined as follows:

$$\mathcal{M}_d = \begin{pmatrix} M_1(P_1) & \cdots & M_N(P_1) \\ \vdots & & \vdots \\ M_1(P_s) & \cdots & M(P_s) \end{pmatrix} .$$

Therefore (1.4) can be written: $\mathcal{M}_d \cdot \begin{pmatrix} c_1 \\ \vdots \\ c_N \end{pmatrix} = 0.$

Now the linear solutions of those equations gives exactly the vector space I_d, and $\dim_K(I_d) =$

$N - \mathrm{rk}(\mathcal{M}_d) = \binom{d+n}{n} - \mathrm{rk}(\mathcal{M}_d)$; hence

$$H(S/I, d) = \binom{d+n}{n} - \dim_K(I_d) = \mathrm{rk}(\mathcal{M}_d).$$

For any s we can choose $P_1, \ldots, P_s$ such that the matrix $\mathcal{M}_d$ has maximal rank for all $d \in \mathbb{Z}^+$, since the s-uples $P_1, \ldots, P_s$ for which the rank of $\mathcal{M}_d$ is not maximum form a closed set (where all the maximal minors are zero):

$$\mathrm{rk}(\mathcal{M}_d) = \min\left\{s, \binom{d+n}{n}\right\}.$$

In conclusion we have:

Proposition 1.3.2. *If $\overline{X} = \{P_1, \ldots, P_s\} \subset \mathbb{P}^n$ is the support of a projective 0-dimensional reduced scheme X where the P_i's are generic, then:*

$$H(X, d) = \min\left\{s, \binom{d+n}{n}\right\}.$$

Now we try to study the non-reduced case, which turns out to be not so simple.

1.4 Non reduced 0-dimensional schemes

Let us introduce the problem of computing the Hilbert function of a non-reduced 0-dimensional scheme with some examples.

1.4.1 Examples

The elements of $S_d(P_1, \ldots, P_s)$, defined as in 1.3.1, correspond to hypersurfaces of degree d which pass through $P_1, \ldots, P_s$.

It is clear that if the points $P_1, \ldots, P_s$ are in general position the dimension of $S_d(P_1, \ldots, P_s)$ is

$$\dim_K(S_d(P_1, \ldots, P_s)) = \left[\binom{n+d}{d} - s\right]^+$$

where $[x]^+ := \max\{x, 0\}$.
If the points are not in general position then:

$$\dim_K(S_d(P_1, \ldots, P_s)) \geq \left[\binom{n+d}{d} - s\right]^+$$

Definition 1.4.1. *We call the number* $\left[\binom{n+d}{d} - s\right]^+$ *the Virtual Dimension of* $S_d(P_1, \ldots, P_s)$.

Notice that this is actually the value of the Hilbert Polynomial of the reduced scheme X given by the P_i's in general position.

Now we want to compute the dimension of the subspace of S_d of hypersurfaces which not only pass through some fixed points P_i but also have some singularities in those P_i: this fact is equivalent to find the polynomials of degree d which vanish in P_i with all their partial derivatives up to a certain order.

Notation: With $S_d(P_1^{\alpha_1}, P_2^{\alpha_2}, \ldots, P_s^{\alpha_s})$ we indicate the subspace of S_d of hypersurfaces of degree d which pass through P_i, $i = 1, \ldots, s$, and which have in those points singularities of multiplicity grater or equal to α_i.

What we expect is that each $P_i^{\alpha_i}$ imposes $\binom{n+\alpha_i-1}{n}$ conditions. So we can ask whether choosing generic $P_1, \ldots, P_s$, we have:

$$\dim_K(S_d(P_1^{\alpha_1}, \ldots, P_s^{\alpha_s})) = \left[\binom{d+n}{n} - \sum_{i=1}^{s} \binom{n+\alpha_i-1}{n}\right]^+ ? \tag{1.5}$$

In this case we do not have an immediate answer as in case of simple points where if $P_1, \ldots, P_s$ are in general position, then the dimension is always the expected one. We can only say that

$$\dim_K(S_d(P_1^{\alpha_1}, \ldots, P_s^{\alpha_s})) \geq \left[\binom{d+n}{n} - \sum_{i=1}^{s} \binom{n+n_i-1}{n}\right]^+ .$$

There are some simple counterexamples:

1. Let us consider $S_2(P_1^2, P_2^2) \subset K[x_0, x_1, x_2]_2$.
   If (1.5) were true it would happen that

   $$\dim_K(S_2(P_1^2, P_2^2)) = \left[\binom{2+2}{2} - 2\binom{3}{2}\right]^+ = 6 - 6 = 0$$

   but this is clearly false: there is always a line through 2 points of $\mathbb{P}^2$ and so the double line through P_1 e P_2 belongs to $S_2(P_1^2, P_2^2)$ and this implies that $\dim_K(S_2(P_1^2, P_2^2)) = 1 \neq 0$ (it is easy to see that there cannot be another conic in the system).

2. Let us consider $S_4(P_1^2, \ldots, P_5^2) \subset K[x_0, x_1, x_2]_4$.
   If (1.5) were true it would happen that

   $$\dim_K(S_4(P_1^2, \ldots, P_5^2)) = \binom{4+2}{5} - 5 \cdot \binom{3}{2} = 15 - 15 = 0$$

but there is always a plane conic passing through 5 points so the double conic through $P_1, \ldots, P_5$ is a plane quartic through $P_1^2, \ldots, P_5^2$ and this implies that $\dim_K(S_4(P_1^2, \ldots, P_5^2)) \geq 1 \neq 0$.

1.4.2 Fat points

Definition 1.4.2. *Let $P_1, \ldots, P_s \in \mathbb{P}^n$; $\wp_1, \ldots, \wp_s \subset S = K[x_0, \ldots, x_n]$ be the associated prime ideals and $\alpha_1, \ldots, \alpha_s \in \mathbb{N}$. The projective scheme defined by the ideal*

$$I = \wp_1^{\alpha_1} \cap \cdots \cap \wp_s^{\alpha_s}$$

is called a scheme of Fat Points in $\mathbb{P}^n$ and we denote it as:

$$X = (P_1, \ldots, P_s; \alpha_1, \ldots, \alpha_s).$$

Remark: Not all zero-dimensional schemes are made of fat points; a 0-dimensional scheme whose coordinate ring is $K[x_0, x_1]/(x_0^3, x_1^2)$ is neither a reduced scheme nor a fat point because it is not possible to write its representative ideal $I = (x_0^3, x_1^2)$ as the intersection of some powers of ideals of points.

Our goal is the study of $H(S/I, d)$ when S/I is the coordinate ring of a fat point, but we think that it can be useful to have a look on what happens also with those non-reduced 0-dimensional schemes which are not fat points.

We will look first at what happens in the affine case, but we want to recall first a few remarks on the degree of a scheme: the degree we have defined in (1.2) is given for a projective scheme but we are going to work with affine schemes. Now we will give a general definition and we will see later that it is equivalent to (1.2), thus implying that the degree of a scheme is independent on the immersion.

1.4.3 Degree of a scheme

Recall that in paragraph 1.3.1 we have defined the degree of a projective n-dimensional scheme X on $\mathbb{P}^r$ as $\deg(X) = n!lc(P(X, d))$ and if X is 0-dimensional then $\deg(X) = P(X, d) = \delta$. Consider a scheme X embedded in $\mathbb{P}^n$; let $S = K[x_0, \ldots, x_n]$ be the polynomial ring. We now denote with X_{red} the reduced scheme whose support is the point $P = [1, 0, \ldots, 0] \in \mathbb{P}^n$. Its representative ideal is the prime ideal $\mathfrak{m} = (x_1, \ldots, x_n)$ and its Hilbert function is:

$$H(X_{red}, d) = \dim(S_d) - \dim(\mathfrak{m}_d) = \binom{d+n}{n} - \left(\binom{d+n}{n} - 1\right) = 1.$$

Suppose now to have a 0-dimensional scheme X whose support is P but X is not necessarily reduced. We want to know its degree. We have to study S/I, where I is the representative ideal of X. Let us introduce the following Proposition (See Propos. I.7.4. on [**Hart**]):

Proposition 1.4.3. *Let M be a finitely generated graded module over a noetherian graded ring S. Then there exists a filtration $0 = M^0 \subseteq M^1 \subseteq \cdots \subseteq M^r = M$ by graded submodules, such that for each i, $M^i/M^{i-1} \simeq (S/\wp_i)(l_i)$ where $\wp_i$ is a homogeneous prime ideal of S, and $l_i \in \mathbb{Z}$. The filtration is not unique, but for any such filtration we have:*

1. *if $\wp$ is a homogeneous prime ideal of S, then $\wp \supseteq Ann(M) \Leftrightarrow \wp \supseteq \wp_i$ for some i. In particular, the minimal elements of the set $\{\wp_1 \ldots, \wp_r\}$ are just the minimal primes of M, i.e., the minimal elements in the set of all primes containing $Ann(M)$;*

2. *for each minimal prime of M, the number of times which $\wp$ occurs in the set $\{\wp_1, \ldots, \wp_r\}$ is equal to the length of $M_\wp$ over the local ring $S_\wp$ (and hence is independent on the filtration).*

Definition 1.4.4. *If $\wp$ is a maximal prime of a graded S-module M, the "multiplicity" of M at $\wp$ is the length of $M_\wp$ over $S_\wp$.*

Considering a 0-dimensional non reduced scheme whose support is only one point and let I be its associated ideal. Then the module S/I has only one associated prime $\wp_i$. Then it must exist a filtration

$$S/I = M = M^r \supseteq \cdots \supseteq M^1 = \{0\} \tag{1.6}$$

such that $M^i/M^{i-1} \simeq S/\mathfrak{m} = K$. So

$$\dim_K(S/I) = r = l(S/I). \tag{1.7}$$

Consider now the following exact sequence

$$0 \to M^{r-1} \to M \to S/\mathfrak{m} \to 0.$$

One has that for a sufficiently large d the Hilbert function $H(M,d) = H(M^{r-1},d) + H(S/\mathfrak{m},d)$; since $H(S/\mathfrak{m},d) = 1$ we have $H(M,d) = H(M^{r-1},d) + 1$.
Also
$H(M^1,d) = H(M^0,d) + 1 = H(\{0\},d) + 1 = 0 + 1 = 1$ and
$H(M^2,d) = H(M^1,d) + 1 = 1 + 1 = 2$
then, with $d >> 0$:

$$H(M,d) = r; \tag{1.8}$$

but when $d >> 0$, if X is a 0-dimensional scheme, it also happens that

$$H(M, d) = P(M, d) = \deg(X). \tag{1.9}$$

By (1.6), (1.7), (1.8) and (1.9) we can conclude that if X is a zero-dimensional scheme whose support is only one point then

$$\deg(X) = H(S/I, d) = r = \dim_K(S/I) = l(S/I)$$

where $l(S/I)$ is the length of the filtration (1.6).

Now we are ready to study the affine case (see [**EH**]).

1.4.4 A few remarks on 0-dimensional schemes

After this digression on the degree of a scheme we want to give some examples of what a non reduced 0-dimensional scheme of low degree can be. We will work in the affine case, since for 0-dimensional schemes this does not make much difference.

Notation: Let X be an affine 0-dimensional scheme. In this paragraph, but only here, we will say that X is a "d-uple" point if $\deg(X) = d$.

A double point in $\mathbb{A}^1$:

We consider the scheme $X = \mathrm{Spec}(K[x]/(x^2))$ viewed as a subscheme of $\mathbb{A}^1$ via the map induced by the quotient map $K[x] \to K[x]/(x^2)$. The support of X is only one point but X is different from $\mathrm{Spec}(K) = \mathrm{Spec}(K[x]/(x))$, both as a subscheme of $\mathbb{A}^1$ and as an abstract scheme.

- As an abstract scheme:

  there exist on X regular functions (for example x) which are not the zero function, but which assume the value 0 at the only point of X.

- As a subscheme of $\mathbb{A}^1$:

  a form $f \in K[x]$ on $\mathbb{A}^1$ vanishes on X if and only if both f and it is first derivative f' vanish at 0. To give a function on X is equivalent to give the values at 0 of both a function on $\mathbb{A}^1$ and of its first derivative. That's why X is called "first order neighbourhood of 0 in $\mathbb{A}^1$".

In general the ideal $(x^n) \subset K[x]$ defines a subscheme $X \subset \mathbb{A}^1$ with coordinate ring $K[x]/(x^n)$; a function $f(x)$ on $\mathbb{A}^1$ becomes zero on X if and only if f vanishes in 0 with all its first $n-1$ derivatives.

A double point in $\mathbb{A}^2$

Let X be the scheme we saw in the previous example: $X = \mathrm{Spec}(K[x]/(x^2))$. Let Y be a subscheme of $\mathbb{A}^2 = \mathrm{Spec}(K[x,y])$ supported at the origin and isomorphic to X. Let R be its coordinate ring and $\varphi : K[x,y] \to R$ the surjection which defines the inclusion $Y \subset \mathbb{A}^2$. Let $\mathfrak{m}$ be the unique maximal ideal of R; its inverse image via φ is the ideal (x,y). By definition of R the square $\mathfrak{m}^2$ is 0, hence the map φ vanishes on $(x,y)^2$ and so it factorizes through a map $\overline{\varphi} : K[x,y]/((x,y)^2) \to R$. Equivalently, Y must be contained in the subscheme $\mathrm{Spec}(K[x,y]/(x^2, xy, y^2))$ but the ring $K[x,y]/(x^2, xy, y^2)$ is a three-dimensional vector space on K, while R is only two-dimensional. Therefore $ker(\varphi)$ contains a non-zero homogeneous linear form $\alpha x + \beta y$ for some $\alpha, \beta \in K$. Let us define

$$X_{\alpha,\beta} := \mathrm{Spec}(K[x,y]/(x^2, xy, y^2, \alpha x + \beta y)) \hookrightarrow \mathbb{A}^2.$$

It can be characterized as:

- the subscheme of $\mathbb{A}^2$ associated to the ideal of the functions $f \in K[x,y]$ vanishing in the origin and having partial derivatives such that $\beta\frac{\partial f}{\partial x} - \alpha\frac{\partial f}{\partial y} = 0$;
- the image of $X \subset \mathbb{A}^1$ by the inclusion $\mathbb{A}^1 \hookrightarrow \mathbb{A}^2,\ x \mapsto (\beta x, -\alpha x)$.

The scheme $X_{\alpha,\beta}$ is classically viewed as the point $(0,0)$ and another point "infinitely near to $(0,0)$" in the direction specified by the line defined by $\alpha x + \beta y = 0$.

This fact leads us to observe that a zero-dimensional scheme of degree 2 in $\mathbb{A}^2$ must be isomorphic to $K[x]/(x^2)$, in fact, as we have just seen, in our case $R = \frac{K[x,y]}{(x^2, xy, y^2, \alpha x + \beta y)}$ with $(\alpha, \beta) \neq (0,0)$; and if we suppose $\beta \neq 0$ then $R = \frac{K[x,y]}{(x^2, xy, y^2, \alpha' x - y)} \simeq \frac{K[x]}{(x^2, x(\alpha' x), (\alpha' x)^2, 0)} \simeq \frac{K[x]}{(x^2)}$.

How can we find schemes as $X_{\alpha,\beta}$? The answer is: as curve intersections or as the limit of reduced subschemes.

- **As curves intersections:**

  Example: Consider a line L and a conic C tangent to each other:

  - if we consider their intersection just set theoretically, we will miss the point that this is a "double intersection";
  - if we try to view $C \cap L$ as a "point of multiplicity 2", this is not satisfactory too, because in this way we miss on which line the scheme is;
  - the satisfactory definition is that $C \cap L$ is the subscheme of $\mathbb{A}^2$ defined by $I_C + I_L$ where I_C and I_L are the defining ideals of C and L respectively. (For example: the ideal $(y) + (y - x^2)$ corresponds to the scheme $X_{0,1} = \mathrm{Spec}(K[x,y]/(x^2, y))$.)

- **As limit of reduced subschemes:**

  Consider the scheme X whose support is a set of two points $(0,0),(a,b)$ in the plane $\mathbb{A}^2$ and $X = \mathrm{Spec}(K[x,y]/((x,y)\cap(x-a,x-b)))$.
  Suppose that (a,b) moves along a curve $(a(t),b(t))$, where $a(t)$ e $b(t)$ are polynomials in the variable t, such that $(a(0),b(0)) = (0,0)$; i.e. we are working with $X_t = \{(0,0),(a(t),b(t))\}$.
  We want to define X such that
  $$X = \lim_{t\to 0} X_t.$$
  Let us define X by imposing that its representative ideal is the limit of $I_t = (x,y)\cap(x-a(t),y-b(t))$ for $t\to 0$. We take this limit as a codimension 2 subspace in $K[x,y]$ viewed as vector space on K.
  We can observe that $I_t = (x^2-a(t)x, xy-b(t)x, xy-a(t)y, y^2-b(t)y)$ where $\lim_{t\to 0}(x^2-a(t)x) = x^2$, $\lim_{t\to 0}(xy-b(t)x) = xy$, $\lim_{t\to 0}(xy-a(t)y) = xy$ and $\lim_{t\to 0}(y^2-b(t)y) = y^2$; so those polynomials belong to $I = \lim_{t\to 0} I_t$.
  We can also observe that I_t contains all linear forms $(a(t)y-b(t)x) = ((xy-b(t)x)-(xy-a(t)y))$, therefore, for $t\neq 0$, also $\frac{a(t)y-b(t)x}{t} = a_1y-b_1x+t(\cdots)$. The ideal I contains the limit $\lim_{t\to 0}\frac{a(t)y-b(t)x}{t} = a_1y+b_1x$ then $I\supset(x^2,xy,y^2,a_1y-b_1x)$ where (x^2,xy,y^2,a_1y-b_1x) has codimension 2 as vector space in $K[x,y]$ then $I = (x^2,xy,y^2,a_1y-b_1x)$ so $\lim_{t\to 0}(X_t) = X_{\alpha,\beta}$, with $\alpha = b_1$ e $\beta = -a_1$.

  The subscheme $X\subseteq\mathbb{A}^2$ "does not forget" the direction that approximates $(a(t),b(t))$; we can look at it as the origin with a tangent direction along the line with equation $a_1y-b_1x=0$. This line is the limit of the set of lines that connect $(0,0)$ and $(a(t),b(t))$ i.e. the tangent line to the curve parameterized by $(a(t),b(t))$ in the origin.

One double points on K are always isomorphic to another since $S\simeq K[x]/(x^2)$; but this is no longer true if we have higher multiplicity.

Triple Point

Let $Z = \mathrm{Spec}(K[x_1,\dots,x_n]/I)$ be a zero-dimensional scheme of degree 3 with support at the origin; then either
$$Z\simeq \mathrm{Spec}(K[x]/(x^3)) =: X$$
or
$$Z\simeq \mathrm{Spec}(K[x,y]/(x^2,xy,y^2)) =: Y$$
and
$$X\ncong Y$$

However, any triple point is isomorphic to either of these. In particular all rings $K[x_1, \ldots, x_n]/I$ of a tridimensional vector space on K can be generated over K by two linear forms in the x_i. From a geometric point of view this means that any triple point in $\mathbb{A}^n$ lies on a $\mathbb{A}^2 \subseteq \mathbb{A}^n$. In $\mathbb{A}^2$ we can realize two kind of triple points: those isomorphic to X which come from 3 points approaching each other along a non-singular curve, and those isomorphic to Y which are realized by the approaching of two points to a third one along two different directions.

Quadruple Point

We have just observed that a triple point can always be contained in a plane: this is no longer true for a quadruple point: for example consider: $K[x, y, z]/(x, y, z)^2$; its maximal ideal cannot be generated by two elements.

Now we come back to the main goal of this section: studying fat points and their postulation (i.e. their Hilbert function).

1.4.5 Fat Points

The main reference of this section is [**Ge**].

Notation: In this section and in the following ones, when we will say "d-uple point" we will mean a fat point (P, d). Sometime we will call (P, d) also a "d-fat point".

We begin with the study of a single point.

One single point

Let us suppose we have a projective scheme whose support is $P = [1, 0, \ldots, 0] \in \mathbb{P}^n$ and let $\wp = (x_1, \ldots, x_n) \subset S = K[x_0, \ldots, x_n]$ be its representative (prime) ideal.
Let $F \in \wp$ be a homogeneous polynomial of degree d; we dehomogenize it with respect to x_0 and we obtain $f \in S$, with $f = f_0 + f_1 + \cdots + f_n$ e $\deg(f_i) = i$. Since $F \in \wp$ then

$$f_0 = 0 \text{ and } P = \underline{0} \in \mathbb{A}^n,$$

$$f_1 = a_1 x_1 + \cdots + a_n x_n = \left(\frac{\partial f}{\partial x_1}\right)\bigg|_0 x_1 + \cdots + \left(\frac{\partial f}{\partial x_n}\right)\bigg|_0 x_n.$$

We recall that if the $\frac{\partial f}{\partial x_i}\Big|_0$'s are not all zero, then P is said to be a "Simple Point" of $V(f)$ and f_1 is the equation of $T_P(V(f))$.

Definition 1.4.5. *If $\wp = (x_1, \ldots, x_n) \subset S$, a polynomial f belongs to $\wp^2$ if and only if $\left(\frac{\partial f}{\partial x_i}\right)\Big|_0 = 0$ for all $i = 1, \ldots, n$ and this is exactly the definition of a singular point of $V(f)$ (in this case it is at least a double point).*

Therefore if $I = \wp^2$ then I_d contains all forms of degree d having a singularity at P. This vector space gives us a classical example of a linear system of hypersurfaces in $\mathbb{P}^n$.

Consider now the Taylor Polynomial of f at 0.
Let $a_{\alpha\beta} y_\alpha y_\beta$ be a term of f_2, then

$$a_{\alpha\beta} = \begin{cases} \left(\frac{\partial f}{\partial x_\alpha \partial x_\beta}\right)\Big|_0 & \text{if } \alpha \neq \beta \\ \frac{1}{2!}\left(\frac{\partial f}{\partial x_\alpha^2}\right)\Big|_0 & \text{if } \alpha = \beta \end{cases}.$$

The polynomial f belongs to $\wp^3$ if and only if all its second partial derivatives vanish in P; that is equivalent to say that P is a singular point of $V(f)$ of multiplicity greater or equal then 3.

More generally:

Definition 1.4.6. *Let $P \in \mathbb{P}^n$, $\wp \subset S$ be its representative prime ideal and $f \in S$. Then the order of all partial derivatives of f vanishing in P is almost t if and only if $f \in \wp^{t+1}$ i.e. iff P is a singular point of $V(f)$ of multiplicity grater or equal than $t+1$.*

Therefore:

$$H(S/\wp^t, d) = \begin{cases} \binom{d+n}{n}, & \text{if } d < t \\ \binom{t-1+n}{n}, & \text{if } d \geq t \end{cases}. \tag{1.10}$$

It is easy to conclude that

Proposition 1.4.7. *One t-fat point of $\mathbb{P}^n$ has the same Hilbert function of $\binom{t-1+n}{n}$ generic distinct points of $\mathbb{P}^n$.*

Remark: By (1.10) we can notice that the degree of a t-fat point in $\mathbb{P}^n$ is not the same of the degree of a t-fat point of $\mathbb{P}^{n+1}$, in fact if $\wp_1 \subset K[x_0, \ldots, x_n]$ and $\wp_2 \subset K[x_0, \ldots, x_{n+1}]$ are two prime ideals representing two points $P_1 \in \mathbb{P}^n$ and $P_2 \in \mathbb{P}^{n+1}$, respectively, then $H(K[x_0, \ldots, x_n]/\wp_1^t, d) \neq H(K[x_0, \ldots, x_{n+1}]/\wp_2^t, d)$ for all $d > 0$.

On the contrary let X be a t-fat point of $\mathbb{P}^n$ with coordinate ring $K[x_0, \ldots, x_n]/I$, then the degree of X is $\dim((K[x_0, \ldots, x_n]/I)_d)$ for $d >> 0$. Suppose now to embed X into $\mathbb{P}^{n+1}$; what happens is that the degree of X does not change, in fact $\deg(X \hookrightarrow \mathbb{P}^n) = \dim((K[x_0, \ldots, x_n]/I)_d) = \dim((K[x_0, \ldots, x_{n+1}]/(I + K[x_0, \ldots, x_n]x_{n+1}))_d) = \deg(X \hookrightarrow \mathbb{P}^{n+1})$; but now $X \hookrightarrow \mathbb{P}^{n+1}$ is no

longer a fat point of $\mathbb{P}^{n+1}$.
Similarly the degree of a scheme X does not change if we consider X embedded before in the affine space $\mathbb{A}^n$ and after in the projective space $\mathbb{P}^n$: we can indifferently study the degree of the scheme we are interested in either in $\mathbb{A}^n$ or in $\mathbb{P}^n$.
Form this observation we can notice that, for example, the degree of $X = \mathrm{Spec}(K[x,y]/(x,y)^2) \subset \mathbb{A}^2$ is the same as the degree $Y = \mathrm{Spec}(K[x]/(x^3)) \subset \mathbb{A}^1$ but X is 2-fat point of $\mathbb{A}^2$ and Y is a 3-fat point of $\mathbb{A}^1$ (we are using an abuse of notation: in the definition 1.4.2 a fat point is a projective scheme; when we say a that X is a t-fat point of $\mathbb{A}^n$ we are meaning that the coordinate ring of X is isomorphic to $K[x_1,\dots,x_n]/\wp^t$, where $\wp$ is the maximal ideal of $K[x_1,\dots,x_n]$).

Example: Let $P_1 = (0,t)$, $P_2 = (t,0)$, $P_3 = (t,t) \in \mathbb{A}^2(K)$ be the support of a scheme with representative ideals $(x_1, x_2-t), (x_1-t, x_2), (x_1-t, x_2-t)$ respectively. The scheme $X = P_1 \cup P_2 \cup P_3$ has as representative ideal $I = (x_1, x_2-t) \cap (x_1-t, x_2) \cap (x_1-t, x_2-t)$ which can be written also

$$I = (x_1(x_1-t), x_2(x_2-t), (x_1-t)(x_2-t)).$$

Now the limit of I for $t \to 0$ is:

$$lim_{t\to 0} I = (x_1^2, x_2^2, x_1x_2) = (x_1, x_2)^2$$

which is the ideal of a double fat point.

Example: Let Y be the scheme of $\mathbb{P}^2$ with support $P = [0,0,1]$ and representative ideal $I = \wp^3$ and so with coordinate ring $A = K[x_0, x_1, x_2]/(x_0^3, x_0^2x_1, x_0x_1^2, x_1^3)$.
Then $\dim_K(A) = \dim < \overline{1}, \overline{x_0}, \overline{x_1}, \overline{x_0^2}, \overline{x_0x_1}, \overline{x_1^2} >= 6$.

In classical algebraic geometry those schemes are called "infinitesimal neighbourhoods" rather then "fat points": a fat point with representative ideal $\wp^t$ was called "$(t-1)$-th infinitesimal neighbourhood".
Let $\wp$ be a prime ideal of $S = K[x_0,\dots,x_n]$ and the representative ideal of a point $P \in \mathbb{P}^n$. An element f of $\wp^2$ is such that $f(P) = 0$ and also $\left(\frac{\partial f}{\partial x_i}\right)\Big|_P = 0$ for $i = 0,\dots,n$. Those $n+1$ conditions can be interpreted as n independent points infinitesimally near to P.
A triple point is characterized by the vanishing of $f(P)$ and of all its first and second partial derivatives. Besides the n conditions on first derivatives, one has also those ones from the vanishing of all the second derivatives that can be viewed as another set of points infinitesimally near to P but not "so near" as those one individuated by the vanishing of first partial derivatives.

Consider $\mathrm{Proj}(S/\wp^k)$, the required order of vanishing for partial derivatives increases; so we can think at those schemes as a series of neighbourhoods around P whose "radius" become grater while the order of vanishing partial derivatives increases.

More than one Fat Point

Let $X = (P_1, \dots, P_s; \alpha_1, \dots, \alpha_s)$ be a scheme made of fat points. We want to study the Hilbert function of such a scheme.
When $s = 1$ then the Hilbert function of $S/\wp^t$ is the same of that of $\binom{t-1+n}{n}$ distinct points of $\mathbb{P}^n$ in general position.

If we have more than one point what happens is not the same: the Hilbert function of s α_i-fat points in general IS NOT equal to the Hilbert function of $\sum_{i=1}^{s} \binom{\alpha_i-1+n}{n}$ distinct points of $\mathbb{P}^n$ in general position.

Examples:

1. Let P_1, P_2 be two points of $\mathbb{P}^2$, $\wp_i \subset S = K[x_0, x_1, x_2]$ their associated prime ideals and let $\alpha_1 = \alpha_2 = 2$ so that $I = \wp_1^2 \cap \wp_2^2$. Is the Hilbert function of I equal to the Hilbert function of 6 points of $\mathbb{P}^2$ in general position? No, because the Hilbert function of 6 general points of $\mathbb{P}^2$ is 1 3 6 6 ... and this means that I should not contain conics, but this is clearly false because the double line through P_1 and P_2 is contained in I (we refer to the first example of Section 1.4.1).

2. Let $P_1, \dots, P_5$ be five points of $\mathbb{P}^2$ in general position and $\wp_1, \dots, \wp_5 \subset S = K[x_0, x_1, x_2]$ the corresponding prime ideals. If $I = \wp_1^2 \cap \dots \cap \wp_5^2$ then its Hilbert function is not equal to the Hilbert function of $5 \cdot 3 = 15$ points of $\mathbb{P}^2$ in general position, which is 1 3 6 10 15 15 ... In fact I contains the double conic (a quartic) through $P_1, \dots P_5$ (we refer to the second example of Section 1.4.1).

3. Another example (see [**Mi**]) is given by plane curves of degree 93 with multiplicity 57 at one point and 28 at other seven. The virtual dimension of $S_{93}(P_0^{57}, P_1^{28}, \dots, P_7^{28})$ is $\left[0, \frac{93(96)}{2} - \frac{57(58)}{2} - \frac{7(28)(29)}{2}\right]^+ = [0, -31]^+ = 0$.
   Then we expect that $\dim(S_{93}(P_0^{57}, P_1^{28}, \dots, P_7^{28})) = 0$. But there is always a plane cubic through seven points which is double in one of them, moreover there is always a sestic through 8 points which is triple at one of them. Let $\mathcal{C}_j$, $j = 1, \dots, 7$, be seven cubics with a double point at P_0 and not passing through P_j. Let also $\mathcal{S}$ be a sestic with a triple point in P_0 and a double one in the other seven P_i. Then $5\mathcal{S} + 3\sum_{j=1}^{7} \mathcal{C}_j$ gives an element of $S_{93}(P_0^{57}, P_1^{28}, \dots, P_7^{28})$, hence $S_{93}(P_0^{57}, P_1^{28}, \dots, P_7^{28}) \neq \{0\}$.

We gave those examples only in order to show that there are many problems in computing the dimension of $S_d(P_1^{\alpha_1}, \dots, P_s^{\alpha_s})$. The general problem is not yet solved: there is only a conjecture due first to Beniamino Segre (rephrased also by B. Harbourne, A. Gimigliano, A. Hirschowitz and others) which describes how the element of $S_d(P_1^{\alpha_1}, \dots, P_s^{\alpha_s})$ should be done when it has not the expected dimension.

Definition 1.4.8. *Let $P_1, \dots, P_s$ be s points of $\mathbb{P}^n$ in general position. If $S_d(P_1^{\alpha_1}, \dots, P_s^{\alpha_s})$ is a linear system whose dimension is not the expected one, it is said to be a Special Linear System.*

Conjecture 1.4.9. *If $S_d(P_1^{\alpha_1}, \dots, P_s^{\alpha_s}) \subset K[x_0, x_1, x_2]$ is a special linear system, then there is a fixed double component for all curves through $(P_1, \dots, P_s; \alpha_1, \dots, \alpha_s)$.*

The considerations on Inverse Systems led us to the equality (1.3). By applying it to an ideal of fat points we can translate the problem of determining the dimension of $S_d(P_1^{\alpha_1}, \dots, P_s^{\alpha_s})$ to a problem of inverse systems. If $I = \wp_1^{\alpha_1+1} \cap \dots \cap \wp_s^{\alpha_s+1} \subset S = K[x_0, \dots, x_n]$ with $\wp_i$ prime ideals of the points $P_1, \dots, P_s \in \mathbb{P}^n$ and $P_i = [p_{i_0}, p_{i_1}, \dots, p_{in}]$, $L_{P_i} = p_{i_0}y_0 + p_{i_1}y_1 + \dots + p_{i_n}y_n \in R = K[y_0, \dots, y_n]$ then

$$\boxed{(I^{-1})_d = \begin{cases} R_d, & \text{for } d \leq \max\{\alpha_i\} \\ L_{P_1}^{d-\alpha_1} R_{\alpha_1} + \dots + L_{P_s}^{d-\alpha_s} R_{\alpha_s}, & \text{for } d \geq \max\{\alpha_i + 1\} \end{cases}}$$

and also

$$\boxed{H(S/I, d) = \dim_K (I^{-1})_d = \begin{cases} \dim_K(R_d), & \text{for } d \leq \max\{\alpha_i\} \\ \dim_K(< L_{P_1}^{d-\alpha_1} R_{\alpha_1}, \dots, L_{P_s}^{d-\alpha_s} R_{\alpha_s} >), & \text{for } d \geq \max\{\alpha_i + 1\} \end{cases}}. \tag{1.11}$$

This last result gives a link between the Hilbert function of a set of fat points and ideals generated by sums of powers of linear forms. This implies that:

Proposition 1.4.10. *If $I = \wp_1^{\alpha_1+1} \cap \dots \cap \wp_s^{\alpha_s+1} \subset S = K[x_0, \dots, x_n]$ then $(I^{-1})_d \subset R_d = K[y_0, \dots, y_n]_d$ is the d-th graded part of the ideal $(L_{P_1}^{d-\alpha_1}, \dots, L_{P_s}^{d-\alpha_s}) \subset R$ for $d \geq \max\{\alpha_i + 1,\ i = 1, \dots, s\}$.*

Finally the link between the big Waring problem and inverse systems is clear. If in (1.11) all the α_i are equal to 1, the dimension of the vector space $< L_{P_1}^{d-1} R_1, \dots, L_{P_s}^{d-1} R_1 >$ is at the same time the Hilbert function of the inverse system of a scheme of s double fat points, and the rank of the differential of the application ϕ defined in (1.1).

Thus we can say:

Theorem 1.4.11. *Let $L_1, \dots, L_s$ be linear forms of $R = K[y_0, \dots, y_n]$ such that:*

$$L_i = a_{i_0}y_0 + \dots + a_{i_n}y_n$$

and let $P_1, \dots, P_s \in \mathbb{P}^n$ *such that:*

$$P_i = [a_{i_0}, \dots, a_{i_n}].$$

Let also $\wp_i \subset S = K[x_0, \dots, x_n]$ *be the prime ideal associated to* P_i *for* $i = 1, \dots, s$ *and*

$$\phi : \underbrace{R_1 \times \cdots \times R_1}_{s} \longrightarrow R_d$$

with

$$\phi(L_1, \dots, L_s) = L_1^d + \cdots + L_s^d;$$

then

$$\mathrm{rk}(d\phi)|_{(L_1,\dots,L_s)} = \dim_K < L_1^{d-1}R_1, \dots, L_s^{d-1}R_1 > .$$

And by (1.3), we have:

$$\dim(< L_1^{d-1}R_1, \dots, L_s^{d-1}R_1 >) = H\left(\frac{S}{\wp_1^2 \cap \cdots \cap \wp_s^2}, d\right).$$

In conclusion solving the big Waring problem is equivalent to finding the minimum $s \in \mathbb{Z}$ such that $H\left(\frac{S}{\wp_1^2 \cap \cdots \cap \wp_s^2}, d\right) = \binom{n+d}{d}$. This problem was completely solved by J.Alexander e A.Hirschowitz (see [**AH**]):

Theorem 1.4.12. (**J. Alexander, A. Hirschowitz**) *Let* $X = \{P_1, \dots, P_s\}$ *be a set of* s *generic points in* $\mathbb{P}^n$. *Let* $\wp_i \subseteq S = K[x_0, \dots, x_n]$ *the prime ideal associated to* P_i *for* $i = 1, \dots, s$ *and let also* $d \geq 3$. *Then:*

$$H\left(\frac{S}{\wp_1^2 \cap \cdots \cap \wp_s^2}, d\right) = \min\{(n+1)s, \dim_K(S_d)\}$$

except for:

- $n = 2$, $d = 4$, $s = 5$;
- $n = 3$, $d = 4$, $s = 9$;
- $n = 4$, $d = 4$, $s = 14$;
- $n = 4$, $d = 3$, $s = 7$.

Another very interesting fact is that the big Waring problem has also a geometric interpretation and the solution via Inverse System allows to solve this other problem too. We are going to present it in the next section.

1.5 The geometric point of view

1.5.1 Veronese variety

The geometric object that is related with the previous problem is the "Veronese variety". We recall that the *Veronese variety* is the image of the following embedding:

$$\begin{array}{rcl} \nu_d : \mathbb{P}^n & \hookrightarrow & \mathbb{P}^{\binom{n+d}{d}-1} \\ (u_0 : \ldots : u_n) & \mapsto & (u_0^d : u_0^{d-1}u_1 : u_0^{d-1}u_2 : \ldots : u_n^d). \end{array}$$

This embedding can also be dually characterized as:

$$\begin{array}{rcccl} \nu_d : & \mathbb{P}(S_1) = (\mathbb{P}^n)^* & \hookrightarrow & \mathbb{P}(S_d) & = \left(\mathbb{P}^{\binom{n+d}{d}-1}\right)^* \\ & [L] & \mapsto & [L^d]. & \end{array}$$

Therefore we can think to the Veronese variety as the variety that parameterizes d-th powers of linear forms. If we want to study the variety that parameterizes sums of s d-powers of linear forms of $K[x_0, \ldots, x_n]$ we have to consider the $(s-1)$-secant variety of $\nu_d(\mathbb{P}^n)$.

In the next section we will study the Secant Variety of a projective variety and the problem of finding its dimension.

1.5.2 Secant Variety

Definition 1.5.1. *Let $X \subset \mathbb{P}^N$ be a projective variety of dimension n; we define $\mathrm{Sec}_{s-1}(X)$ the $(s-1)$-secant variety of X as follows:*

$$\mathrm{Sec}_{s-1}(X) := \overline{\bigcup_{P_1,\ldots,P_s \in X} < P_1, \ldots, P_s >}$$

where $< P_1, \ldots, P_s >$ is the $(s-1)$-projective space containing $P_1, \ldots, P_s \in X$.

In other words $\mathrm{Sec}_{s-1}(X)$ parameterizes sums of s elements of X and moreover the $(s-1)$-secant variety of X is a projective variety.

By definition $\mathrm{Sec}_0(X) = X$. It is clear that if X is not degenerate than $X \subsetneq \mathrm{Sec}_1(X)$. The first secant variety of X is obtained by adding to X all the points which are linearly spanned by a pair of points of X and then taking closure of this set. If $\mathrm{Sec}_1(X)$ is not linear we can continue in this process of partial linearization of X and we construct $\mathrm{Sec}_2(X)$, and so on, until we find an $s \in \mathbb{N}$ such that $\mathrm{Sec}_{s-1}(X) = \mathbb{P}^N$. Finally we have the following obvious chain of inclusions (if X is not degenerate):

$$X = \mathrm{Sec}_0(X) \subset \mathrm{Sec}_1(X) \subset \mathrm{Sec}_2(X) \subset \cdots \subset \mathrm{Sec}_{s-1}(X) = \mathbb{P}^N.$$

As consequence:

$$n = \dim(X) < \dim(\mathrm{Sec}_1(X)) < \dim(\mathrm{Sec}_2(X)) < \cdots < \dim(\mathrm{Sec}_{s-1}(X)) = N.$$

Definition 1.5.2. *The smallest $s \in \mathbb{Z}$ such that $\mathrm{Sec}_{s-1}(X) = \mathbb{P}^N$ is the Typical Rank of X.*

The typical rank of X is an invariant of the embedded variety X.

Example: If we consider the d-uple Veronese embedding of $\mathbb{P}^n$ it can be viewed as the subset of $K[x_0, \ldots, x_n]_d$ made by all forms which can be written as d-powers of linear forms. From this point of view the typical rank s of the Veronese variety is the minimum integer such that the generic form of degree d in $n+1$ variables is a linear combination of s powers of linear forms in the same number of variables.

Example: Let us consider the Segre product as the image of the following map:

$$\begin{gathered} \nu_{n,m} : \mathbb{P}^n \times \mathbb{P}^m \hookrightarrow \mathbb{P}^{nm+n+m} \\ \nu_{n,m}((x_0, \ldots, x_n), (y_0, \ldots, y_m)) = (x_0y_0, x_0y_1, \ldots, x_ny_m). \end{gathered} \tag{1.12}$$

The Segre product is then the subset of $((n+1) \times (m+1))$-matrices having rank equal to 1. Therefore the typical rank s of $\nu_{n,m}(\mathbb{P}^n \times \mathbb{P}^m)$ is the minimum integer s such that the generic matrix of order $(n+1) \times (m+1)$ is a linear combination of s matrices of rank 1, i.e. it has rank equal to s. Hence the value of s for all n and m is completely solved: $s = \min\{n+1, m+1\}$.
Something more complicated occurs if we consider Segre product with more factors: $\mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_m}$, which can be viewed as the set of m-dimensional $((a_1+1) \times \cdots \times (a_m+1))$-tensors of "rank 1" (admitting we know what "rank 1" means for a generic tensor). For more details (but not complete answers) on the problem of finding this typical rank we refer to [**CGG1**]. We will come back later to this example.

The following analysis is from [**Ch**].
Let $X \subset \mathbb{P}^N$ be a non-linear, reduced, non-degenerate projective variety, let $P \in \mathbb{P}^N \smallsetminus X$ be a fixed point of $\mathbb{P}^N$ and $A, B \in X$. Let also $p : X \to \mathbb{P}^{N-1}$ be the projection of X from P to a generic hyperplane of $\mathbb{P}^N$. Now if $P \in < A, B >$, it is clear that $p(A) = p(B)$, i.e. if $P \in \mathrm{Sec}_1(X) \smallsetminus X$ than the projection p is not injective. The viceversa is obviously true. This proves the following proposition.

Proposition 1.5.3. *Let X be a projective variety of $\mathbb{P}^N$ and $p : X \to \mathbb{P}^{N-1}$ the projection of X from a generic point $P \notin X$ to a generic hyperplane, then $p(X) \simeq X$ if and only if $P \notin \mathrm{Sec}_1(X)$. This result is equivalent to the following statement: $p(X) \simeq X$ iff the typical rank s is bigger then 2.*

If we iterate this idea we obtain that if $P \notin \mathrm{Sec}_2(X)$ and if $A, B, C \in X$ are three independent points of X, they remain independent after the projection.

Proposition 1.5.4. *If the typical rank of a projective variety $X \subset \mathbb{P}^r$ is bigger than 3, then the projection $p : X \to \mathbb{P}^{N-1}$ from a generic point $P \notin X$ preserves the linear independence of any three points of X, i.e. $p(X)$ has no new trisecant lines.*

One can generalize.

Theorem 1.5.5. *The typical rank s of a projective variety $X \subset \mathbb{P}^N$ is the maximum integer such that the projection $p : X \to \mathbb{P}^{N-1}$ from a generic fixed point $P \in \mathbb{P}^N \smallsetminus X$ to a generic hyperplane preserves the independence of elements of a set of s points of X.*

We want to study the problem of determining the dimension of $(s-1)$-secant varieties of an n-dimensional projective variety $X \subset \mathbb{P}^N$.

Let $X^s := \underbrace{X \times \cdots \times X}_{s}$, $X_0 \subset X$ be the open subset of regular points of X and $U_{s-1}(X)$ be the subset of X^s defined as

$$U_{s-1}(X) = \{(P_1, \ldots, P_s) \in X^s \mid P_i \in X_0 \ \forall\, i \text{ and the } P_i\text{'s are independent}\}.$$

Therefore for all $(P_1, \ldots, P_s) \in U_{s-1}(X)$ the span $< P_1, \ldots, P_s >$ is a $\mathbb{P}^{s-1}$.

Definition 1.5.6. *The $(s-1)$-abstract secant variety of X is the incidence variety:*

$$AbS^{s-1}(X) = \{(Q, \pi) \in \mathbb{P}^N \times U_{s-1}(X) \mid Q \in \pi\}$$

The dimension of the variety $AbS^{s-1}(X)$ is

$$\dim(AbS^{s-1}(X)) = n(s-1) + n + s - 1.$$

With this definition we can consider the usual projection

$$p_1 : AbS^{s-1}(X) \to \mathbb{P}^N;$$

the $(s-1)$-secant variety of X is just the image of the map p_1:

$$\mathrm{Sec}_{s-1}(X) = \overline{Im(p_1 : AbS^{s-1}(X) \to \mathbb{P}^N)}.$$

Now, if $\dim(X) = n$, it is clear that, while $\dim(AbS^{s-1}(X)) = ns + s - 1$, the dimension of $\mathrm{Sec}_{s-1}(X)$ can be smaller: it suffices that the generic fiber of p_1 has positive dimension to impose $\dim(\mathrm{Sec}_{s-1}(X)) < n(s-1) + n + s - 1$. So it is a general fact that if $X \subset \mathbb{P}^N$ and $\dim(X) = n$ then:

$$\dim(\mathrm{Sec}_{s-1}(X)) \leq \min\{N, sn + s - 1\}.$$

Definition 1.5.7. *A projective variety $X \subset \mathbb{P}^N$ of dimension n is said to be $(s-1)$-defective if* $\dim(\mathrm{Sec}_{s-1}(X)) < \min\{N, sn+s-1\}$ *and* $\delta_{s-1}(X) := \min\{N, sn+s-1\} - \dim(\mathrm{Sec}_{s-1}(X))$ *is called the $(s-1)$-defect of X.*

Example: A classical example of defective projective variety is the quadric Veronese surface: $dim(AbS^1(\nu_2(\mathbb{P}^2))) = \min\{5, 2\cdot 2+1\} = 5$ but $\dim(\mathrm{Sec}_1(\nu_2(\mathbb{P}^2))) = 4$.

1.5.3 Secant varieties of Veronese varieties and fat points

With this new point of view, it is not difficult to understand that if the variety X is precisely a Veronese variety $\nu_d(\mathbb{P}^n)$, then $\mathrm{Sec}_{s-1}(\nu_d(\mathbb{P}^n))$ parameterizes sums of s d-powers of linear forms of $K[x_0, \ldots, x_n]$.

As an easy consequence of this fact we have the following proposition that is another way to attack the big Waring problem:

Proposition 1.5.8. *The generic element of $S_d = K[x_0, \ldots, x_n]_d$ can be written as a sum of s d-th powers of linear forms if and only if* $\mathrm{Sec}_{s-1}(\nu_d(\mathbb{P}^n)) = \mathbb{P}^{N-1}$, *with* $N = \binom{n+d}{d}$.

What about the dimension of $\mathrm{Sec}_{s-1}(\nu_d(\mathbb{P}^n))$? The link between the big Waring problem and the secant variety of Veronese variety shows that the differential of the map ϕ defined in (1.1) gives the parameterization of the tangent space to $\mathrm{Sec}_{s-1}(\nu_d(\mathbb{P}^n))$ at a point on $< P_1, \ldots, P_s >$ where each P_i has as representative prime ideal $\wp_i = (L_i) \subset S$ with $L_i \in S_1$ for $i = 1, \ldots, s$. So the problem can be rephrased in terms of Hilbert functions of 2-fat points as follows:

Theorem 1.5.9. *If ν_d is the d-uple Veronese embedding of $\mathbb{P}^n$ into $\mathbb{P}^{\binom{n+d}{d}-1}$ and $\wp_i \subset S = K[x_0, \ldots, x_n]$, $i = 1, \ldots, s$, are prime ideals of points $P_1, \ldots, P_s$, then*

$$\boxed{\dim(\mathrm{Sec}_{s-1}(\nu_d(\mathbb{P}^n))) = H\left(\frac{S}{\wp_1^2 \cap \cdots \cap \wp_s^2}, d\right) - 1}\ .$$

Corollary 1.5.10. *The $(s-1)$-secant variety of the d-th Veronese variety of $\mathbb{P}^{\binom{n+d}{d}-1}$ fills up the whole $\mathbb{P}^{\binom{n+d}{d}-1}$ if and only if $\wp_1^2 \cap \cdots \cap \wp_s^2 = \{0\}$.*

Thanks to Alexander Hirschowitz's Theorem (see 1.4.12), Theorem 1.5.9 allows us to know the dimension of $\mathrm{Sec}_{s-1}(\nu_d(\mathbb{P}^n))$. If we try to compute $\dim(\mathrm{Sec}_{s-1}(\nu_d(\mathbb{P}^n)))$ in an heuristic way, we have to

- consider an s-uple $(P_1, \dots, P_s) \in \overbrace{\nu_d(\mathbb{P}^n) \times \cdots \times \nu_d(\mathbb{P}^n)}^{s} = (\nu_d(\mathbb{P}^n))^s$ (then $\dim(\nu_d(\mathbb{P}^n))^s = ns$);
- consider s generic points of $\mathbb{P}^n$ (they span a $\mathbb{P}^{s-1} \subset \mathbb{P}^{\binom{n+d}{d}-1}$);

so we expect that $\dim(\mathrm{Sec}_{s-1}(\nu_d(\mathbb{P}^n))) = ns + s - 1$ unless $ns + s - 1 \geq \binom{n+d}{d} - 1$ where we expect that $\mathrm{Sec}_{s-1}(\nu_d(\mathbb{P}^n)) = \mathbb{P}^{\binom{n+d}{d}-1}$. In other words

$$\mathrm{expdim}(\mathrm{Sec}_{s-1}(\nu_d(\mathbb{P}^n)) = \min\left\{\binom{n+d}{d} - 1, (n+1)s - 1\right\} = \dim(AbS^{s-1}(\nu_d(\mathbb{P}^n))).$$

Now the Alexander Hirschowitz Theorem tells that the dimension of the s-secant variety to the Veronese variety is not always the expected one ad we will be able to list all of them:

Theorem 1.5.11. (via Alexander-Hirschowitz) *If $X = \mathrm{Sec}_{s-1}(\nu_d(\mathbb{P}^n))$, for $d \geq 2$. Then:*

$$\dim(X) = \min\left\{\binom{n+d}{d} - 1, s(n+1) - 1\right\}$$

except for:

- $d = 2$, $n \geq 2$, $s \leq n$;
- $d = 3$, $n = 4$, $s = 7$, $(\delta = 1)$;
- $d = 4$, $n = 2$, $s = 5$, $(\delta = 1)$;
- $d = 4$, $n = 3$, $s = 9$, $(\delta = 2)$;
- $d = 4$, $n = 4$, $s = 14$, $(\delta = 1)$.

Proof. Cases with $d \geq 3$ come directly from 1.4.12. The case $d = 2$ is classically known, and a proof can run as follows.
For all $s \leq n$ we should have $\binom{n+2}{2} - s\binom{n+1}{n}$ quadrics through s 2-fat points. Consider the $\mathbb{P}^{s-1}$ containing the s simple points; there are $(n-s+1)$ linear forms through it, let them be $L_1, \dots, L_{n-s+1}$. Then $L_1^2, L_1L_2, \dots, L_{n-s+1}^2$ are quadrics in $\wp_1^2 \cap \cdots \cap \wp_s^2$ and they are $\binom{n-s+2}{2}$ in number, which is always bigger than $\binom{n+2}{2} - s\binom{n+1}{n}$. $\square$

Chapter 2

Algebraic generalization

2.1 Definition of canonical forms

We have seen how determining the Postulation of s double fat points in $\mathbb{P}^n$ can solve the big Waring problem and compute the dimension of the $(s-1)$-secant variety to the Veronese variety. If we study the postulation of other zero-dimensional schemes, we will be able to solve more general problems. In this section we want to study some known results in terms of the algebraic generalization of the big Waring problem and to describe some varieties related with this algebraic problem. Let us consider the following question:

> "Which is the least integer $G(d)$ such that the generic element of $K[x_0, \ldots, x_n]_d$ can be written as
> $$F = N_1 + \cdots + N_{G(d)} \tag{2.1}$$
> where each $N_i = M^{(i)}_{1,j(1)} \cdots M^{(i)}_{k,j(k)}$ and $M^{(i)}_{1,j(1)}, \ldots, M^{(i)}_{k,j(k)}$ belong to $K[x_0, \ldots, x_n]_{j(1)}$, $\ldots$, $K[x_0, \ldots, x_n]_{j(k)}$, respectively?"

Definition 2.1.1. *We will say that (2.1) is a "Canonical Form" in $K[x_0, \ldots, x_n]_d$ if the generic element of $K[x_0, \ldots, x_n]_d$ can be written as $F = N_1 + \cdots + N_s$ as above.*

Example: For the proves of the following three examples see [**ER**].

1. If $Q_1, Q_2, Q_3 \in K[x_0, x_1, x_2]_2$, then $F = Q_1Q_2 + Q_3^2 \in K[x_0, x_1, x_2]_4$ is a canonical form;
2. if $L_1, L_2, L_3 \in K[x_0, x_1, x_2]_1$, then there exists $c \in K$ such that $F = L_1^3 + L_2^3 + L_3^3 + cL_1L_2L_3 \in K[x_0, x_1, x_2]_3$ is a canonical form;

3. if $L_1, \dots, L_{2s} \in K[x_0, \dots, x_{2s-1}]_1$ then $F = L_1L_2 + L_3L_4 + \cdots + L_{2s-1}L_{2s} \in K[x_0, \dots, x_{2s-1}]_2$ is a canonical form.

With this definition the big Waring problem can be rephrased:

> "If $L_1, \dots, L_s$ are linear forms of $K[x_0, \dots, x_n]$, which is the least integer s such that the form $F = L_1^d + \cdots + L_s^d \in S_d$ is canonical?"

We have seen that the geometric equivalence of the big Waring problem is:

> "Which is the least integer s such that the $(s-1)$-secant variety to the d-uple Veronese embedding of $\mathbb{P}^n$ fills up the whole $\mathbb{P}^{\binom{n+d}{d}-1}$?"

We have already defined (see Definition 1.5.2) the typical rank of a projective variety $X \subset \mathbb{P}^N$, as the least integer s such that $\mathrm{Sec}_{s-1}(X) = \mathbb{P}^N$.

Consider $F, N_i, M_{l,j(l)}^{(i)} \in S_d = K[x_0, \dots, x_n]_d$ defined as in (2.1), for $i = 1, \dots, G(d)$ and $l = 1, \dots, k$. Let ϕ be the map defined as follows:

$$\begin{array}{rcl} \phi : \mathbb{P}(S_{j(1)}) \times \cdots \times \mathbb{P}(S_{j(k)}) & \to & \mathbb{P}(S_d) \\ ([M_{1,j(1)}], \dots, [M_{k,j(k)}]) & \mapsto & [M_{1,j(1)} \cdots M_{k,j(k)}] \end{array}$$

where $\sum_{l=1}^k j(l) = d$. We define now a variety X as the closure of the image of this map:

$$X := \overline{Im(\phi)}. \tag{2.2}$$

We will also say that such a X is the projective variety that parameterizes forms like the N_i's, i.e.:

$$X = \left\{ [f] \in \left(\mathbb{P}^{\binom{n+d}{d}-1}\right)^* | f = M_{1,j(1)} \cdots M_{k,j(k)}, M_{l,j(l)} \in K[x_0, \dots, x_n]_{j(l)}, l = 1, \dots, k, \sum_{l=1}^k j(l) = d \right\}.$$

Therefore the form $F = N_1 + \cdots + N_{G(d)}$ defined in (2.1) is canonical if and only if the $(s-1)$-secant variety of X fills up the whole $\left(\mathbb{P}^{\binom{n+d}{d}-1}\right)^*$.

We will come back later on this problem. Now we do a little digression on the history on the study of canonical forms.

2.2 Some known results on canonical forms

The problem to check if a form is canonical or not is an old problem. Many mathematicians in the past have tried to find ways to establish some criteria.

Looking at the references about the study of canonical forms, we found some interesting results in following papers: [**Du**], [**ER**], [**Gu**], [**KR**] and [**Wak**]. In the following tables we summarize some of those. The tables are made in the following way:

- in the first column there is the polynomial ring we are working in,
- in the second column there is the degree of the form we are considering,
- in the third column there is a not canonical (in Table 1) or canonical (in Table 2) form,
- in the fourth column there is one reference.
- In our notation L_i and Q_i are always forms of degree 1 and 2 respectively.
- When we write $d_m(n) = h$, in the third column, we mean that the generic form in $K[x_0, \ldots, x_m]_n$ can be written as a sum of $h + 1$ n-th powers of linear forms.
- The binary forms of degree 3 and 5 that are in Table 2 are all the possible canonical ones.
- An "(S)" in the last column means that the corresponding result is due to Sylvester.

Table 1.

Polynomial ring	degree	NOT canonical forms	references
$K[x_1, x_2]$	2	L_1^2	[**Wak**]
$K[x_1, x_2, x_3]$	2	$L_1^2 + L_2^2$	[**Wak**]
$K[x_1, x_2, x_3]$	4	$L_1^4 + \cdots + L_5^4$	[**ER**],[**Wak**]
$K[x_1, \ldots, x_5]$	3	$L_1^3 + \cdots + L_7^3$	[**ER**], [**Wak**]

Table 2.

Polynomial ring	degree	canonical form	references
$K[x_1, x_2]$	$2j = p \geq 4$	$L_1^p + \cdots + L_j^p + cL_1^2 \cdots L_j^2$	[**ER**]
$K[x_1, x_2]$	$d = 2r + 1$	$\sum_{i=1}^m h_i(\mathbf{x}) L_i(\mathbf{x})^{2r-k_i}$ con $h_i \in R_{k_i-1}, \sum_{i=1}^m k_i = r$	[**ER**]
$K[x_1, x_2]$	$2j - 1$	$L_1^{2j-1} + \cdots + L_j^{2j-1}$	(S) [**ER**],[**Wak**]
$K[x_1, x_2]$	$2n$ con $n \neq 1$	$L_1^{2n} + \cdots + L_n^{2n} + \frac{2n!}{n!} m L_1^2 \cdots L_n^2$	[**Wak**]
$K[x_1, x_2]$	n	$L_1^n + \cdots + L^n_{\lceil \frac{n}{2} \rceil + 1}$	[**Du**]
$K[x_1, x_2, x_3]$	1	L, $(d_3(1) = 0)$	[**Du**]
$K[x_1, x_2, x_3]$	2	$L_1^2 + L_2^2 + L_3^2$, $(d_3(2) = 2)$	[**Du**]
$K[x_1, x_2, x_3]$	3	$L_1^3 + \ldots + L_4^3$, $(d_3(3) = 3)$	[**Du**]
$K[x_1, x_2, x_3]$	3	$L_1^3 + L_2^3 + L_3^3 + cL_1L_2L_3$	[**ER**],[**Gu**]
$K[x_1, x_2, x_3]$	3	$L_1L_2L_3 + L_4L_5L_6$	[**Wak**]
$K[x_1, x_2, x_3]$	4	$L_1^4 + \cdots + L_6^4$, $(d_3(4) = 5)$	[**Du**]
$K[x_1, x_2, x_3]$	4	$Q_1Q_2 + Q_3^2$	[**ER**],[**Du**]
$K[x_1, x_2, x_3]$	4	$Q_1^2 + Q_2^2 + Q_3^2$	[**Du**]
$K[x_1, x_2, x_3]$	5	$L_1^5 + \cdots + L_7^5$	[**ER**],[**Wak**],[**Du**]
$K[x_1, x_2, x_3]$	$2h$	$L_1^{2h} + \cdots + L^{2h}_{\frac{1}{2}h(h+3)} + \cdots$ $(d_3(2h) \geq \frac{1}{2}h(h+3))$	[**Du**]
$K[x_1, x_2, x_3]$	n	$L_1^n + \cdots + L^n_{\frac{1}{6}(n+4)(n-1)+1} + \cdots$ $(d_3(n) \geq \frac{1}{6}(n+4)(n-1)+1)$	[**Du**]
$K[x_1, x_2, x_3, x_4]$	1	L, $(d_4(1) = 0)$	[**Du**]
$K[x_1, x_2, x_3, x_4]$	2	$L_1^2 + \cdots + L_4^2$, $(d_4(2) = 3)$	[**Du**]
$K[x_1, x_2, x_3, x_4]$	3	$L_1L_2L_3 + L_4L_5L_6$	[**ER**]
$K[x_1, x_2, x_3, x_4]$	3	$L_1^3 + \cdots + L_5^3$	(S) [**ER**],[**Wak**],[**Du**]
$K[x_1, x_2, x_3, x_4]$	3	$\sum_{i=1}^4 L_i^3 + 6\sum_{p=1}^4 \lambda_p L_q L_r L_s$ $\{p, q, r, s\} = \{1, 2, 3, 4\}$ $\forall p \in \{1, 2, 3, 4\}$	[**Gu**]
$K[x_1, \ldots, x_q]$ with even q	2	$L_1L_2 + L_3L_4 + \cdots + L_{q-1}L_q$	[**ER**]
$K[x_1, \ldots, x_q]$	1	L, $(d_q(1) = 0)$	[**Du**]
$K[x_1, \ldots, x_q]$	2	$L_1^2 + \cdots + L_q^2$, $(d_q(2) = q - 1)$	[**ER**], [**Du**]

2.3 Join variety

The problem of canonical forms can be generalized with respect to (2.1). Suppose that N and M are two forms of S_d such that the projective varieties X and Y parameterizing forms like N and M respectively are two different varieties. A form $F = N + M$ is canonical if and only if the projective variety parameterizing forms like $N + M$ fills up $\mathbb{P}^{\binom{n+d}{d}-1}$. Such a variety is called "Join variety" of X and Y (if $X = Y$ the join variety is the first secant variety of X).

Let $X, Y \subset \mathbb{P}^n$ be two disjoint proiective varieties and let $\mathbb{G}(k,n)$ the projective Grassmannian of subspaces of $\mathbb{P}^n$ of dimension k. Let us define j as:

$$\begin{array}{rcl} j : X \times Y & \rightarrow & \mathbb{G}(1,n) \\ (P,Q) & \mapsto & < P, Q > \end{array} \tag{2.3}$$

Remark: If $X \cap Y \neq \emptyset$ then j is a rational map: $j : X \times Y \dashrightarrow \mathbb{G}(1,n)$. The image of this map is the closure of the locus of lines $\overline{PQ}$ with $P \in X$, $Q \in Y$ and $P \neq Q$.

Definition 2.3.1. *The image of the map j defined as in (2.3) is the "Variety of lines joining X and Y" and it is denoted by $\mathcal{J}(X,Y)$.*

Definition 2.3.2. *The "Join of X and Y", denoted by $J(X,Y) \subset \mathbb{P}^n$, is closure of the union of all the lines $L \in \mathcal{J}(X,Y)$.*

We can observe that $\mathcal{J}(X,Y)$ is a subvariety of the Grassmanian, and $J(X,Y)$ is a subvariety of $\mathbb{P}^n$ (for details see [**Harr**]).

This is just a digression because we are interested in the case of forms whose geometric associated problem is in terms of secant varieties.

2.4 Inverse Systems

In the previous chapter we have introduced the concept of "Inverse Systems" that gave a way of solving the big Waring problem and, more generally, of finding the dimension of all secant varieties to the Veronese variety. This procedure can give some results also in a general case. Suppose that X is a projective variety of dimension n that parameterizes forms like $F \in K[x_0, \ldots, x_n]_d$ i.e. X is the set of all classes of forms $[f]$ in $n+1$ variables and of degree d for which there exists a change of coordinate $\phi_f : K^{n+1} \rightarrow K^{n+1}$ such that $\phi_f(f) = F$. Then $\mathrm{Sec}_{s-1}(X)$ parameterizes all forms that are linear combinations of s elements of X, i.e. if there exist $[F_1], \ldots, [F_s] \in X$ such that $f = F_1 + \cdots + F_s$ then $[f] \in \mathrm{Sec}_{s-1}(X)$. Now if we are interested in the dimension of $\mathrm{Sec}_{s-1}(X)$ the most natural thing is to study the affine dimension of its tangent space. If we know explicitly the form F we can apply the same procedure we used in the paragraph 1.1 in order to compute the elements of $T_{[f]}(\mathrm{Sec}_{s-1}(X))$. Now if it is possible to find a projective scheme Z with representative ideal $I(Z) \subset K[y_0, \ldots, y_n]$ whose inverse system is, in some degree d, the space $T_{[f]}(\mathrm{Sec}_{s-1}(X))$, it is sufficient to compute the postulation of such a scheme Z in order to compute the dimension of $\mathrm{Sec}_{s-1}(X)$:

$$H(Z,d) = \dim(S_d) - \dim(I(Z)_d) = \dim((I(Z)^{-1})_d) = \dim(T_{[f]}(\mathrm{Sec}_{s-1}(X))).$$

This procedure seems to be very complicated but in many cases it is easier to compute the projective scheme Z such that $(I(Z))_d^{-1} = T_{[f]}(\mathrm{Sec}_{s-1}(X))$ rather than to compute directly the dimension of $T_{[f]}(\mathrm{Sec}_{s-1}(X))$.

2.4.1 An example of how to use apolarity

We have seen that the problem (2.1) is equivalent to find the least integer $G(d)$ such that the $(G(d)-1)$-secant variety to the variety X defined in (2.2) fills $\mathbb{P}^N$, with $N = \binom{n+d}{d} - 1$.

Suppose now we fix s points $[f_1], \ldots, [f_s]$ of X; the question:

"is $F = f_1 + \cdots + f_s$ a canonical form of $K[x_0, \ldots, x_n]_d$?"

is a "stronger" question than:

"is $\mathrm{Sec}_{s-1}(X)$ equal to $\mathbb{P}^N$?".

Since $[f_1], \ldots, [f_s] \in X$ are fixed, then F is canonical if the linear space $V =< [f_1], \ldots, [f_s] >$ is equal to $\mathbb{P}^N$. Clearly if $V = \mathbb{P}^N$ then also $\mathrm{Sec}_{s-1}(X) = \mathbb{P}^N$ because $V \subsetneq \mathrm{Sec}_{s-1}(X)$, but the least $s \in \mathbb{N}$ such that the fixed $F \in K[x_0, \ldots, x_n]_d$ is canonical could be bigger than the least $G(d)$ such that $\mathrm{Sec}_{G(d)-1}(X) = \mathbb{P}^N$.

Example: Let L_i and Q_i, for $i = 1, \ldots, s$, be linear and quadratic forms of $K[x_0, x_1, x_2]$, respectively; for which s the form $F = L_1^{d-2}Q_1 + \cdots + L_s^{d-2}Q_s$ is canonical?

It is not very difficult to find out that there exist $P_1, \ldots, P_s \in \mathbb{P}^2$ such that the degree d part of the representing ideal $I(X)$ of the 0-dimensional scheme $X = (P_1, \ldots, P_s; 3, \ldots, 3)$ has inverse system: $< L_1^{d-2}Q_1, \ldots, L_s^{d-2}Q_s >$.

Proposition 2.4.1. *If $s(d) = \left\lceil \frac{(d+2)(d+1)}{12} \right\rceil$, $L_i \in K[x_0, x_1, x_2]_1$ and $Q_i \in K[x_0, x_1, x_2]_2$ for $i = 1, \ldots, s$, then $\forall s \geq s(d)$ the form $\sum_{i=1}^{s} L_i^{d-2}Q_i$ is canonical except for*

1. *$d = 1, 2$, when $s(d) = 1$;*
2. *$d = 3$, when $s(d) = 3$;*
3. *$d = 6$, when $s(d) = 6$.*

Proof. This proposition is proved in [**Hi**]. We show why these three cases are not expected.

1. If $d = 1, 2$, if $\wp \subset S = K[x_0, x_1, x_2]$ is the prime ideal associated to $P \in \mathbb{P}^2$, then $H(S/\wp^3, 1) = H(S/\wp^3, 2) = 0$.
2. If $d = 3$ and $s = 2$ we expect that $H((P_1, P_2; 3, 3), 3)$ is equal to he Hilbert function in degree 3 of 6 generic points of $\mathbb{P}^2$, that is zero. This is false because if $l = 0$ is the equation of the line $< P_1, P_2 > \subset \mathbb{P}^2$ and $\wp_1, \wp_2 \subset K[x_0, x_1, x_2]$ are the prime ideals associated to P_1, P_2 respectively, then $l^3 \in (\wp_1^3 \cap \wp_2^3)_3$.

3. If $d = 6$ and $s = 5$ we expect that $H((P_1, \dots, P_5; 3, \dots, 3), 6) = H((P_1, \dots, P_{30}; 1, \dots, 1), 6) = 0$, but if $\mathcal{C} = 0$ is the equation of the conic passing through $P_1, \dots, P_5$ and $\wp_1, \dots, \wp_5 \subset K[x_0, x_1, x_2]$ are the prime ideals associated to $P_1, \dots, P_5 \in \mathbb{P}^2$, then $\mathcal{C}^3 \in (\wp_1^3 \cap \dots \cap \wp_5^3)_6$.

$\square$

Example: The study of the Hilbert function of a projective scheme $X = (P_1, \dots, P_s; n, \dots, n) \subset \mathbb{P}^2$ of s n-fat points on $\mathbb{P}^2$ leads to the study of the canonicity of the form $F = L_1^{d-n} N_1 + \dots L_s^{d-n} N_s \in K[x_0, x_1, x_2]_d$ where L_i and N_i are ternary forms of degree one and n respectively. The problem is not completely solved because it is only possible to know the expected dimension of X but the exceptions are not all known yet (there is a conjecture about that, but we will consider this later).

Remark: If $s(d) = \left\lceil \frac{(d+2)(d+1)}{n(n+1)} \right\rceil$, $L_i \in K[x_0, x_1, x_2]_1$ and $N_i \in K[x_0, x_1, x_2]_n$ for $i = 1, \dots, s$, then $\forall s \geq s(d)$ the form $\sum_{i=1}^{s} L_i^{d-n} N_i$ is canonical, if and only if the Hilbert function of $\mathrm{Proj}(K[x_0, x_1, x_2]/(\wp_1^n \cap \dots \cap \wp_s^n))$ has the expected dimension in degree d.

2.5 Inverse Systems and Canonical Forms

In this section we want to work out some examples which show the use of Inverse Systems to compute Canonical Forms. We will look at some particular cases in degrees 2, 3 and 4.

2.5.1 Degree two

Let L_i, M_j be generic linear forms of $K[x_0, \dots, x_n]$ for $i = 0, \dots, 2s-1$ and $j = 0, \dots, 2k-1$; the possible kinds of forms we can find in $K[x_0, \dots, x_n]_2$ are of the following three types:

1. $L_0^2 + \dots + L_s^2$,
2. $L_0 L_1 + \dots + L_{2s-2} L_{2s-1}$,
3. $M_0 M_1 + \dots + M_{2k-2} M_{2k-1} + L_0^2 + \dots + L_s^2$.

The first case has been throughly analyzed in the previous chapter via the study of the (s)-secant varieties to the Veronese varieties $\nu_2(\mathbb{P}^n)$ (see Theorem 1.5.11).

The second case corresponds to $\mathrm{Sec}_{(s-1)}(\mathrm{Split}_2(\mathbb{P}^n))$. We will show in Section 4.1 that it is also the $(s-1)$-secant variety to the tangential variety to the Veronese variety $\nu_2(\mathbb{P}^n)$; Their dimensions can be found in [**CGG2**], Proposition 3.3.

For the third case we need the notion of Join variety introduced in Section 2.3: the variety that parameterizes forms like

$$F := L_0^2 + \cdots + L_s^2 + L_{s+1}L_{s+2} + \cdots + L_{k-1}L_k$$

is

$$J\left(\mathrm{Sec}_s(\nu_2(\mathbb{P}^n)), \mathrm{Sec}_{\frac{k-s}{2}-1}(\mathrm{Split}_2(\mathbb{P}^n))\right).$$

Proposition 2.5.1. *If $s < n$ and $n - s$ is even, then*

$$F = L_0^2 + \cdots + L_s^2 + L_{s+1}L_{s+2} + \cdots + L_{n-1}L_n \tag{2.4}$$

is a canonical form.

Proof. We denote with X the variety $J\left(\mathrm{Sec}_s(\nu_2(\mathbb{P}^n)), \mathrm{Sec}_{\frac{n-s}{2}-1}(\mathrm{Split}_2(\mathbb{P}^n))\right)$ and with W the affine cone over the tangent space $T_P(X)$ at a smooth point $P = [L_0^2 + \cdots + L_s^2 + L_{s+1}L_{s+2} + \cdots + L_{n-1}L_n] \in X$. It turns out that $W =< L_0S_1, \ldots, L_nS_1 >= S_2$, then $X = \mathbb{P}^{\binom{n+2}{2}-1}$, so the form (2.4) is canonical. □

We want to see what happens if we eliminate some terms from (2.4).

Without one square

Along all this section we will always assume that $(n - s)$ is a positive even integer. Let us take out from (2.4) the first term L_0^2; we obtain the form:

$$F_{0,s,n} = L_1^2 + \cdots + L_s^2 + L_{s+1}L_{s+2} + \cdots + L_{n-1}L_n \tag{2.5}$$

which is parameterized by

$$J_{0,s,n} = J\left(\mathrm{Sec}_{s-1}(\nu_2(\mathbb{P}^n)), \mathrm{Sec}_{\frac{n-s}{2}-1}(\mathrm{Split}_2(\mathbb{P}^n)\right).$$

Then the affine cone W on the tangent space $T_P(J_{0,s,n})$ at a smooth point $P = [L_1^2 + \cdots + L_s^2 + L_{s+1}L_{s+2} + \cdots + L_{n-1}L_n] \in J_{0,s,n}$ is

$$W =< L_1S_1, \ldots, L_nS_1 > .$$

Since the number of forms that appear in F is less then $n+1$ we can choose $L_i \in S_1$ to be x_i for all $i = 1, \dots, n$. So the ideal $I \subset R = K[y_0, \dots, y_n]$ such that $(I^{-1})_2 = W$ turns out to be:

$$I = (y_0^2),$$

hence

$$\dim(J_{0,s,n}) = \dim(W) - 1 = \binom{n+2}{2} - 2$$

but

$$\operatorname{expdim}(J_{0,s,n}) = \min\left\{N-1, sn+s-1+2n\frac{n-s}{2} + \frac{n-s}{2} - 1 + 1\right\} =$$

$$= \min\left\{N-1, n^2 + \frac{n+s}{2} - 1\right\}$$

with $N = \binom{n+2}{2}$.
Observe that $\min\left\{N-1, n^2 + \frac{n+s}{2} - 1\right\} = N-1$ for all $s \geq -n^2+2n+2$, but $(-n^2+2n+4) \leq 0$ for all $n \geq 3$, so, since $s > 0$, $\min\left\{N-1, n^2 + \frac{n+s}{2} - 1\right\} = N-1$ for all $n \geq 3$. Therefore for $n \geq 3$ the variety $\operatorname{expdim}(J_{0,s,n}) = N-1 = \dim(J_{0,s,n}) + 1$.

Hence we have proved the following:

Proposition 2.5.2. *If $0 < s \leq n-2$ and $(n-s) \in \mathbb{Z}^+$ is even, then*

$$\dim\left(J\left(\operatorname{Sec}_{s-1}(\nu_2(\mathbb{P}^n)), \operatorname{Sec}_{\frac{n-s}{2}-1}(\operatorname{Split}_2(\mathbb{P}^n))\right)\right) = \binom{n+2}{2} - 2$$

and

$$\delta\left(J\left(\operatorname{Sec}_{s-1}(\nu_2(\mathbb{P}^n)), \operatorname{Sec}_{\frac{n-s}{2}-1}(\operatorname{Split}_2(\mathbb{P}^n))\right)\right) = 1.$$

General case

Assume for all this section that $0 \leq i < s \leq j-2 < n-2$ and that both $(n-j)$ and $(j-s)$ are even positive integers. Consider a form obtained from (2.4) by taking out the terms $L_0^2, \dots, L_i^2$ and $L_{s+1}L_{s+2}, \dots, L_{j-1}L_j$:

$$\boxed{F_{i,s,j,n} = \widehat{L_0^2} + \cdots + \widehat{L_i^2} + L_{i+1}^2 + \cdots + L_s^2 + \widehat{L_{s+1}L_{s+2}} + \cdots + \widehat{L_{j-1}L_j} + L_{j+1}L_{j+2} + \cdots + L_{n-1}L_n}$$

Such a form is parameterized by

$$J_{i,s,j,n} := J\left(\mathrm{Sec}_{s-i-1}(\nu_2(\mathbb{P}^n)), \mathrm{Sec}_{\frac{n-j}{2}-1}(\mathrm{Split}_2(\mathbb{P}^n))\right). \tag{2.6}$$

The affine cone W over the tangent space at a smooth point $P = [\widehat{L_0^2} + \cdots + \widehat{L_i^2} + L_{i+1}^2 + \cdots + L_s^2 + \widehat{L_{s+1}L_{s+2}} + \cdots + \widehat{L_{j-1}L_j} + L_{j+1}L_{j+2} + \cdots + L_{n-1}L_n]$ of this variety is

$$W =< L_{i+1}S_1, \ldots, L_sS_1, L_{j+1}S_1, \ldots, L_nS_1 > .$$

Again, since the number of independent forms that appear in $F_{i,s,j,n}$ is less than the number of variables, we can choose each $L_i \in K[x_0, \ldots, x_n]_1$ to be x_i, hence the ideal $I \subset R$ such that $(I^{-1})_2 = W$ is

$$I = (y_0, \ldots, y_i, y_{s+1}, \ldots, y_j)^2,$$

then

$$\dim(J_{i,s,j,n}) = \dim(W) - 1 = N - \binom{i+1+j-s+2}{2} - 1 = N - \binom{i+j-s+3}{2} - 1$$

while

$$\mathrm{expdim}(J_{i,s,j,n}) = \min\left\{N-1, (s-i)n + (s-i) - 1 + n(n-j) + \frac{n-j}{2} - 1 + 1\right\}.$$

Observe that $\min\left\{N-1, N-1, (s-i)n + (s-i) - 1 + n(n-j) + \frac{n-j}{2}\right\} = N-1$ if and only if

$$n \geq 1 + i + j - s + \sqrt{(s-1-i-j)^2 - 2s + 2i + j + 2} := n(i,s,j). \tag{2.7}$$

The defect δ of $J_{i,s,j,n}$ is

$$\delta(J_{i,s,j,n}) = \min\left\{N-1, (s-i)n + (s-i) - 1 + n(n-j) + \frac{n-j}{2}\right\} - \left(N - \binom{i+1+j-s+2}{2} - 1\right)$$

$$= \begin{cases} \binom{i+j-s+3}{2} & \text{if } n \geq n(i,s,j); \\ ns - ni - nj - n + 2j + ij - is - js + \frac{3i-3s+n^2+i^2+j^2+s^2}{2} + 2 & \text{if } n \leq n(i,s,j). \end{cases}$$

We can state the following:

Proposition 2.5.3. *If $0 \leq i < s \leq j-2 < n-2$ and, $(n-j)$ and $(j-s)$ are even positive integers and $n(s,i,j)$ is defined as in (2.7), then*

$$\dim\left(J\left(\mathrm{Sec}_{s-i-1}(\nu_2(\mathbb{P}^n)), \mathrm{Sec}_{\frac{n-j}{2}-1}(\mathrm{Split}_2(\mathbb{P}^n))\right)\right) = \binom{n+2}{2} - \binom{i+j-s+3}{2} - 1$$

and the defect is

$$\delta(J_{i,s,j,n}) = \begin{cases} \binom{i+j-s+3}{2} & \text{if } n \geq n(i,s,j); \\ ns - ni - nj - n + 2j + ij - is - js + \frac{3i-3s+n^2+i^2+j^2+s^2}{2} + 2 & \text{if } n \leq n(i,s,j). \end{cases}$$

Example: Suppose that $i = 0$, then

$$F_{0,s,j,n} = L_1^2 + \cdots + L_s^2 + \widehat{L_{s+1}L_{s+2}} + \cdots + \widehat{L_{j-1}L_j} + L_{j+1}L_{j+2} \cdots + L_{n-1}L_n,$$

the value $n(i,s,j)$ defined in (2.7) is:

$$n(0,s,j) = 1 - s + j + \sqrt{3 - 4s - 2js + 3j + s^2 + j^2}$$

and

$$\delta(J_{0,s,j,n}) = \begin{cases} 3 - js + \frac{j^2+5j+s^2-5s}{2}, & \text{if } n \geq n(0,s,j); \\ ns - nj - n + 2j - js + \frac{n^2-3s+j^2+s^2}{2} + 2, & \text{if } n \leq n(0,s,j). \end{cases}$$

- If $i = 0$, $s = 1$ and $j = 2h + 1 \geq 3$, for $h \in \mathbb{N}$, then

$$F_{0,1,j,n} = L_1^2 + \widehat{L_2L_3} + \cdots + \widehat{L_{j-1}L_j} + L_{j+1}L_{j+2} + \cdots + L_{n-1}L_n,$$

$$n(0,1,j) = j + \sqrt{j^2 + j}$$

and

$$\delta(J_{0,1,j,n}) = \begin{cases} 1 + \frac{j^2+3j}{2}, & \text{if } n \geq n(0,1,j); \\ 1 - nj + j + \frac{n^2+j^2}{2}, & \text{if } n \leq n(0,1,j). \end{cases}$$

  - If $i = 0$, $s = 1$, $j = 3$ and $n = 2k + 1 \geq 5$, for $k \in \mathbb{N}$, then $6 < n(0,1,3) < 7$, the form we are considering is

$$F_{0,1,3,n} = L_1^2 + L_4L_5 + \cdots + L_{n-1}L_n$$

    and the defect $\delta(J_{0,1,3,n})$ is

$$\delta(J_{0,1,3,n}) = \begin{cases} 10, & \text{if } n \geq 7; \\ 6 & \text{if } n = 5. \end{cases}$$

- If $i = 0$, $s = 1$, $j = 5$ and $n = 2k + 1 \geq 7$, for $k \in \mathbb{N}$, then $10 < n(0, 1, 5) < 11$; the form we are considering is
$$F_{0,1,5,n} = L_1^2 + L_6 L_7 + \cdots + L_{n-1} L_n$$
and the defect of $J_{0,1,5,n}$ is
$$\delta(J_{0,1,5,n}) = \begin{cases} 21, & \text{if } n \geq 11; \\ \frac{n^2+37}{2} - 5n, & \text{if } n \leq 10. \end{cases}$$
Therefore:
  * if $n = 7$, then $F_{0,1,5,7} = L_1^2 + L_6 L_7 \in K[x_0, \ldots, x_7]_2$ have $\delta(J_{0,1,5,7}) = 8$.
  * If $n = 9$, then $F_{0,1,5,9} = L_1^2 + L_6 L_7 + L_8 L_9 \in K[x_0, \ldots, x_9]_2$ have $\delta(J_{0,1,5,9}) = 14$.
  * If $n \geq 11$, then $\delta(J_{0,1,5,n}) = 21$.
- If $i = 0$, $s = 1$, $j = 7$ and $n = 2k + 1 \geq 9$, for $k \in \mathbb{N}$, then $14 < n(0, 1, 7) < 15$, the form we are considering is
$$F_{0,1,7,n} = L_1^2 + L_8 L_9 + \cdots + L_{n-1} L_n$$
and the defect $\delta(J_{0,1,7,n})$ is
$$\delta(J_{0,1,7,n}) = \begin{cases} 36, & \text{if } n \geq 15; \\ \frac{n^2+65}{2} - 7n, & \text{if } n \leq 14. \end{cases}$$
Therefore:
  * if $n = 9$, then $F_{0,1,7,9} = L_1^2 + L_8 L_9 \in K[x_0, \ldots, x_9]_2$ have $\delta(J_{0,1,7,9}) = 10$.
  * If $n = 11$, then $F_{0,1,7,11} = L_1^2 + L_8 L_9 + L_{10} L_{11} \in K[x_0, \ldots, x_{11}]_2$ have $\delta(J_{0,1,7,11}) = 16$.
  * If $n = 13$, then $F_{0,1,7,13} = L_1^2 + L_8 L_9 + L_{10} L_{11} L_{12} L_{13} \in K[x_0, \ldots, x_{13}]_2$ have $\delta(J_{0,1,7,13}) = 26$.
  * If $n \geq 15$, then $\delta(J_{0,1,7,n}) = 36$.

Example: Suppose that $i = 1$, then
$$F_{1,s,j,n} = L_2^2 + \cdots + L_s^2 + \widehat{L_{s+1} L_{s+2}} + \cdots + \widehat{L_{j-1} L_j} + L_{j+1} L_{j+2} + \cdots + L_{n-1} L_n,$$
the value $n(i, s, j)$ defined in (2.7) is
$$n(1, s, j) = 2 - s + j + \sqrt{8 - 6s + 5j + s^2 - 2js + j^2}$$
and
$$\delta(J_{1,s,j,n}) = \begin{cases} 6 - js + \frac{7j-7s+j^2+s^2}{2}, & \text{if } n \geq n(1, s, j); \\ ns - 2n + 4 - nj + 3j - js + \frac{n^2-5s+j^2+s^2}{2}, & \text{if } n \leq n(1, s, j). \end{cases}$$

Example: Suppose that $i = 2$, then

$$F_{2,s,j,n} = L_3^2 + \cdots + L_s^2 + \widehat{L_{s+1}L_{s+2}} + \cdots + \widehat{L_{j-1}L_j} + L_{j+1}L_{j+2} + \cdots + L_{n-1}L_n,$$

the value $n(i,s,j)$ defined in (2.7) is

$$n(2,s,j) = 3 - s + j + \sqrt{15 - 8s + 7j + s^2 - 2js + j^2}$$

and

$$\delta(J_{2,s,j,n}) = \begin{cases} 10 - js + \frac{9j-9s+j^2+s^2}{2}, & \text{if } n \geq n(2,s,j); \\ ns - 3n + 7 - nj + 4j - js + \frac{n^2-7s+j^2+s^2}{2}, & \text{if } n \leq n(2,s,j). \end{cases}$$

Example: Suppose we want to compute the defectivity of $J(\mathrm{Sec}_{40}(\nu_2(\mathbb{P}^{80})), \mathrm{Sec}_{12}(\mathrm{Split}_2(\mathbb{P}^{80})))$. We have that $n = 80, j = 54$ and $s = 41+i$, hence $F_{i,41+i,54,80} = L_{i+1}^2 + \cdots + L_{41+i}^2 + L_{55}L_{56} + \cdots + L_{79}L_{80}$. Now, $27 < n(i, 41+i, 54) < 28 \Rightarrow n = 80 > 28 \Rightarrow \delta(J_{i,s+i,54,80}) = 120$.

2.5.2 Degree three

For the degree 3 case we study in this section two different kinds of forms:

1. in the first case $F \in S_3 = K[x_0, \ldots, x_n]_3$ is a form involving exactly $n+1$ linear forms without any repeated one, i.e. if $1 < a+1 < b < n-2$ and $(b-a)$ even, then

$$F = L_0^3 + \cdots + L_{a-1}^3 + L_a^2 L_{a+1} + \cdots + L_{b-2}^2 L_{b-1} + L_b L_{b+1} L_{b+2} + \cdots + L_{n-2} L_{n-1} L_n \quad (2.8)$$

   where L_i are all independent linear forms for $i = 0, \ldots, n$.

2. In the second case $F \in S_3$ is such that

$$F = L_0(L_1L_2 + \cdots + L_{2s-1}L_s)$$

   where $s \leq n$, s even and $L_i \in S_1$ are all independent.

Cubic forms involving exactly $n+1$ independent linear forms and without any repeated term

In the list below we enumerate all the cases we have studied for the first kind of forms. Consider the map

$$\begin{array}{rcl} \phi : \underbrace{S_1 \times \cdots \times S_1}_{n+1} & \to & S_3 \\ (L_0, \ldots, L_n) & \mapsto & F, \end{array} \quad (2.9)$$

where F is as in (2.8) for some $a, b \in \mathbb{N}$ such that $a < b < n-2$.

Let X be the projective variety obtained as the closure of the image of the map ϕ defined in (2.9).

For all this section the space W will be the affine cone over the tangent space to X at a smooth point $P = [L_0^3 + \cdots + L_{a-1}^3 + L_a^2 L_{a+1} + \cdots + L_{b-2}^2 L_{b-1} + L_b L_{b+1} L_{b+2} + \cdots + L_{n-2} L_{n-1} L_n] \in X$.

Since the number of linear forms involved in (2.8) is exactly $n+1$ we can suppose, without loss of generality, that each $L_i = x_i \in K[x_0, \ldots, x_n]_1$ for $i = 0, \ldots, n$.

For all the cases we are going to list below, the ideal $I \subset R[y_0, \ldots, y_n]$ will be the ideal such that $(I^{-1})_3 = W$.

- $\mathbf{F = L_0^3 + \cdots + L_n^3}$
  - $W =< L_0^2 S_1, \ldots, L_n^2 S_1 >$
  - $I = (y_i y_j y_k)$,

    with $i \neq j, i \neq k, j \neq k$ and $i, j, k = 0, \ldots, n$
- $\mathbf{F = L_0^3 + \cdots + L_{n-2}^3 + L_{n-1}^2 L_n}$
  - This case makes sense if and only if $n \geq 2$
  - $W =< L_0^2 S_1, \ldots, L_{n-2}^2 S_1, L_{n-1}^2 S_1, L_{n-1} L_n S_1 >$
  - $I = (y_n^3, y_n^2 y_i, y_j y_k y_h)$,

    with: $i = 0, \ldots, n-2$;
    $\{j, h\}, \{j, k\}, \{h, k\} \neq \{n-1, n\}$ and $j \neq k$, $j \neq h$, $k \neq h$

  $\mathbf{F = L_0^3 + \cdots + L_{n-4}^3 + L_{n-3}^2 L_{n-2} + L_{n-1}^2 L_n}$
  - $W =< L_0^2 S_1, \ldots, L_{n-4}^2 S_1, L_{n-3} L_{n-2} S_1, L_{n-3}^2 S_1, L_{n-1} L_n S_1, L_{n-1}^2 S_1 >$
  - $I = (y_{n-2}^3, y_n^3, y_{n-2}^2 y_i, y_n^2 y_j, y_h y_k y_l)$,

    with: $i \neq n-3, n-2$ and $j \neq n, n-1$,
    $\{h, k\}, \{h, l\}, \{k, l\} \neq \{n-3, n-2\}, \{n-1, n\}$ and $h \neq k$, $h \neq l$, $k \neq l$

  ...

  $\mathbf{F = L_0^2 L_1 + \cdots + L_{n-1}^2 L_n}$
  - This case makes sense if and only if $n \in \mathbb{N}^+$ is odd

- $W =< L_0L_1S_1, L_0^2S_1, \ldots, L_{n-1}L_nS_1, L_{n-1}^2S_1 >$
- $I = (y_{2k+1}^3, y_{2k+1}^2y_i, y_ay_by_c)$,

  with: $i \neq 2k+1$ for $k = 0, \ldots, \frac{n-1}{2}$,
  $\{a,b\}, \{a,c\}, \{b,c\} \neq \{2h, 2h+1\}$ for $h = 0, \ldots, \frac{n-1}{2}$, and $a \neq b$, $a \neq c$, $b \neq c$
- We will see in the next chapter that the variety X which parameterizes forms $F = L_0^2L_1 + \cdots + L_{n-1}^2L_n$ is the $(\frac{n+1}{2} - 1)$-secant variety to the tangential variety to the Veronese $\nu_3(\mathbb{P}^n)$.

- $\mathbf{F = L_0^3 + \cdots + L_{n-3}^3 + L_{n-2}L_{n-1}L_n}$
  - This case makes sense if and only if $n \geq 3$
  - $W =< L_0^2S_1, \ldots, L_{n-3}^2S_1, L_{n-1}L_nS_1, L_{n-2}L_nS_1, L_{n-2}L_{n-1}S_1 >$
  - $I = (y_{n-2}^3, y_{n-1}^3, y_n^3, y_{n-2}^2y_i, y_{n-1}^2y_j, y_n^2y_k, y_ay_by_c)$,

    with: $i, j, k \neq n, n-1, n-2$,
    $\{a,b\}, \{a,c\}, \{b,c\} \neq \{n-2, n-1\}, \{n-2, n\}, \{n-1, n\}$ and $a \neq b$, $a \neq c$, $b \neq c$

  $\mathbf{F = L_0^3 + \cdots + L_{n-6}^3 + L_{n-5}L_{n-4}L_{n-3} + L_{n-2}L_{n-1}L_n}$
  - This case makes sense if and only if $n \geq 6$
  - $W =< L_0^2S_1, \ldots, L_{n-6}^2S_1,$
    $L_{n-4}L_{n-3}S_1, L_{n-5}L_{n-3}S_1, L_{n-5}L_{n-4}S_1, L_{n-1}L_nS_1, L_{n-2}L_nS_1,\ L_{n-2}L_{n-1}S_1 >$
  - $I = (y_i^3, y_{n-5}^2y_a, y_{n-4}^2y_b, y_{n-3}^2y_c, y_{n-2}^2y_d, y_{n-1}^2y_e, y_n^2y_f, y_jy_hy_k)$,

    with: $i = n-5, \ldots, n$,
    $a, b, c \neq n-5, n-4, n-3$,
    $d, e, f \neq n-2, n-1, n$,
    $\{j,h\}, \{j,k\}, \{h,k\} \neq \{n-5, n-4\}, \{n-5, n-3\}, \{n-4, n-3\}, \{n-2, n-1\}, \{n-2, n\}, \{n-1, n\}$ and $j \neq h$, $j \neq k$, $h \neq k$

  ...

  $\mathbf{F = L_0L_1L_2 + \cdots + L_{n-2}L_{n-1}L_n}$
  - This case makes sense if and only if $n + 1 = 3k$.
  - $W =< L_1L_2S_1, L_0L_2S_1, L_0L_1S_1, \ldots, L_{n-2}L_{n-1}S_1, L_{n-2}L_nS_1, L_{n-1}L_nS_1 >$

- $I = (y_h^3, y_i^2 y_j, y_a y_b y_c)$,

  with: $0 \leq h, i \leq n$,
  if $i \equiv 0 \bmod 3 \Rightarrow j \neq i+1, i+2$,,
  if $i \equiv 1 \bmod 3 \Rightarrow j \neq i-1, i+1$,
  if $i \equiv 2 \bmod 3 \Rightarrow j \neq i-1, i-2$,
  $\{a,b\}, \{a,c\}, \{b,c\} \neq \{3\alpha, 3\alpha+1\}, \{3\alpha, 3\alpha+2\}, \{3\alpha+1, 3\alpha+2\}$ for $3\alpha = 0, \ldots, n-2$ and $a \neq b$, $a \neq c$, $b \neq c$

- We will show in Chapter 4 that the projective variety X that parameterizes forms $F = L_0 L_1 L_2 + \cdots + L_{n-2} L_{n-1} L_n$ is the $\left(\frac{n+1}{3} - 1\right)$-secant variety to the variety that we will call the Split variety $\mathrm{Split}_3(\mathbb{P}^n)$.

- $\mathbf{F = L_0^3 + \cdots + L_{n-5}^3 + L_{n-4}^2 L_{n-3} + L_{n-2} L_{n-1} L_n}$

  - This case makes sense if and only if $n \geq 5$
  - $W =< L_0^2 S_1, \ldots, L_{n-5}^2 S_1, L_{n-4} L_{n-3} S_1, L_{n-4}^2 S_1, L_{n-2} L_{n-1} S_1, L_{n-2} L_n S_1, L_{n-1} L_n S_1 >$
  - $I = (y_i^3, y_{n-3}^2 y_a, y_{n-2}^2 y_b, y_{n-1}^2 y_c, y_n^2 y_d, y_h y_j y_k)$,

    with: $i = n-3, \ldots, n$
    $a \neq n-4, n-3$, and $b, c, d \neq n-2, n-1, n$
    $\{h,j\}, \{h,k\}, \{j,k\} \neq \{n-4, n-3\}, \{n-2, n-1\}, \{n-2, n\}, \{n-1, n\}$ and $j \neq h$, $j \neq k$, $h \neq k$

  $\mathbf{F = L_0^3 + \cdots + L_{n-7}^3 + L_{n-6}^2 L_{n-5} + L_{n-4}^2 L_{n-3} + L_{n-2} L_{n-1} L_n}$

  - This case makes sense if and only if $n \geq 7$.
  - $W =< L_0^2 S_1, \ldots, L_{n-7}^2 S_1, L_{n-6} L_{n-5} S_1, L_{n-6}^2 S_1, L_{n-4} L_{n-3} S_1, L_{n-4}^2 S_1,$
    $L_{n-2} L_{n-1} S_1,\ L_{n-2} L_n S_1, L_{n-1} L_n S_1 >$
  - $I = (y_i^3, y_{n-5}^2 y_a, y_{n-3}^2 y_b, y_{n-2}^2 y_c, y_{n-1}^2 y_d, y_n^2 y_e, y_h y_j y_k)$,

    with: $i = n-5, n-3, n-2, n-1, n$,
    $a \neq n-5, n-6$, $b \neq n-4, n-3$, $c, d, e \neq n-2, n-1, n$,
    $\{h,j\}, \{h,k\}, \{j,k\} \neq \{n-6, n-5\}, \{n-4, n-3\}, \{n-2, n-1\}, \{n-2, n\}, \{n-1, n\}$ and $h \neq j$, $h \neq k$, $j \neq k$

  $\ldots$

  $\mathbf{F = L_0^2 L_1 + \cdots + L_{n-4}^2 L_{n-3} + L_{n-2} L_{n-1} L_n}$

- This case makes sense if and only if $n \geq 4$ and $n \in \mathbb{N}$ even.
- $W =< L_0L_1S_1, L_0^2S_1, \ldots, L_{n-4}L_{n-3}S_1, L_{n-4}^2S_1, L_{n-2}L_{n-1}S_1, L_{n-2}L_nS_1, L_{n-1}L_nS_1 >$
- $I = (y_i^3, y_j^2y_h, y_ay_by_c)$,

  with: $i = n-2, n, 2k+1$ for $k = 0, \ldots, \frac{n}{2}-1$;
  if $j \leq n-3 \Rightarrow j = 2k+1, h \neq j-1$,
  if $j = n-2 \Rightarrow h \neq n-1, n$,
  if $j = n-1 \Rightarrow h \neq n-2, n$,
  if $j = n \Rightarrow h \neq n-2, n-1$;
  $\{a,b\}, \{a,c\}, \{b,c\} \neq \{2k, 2k+1\}, \{n-2, n-1\}, \{n-2, n\}, \{n-1, n\}$ for $k = 0, \ldots, \frac{n}{2}-2$ and $a \neq b$, $a \neq c$, $b \neq c$
- The projective variety that parameterizes forms $F = L_0^2L_1 + \cdots + L_{n-4}^2L_{n-3} + L_{n-2}L_{n-1}L_n$ is $J\left(\mathrm{Sec}_{\left(\frac{n-2}{2}-1\right)}(T(\nu_3(\mathbb{P}^n))), \mathrm{Split}_3(\mathbb{P}^n)\right)$ where $T(\nu_3(\mathbb{P}^n))$ is the tangential variety to $\nu_3(\mathbb{P}^n)$.

- $\mathbf{F = L_0^3 + \cdots + L_{n-8}^3 + L_{n-7}^2L_{n-6} + L_{n-5}L_{n-4}L_{n-3} + L_{n-2}L_{n-1}L_n}$
  - This case makes sense if and only if $n \geq 8$.
  - $W =< L_0^2S_1, \ldots, L_{n-8}^2S_1, L_{n-7}L_{n-6}S_1, L_{n-7}^2S_1, L_{n-5}L_{n-4}S_1, L_{n-5}L_{n-3}S_1, L_{n-4}L_{n-3}S_1, L_{n-2}L_{n-1}S_1, L_{n-2}L_nS_1, L_{n-1}L_nS_1 >$
  - $I = (y_i^3, y_{n-6}^2y_a, y_{n-5}^2y_b, y_{n-4}^2y_c, y_{n-3}^2y_d, y_{n-2}^2y_e, y_{n-1}^2y_f, y_n^2y_g, y_jy_hy_k)$,

    with: $i \geq n-6$;
    $a \neq n-7, n-6$, $b, c, d \neq n-5, n-4, n-3$, $e, f, g \neq n-2, n-1, n$;
    $\{j,h\}, \{j,k\}, \{h,k\} \neq \{n-6, n-7\}, \{n-5, n-4\}, \{n-5, n-3\}, \{n-4, n-3\}, \{n-2, n-1\}, \{n-2, n\}, \{n-1, n\}$ and $j \neq h$, $j \neq k$, $h \neq k$

$\mathbf{F = L_0^3 + \cdots + L_{n-10}^3 + L_{n-9}^2L_{n-8} + L_{n-7}^2L_{n-6} + L_{n-5}L_{n-4}L_{n-3} + L_{n-2}L_{n-1}L_n}$

  - This case makes sense if and only if $n \geq 10$.
  - $W =< L_0^2S_1, \ldots, L_{n-10}^2S_1, L_{n-9}L_{n-8}S_1, L_{n-9}^2S_1, L_{n-7}L_{n-6}S_1, L_{n-7}^2S_1, L_{n-5}L_{n-4}S_1, L_{n-5}L_{n-3}S_1, L_{n-4}L_{n-3}S_1, L_{n-2}L_{n-1}S_1, L_{n-2}L_nS_1, L_{n-1}L_nS_1 >$
  - $I = (y_i^3, y_{n-8}^2y_a, y_{n-6}^2y_b, y_{n-5}^2y_c, y_{n-4}^2y_d, y_{n-3}^2y_e, y_{n-2}^2y_f, y_{n-1}^2y_g, y_n^2y_h, y_jy_ky_l)$,

    with: $i = n-8, n-6, n-5, n-4, n-3, n-2, n-1, n$,
    $a \neq n-9, n-8$; $b \neq n-7, n-6$;

$c, d, e \neq n-5, n-4, n-3$,
$\{j,k\}, \{j,l\}, \{k,l\} \neq \{n-9, n-8\}, \{n-7, n-6\}, \{n-5, n-4\}, \{n-5, n-3\}, \{n-4, n-3\}, \{n-2, n-1\}, \{n-2, n\}, \{n-1, n\}$ and $j \neq k$, $j \neq l$, $k \neq l$

...

$\mathbf{F = L_0^2 L_1 + \cdots + L_{n-7}^2 L_{n-6} + L_{n-5} L_{n-4} L_{n-3} + L_{n-2} L_{n-1} L_n}$

- This case makes sense if and only if $n \geq 7$ and $n \in \mathbb{N}$ is odd.
- $W =< L_0 L_1 S_1, L_0^2 S_1, \ldots, L_{n-7} L_{n-6} S_1, L_{n-7}^2 S_1, L_{n-5} L_{n-4} S_1, L_{n-5} L_{n-3} S_1, L_{n-4} L_{n-3}, L_{n-2} L_{n-1} S_1, L_{n-2} L_n S_1, L_{n-1} L_n S_1 >$
- $I = (y_i^3, y_l^2 y_j, y_{n-5}^2 y_a, y_{n-4}^2 y_b, y_{n-3}^2 y_c, y_{n-2}^2 y_d, y_{n-1}^2 y_e, y_n^2 y_f, y_m y_r y_q)$,

  with: $i = 2k+1, n-5, n-3, n-1$; for $k = 0, \ldots, \frac{n-1}{2}$,
  $l = 2h+1$, $j \neq l-1$ and $h = 0, \ldots, \frac{n-6}{2}$;
  $a, b, c \neq n-5, n-4, n-3$; $d, e, f \neq n-2, n-1, n$
  $\{m,r\}, \{m,q\}, \{r,q\} \neq \{n-2, n-1\}, \{n-2, n\}, \{n-1, n\}, \{n-5, n-4\}, \{n-5, n-3\}, \{n-4, n-3\}\{2k, 2k+1\}$ for $k = 0, \ldots, \frac{n-7}{2}$ and $m \neq r$, $m \neq q$, $r \neq q$
- The projective variety X that parameterizes forms $F = L_0^2 L_1 + \cdots + L_{n-7}^2 L_{n-6} + L_{n-5} L_{n-4} L_{n-3} + L_{n-2} L_{n-1} L_n$ is $J\left(\mathrm{Sec}_{\left(\frac{n-s}{2}-1\right)}(T(\nu_3(\mathbb{P}^n))), \mathrm{Sec}_1(\mathrm{Split}_3(\mathbb{P}^n))\right)$.

...

...

- $\mathbf{F = L_0^3 + \cdots + L_{n-\alpha-3}^3 + L_{n-\alpha-2}^2 L_{n-\alpha-1} + L_{n-\alpha} L_{n-\alpha+1} L_{n-\alpha+2} + \cdots + L_{n-2} L_{n-1} L_n}$
  - This case makes sense if and only if $\alpha \geq 2$ and $\alpha + 1 = 3k$.
  - $W =< L_0^2 S_1, \ldots, L_{n-\alpha-3}^2 S_1, L_{n-\alpha-2} L_{n-\alpha-1} S_1, L_{n-\alpha-2}^2 S_1, L_{n-\alpha} L_{n-\alpha+1} S_1, L_{n-\alpha} L_{n-\alpha+2} S_1, L_{n-\alpha+1} L_{n-\alpha+2} S_1, \ldots, L_{n-2} L_{n-1} S_1, L_{n-2} L_n S_1, L_{n-1} L_n S_1 >$
  - $I = (y_i^3, y_{n-\alpha-1}^2 y_j, y_h^2 y_k, y_a y_b y_c)$,

    with: $i = n-\alpha-1, \ldots, n$ and $j \neq n-\alpha-2, n\alpha-1$;
    $h \neq k$, $\{h,k\} \neq \{n-\alpha-2, n-\alpha-1\}, \{n-\alpha+3l, n-\alpha+3l+1\}, \{n-\alpha+3l, n-\alpha+3l+2\}, \{n-\alpha+3l+1, n-\alpha+3l+2\}$ for $l = 0, \ldots, \frac{\alpha-2}{3}$

$a \neq b$, $a \neq c$, $b \neq c$
or $a, b, c \leq n - \alpha - 3$,
or $\{a, b\}, \{b, c\}, \{a, c\} \neq \{n - \alpha + 3l, n - \alpha + 3l + 1\}, \{n - \alpha + 3l, n - \alpha + 3l + 2\}, \{n - \alpha + 3l + 1, n - \alpha + 3l + 2\}, \{n - \alpha - 2, n - \alpha - 1\}$ for $l = 0, \ldots, \frac{\alpha-2}{3}$

$\mathbf{F = L_0^3 + \cdots + L_{n-\alpha-5}^3 + L_{n-\alpha-4}^2 L_{n-\alpha-3} + L_{n-\alpha-2}^2 L_{n-\alpha-1} +}$
$\mathbf{+ L_{n-\alpha} L_{n-\alpha+1} L_{n-\alpha+2} + \cdots + L_{n-2} L_{n-1} L_n}$

- This case makes sense if and only if $n \geq \alpha + 5$, $\alpha + 1 = 3k$ and $\alpha \geq 2$.
- $W =< L_0^2 S_1, \ldots, L_{n-\alpha-5}^2 S_1, L_{n-\alpha-4} L_{n-\alpha-3} S_1, L_{n-\alpha-4}^2 S_1, L_{n-\alpha-2} L_{n-\alpha-1} S_1, L_{n-\alpha-2}^2 S_1,$
  $L_{n-\alpha} L_{n-\alpha+1} S_1, L_{n-\alpha} L_{n-\alpha+2} S_1, L_{n-\alpha+1} L_{n-\alpha+2} S_1, \ldots, L_{n-2} L_{n-1} S_1, L_{n-2} L_n S_1, L_{n-1} L_n S_1 >$
- $I = (y_i^3, y_k^2 y_j, y_a y_b y_c)$,

  with: $i = n - \alpha - 3, n - \alpha - 1, n - \alpha, n - \alpha + 1, \ldots, n$;
  $k \geq n - \alpha - 3$, $k \neq j$, if $k = n - \alpha - 3 \Rightarrow j \neq n - \alpha - 4, n - \alpha - 3$, if $k = n - \alpha - 1 \Rightarrow j \neq n - \alpha - 2, n - \alpha - 1$,
  $\{k, j\} \neq \{n - \alpha + 3l, n - \alpha + 3l + 1\}, \{n - \alpha + 3l, n - \alpha + 3l + 2\}, \{n - \alpha + 3l + 1, n - \alpha + 3l + 2\}$ for $l = 0, \ldots, \frac{\alpha-2}{3}$
  $a \neq b$, $a \neq c$, $b \neq c$,
  or $a, b, c \leq n - \alpha - 5$
  or $\{a, b\}, \{b, c\}, \{a, c\} \neq \{n - \alpha + 3l, n - \alpha + 3l + 1\}, \{n - \alpha + 3l, n - \alpha + 3l + 2\}, \{n - \alpha + 3l + 1, n - \alpha + 3l + 2\}, \{n - \alpha - 2, n - \alpha - 1\}, \{n - \alpha - 4, n - \alpha - 3\}$ for $l = 0, \ldots, \frac{\alpha-2}{3}$

...

$\mathbf{F = L_0^2 L_1 + \cdots + L_{n-\alpha-2}^2 L_{n-\alpha-1} + L_{n-\alpha} L_{n-\alpha+1} L_{n-\alpha+2} + \cdots + L_{n-2} L_{n-1} L_n}$

- This case makes sense if and only if $n \geq \alpha + 2$, $\alpha \geq 2$, $(n - \alpha)$ is even and $n - \alpha + 1 = 3k$.
- $W =< L_0 L_1 S_1, L_0^2 S_1, \ldots, L_{n-\alpha-2} L_{n-\alpha-1} S_1, L_{n-\alpha-2}^2 S_1,$
  $L_{n-\alpha} L_{n-\alpha+1} S_1, L_{n-\alpha} L_{n-\alpha+2} S_1, L_{n-\alpha+1} L_{n-\alpha+2} S_1, \ldots, L_{n-2} L_{n-1} S_1, L_{n-2} L_n S_1, L_{n-1} L_n S_1 >$
- $I = (y_i^3, y_h^2 y_j, y_a y_b y_c)$,

  with: $i > n - \alpha$ or $i = 2k + 1$ for $k = 0, \ldots, \frac{n-\alpha}{2} - 1$;
  if $h \leq n - \alpha - 1 \Rightarrow h = 2k + 1$ and $j \neq h - 1, h$,
  but if $h \geq n - \alpha \Rightarrow \{h, j\} \neq \{n - \alpha + 3k, n - \alpha + 3k + 1\}, \{n - \alpha + 3k, n - \alpha + 3k + 2\}, \{n - \alpha + 3k + 1, n - \alpha + 3k + 2\}$ for $k = 0, \ldots, \frac{\alpha-2}{3}$ and $h \neq j$
  $\{a, b\}, \{b, c\}, \{a, c\} \neq \{n - \alpha + 3k, n - \alpha + 3k + 1\}, \{n - \alpha + 3k, n - \alpha + 3k + 2\}, \{n -$

$\alpha+3k+1, n-\alpha+3k+2\}, \{2l, 2l+1\}$ for $k=0,\dots,\frac{\alpha-2}{3}$ and $l=0,\dots,\frac{n-\alpha}{2}-1$ and $a\neq b$, $a\neq c$, $b\neq c$.

- The projective variety X that parameterizes forms $F=L_0^2L_1+\cdots+L_{n-\alpha-2}^2L_{n-\alpha-1}+L_{n-\alpha}L_{n-\alpha+1}L_{n-\alpha+2}+\cdots+L_{n-2}L_{n-1}L_n$ is $J\left(\mathrm{Sec}_{\left(\frac{n-\alpha-1}{2}-1\right)}(T(\nu_3(\mathbb{P}^n))), \mathrm{Sec}_{\left(\frac{\alpha+2}{3}-1\right)}(\mathrm{Split}_3(\mathbb{P}^n)\right.$

...

- $\mathbf{F=L_0^2L_1+L_2L_3L_4+\cdots+L_{n-2}L_{n-1}L_n}$
  - This case makes sense if and only if $n-1=3k$ and $n\geq 4$.
  - $W=< L_0^2S_1, L_0L_1S_1, L_2L_3S_1, L_2L_4S_1, L_3L_4S_1,\dots,L_{n-2}L_{n-1}S_1, L_{n-2}L_nS_1, L_{n-1}L_nS_1 >$
  - $I=(y_i^3, y_1^2y_j, y_a^2y_b, y_0y_cy_d, y_ey_fy_g)$,

    with: $i=1,\dots,n$, $j\neq 0,1$,
    if $a\equiv 0 \bmod 3 \Rightarrow b\neq a-1, a+1$,
    if $a\equiv 1 \bmod 3 \Rightarrow b\neq a-1, a-2$;
    if $a\equiv 2 \bmod 3 \Rightarrow b\neq a+1, a+2$,
    $c\neq d\geq 2 \Rightarrow \{c,d\}\neq(2+3k, 2+3k+1)$ for $k=0,\dots,\frac{n-4}{3}$
    $\{ef\}, \{f,g\}, \{e,g\}\neq\{2+3k, 2+3k+1\}, \{2+3k, 2+3k+2\}, \{2+3k+1, 2+3k+2\}, \{0,1\}$ for $k=0,\dots,\frac{n-4}{3}$
  - The projective variety X that parameterizes forms $F=L_0^2L_1+L_2L_3L_4+\cdots+L_{n-2}L_{n-1}L_n$ is $J\left(T(\nu_3(\mathbb{P}^n)), \mathrm{Sec}_{\left(\frac{n-1}{3}-1\right)}(\mathrm{Split}_3(\mathbb{P}^n))\right)$.

Since no one of the previous ideal $I\subset R$ is (0) we can conclude that no one of the forms F written above is canonical.

We can summarize the long list of the previous pages as follows.

A degree 3 form F involving exactly $n+1$ linear forms and without any repeated term can be written as (2.8), i.e.:

$$F=L_0^3+\cdots+L_{a-1}^3+L_a^2L_{a+1}+\cdots+L_{b-2}^2L_{b-1}+L_bL_{b+1}L_{b+2}+\cdots+L_{n-2}L_{n-1}L_n.$$

Then the affine cone over the tangent space to the projective variety X defined as the closure of the image of the map ϕ defined in (2.9) at a smooth point $P=[L_0^3+\cdots+L_{a-1}^3+L_a^2L_{a+1}+\cdots+L_{b-2}^2L_{b-1}+L_bL_{b+1}L_{b+2}+\cdots+L_{n-2}L_{n-1}L_n]\in X$ is

$$W =< L_0^2 S_1, \dots, L_{a-1}^2 S_1, L_a^2 S_1, L_a L_{a+1} S_1, \dots, L_{b-2}^2 S_1, L_{b-2} L_{b-1} S_1,$$
$$L_b L_{b+1} S_1, L_b L_{b+2} S_1, L_{b+1} L_{b+2} S_1, \dots L_{n-2} L_{n-1} S_1, L_{n-2} L_n S_1, L_{n-1} L_n S_1 > \tag{2.10}$$

and the ideal $I \subset R$ such that $(I^{-1})_3 = W$ is

$$I = (y_i^3, y_j^2 y_h, y_l y_m y_r)$$

with:

- $i = b, \dots, n$;
  or

  $$\begin{cases} i = a + 2k + 1 \\ k \in \mathbb{Z} \\ 0 \leq k \leq \frac{b-a-2}{2} \end{cases}$$

- $$\begin{cases} j = a + 2k + 1 \\ k \in \mathbb{Z} \\ 0 \leq k \leq \frac{b-a-2}{2} \\ h \neq j - 1 \end{cases}$$

  or

  $j = b, \dots, n$; $j \neq h$; $\{j, h\} \neq \{b + 3k, b + 3k + 1\}, \{b + 3k, b + 3k + 2\}, \{b + 3k + 1, b + 3k + 2\}$; $k \in \mathbb{Z}$; $k = 0, \dots, \frac{n-b-2}{3}$

- $l, m, r \leq a - 1$;

  or

  $\{l, m\}, \{l, r\}, \{m, r\} \neq \{a + 2k, a + 2k + 1\}, \{b + 3l, b + 3l + 1\}, \{b + 3l, b + 3l + 2\}, \{b + 3l + 1, b + 3l + 2\}$, for $l = 0, \dots, \frac{n-b-2}{2}$ and $k = 0, \dots, \frac{b-a-2}{2}$;

  and $l \neq m$, $l \neq r$, $m \neq r$.

Cubic form obtained as the product of a linear form and a quadric without repeated terms

Now we want to study the forms of type:

$$F = L_0(L_1L_2 + \cdots + L_{2s-1}L_{2s}) \tag{2.11}$$

where L_i are generic linear forms for $i = 0, \ldots, 2s$ and $2s \leq n$.

How can we view the variety that parameterizes forms F? Consider the following two maps:

$$\begin{array}{rcl} \alpha : \underbrace{\mathbb{P}(S_1) \times \cdots \times \mathbb{P}(S_1)}_{2s+1} & \to & \mathbb{P}(S_1) \times \mathbb{P}(S_2) \\ ([L_0]; [L_1], \ldots, [L_{2s}]) & \mapsto & ([L_0]; [L_1L_2 + \cdots + L_{2s-1}L_{2s}]) \end{array}$$

and

$$\begin{array}{rcl} \beta : \mathbb{P}(S_1) \times \mathbb{P}(S_2) & \to & \mathbb{P}(S_3), \\ ([L], [Q]) & \mapsto & [LQ]. \end{array}$$

Their composition turns out to be:

$$\begin{array}{ccccc} \underbrace{\mathbb{P}(S_1) \times \cdots \times \mathbb{P}(S_1)}_{2s+1} & \xrightarrow{\alpha} & \mathbb{P}(S_1) \times \mathbb{P}(S_2) & \xrightarrow{\beta} & \mathbb{P}(S_3), \\ ([L_0]; [L_1], \ldots, [L_{2s}]) & \mapsto & ([L_0]; [L_1L_2 + \cdots + L_{2s-1}L_{2s}]) & \mapsto & [L_0(L_1L_2 + \cdots + L_{2s-1}L_{2s})]. \end{array}$$

The closure of the image of $\beta \circ \alpha$ is the variety parameterizing forms of type (2.11). We view it as $\mathbb{P}^n \times \mathrm{Sec}_{s-1}(\mathrm{Split}_2(\mathbb{P}^n))$. Which is its expected dimension? The dimension of $\mathrm{Split}_2(\mathbb{P}^n)$ is $2n$, then the expected dimension of $\mathrm{Sec}_{s-1}(\mathrm{Split}_2(\mathbb{P}^n))$ is $\min\left\{\binom{n+2}{2} - 1, 2ns + s - 1\right\}$; hence

$$\mathrm{expdim}(\mathbb{P}^n \times \mathrm{Sec}_{s-1}(\mathrm{Split}_2(\mathbb{P}^n))) = \min\left\{\binom{n+2}{2} - 1, 2ns + s - 1 + n\right\}.$$

Let W be the affine cone over the tangent space to $\mathbb{P}^n \times \mathrm{Sec}_{s-1}(\mathrm{Split}_2(\mathbb{P}^n))$ at a smooth point $P = [L_0(L_1L_2 + \cdots + L_{2s-1}L_{2s})]$, and let $I \subset R = K[y_0, \ldots, y_n]$ the ideal such that $(I^{-1})_3 = W$.

The form (2.11) can be written as

$$F = L_0L_1L_2 + L_0L_3L_4 + \cdots + L_0L_{2s-1}L_{2s}.$$

Consider the variety $\mathrm{Split}_3(\mathbb{P}^n)$. If $n \geq 3$ we can choose the forms $L_0, L_i, L_j \in K[x_0, \ldots, x_n]_1$ to be x_0, x_i, x_j respectively. We will see in Section 4.1.1 that if $n \geq 3$ then the ideal $I_{0,i,j} \subset R$ such that $(I_{0,i,j}^{-1})_3$ is the affine cone over the tangent space to $\mathrm{Split}_3(\mathbb{P}^n)$ at the point $P = [L_0L_iL_j]$ is

$$I_{0,i,j} = (y_0^3, y_i^3, y_j^3) + (y_0^2, y_i^2, y_j^2)(y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_{j-1}, y_{j+1} \ldots, y_n) +$$

$$+ (y_0, y_i, y_j)(y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_{j-1}, y_{j+1} \ldots, y_n)^2 +$$

$$+(y_1,\dots,y_{i-1},y_{i+1},\dots,y_{j-1},y_{j+1}\dots,y_n)^3.$$

Hence, since $n \geq 2s$, we can choose the forms L_i to be x_i for $i = 0,\dots,2s$; moreover we can write the ideal I as:

$$I = I_{0,1,2} \cap I_{0,3,4} \cap \cdots \cap I_{0,2s-1,2s}.$$

If we compute it we obtain:

$$I = (y_0^3,\dots,y_{2s}^3) + (y_0)^2(y_{2s+1},\dots,y_n) + \sum_{\substack{[i]_2=[1]_2\\ i=1}}^{2s-1} (y_i^2,y_{i+1}^2)(y_1,\dots,y_{i-1},\hat{y}_i,\hat{y}_{i+1},y_{i+2},\dots,y_n)+$$

$$+(y_0,\dots,y_{2s})(y_{2s+1},\dots,y_n)^2 + \sum_{\substack{[i]_2=[j]_2=[k]_2=[1]_2\\ i\neq j,\, i\neq k,\, j\neq k\\ i,j,k=1,\dots,2s-1}} (y_i,y_{i+1})(y_j,y_{j+1})(y_k,y_{k+1})+$$

$$+\left(\sum_{\substack{[i]_2=[j]_2=[1]_2\\ i\neq j\\ i,j=1,\dots,2s-1}} (y_i,y_{i+1})(y_j,y_{j+1})\right)(y_{2s+1},\dots,y_n) + (y_{2s+1},\dots,y_n)^3+$$

$$+\left(\sum_{\substack{[i]_2=[j]_2=[1]_2\\ i\neq j\\ i,j=1,\dots,2s-1}} (y_iy_{i+1}y_jy_{j+1})\right) + (y_0)^2 + \left(\sum_{\substack{[i]_2=[j]_2=[1]_2\\ i\neq j\\ i,j=1,\dots,2s-1}} (y_i,y_{i+1})(y_j,y_{j+1})\right).$$

Now the dimension of $\mathbb{P}^n \times \mathrm{Sec}_{s-1}(\mathrm{Split}_2(\mathbb{P}^n))$ is $\dim(W) - 1 = H(R/I,3) - 1$ that is $\binom{n+3}{3} - 1 - [2s+1+n-2s+2s(n-2)+(2s+1)\binom{n-2s-1+2}{2} + 8\binom{s}{3} + 4(n-s)\binom{s}{2} + \binom{n-2s-1+3}{3}]$, hence

$$\dim(\mathbb{P}^n \times \mathrm{Sec}_{s-1}(\mathrm{Split}_2(\mathbb{P}^n))) = 3s - 2s^3 + 3ns - 1.$$

Therefore we can compute the defect δ:

$$\delta(\mathbb{P}^n \times \mathrm{Sec}_{s-1}(\mathrm{Split}_2(\mathbb{P}^n))) = \begin{cases} 2s^3 - ns - 2s + n, & \text{if } n \geq \frac{4s-1+\sqrt{16s^2-7}}{2},\\ \frac{n^2+3n}{2} + 2s^3 - 3s - 3ns + 1, & \text{otherwise.}\end{cases}$$

2.5.3 Degree four

For the degree 4 forms we study the general case of a quartic involving exactly $n+1$ linear forms but without any repeated term and the case of a quartic obtained as a product of a linear form and a cubic which involves exactly n terms and without any repeated one. We will study also the particular case of the forms that can be written as $L_0^2L_1^2 + \cdots + L_{2s-1}^2L_{2s}^2$ where $L_i \in S_1$ for $i = 0,\dots,2s$ and $2s \leq n$.

Quartics involving $n+1$ terms and without repeated terms

We can work in the same way as we did for the case of degree 3 forms.

Let us consider the map:

$$\begin{array}{rcl} \phi : \underbrace{\mathbb{P}(S_1) \times \cdots \times \mathbb{P}(S_1)}_{n+1} & \to & \mathbb{P}(S_4) \\ ([L_0], \ldots, [L_n]) & \mapsto & [F] \end{array} \tag{2.12}$$

where F is

$$F = L_0^4 + \cdots + L_{a-1}^4 + L_a^3 L_{a+1} + \cdots + L_{b-2}^3 L_{b-1} + L_b^2 L_{b+1}^2 + \cdots + L_{c-2}^2 L_{c-1}^2 + \tag{2.13}$$

$$+ L_c^2 L_{c+1} L_{c+2} + \cdots + L_{d-3}^2 L_{d-2} L_{d-1} + L_d L_{d+1} L_{d+2} L_{d+3} + \cdots + L_{n-3} L_{n-2} L_{n-1} L_n$$

for $0 < a < b < c < d < n$; $(c-b), (b-a)$ even; $(d-c) = 3\alpha$ and $(n+1-d) = 4\beta$.
Let X be the projective variety obtained as the closure of the image of the map ϕ defined in (2.12). Let W be the affine cone over the tangent space $T_P(X)$ at a smooth point $P = [F]$ where F is that one of (2.13). Let also $I \subset R$ be the ideal such that $(I^{-1})_4 = W$.
We observe that:

$$\begin{gathered} W =< L_0^3 S_1, \ldots, L_{a-1}^3 S_1, L_a^2 L_{a+1} S_1, L_a^3 S_1, \ldots, L_{b-2}^2 L_{b-1} S_1, L_{b-2}^3 S_1, \\ L_b L_{b+1}^2 S_1, L_b^2 L_{b+1} S_1, \ldots L_{c-2} L_{c-1}^2 S_1, L_{c-2}^2 L_{c-1} S_1, \\ L_c L_{c+1} L_{c+2} S_1, L_c^2 L_{c+1} S_1, L_c^2 L_{c+2} S_1, \ldots, L_{d-3} L_{d-2} L_{d-1} S_1, L_{d-3}^2 L_{d-1} S_1, L_{d-3}^2 L_{d-2} S_1, \\ L_{d+1} L_{d+2} L_{d+3} S_1, L_d L_{d+2} L_{d+3} S_1, L_d L_{d+1} L_{d+3} S_1, L_d L_{d+1} L_{d+2} S_1, \ldots, \\ L_{n-2} L_{n-1} L_n S_1, L_{n-3} L_{n-1} L_n S_1, L_{n-3} L_{n-2} L_n S_1, L_{n-3} L_{n-2} L_{n-1} S_1 > . \end{gathered}$$

Since the number of liner forms that appear in (2.13) is exactly $n+1$ we can choose each $L_i \in S_1 = K[x_0, \ldots, x_n]_1$ with the monomials x_i for $i = 0, \ldots, n$, then the ideal I is:

$$I = (y_i^4, y_j^3 y_h, y_k^2 y_l^2, y_m^2 y_p y_q, y_r y_s y_t y_u)$$

with

- – $a \leq i \leq b-1$ and $(i-a) \equiv 1 \pmod 2$
  – $i \geq b$
- $j \neq h$ and
  – $a \leq j \leq b-1$ and $(j-a) \equiv 1 \pmod 2$

- $\begin{cases} b \leq j \leq c-1 \\ \{j,h\} \neq \{b+2\alpha, b+2\alpha+1\} \\ \alpha = 0, \ldots, \frac{c-b}{2} - 1 \end{cases}$
- if $c \leq j \leq d-1 \Rightarrow \left[\begin{array}{l} \text{if } (j-c) \equiv 0 \pmod 3 \Rightarrow h \neq j, j+1, j+2 \\ \text{if } (j-c) \equiv 1, 2 \pmod 3 \Rightarrow \text{any } h \end{array} \right.$
- $j \geq d$

- $k \neq l$ and
  - $k \leq a-1$
  - $l \leq a-1$
  - $\begin{cases} \{k,l\} \neq \{a+2\alpha, a+2\alpha+1\} \\ \alpha = 0, \ldots, \frac{c-a}{2} - 1 \end{cases}$
  - $\begin{cases} (k,l) \neq (c+2\alpha, c+2\alpha+1), (c+2\alpha, c+2\alpha+2) \\ \alpha = 0, \ldots, \frac{d-c-3}{2} \end{cases}$
  - $k \geq d$
  - $l \geq d$

- $p \neq q, p \neq m, q \neq m$
  - $m \leq a-1$
  - if $a \leq m \leq b-1 \Rightarrow \left[\begin{array}{l} \text{if } (m-a) \equiv 0 \pmod 2 \Rightarrow p, q \neq m+1 \\ \text{if } (m-a) \equiv 1 \pmod 2 \Rightarrow \text{for all } p, q \end{array} \right.$
  - if $b \leq m \leq c-1 \Rightarrow \left[\begin{array}{l} \text{if } (m-b) \equiv 0 \pmod 2 \Rightarrow p, q \neq m+1 \\ \text{if } (m-b) \equiv 1 \pmod 2 \Rightarrow p, q \neq m-1 \end{array} \right.$
  - if $c \leq m \leq d-1 \Rightarrow \left[\begin{array}{l} \text{if } (m-c) \equiv 0 \pmod 3 \Rightarrow p, q \neq m+1, m+2 \\ \text{if } (m-c) \equiv 1 \pmod 3 \Rightarrow p, q \neq m-1, m+1 \\ \text{if } (m-c) \equiv 2 \pmod 3 \Rightarrow p, q \neq m-1, m-2 \end{array} \right.$
  - if $m \geq d \Rightarrow \left[\begin{array}{l} \text{if } (m-d) \equiv 0 \pmod 4 \Rightarrow p, q \neq m+1, m+2, m+3 \\ \text{if } (m-d) \equiv 1 \pmod 4 \Rightarrow p, q \neq m-1, m+1, m+2 \\ \text{if } (m-d) \equiv 2 \pmod 4 \Rightarrow p, q \neq m-1, m-2, m+1 \\ \text{if } (m-d) \equiv 3 \pmod 4 \Rightarrow p, q \neq m-1, m-2, m-3 \end{array} \right.$

- r, s, t, u different each other,

- $\{r,s,t\},\{r,s,u\},\{s,t,u\},\{r,t,u\} \neq \{c+2\alpha, c+2\alpha+1, c+2\alpha+2\}, \{d+4\beta, d+4\beta+1, d+4\beta+2\}, \{d+4\beta, d+4\beta+1, d+4\beta+3\}, \{d+4\beta, d+4\beta+2, d+4\beta+3\}, \{d+4\beta+1, d+4\beta+2, d+4\beta+3\}$ for $\alpha = 0,\dots,\frac{d-c-3}{2}$ and $\beta = 0,\dots,\frac{n-d-3}{4}$

The particular case of $L_0^2L_1^2 + \cdots + L_{2s-1}^2L_{2s}^2$

The variety that parameterizes a quartic that can be written as $L_0^2L_1^2$ can be viewed by looking at the composition of the following two maps:

$$\begin{array}{rcl} \alpha : \mathbb{P}(S_1)\times\mathbb{P}(S_1) & \to & \mathbb{P}(S_2) \\ ([L_0],[L_1]) & \mapsto & [L_0L_1] \end{array}$$

and

$$\begin{array}{rcl} \nu_2 : \mathbb{P}(S_2) & \to & \mathbb{P}(S_4) \\ [Q] & \mapsto & [Q^2]; \end{array}$$

i.e.

$$\begin{array}{ccccc} \mathbb{P}(S_1)\times\mathbb{P}(S_1) & \xrightarrow{\alpha} & \mathbb{P}(S_2) & \xrightarrow{\nu_2} & \mathbb{P}(S_4) \\ ([L_0],[L_1]) & \mapsto & [L_0L_1] & \mapsto & [L_0^2L_1^2]. \end{array}$$

The closure of the image of $\nu_2 \circ \alpha$ is $\nu_2(\mathrm{Split}_2(\mathbb{P}^n))$; its dimension is $2n$. The variety that parameterizes forms $L_0^2L_1^2 + \cdots + L_{2s-1}^2L_{2s}^2$ is $\mathrm{Sec}_{s-1}(\nu_2(\mathrm{Split}_2(\mathbb{P}^n)))$ whose expected dimension is $\min\left\{\binom{n+4}{4}-1, 2sn+s-1\right\}$.

Proposition 2.5.4. *If $2s-1 \leq n$ then*

$$\dim(\mathrm{Sec}_{s-1}(\nu_2(\mathrm{Split}_2(\mathbb{P}^n)))) = \mathrm{expdim}(\mathrm{Sec}_{s-1}(\nu_2(\mathrm{Split}_2(\mathbb{P}^n)))) = 2sn+s-1.$$

Proof. By hypothesis $2s-1 \leq n$, then

$$\mathrm{expdim}(\mathrm{Sec}_{s-1}(\nu_2(\mathrm{Split}_2(\mathbb{P}^n)))) = \min\left\{\binom{n+4}{4}-1, 2sn+s-1\right\} = 2sn+s-1.$$

Let us consider the following parameterization:

$$\phi : S_1 \times \cdots \times S_1 \to S_4$$

$$\phi(L_0,\dots,L_{2s-1}) = L_0^2L_2^2 + \cdots + L_{2s-2}^2L_{2s-1}^2$$

where, as usually, $S = K[x_0,\dots,x_n]$ and L_j are linear forms.

Let $A_0, \dots, A_{2s-1} \in S_1$, we define $2F'$ as $\lim_{\lambda\to 0}\left[\frac{d}{d\lambda}((L_0+\lambda A_0)^2(L_0+\lambda A_1)^2+\cdots+ +(L_{2s-2}+\lambda A_{2s-2})^2)(L_{2s-1}+\lambda A_{2s-1})^2\right] =$
$= \lim_{\lambda\to 0}\left[2(L_0+\lambda A_0)A_0(L_0+\lambda A_1)^2+2(L_0+\lambda A_0)^2A_1(L_0+\lambda A_1)+\cdots+ +2(L_{2s-2}+\lambda A_{2s-2})A_{2s-2}(L_{2s-1}+\lambda A_{2s-1})^2+2(L_{2s-2}+\lambda A_{2s-2})^2A_{2s-1}(L_{2s-1}+\lambda A_{2s-1})\right] =$
$= 2(L_0L_0^2A_0+L_0^2L_0A_1+L_2L_3^2A_2+L_2^2L_3A_3+\cdots+L_{2s-2}L_{2s-1}^2A_{2s-2}+L_{2s-2}^2L_{2s-1}A_{2s-1})$.

Let W be the space spanned by the forms that appear in F'.
Since $n \geq 2s-1$, then we can assume that $L_i = x_i$ for all $i = 0, \dots, 2s-1$, hence

$$W =< x_0^2x_1S_1, x_0x_1^2S_1, x_2^2x_3S_1, x_2x_3^2S_1, \dots, x_{2s-2}^2x_{2s-1}S_1, x_{2s-2}x_{2s-1}^2S_1 > .$$

By construction

$$\dim(\mathrm{Sec}_{s-1}(\nu_2(\mathrm{Split}_2(\mathbb{P}^n)))) = \dim(W) - 1.$$

In order to study the dimension of W we consider the ideal $I \subset R = K[y_0, \dots, y_n]$ such that $(I^{-1})_4 = W$. By inverse system theory it is clear that $\dim(W) = \dim(S_4) - H(R/I, 4)$ where $H(R/I, 4)$ is the Hilbert function of I in degree 4. The ideal I turns out to be the following:

$$I = (y_i^4, y_j^3y_h, y_k^2y_l^2, y_m^2y_py_q, y_ry_vy_ty_u)$$

where $i, j, h, k, l, m, p, q, r, v, t, u$ are chosen in the following way:

1. y_i with $i = 0, \dots, n$;
2. $y_j^3y_h$ with:
   - $h \neq j$,
   - if $j \leq 2s-1$ then $\begin{cases} h \neq j+1, & \text{if } j \text{ is even;} \\ h \neq j-1, & \text{if } j \text{ is odd;} \end{cases}$
   - $j = 2s, \dots, n$ and any h;
3. $y_k^2y_l^2$ with:
   - $k \neq l$,
   - if $k \leq 2s-1$ then $\begin{cases} l \neq k+1, & \text{if } k \text{ is even;} \\ l \neq k-1, & \text{se } k \text{ is odd;} \end{cases}$
   - $k = 2s, \dots, n$ and any l;
4. $y_m^2y_py_q$ with:
   - m, p, q different each other,

- and if $m \leq 2s-1$ then $\begin{cases} p,q \neq m+1, & \text{if } m \text{ is even;} \\ p,q \neq m-1, & \text{if } m \text{ is odd;} \end{cases}$
- $m = 2s, \dots, n$ and for all $p \neq q$;

5. $y_r y_v y_t y_u$ with r, v, t, u different each other.

Let now $A_1, \dots, A_5$ be the sets of the element of R_4 previously described at the "points" from 1. up to 5.. We can now verify that:

1. $\sharp A_1 = n+1$,
2. $\sharp A_2 = 2s(n-1) + (n-2s+1)n$,
3. $\sharp A_3 = \binom{n+1}{2} - s$,
4. $\sharp A_4 = \binom{n}{2}(n+1) - 2s(n-1)$,
5. $\sharp A_5 = \binom{n+1}{4}$.

Therefore $H(R/I,4) = \frac{1}{24}(n^4 + 10n^3 + 35n^2 + 50n) - 2ns - s + 1$ that is equivalent to:

$$H(R/I,4) = \binom{n+4}{4} - 1 - (2ns + s - 1)$$

so $\dim(W) = 2ns + s$ which proves

$$\boxed{\dim(\mathrm{Sec}_{s-1}(\nu_2(\mathrm{Split}_2(\mathbb{P}^n)))) = \mathrm{expdim}(\mathrm{Sec}_{s-1}(\nu_2(\mathrm{Split}_2(\mathbb{P}^n)))) = 2sn + s - 1.}$$

□

Quartics obtained as a product of a linear form and a cubic form involving n terms and without repeated terms

We want to follow a procedure similar to the one we used for cubics.

Let $2 \leq a \leq b-2 \leq n-4$, $(b-a) \in \mathbb{N}$ even and $(n-b) = 3\alpha + 2$ for $\alpha \in \mathbb{N}$. Consider first the composition of the two following maps:

$$\begin{aligned} \alpha : \underbrace{\mathbb{P}(S_1) \times \cdots \times \mathbb{P}(S_1)}_{n+1} &\to \mathbb{P}(S_1) \times \mathbb{P}(S_3) \\ ([L_0]; [L_1], \dots, [L_n]) &\mapsto ([L_0], [F_3]) \end{aligned}$$

where

$$F_3 = L_1^3 + \cdots + L_{a-1}^3 + L_a^2 L_{a+1} + \cdots + L_{b-2}^2 L_{b-1} + L_b L_{b+1} L_{b+2} + \cdots + L_{n-2} L_{n-1} L_n \in S_3 \quad (2.14)$$

and

$$\begin{array}{rcl} \beta : \mathbb{P}(S_1) \times \mathbb{P}(S_3) & \to & \mathbb{P}(S_4) \\ ([L],[C]) & \mapsto & [LC]. \end{array}$$

Let $\phi := \beta \circ \alpha$:

$$\begin{array}{rcccl} \phi : \underbrace{\mathbb{P}(S_1) \times \cdots \times \mathbb{P}(S_1)}_{n+1} & \xrightarrow{\alpha} & \mathbb{P}(S_1) \times \mathbb{P}(S_3) & \xrightarrow{\beta} & \mathbb{P}(S_4), \\ (L_0; L_1, \ldots, L_n) & \mapsto & (L_0; F_3) & \mapsto & L_0 F_3. \end{array}$$

Let X be the projective variety obtained as the closure of $Im(\phi)$.
Let W be the affine cone over the tangent space $T_P(X)$ at a smooth point $P = [F_3]$ where F_3 is as in (2.14).
Let also $I \subset R$ be the ideal such that: $(I^{-1})_4 = W$.
Therefore:

$$F = L_0 F_3 = L_0(L_1^3 + \cdots + L_{a-1}^3 + L_a^2 L_{a+1} + \cdots + L_{b-2}^2 L_{b-1} + L_b L_{b+1} L_{b+2} + \cdots + L_{n-2} L_{n-1} L_n),$$

$$W =< S_1 F, L_0 \mathcal{W} >$$

where F_3 is the form (2.14) and

$$\mathcal{W} =< L_1^2 S_1, \ldots, L_{a-1}^2 S_1, L_a^2 S_1, L_a L_{a+1} S_1, \ldots, L_{b-2}^2 S_1, L_{b-2} L_{b-1} S_1,$$

$$L_b L_{b+1} S_1, L_b L_{b+2} S_1, L_{b+1} L_{b+2} S_1, \ldots L_{n-2} L_{n-1} S_1, L_{n-2} L_n S_1, L_{n-1} L_n S_1 >;$$

so:

$$W =< L_1^3 S_1, \ldots, L_{a-1}^3 S_1, L_a^2 L_{a+1} S_1, \ldots, L_{b-2}^2 L_{b-1} S_1, L_b L_{b+1} L_{b+2} S_1, \ldots, L_{n-2} L_{n-1} L_n S_1,$$

$$L_0 L_1^2 S_1, \ldots, L_0 L_{a-1}^2 S_1, L_0 L_a^2 S_1, L_0 L_a L_{a+1} S_1, \ldots, L_0 L_{b-2}^2 S_1, L_0 L_{b-2} L_{b-1} S_1,$$

$$L_0 L_b L_{b+1} S_1, L_0 L_b L_{b+2} S_1, L_0 L_{b+1} L_{b+2} S_1, \ldots, L_0 L_{n-2} L_{n-1} S_1, L_0 L_{n-2} L_n S_1, L_0 L_{n-1} L_n S_1 > .$$

We can assume that $L_i = x_i \in S_1$, for $i = 0, \ldots, n$, then

$$I = (y_i^4, y_j^3 y_h, y_k^2 y_l^2, y_m^2 y_p y_q, y_r y_s y_t y_u)$$

with $j \neq h$; $k \neq l$; $m \neq p, m \neq q, p \neq q$; r, s, t, u different each other:

- $i = 0$ and $i \geq a$

- – $j = 0$
  – if $a \leq j \leq b-1 \Rightarrow \left[\begin{array}{l} \text{if } (j-a) \equiv 0 \pmod 2 \Rightarrow h \neq 0, j+1 \\ \text{if } (j-a) \equiv 1 \pmod 2 \Rightarrow \text{any } h \end{array} \right.$
  – $j \geq b$
- – if $k = 0 \Rightarrow \left[\begin{array}{l} a \leq l \leq b-1 \text{ and } (l-a) \equiv 1 \pmod 2 \\ l \geq b \end{array} \right.$
  – if $l = 0 \Rightarrow \left[\begin{array}{l} a \leq k \leq b-1 \text{ and } (k-a) \equiv 1 \pmod 2 \\ k \geq b \end{array} \right.$
  – if $0 < k \leq a-1 \Rightarrow l \neq 0$,
  – if $0 < l \leq a-1 \Rightarrow k \neq 0$,
  – if $a \leq k \leq b-1 \Rightarrow \left[\begin{array}{l} \text{if } (k-a) \equiv 0 \pmod 2 \Rightarrow l \neq 0, k+1 \\ \text{if } (k-a) \equiv 1 \pmod 2 \Rightarrow l \neq k-1 \end{array} \right.$
  – if $a \leq l \leq b-1 \Rightarrow \left[\begin{array}{l} \text{if } (l-a) \equiv 0 \pmod 2 \Rightarrow k \neq 0, l+1 \\ \text{if } (l-a) \equiv 1 \pmod 2 \Rightarrow k \neq l-1 \end{array} \right.$
  – if $k \geq b \Rightarrow$ any l,
  – if $l \geq b \Rightarrow$ any k;
- – if $m = 0 \Rightarrow$
  * if $p \leq a-1 \Rightarrow$ any q,
  * if $q \leq a-1 \Rightarrow$ any p,
  * if $a \leq p \leq b-1 \Rightarrow$
    · if $(p-a) \equiv 0 \pmod 2 \Rightarrow q \neq p+1$
    · if $(p-a) \equiv 1 \pmod 2 \Rightarrow q \neq p-1$
  * if $a \leq q \leq b-1 \Rightarrow$
    · if $(q-a) \equiv 0 \pmod 2 \Rightarrow p \neq q+1$
    · if $(q-a) \equiv 1 \pmod 2 \Rightarrow p \neq q-1$
  * if $p \geq b \Rightarrow$
    · if $(p-b) \equiv 0 \pmod 3 \Rightarrow q \neq p+1, p+2$
    · if $(p-b) \equiv 1 \pmod 3 \Rightarrow q \neq p-1, p+1$
    · if $(p-b) \equiv 2 \pmod 3 \Rightarrow q \neq p-1, p-2$
  * if $q \geq b \Rightarrow$
    · if $(q-b) \equiv 0 \pmod 3 \Rightarrow p \neq q+1, q+2$

  - · if $(q-b) \equiv 1 \pmod 3 \Rightarrow p \neq q-1, q+1$
  - · if $(q-b) \equiv 2 \pmod 3 \Rightarrow p \neq q-1, q-2$

- if $0 < m \leq a-1 \Rightarrow p, q \neq 0$;
- if $a \leq m \leq b-1 \Rightarrow$
  - $*$ if $(m-a) \equiv 0 \pmod 2 \Rightarrow p, q \neq 0, m+1$
  - $*$ if $(m-a) \equiv 1 \pmod 2 \Rightarrow \{p, q\} \neq \{0, m-1\}$
- if $m \geq b \Rightarrow$
  - $*$ if $(m-b) \equiv 0 \pmod 3 \Rightarrow \{p, q\} \neq \{0, m+1\}, \{0, m+2\}, \{m+1, m+2\}$
  - $*$ if $(m-b) \equiv 1 \pmod 3 \Rightarrow \{p, q\} \neq \{0, m-1\}, \{0, m+1\}, \{m-1, m+1\}$
  - $*$ if $(m-b) \equiv 2 \pmod 3 \Rightarrow \{p, q\} \neq \{0, m-1\}, \{0, m-2\}, \{m-1, m-2\}$

- $\{r, s, t\}, \{r, s, u\}, \{r, t, u\}, \{s, t, u\} \neq \{0, a+2\alpha, a+1+2\alpha\}, \{b+3\beta, b+1+3\beta, b+2+3\beta\}, \{0, b+3\beta, b+1+3\beta\}, \{0, b+3\beta, b+2+3\beta\}, \{0, b+1+3\beta, b+2+3\beta\}$ for $\alpha = 0, \ldots, \frac{b-a}{2} - 1$ and for $\beta = 0, \ldots, \frac{n-b-2}{3}$.

2.6 Three classes of canonical forms

In this section we want to present the three problems that we will study along all this thesis. We will consider three classes of forms (in the last case they will actually be tensors, whose case corresponds to studying multi-degree $(1, \ldots, 1)$ forms in a multi-graded ring). We will be interested in discovering when they are canonical. We will study this problem also from a geometric point of view and the most important result we will need is the so called "Terracini's Lemma".

We have previously described how to move the problem for the computation of the dimension of the secant varieties to a variety X to the problem of the evaluation of the Hilbert function of a projective scheme (and vice versa). This passage can be made easier by using Terracini's lemma (see [**Te**], or [**Ad**]), which we give here in the following form:

Lemma 2.6.1. (**Terracini's Lemma**) *Let X be an irreducible variety in $\mathbb{P}^N$, and let $P_1, \ldots, P_s$ be s generic points on X. Then, the projectivised tangent space to $\mathrm{Sec}_{s-1}(X)$ at a generic point $Q \in < P_1, \ldots, P_s >$ is the linear span in $\mathbb{P}^N$ of the tangent spaces $T_{P_i}(X)$ to X at P_i, $i = 1, \ldots, s$, i.e.*

$$T_Q(\mathrm{Sec}_{s-1}(X)) = < T_{P_1}(X), \ldots, T_{P_s}(X) > .$$

This "Lemma" can be proved in many ways, we present here a proof "made by hands".

Proof. We have already used the notation X^s for $X \times \cdots \times X$ taken s times. Suppose that $\dim(X) = n$. Let us consider the following incidences variety:

$$I = \{(P; P_1, \ldots, P_s) \in \mathbb{P}^n \times X^s \mid P \in < P_1, \ldots, P_s >, \text{ with } P_1, \ldots, P_s \text{ generic in } X\} \subset \mathbb{P}^n \times X^s,$$

and the two following projections:

$$\pi_1 : I \to \mathrm{Sec}_{s-1}(X)$$

and

$$\pi_2 : I \to X^s.$$

The dimension of X^s is clearly sn. If $(P_1, \ldots, P_s) \in X^s$ the fiber $\pi_2^{-1}((P_1, \ldots, P_s))$ is generically a $\mathbb{P}^{s-1}$, $s < N$. Then $\dim(I) = sn + s - 1$. If π_1 has finite fibers the $(s-1)$-secant variety to X is regular, otherwise it is defective with defect equal to the dimension of the generic fiber.

Suppose that each $P_i \in X \subset \mathbb{P}^N$ has coordinates $P_i = [a_{i,0}, \ldots, a_{i,N}]$ for $i = 1, \ldots, s$; around each P_i the variety X can be locally parameterized with some functions $f_{i,j} : K^{n+1} \to K^{n+1}$ for $i = 1, \ldots, s$ and $j = 0, \ldots, N$ that are zero at the origin:

$$X : \begin{cases} x_0 = a_{i,0} + f_{i,0}(u_{i,0}, \ldots, u_{i,n}) \\ \vdots \\ x_N = a_{i,N} + f_{i,N}(u_{i,0}, \ldots, u_{i,n}) \end{cases} .$$

Now we need a parameterization φ for $\mathrm{Sec}_{s-1}(X)$. Consider a point in the subspace spanned by s points of X (for simplicity of notation we omit the dependence of the $f_{i,j}$ from the variables $u_{i,j}$): $< (a_{1,0}+f_{1,0}, \ldots, a_{1,N}+f_{1,N}), \ldots, (a_{s,0}+f_{s,0}, \ldots, a_{s,N}+f_{s,N}) >$; an element of this subspace is of the form: $\lambda_1(a_{1,0}+f_{1,0}, \ldots, a_{1,N}+f_{1,N})+\lambda_2(a_{2,0}+f_{2,0}, \ldots, a_{2,N}+f_{2,N})+\cdots+\lambda_s(a_{s,0}+f_{s,0}, \ldots, a_{s,N}+f_{s,N})$ for some $\lambda_1, \ldots, \lambda_s \in K$ (we can assume that $\lambda_1 = 1$). Therefore a parameterization of the $(s-1)$-secant variety to X can be obtained by $(a_{1,0} + f_{1,0}, \ldots, a_{1,N} + f_{1,N}) + (\lambda_2 + t_2)(a_{2,1} - a_{1,0} + f_{2,1} - f_{1,0}, \ldots, a_{2,N} - a_{1,N} + f_{2,N} - f_{1,N}) + \cdots + (\lambda_s + t_s)(a_{s,1} - a_{1,0} + f_{s,1} - f_{1,0}, \ldots, a_{s,N} - a_{1,N} + f_{s,N} - f_{1,N})$ for some parameters $t_2, \ldots, t_s$, i.e. in coordinates the parameterization φ that we are looking for is that one that sends an element $(u_{1,0}, \ldots u_{1,n}, u_{2,0}, \ldots, u_{2,n}, \ldots\ldots, u_{s,0}, \ldots, u_{s,n}, t_2, \ldots, t_s) \in K^{s(n+1)+s-1}$ into

$$(\ldots, a_{1,j} + f_{1,j} + (\lambda_2 + t_2)(a_{2,j} - a_{1,j} + f_{2,j} - f_{1,j}) + \cdots + (\lambda_s - t_s)(a_{s,j} - a_{1,j} + f_{s,j} - f_{1,j}), \ldots) \in K^{N+1}.$$

For simplicity we have written only the j-th element of the image. Therefore we are able to write the Jacbian of φ. We are writing it in three blocks: the first one is $(N+1) \times (n+1)$, the second one is $(N+1) \times (s-1)(n+1)$ and the third one is $(N+1) \times (s-1)$:

$$J_{\underline{0}}(\varphi) = \left(\ (1 - \lambda_2 - \cdots - \lambda_s)\frac{\partial f_{1,j}}{\partial u_{1,k}} \ \mid \ \lambda_i \frac{\partial f_{i,j}}{\partial u_{i,k}} \ \mid \ a_{i,j} - a_{1,j} \ \right),$$

with $i = 2, \ldots, s$; $j = 0, \ldots, N$ and $k = 0, \ldots, n$. Now the first block is a base of the tangent space to X at P_1, and in the second block we can find the bases for the tangent spaces to X at $P_2, \ldots, P_s$; the rows of

$$\begin{pmatrix} \frac{\partial f_{i,0}}{\partial u_{i,0}} & \cdots & \frac{\partial f_{i,0}}{\partial u_{i,N}} \\ \vdots & & \vdots \\ \frac{\partial f_{i,N}}{\partial u_{i,0}} & \cdots & \frac{\partial f_{i,N}}{\partial u_{i,N}} \end{pmatrix}$$

give a base for $T_{P_i}(X)$. □

Corollary 2.6.2. *Let $(X, \mathcal{L})$ be an integral, polarized scheme. If $\mathcal{L}$ embeds X as a closed scheme in $\mathbb{P}^N$, then*

$$\dim(\mathrm{Sec}_{s-1}(X)) = N - \dim(h^0(\mathcal{I}_{Z,X} \otimes \mathcal{L}))$$

where Z is the union of s generic 2-fat points in X.

Proof. By Terracini's Lemma, $\dim(\mathrm{Sec}_{s-1}(X)) = \dim(< T_{P_1}(X), \ldots, T_{P_s}(X) >)$, with $P_1, \ldots, P_s$ generic points on X. Since X is embedded in $\mathbb{P}^N = \mathbb{P}(H^0(X, \mathcal{L})^*)$, we can view the elements of $H^0(X, \mathcal{L})$ as hyperplanes in $\mathbb{P}^N$; the hyperplanes which contain a space $T_{P_i}(X)$ correspond to elements in $H^0(\mathcal{I}_{2P_i,X} \otimes \mathcal{L})$, since they intersect X in a subscheme containing the first infinitesimal neighborhood of P_i. Hence the hyperplanes of $\mathbb{P}^N$ containing the subspace $< T_{P_1}(X), \ldots, T_{P_s}(X) >$ are the sections of $H^0(\mathcal{I}_{Z,X} \otimes \mathcal{L})$, where Z is the scheme union of the first infinitesimal neighborhoods in X of the points P_i's. □

Remark: A hyperplane H contains the tangent space to a projective variety X at a smooth point P if and only if the intersection $X \cap H$ has a singular point at P.

In fact the tangent space $T_P(X)$ to X at P has the same dimension of X and $T_P(X \cap H) = H \cap T_P(X)$. Moreover P is singular in $H \cap X$ if and only if $\dim(T_P(X \cap H)) \geq \dim(X \cap H) = \dim(X) - 1$ and this happens if and only if $H \supset T_P(X)$.

Example: Consider the Veronese surface of $\mathbb{P}^5$. Let P be a general point of $\mathrm{Sec}_1(\nu_2(\mathbb{P}^2))$ and suppose that $P \in < R, Q >$ where $R, Q \in \nu_2(\mathbb{P}^2)$. By Terracini's Lemma $T_P(\mathrm{Sec}_1(\nu_2(\mathbb{P}^2))) =< T_R(\nu_2(\mathbb{P}^2)), T_Q(\nu_2(\mathbb{P}^2)) >$. The expected dimension for $\mathrm{Sec}_1(\nu_2(\mathbb{P}^2))$ is 5, so $\dim(T_P(\mathrm{Sec}_1(\nu_2(\mathbb{P}^2)))) < 5$ if and only if there exists a hyperplane H containing $T_P(\mathrm{Sec}_1(\nu_2(\mathbb{P}^2)))$. The Remark above tells us that this happens if and only if there exists a hyperplane H such that $H \cap \nu_2(\mathbb{P}^2)$ is singular at R, Q.
Now $\nu_2(\mathbb{P}^2)$ is the image of $\mathbb{P}^2$ via the map defined by complete linear system of quadrics hence $\nu_2(\mathbb{P}^2) \cap H$ is the image of plane conics. Let R', Q' be the pre-images via ν_2 of R, Q respectively.

Then $2 < R', Q' >$ is a plane conic singular at R' and Q'; it corresponds to the hyperplane section of $\nu_2(\mathbb{P}^2)$ which is singular at R, Q. Since $2 < R', Q' >$ is the only one plane conic singular at R', Q' we can say that $\dim(T_P(\mathrm{Sec}_1(\nu_2(\mathbb{P}^2)))) = 4 < 5$.
Since the 2-Veronese surface is defined by the complete linear system of quadrics, the Corollary 2.6.2 allows to rephrase the defectivity of $\mathrm{Sec}_1(\nu_2(\mathbb{P}^2))$ in terms of number of conditions imposed by 2-fat points to forms of degree 2; i.e. "two 2-fat points of $\mathbb{P}^2$ do not impose independent conditions to the degree 2 forms of $K[x_0, x_1, x_2]$".

Corollary 2.6.2 can be generalized to non complete linear systems on X.

Notation: Let D be any divisor of an irreducible projective variety X. With $|D|$ we indicate the complete linear system defined by D. Let $V \subset |D|$ be a linear system. We use the notation

$$V(m_1P_1, \ldots, m_sP_s)$$

for the subsystem of divisors of V passing through the fixed points $P_1, \ldots, P_s$ with multiplicities at least $m_1, \ldots, m_s$ respectively.

When the multiplicities m_i are equal to 2 for $i = 1, \ldots, s$, the problem of the knowledge of $\dim(V(2P_1, \ldots, 2P_s))$ is equivalent to that of the dimension of the $(s-1)$-secant variety to a variety obtained as the closure of the image of the map we are going to define.
Suppose that V is associated to a morphism $\varphi_V : X_0 \to \mathbb{P}^r$ (if $\dim(V) = r$) which is an embedding on a dense open set $X_0 \subset X$. We will consider the variety $\overline{\varphi_V(X_0)}$.
In general we expect that if $\dim(X) = n$ then

$$\mathrm{expdim}(V(2P_1, \ldots, 2P_s)) = \dim(V) - s(n+1).$$

Proposition 2.6.3. *Let X be an integral scheme and V be a linear system on X such that the rational function $\varphi_V : X \dashrightarrow \mathbb{P}^r$ associated to V, is an embedding on a dense open subset X_0 of X. Then $\mathrm{Sec}_{s-1}\left(\overline{\varphi_V(X_0)}\right)$ is defective if and only if for general points $P_1, \ldots, P_s \in X$*

$$\dim(V(2P_1, \ldots, 2P_s)) > \min\{-1, r - s(n+1)\}.$$

2.6.1 Three questions

In this work we want to focalize our attention on three particular questions.

Osculating varieties to Veronese varieties

Let $L_1, \dots, L_s$ be generic linear forms of $S = K[x_0, \dots, x_n]$ and $F_1, \dots, F_s$ be generic forms belonging to S_k; the first case we are interested in is:

> "which are the conditions on $s, k, d \in \mathbb{Z}$ such that the following form is canonical:
>
> $$L_1^{d-k} F_1 + \cdots + L_s^{d-k} F_s?"$$

The results which we are going to present about this problem are for the most part in the joint works [**BCGI**] and [**BC**] (we will prove in the next chapter that a form $F = L_1^{d-k} F_1 + \cdots + L_s^{d-k} F_s \in S_d$ is canonical if and only if the $(s-1)$-secant variety to the k-th Osculating variety to a Veronesean fills up the ambient space). Let us first look at some peculiar examples.

Example: If $d = 3$, $k = 2$ and $n = 4$ one would expect that a generic $f \in K[x_0, \dots, x_4]_3$ could be written as $f = L_1 F_1 + L_2 F_2$ with $L_i \in S_1$ and $F_i \in S_2$, but actually we need three addenda: $f = L_1 F_1 + L_2 F_2 + L_3 F_3$.

Example: If $d = 3$, $k = 2$ and $n = 5$ we cannot write a generic $f \in K[x_0, \dots, x_5]_3$ as $f = L_1 F_1 + L_2 F_2 + L_3 F_3$, but only as $f = L_1 F_1 + \cdots + L_4 F_4$ for $L_i \in S_1$ and $F_i \in S_2$.

Example: If $d = 4$, $k = 3$ and $n = 6$ one would expect that a generic $f \in K[x_0, \dots, x_6]_4$ could be written as $f = L_1 F_1 + L_2 F_2 + L_3 F_3$, with $L_i \in S_1$ and $F_i \in S_3$, but not only it is not possible to write a generic f as a sum of three addenda, but it is not even possible to write it as a sum of four. In fact f can only be written as $f = L_1 F_1 + \cdots + L_5 F_5$.

Split varieties

Let $L_i^{(j)}$ be generic linear forms of $S = K[x_0, \dots, x_n]$ for $i = 1, \dots, d$ and $j = 1, \dots, s$;

> "which is the least integer s such that the following form is canonical:
>
> $$L_1^{(1)} \cdots L_d^{(1)} + \cdots + L_1^{(s)} \cdots L_d^{(s)}?"$$

The motivation of this study comes from a conjecture in [**Eh**]. Let $\mathbb{G}(k, n)$ be the Grassmannian of k-spaces of $\mathbb{P}^n$; with $\mathrm{Split}_d(\mathbb{P}^n)$ we indicate the variety that is obtained as the closure of the image of the following map:

$$\begin{aligned} \phi : \underbrace{\mathbb{P}(S_1) \times \cdots \times \mathbb{P}(S_1)}_{d} &\to \mathbb{P}(S_d), \\ ([L_1], \dots, [L_d]) &\mapsto [L_1 \cdots L_d]; \end{aligned} \tag{2.15}$$

hence the Split variety can be viewed as the locus:

$$\mathrm{Split}_d(\mathbb{P}^n) := \{[f] \in K[x_0, \dots, x_n]_d \mid f = L_1 \cdots L_d \text{ with } L_i \in K[x_0, \dots, x_n]_1\}.$$

In the paper we cited above Ehrenborg observed that for a positive integer d, the varieties $\mathrm{Split}_d(\mathbb{P}^n)$ and $\mathbb{G}(n-1, n+d-1)$ are embedded in the same $\mathbb{P}^{\binom{n+d}{d}-1}$, and moreover he found many examples where the typical rank of the two variety is the same. Therefore he stated the following conjecture.

Conjecture 2.6.4. (**Eherenborg**) The typical ranks of $\mathbb{G}(n-1, n+d-1)$ and of $\mathrm{Split}_d(\mathbb{P}^n)$ are the same.

If this conjecture were true, we would be able to compute the dimension of $\mathrm{Sec}_{s-1}(\mathrm{Split}_d(\mathbb{P}^n))$ in many cases. Unfortunately things are not so simple and the following example shows that the conjecture is false.

Example: It is a known result (see for example [**CGG4**]) that $\mathrm{Sec}_{3-1}(\mathbb{G}(3,6))$ is defective with defect $\delta_3 = 1$, i.e one expects that $\mathrm{Sec}_2(\mathbb{G}(3,6)) = \mathbb{P}^{34}$ but $\dim(\mathrm{Sec}_2(\mathbb{G}(3,6))) = 33$; we need $\mathrm{Sec}_3(\mathbb{G}(3,6))$ in order to fill up $\mathbb{P}^{34}$. This means that the typical rank of $\mathbb{G}(3,6)$ is not 3, as expected, but 4. Unfortunately this fact does not imply that the least integer s such that $\mathrm{Sec}_{s-1}(\mathrm{Split}_4(\mathbb{P}^3))$ fills up the ambient space is 4 too; in fact $\mathrm{Sec}_2(\mathrm{Split}_4(\mathbb{P}^3)) = \mathbb{P}^{34}$ (we made computations with [**CoCoA**]).

Anyway, we have that Ehremborg's conjecture is true for $d = 2$.

Proposition 2.6.5. *The $(s-1)$-secant varieties of $\mathbb{G}(1, n+1)$ and of $\mathrm{Split}_2(\mathbb{P}^n)$ have the same dimension for all s.*

Proof. The embedding of $\mathbb{G}(1, n+1)$ into $\mathbb{P}^{\binom{n+2}{2}-1} \simeq \mathbb{P}(K[x_0, \dots, x_n]_2)$ allows us to view the Grassmannian as the set of quadrics whose representative $(n+2) \times (n+2)$ matrices are skew symmetric and of rank at most 2 (we will present this construction in details in Section 4.4, in particular see (4.11) and (4.12)). Therefore

$$\mathrm{Sec}_{s-1}(\mathbb{G}(1, n+1)) \simeq \{M \in M_{n+2}(K) \mid M \text{ is skew symmetric and } \mathrm{rk}(M) \leq 2s\},$$

then

$$\mathrm{codim}(\mathrm{Sec}_{s-1}(\mathbb{G}(1, n+1))) = \binom{n+2-2s}{2}.$$

In the same way $\mathrm{Split}_2(\mathbb{P}^n) \simeq \{M \in M_{n+1}(K) \mid M \text{ is symmetric and } \mathrm{rk}(M) \leq 2\}$; therefore

$$\mathrm{Sec}_{s-1}(\mathrm{Split}_2(\mathbb{P}^n)) \simeq \{M \in M_{n+1}(K) \mid M \text{ is symmetric and } \mathrm{rk}(M) \leq 2s\},$$

then

$$\operatorname{codim}(\operatorname{Sec}_{s-1}(\operatorname{Split}_d(\mathbb{P}^n)) = \binom{n+2-2s}{2}.$$

□

Notice also that when $d = 2$ the variety $\operatorname{Split}_2(\mathbb{P}^n)$ is the variety that parameterizes forms of the type $L^{d-k}F$ with $L \in S_1$ and $F \in S_k$, $1 \leq k < d = 2$.

Segre Varieties

Let us give the following definition:

Definition 2.6.6. *Let $V_1, \ldots, V_t$ vector spaces on K, a tensor $T \in V_1^* \otimes \cdots \otimes V_t^*$ is said to be "decomposable" if there exist vectors $v_i^* \in V_i^*$ such that $T = v_1^* \otimes \cdots \otimes v_t^*$.*

A well known problem is:

> "which is the minimum integer s such that the generic tensor T of $V_1^* \otimes \cdots \otimes V_t^*$ is the sum of s decomposable tensors? This minimum integer s is called the "typical rank" of T."

The geometric problem associated at this last algebraic problem is the study of the dimension of the secant varieties to the Segre varieties. Let $\mathbb{P}^{n_i} = \mathbb{P}(V_i)$ for $i = 1, \ldots, k$, be the Segre variety which is defined as the image of the following map:

$$\begin{array}{rcl} \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k} & \longrightarrow & \mathbb{P}^{(n_1+1)\cdots(n_k+1)-1} \\ ((x_0^{(1)}, \ldots, x_{n_1}^{(1)}), \ldots, (x_0^{(k)}, \ldots, x_{n_k}^{(k)})) & \mapsto & (\ldots, x_{i_1}^{(1)} \cdots x_{i_k}^{(k)}, \ldots) \end{array}.$$

In the last chapter, two different ways to approach the study of secant varieties of Segre varieties: the first one uses Inverse System theory and it is due to M.V. Catalisano, A.V. Geramita and A. Gimigliano (see [**CGG1**]); the second one is strictly connected to Representation Theory and it is due to J.M. Landsberg and L. Manivel (see [**LM1**]). In this last paper the authors give two different algorithms to compute the equations of the secant varieties to Segre varieties. The most important result contained in [**LM1**] that we will present is the solution of the Garcia, Stillman, Sturmfels conjecture (see Conjecture 5.6.32) on the generation of the ideal of the chordal variety to Segre variety in the case of three factors.

Chapter 3

Secant varieties to the Osculating varieties of Veronese varieties

In this chapter we want to give a partial solution to the first problem presented in Section 2.6.1:

> "which is the least integer $G(d)$ such that the generic form of $S_d = K[x_0, \ldots, x_n]_d$ can be written as sums of $G(d)$ forms of the type $L^{d-k}F$ where $L \in S_1$ and $F \in S_k$?".

Accordingly with the technique presented in Section 2.4, we look for a variety parameterizing those kind of forms and, after that, the knowledge of the dimensions of its secant varieties, when we are able to compute them, will solve the problem. We will see that this variety will be given by osculating spaces to the Veronese $\nu_d(\mathbb{P}^n)$; in the case $k = 1$ the tangential variety is the one involved.

Definition 3.0.7. *Let $X \subset \mathbb{P}^N$ be a projective, reduced and irreducible variety. The tangent star to X at P is defined as follows:*

$$T_P^*(X) = \overline{\bigcup_{\substack{y(t), z(t) \in X, \\ y(0) = z(0) = P}} lim_{t \to 0} < y(t), z(t) >}.$$

Definition 3.0.8. *Let $X \subset \mathbb{P}^N$ be a projective, reduced and irreducible variety. Define the tangential variety of X, $\tau(X) \subset \mathbb{P}^N$ by*

$$\tau(X) := \bigcup_{P \in X} T_P^* X.$$

We can observe that if the variety X is smooth the definition of Tangential variety that we have just introduced coincide with the following:

Definition 3.0.9. *Let $X \subset \mathbb{P}^N$ be a projective, reduced and irreducible variety. Let $X_0 \subset X$ be the dense subset of regular points of X. We define the tangential variety to X as*

$$T(X) := \overline{\bigcup_{P \in X_0} T_P(X)}$$

where $T_P(X)$ is the tangent space to X at P.

A reason for using Definition 3.0.8 rather than Definition 3.0.9 is given by Fulton Hansen Theorem (see [**FHan**]) that can be applied for $\tau(X)$ of Definition 3.0.8 and not for $T(X)$ of Definition 3.0.9.

Theorem 3.0.10. (**Fulton Hansen**) *Let $X \subset \mathbb{P}^N$ be a projective variety. Then either:*

- $\dim(\tau(X)) = 2n$ *and* $\dim(\mathrm{Sec}_1(X)) = 2n+1$, *or*
- $\tau(X) = \mathrm{Sec}_1(X)$.

We have already observed that when X is smooth, $T^*_P(X)$ is just $T_P(X)$, so $T(X) = \tau(X)$. When X is singular, for $\tau(X)$ Theorem 3.0.10 holds, while for $T(X)$ it does not.

Example: Consider the Del Pezzo surface $X := \nu_3(\mathbb{P}^2) \subset \mathbb{P}^9$ which parameterizes cubics of $\mathbb{P}^2$ made of a single triple line. Then $T(X)$ is a 4-fold in $\mathbb{P}^9$ and we will see that it parameterizes forms of type $[LM^2]$, where L, M are linear forms in $S := K[x_0, x_1, x_2]$; such variety is singular along X. Let us consider the variety $T(T(X))$. At every point $[LM^2]$, we have that $T_{[LM^2]}(T(X))$ corresponds to $< M^2S_1, MLS_1 >$, hence $T(T(X))$ parameterizes all cubic forms which are limit of something of the form $[MF]$, where L is a line and F a conic.
Notice that this shows that $T(T(X)) = O_2(X)$, the second osculating variety to X, which has dimension 7, hence $T(T(X))$ is defective (it should have dimension 8), and this defectivity is not surprising, since along any tangent space $T_P(X)$ we have that all $T_Q(T(X))$, $Q \in T_P(X)$, have $T_P(X)$ in common.
If we consider $\tau(T(X))$, instead, Hansen-Fulton theorem 3.0.10 gives us that $\tau(T(X)) = \mathrm{Sec}_1(T(X))$, since it is known that $\mathrm{Sec}_1(T(X))$ is defective (see [**CGG2**]) and has dimension 8 (it actually parameterizes all singular cubics).

Definition 3.0.11. *A $(2,3)$-point in $\mathbb{P}^n$ is a 0-dimensional scheme in $\mathbb{P}^n$ with support at one point P, whose ideal is of the type $\wp^3 + I_l^2$ where $l \subset \mathbb{P}^n$ is a line through P with defining ideal I_l and $\wp$ is the ideal of P.*

If $X = \nu_d(\mathbb{P}^n)$, the tangential variety to X can be dually viewed as the locus $\{[L^{d-1}M] \in \mathbb{P}(S_d) \mid L, M \in S_1\} \subset \left(\mathbb{P}^{\binom{n+d}{d}-1}\right)^*$. In [**CGG2**] it is shown, via inverse systems' theory, that $\dim(\mathrm{Sec}_{s-1}(T(\nu_d(\mathbb{P}^n)))) = H(Z_1, 1) - 1$, where Z_1 is the union of s generic "$(2,3)-$points" (i.e. the intersection of a 3-fat point with a double line). In that paper the defectivity of $T(\nu_d(\mathbb{P}^n))$ was studied and a conjecture regarding all defective cases was stated (see also [**Ba**]). The authors proved that if $P_1, \ldots, P_s \in \nu_d(\mathbb{P}^n)$, $\wp_i \subset R = K[y_0, \ldots, y_n]$ are the prime ideals associated to P_i, and $Q \in < P_1, \ldots, P_s >$ then it is possible to find s prime ideals $l_i \subset R$ representing lines through P_i such that the dimension of $T_Q(\mathrm{Sec}_{s-1}(T(\nu_d(\mathbb{P}^n))))$ is equal to $H(R/(\wp_1^3 \cap l_1^2) \cap \cdots \cap (\wp_s^3 \cap l_s^2), d)$.

The forms parameterized by $T(\nu_d(\mathbb{P}^n))$ suggest that, if we want to find the variety parameterizing forms $L^{d-k}F$ with $L \in S_1$ and $F \in S_k$, we have to look at the k-osculating variety to $\nu_d(\mathbb{P}^n)$.

3.1 The k-th osculating space

3.1.1 Definition and remark

Let $X \subset \mathbb{P}^r$ be a projective variety of dimension n.
Let $U_0 \subset \mathbb{C}^n$ be an open neighborhood of $\underline{0} := (0, \ldots, 0) \in \mathbb{C}^n$ in the Euclidian topology; and let $\{u_1, \ldots, u_n\}$ be a set of coordinates in $\mathbb{C}^n$.
Now let $F : U_0 \to X$ such that $F(\underline{0}) = P$ is a local parametrization.

Definition 3.1.1. *Let $X \subset \mathbb{P}^r$ be a projective n-dimensional variety, and let F be a local parameterization as above. The k-th osculating space to X at a regular point $P \in X$ is the linear projective space obtained as the projectivization of the following affine subspace of $\mathbb{C}^{r+1}$:*

$$\begin{aligned} O_{k,X,P} := \; & < F_{u_1}(\underline{0}), \ldots, F_{u_n}(\underline{0}); \\ & F_{u_1u_1}(\underline{0}), F_{u_1u_2}(\underline{0}), \ldots, F_{u_nu_n}(\underline{0}); \\ & \vdots \\ & F_{\underbrace{u_1\cdots u_1}_{k}}(\underline{0}), \ldots, F_{\underbrace{u_1\cdots u_1u_n}_{k}}(\underline{0}); \ldots\ldots, F_{\underbrace{u_n\ldots u_n}_{k}}(\underline{0}) > \end{aligned}$$

where $F_{u_i}(\underline{0}) = \frac{\partial F}{\partial u_i}(\underline{0})$, and $F_{u_{j_1}\cdots u_{j_h}}(\underline{0}) = \frac{\partial^h F}{\partial u_{j_1}\cdots \partial u_{j_h}}(\underline{0})$.

An equivalent definition of k-th osculating space can be given as follows:

Definition 3.1.2. *Let $X \subset \mathbb{P}^r$ be a projective n-dimensional variety, consider all the curves $t \mapsto x(t)$ such that $x(t) \in X$ for all $t \in K$ and $x(0) = P$, then the k-th osculating space to X at P can*

be defined as

$$\bigcup_{\substack{x(t) \subset X \\ x(0) = P}} < x(0), x'(0), x''(0), \ldots, x^{(k)}(0) > .$$

Example: Let $U_0 \subset \mathbb{C}^1$ and X the 1-dimensional projective variety locally defined by the following parameterization:

$$F : U_0 \to X, \quad F(u) = (u, u^2, u^3)$$

then
$O_{0,X,0} = \{0\} = P$,
$O_{1,X,0} =< (1, 2u, 3u^2)|_0 >=< (1, 0, 0) >$,
$O_{2,X,0} =< (1, 0, 0), (0, 2, 6u)|_0 >=< (1, 0, 0), (0, 2, 0) >$,
$O_{3,X,0} =< (1, 0, 0), (0, 2, 0), (0, 0, 6) >$.

3.1.2 Intersection between a projective variety X and its k-th osculating space at one point

The goal of this section is the following proposition:

Proposition 3.1.3. *The intersection between a projective variety $X \subset \mathbb{P}^r$ and its k-th osculating space $O_{k,X,P}$ at a regular point P is at least a $(k+1)$-fat point, i.e. there exists $r' \in \mathbb{Z}$, $r' \leq r$ such that* $\operatorname{Spec}\left(\frac{K[x_1,\ldots,x_r]}{I(X \cap O_{k,X,P})}\right) \supseteq \operatorname{Spec}\left(\frac{K[x_1,\ldots,x_{r'}]}{(I(P))^{k+1}}\right)$.

Let $P = \underline{0}$, $F : U_0 \to X$ be a local parametrization such that $F(\underline{0}) = \underline{0}$ and

$$F(u_1, \ldots, u_n) = (f_1(u_1, \ldots, u_n), \ldots, f_r(u_1, \ldots, u_n)).$$

We want to start by studying the case of the tangent space $T_{\underline{0}}(X)$.

Tangent space

According with the previous definition, the affine tangent space to X at the point $P = \underline{0}$ is the first osculating space $O_{1,X,\underline{0}}$:

$$T_{\underline{0}}(X) =< \frac{\partial F}{\partial u_1}(\underline{0}), \ldots, \frac{\partial F}{\partial u_n}(\underline{0}) >=$$
$$=< \left(\frac{\partial f_1}{\partial u_1}(\underline{0}), \ldots, \frac{\partial f_r}{\partial u_1}(\underline{0})\right), \ldots, \left(\frac{\partial f_1}{\partial u_n}(\underline{0}), \ldots, \frac{\partial f_r}{\partial u_n}(\underline{0})\right) > .$$

Let us consider the following system:

$$\begin{cases} x_1 = f_1(u_1, \dots, u_n) \\ \vdots \\ x_r = f_r(u_1, \dots, u_n) \end{cases} . \tag{3.1}$$

By elimination of the u_i's we can find polynomials $F_1, \dots, F_s \in K[x_1, \dots, x_r]$ such that the variety X is defined (locally, at $\underline{0}$), by:

$$\begin{cases} F_1(x_1, \dots, x_r) = 0 \\ \vdots \\ F_s(x_1, \dots, x_r) = 0 \end{cases} \tag{3.2}$$

and so its defining ideal $I(X) \subset K[x_1, \dots, x_r]$ is, locally, $I(X) = (F_1, \dots, F_s)$. If we substitute the equations (3.1) into (3.2) we find out

$$\begin{cases} F_1(f_1(u_1, \dots, u_n), \dots, f_r(u_1, \dots, u_n)) = 0 \\ \vdots \\ F_s(f_1(u_1, \dots, u_n), \dots, f_r(u_1, \dots, u_n)) = 0 \end{cases} \tag{3.3}$$

that implies $0 = \frac{\partial F_j}{\partial u_i} = \sum_{h=1}^{r} \frac{\partial F_j}{\partial x_h} \frac{\partial x_h}{\partial u_i} = \sum_{h=1}^{r} \frac{\partial F_j}{\partial x_h} \frac{\partial f_h}{\partial u_i}$ for all $j = 1, \dots, s$ and therefore the equations of $T_{\underline{0}}(X)$ are determined by the following system:

$$\begin{cases} \sum_{h=1}^{r} \frac{\partial F_1}{\partial x_h}(\underline{0}) x_h = 0 \\ \vdots \\ \sum_{h=1}^{r} \frac{\partial F_s}{\partial x_h}(\underline{0}) x_h = 0 \end{cases} \tag{3.4}$$

and so

$$I(T_{\underline{0}}(X)) = \left(\frac{\partial F_1}{\partial x_1}(\underline{0}) x_1 + \dots + \frac{\partial F_1}{\partial x_r}(\underline{0}) x_r, \dots, \frac{\partial F_s}{\partial x_1}(\underline{0}) x_1 + \dots + \frac{\partial F_s}{\partial x_r}(\underline{0}) x_r \right) \subset K[x_1, \dots, x_r]$$

Let us write $F_j = F_{j0} + F_{j1} + \dots + F_{jd_j}$ where F_{jl} is the homogeneous part of degree l of F_j for all $j = 1, \dots, s$. Since $F_j(\underline{0}) = \underline{0}$, we have $F_{j0} = 0$ for all $j = 1, \dots, s$ then $F_j = F_{j1} + \dots + F_{jd_j}$. Let $F_{j1} = a_{j1}x_1 + \dots + a_{jr}x_r$ then

$$F_j = a_{j1}x_1 + \dots + a_{jr}x_r + F_{j2} + \dots + F_{jd_j}.$$

It is clear that $\frac{\partial F_j}{\partial x_h}(\underline{0}) = a_{jh}$ and so

$$I(T_{\underline{0}}(X)) = (a_{11}x_1 + \cdots + a_{1r}x_r, \ldots, a_{s1}x_1 + \cdots + a_{sr}x_r).$$

Let us consider now the following intersection:

$$\operatorname{Spec}\left(\frac{K[x_1,\ldots,x_r]}{I(X)}\right) \cap \operatorname{Spec}\left(\frac{K[x_1,\ldots,x_r]}{I(T_{\underline{0}}(X))}\right)$$

this is equal to

$$\operatorname{Spec}\left(\frac{K[x_1,\ldots,x_r]}{(I(X)+I(T_{\underline{0}}(X)))}\right) = \operatorname{Spec}\left(\frac{K[x_1,\ldots,x_r]}{(F_1,\ldots,F_r,\sum_{h=1}^r \frac{\partial F_1}{\partial x_h}(\underline{0})x_h,\ldots,\sum_{h=1}^r \frac{\partial F_s}{\partial x_h}(\underline{0})x_h)}\right) =$$

$$= \operatorname{Spec}\left(\frac{K[x_1,\ldots,x_r]}{((\sum_{i=1}^r a_{1i}x_i)+(\sum_{i=2}^{d_1} F_{1i}),\ldots,(\sum_{i=1}^r a_{si}x_i)+(\sum_{i=2}^{d_s} F_{si}),\sum_{i=1}^r a_{1i}x_i,\ldots,\sum_{i=1}^r a_{si}x_i)}\right) \simeq$$

$$\simeq \operatorname{Spec}\left(\frac{K[x_1,\ldots,x_r]}{(\sum_{i=2}^{d_1} F_{1i},\ldots,\sum_{i=2}^{d_s} F_{si},\sum_{i=1}^r a_{1i}x_i,\ldots,\sum_{i=1}^r a_{si}x_i)}\right)$$

that is isomorphic, for some $F'_j \in K[x_1,\ldots,x_{r-1}]$ and $a'_{ij} \in K$ (if $(a_{s1},\ldots,a_{sr}) \neq (0,\ldots,0)$), to

$$\operatorname{Spec}\left(\frac{K[x_1,\ldots,x_{r-1}]}{(\sum_{i=2}^{d_1} F'_{1i},\ldots,\sum_{i=2}^{d_s} F'_{si},\sum_{i=1}^{r-1} a'_{1i}x_i,\ldots,\sum_{i=1}^{r-1} a'_{s-1i}x_i)}\right) \simeq$$

$$\simeq \cdots \simeq \operatorname{Spec}\left(\frac{K[x_1,\ldots,x_l]}{(\sum_{i=2}^{d_1} \tilde{F}_{1i},\ldots,\sum_{i=2}^{d_s} \tilde{F}_{si})}\right) = \operatorname{Spec}\left(\frac{K[x_1,\ldots,x_l]}{(\tilde{F}_{12}+\cdots+\tilde{F}_{1d_1},\ldots,\tilde{F}_{s2}+\cdots+\tilde{F}_{sd_s})}\right)$$

for some $\tilde{F}_i \in K[x_1,\ldots,x_l]$ and $r-s \leq l \leq r$ and any $i = 1,\ldots,s$.
Now, since the ideal $(\tilde{F}_{12}+\cdots+\tilde{F}_{1d_1},\ldots,\tilde{F}_{s2}+\cdots+\tilde{F}_{sd_s}) \subset K[x_1,\ldots,x_l]$ is generated in degree at least 2, it means that

$$\operatorname{Spec}\left(\frac{K[x_1,\ldots,x_l]}{(\tilde{F}_{12}+\cdots+\tilde{F}_{1d_1},\ldots,\tilde{F}_{s2}+\cdots+\tilde{F}_{sd_s})}\right) \supseteq \operatorname{Spec}\left(\frac{K[x_1,\ldots,x_l]}{(x_1,\ldots,x_l)^2}\right).$$

This means that the intersection between a variety X and its tangent space at the point $P = \underline{0}$ is at least a double fat point.

Second osculating space

Now we want to study the intersection between a projective variety X and its second osculating space at $P = \underline{0}$.

By definition

$$O_{2,X,\underline{0}} =< F_{u_1}(\underline{0}), \dots, F_{u_n}(\underline{0}), F_{u_1u_1}(\underline{0}), F_{u_1u_2}(\underline{0}), \dots, F_{u_nu_n}(\underline{0}) >=$$

$$=< \left(\frac{\partial f_1}{\partial u_1}(\underline{0}), \dots, \frac{\partial f_r}{\partial u_1}(\underline{0})\right), \dots, \left(\frac{\partial f_1}{\partial u_n}(\underline{0}), \dots, \frac{\partial f_r}{\partial u_n}(\underline{0})\right);$$

$$\left(\frac{\partial^2 f_1}{\partial u_1^2}(\underline{0}), \dots, \frac{\partial^2 f_r}{\partial u_1^2}(\underline{0})\right), \left(\frac{\partial^2 f_1}{\partial u_1 \partial u_2}(\underline{0}), \dots, \frac{\partial^2 f_r}{\partial u_1 \partial u_2}(\underline{0})\right), \dots, \left(\frac{\partial^2 f_1}{\partial u_1 \partial u_n}(\underline{0}), \dots, \frac{\partial^2 f_r}{\partial u_1 \partial u_n}(\underline{0})\right),$$

$$\left(\frac{\partial^2 f_1}{\partial u_2 \partial u_1}(\underline{0}), \dots, \frac{\partial^2 f_r}{\partial u_2 \partial u_1}(\underline{0})\right), \left(\frac{\partial^2 f_1}{\partial u_2^2}(\underline{0}), \dots, \frac{\partial^2 f_r}{\partial u_2^2}(\underline{0})\right), \dots, \left(\frac{\partial^2 f_1}{\partial u_2 \partial u_n}(\underline{0}), \dots, \frac{\partial^2 f_r}{\partial u_2 \partial u_n}(\underline{0})\right),$$

$$\dots$$

$$\left(\frac{\partial^2 f_1}{\partial u_n \partial u_1}(\underline{0}), \dots, \frac{\partial^2 f_r}{\partial u_n \partial u_1}(\underline{0})\right), \dots, \left(\frac{\partial^2 f_1}{\partial u_n \partial u_{n-1}}(\underline{0}), \dots, \frac{\partial^2 f_r}{\partial u_n \partial u_{n-1}}(\underline{0})\right), \left(\frac{\partial^2 f_1}{\partial u_n^2}(\underline{0}), \dots, \frac{\partial^2 f_r}{\partial u_n^2}(\underline{0})\right) >=$$

$$\begin{aligned} =< \quad & \left(\tfrac{\partial f_1}{\partial u_1}(\underline{0}), \dots, \tfrac{\partial f_r}{\partial u_1}(\underline{0})\right), \dots \left(\tfrac{\partial f_1}{\partial u_n}(\underline{0}), \dots, \tfrac{\partial f_r}{\partial u_n}(\underline{0})\right); \\ & \left(\tfrac{\partial^2 f_1}{\partial u_1^2}(\underline{0}), \dots, \tfrac{\partial^2 f_r}{\partial u_1^2}(\underline{0})\right), \\ & \left(\tfrac{\partial^2 f_1}{\partial u_2 \partial u_1}(\underline{0}), \dots, \tfrac{\partial^2 f_r}{\partial u_2 \partial u_1}(\underline{0})\right), \left(\tfrac{\partial^2 f_1}{\partial u_2^2}(\underline{0}), \dots \tfrac{\partial^2 f_r}{\partial u_2^2}(\underline{0})\right), \\ & \dots \\ & \left(\tfrac{\partial^2 f_1}{\partial u_n \partial u_1}(\underline{0}), \dots, \tfrac{\partial^2 f_r}{\partial u_n \partial u_1}(\underline{0})\right), \dots, \left(\tfrac{\partial^2 f_1}{\partial u_{n-1} \partial u_n}(\underline{0}), \dots, \tfrac{\partial^2 f_r}{\partial u_{n-1} \partial u_n}(\underline{0})\right), \left(\tfrac{\partial^2 f_1}{\partial u_n^2}(\underline{0}), \dots, \tfrac{\partial^2 f_r}{\partial u_n^2}(\underline{0})\right) > . \end{aligned}$$

Therefore the affine dimension of $O_{2,X,\underline{0}}$ is at most $\binom{n+1}{2} + n$. Since $O_{2,X,\underline{0}}$ is an affine linear vector space through the origin and the ideal $I(X)$ is contained in $K[x_1, \dots, x_r]$, there exist m polynomials in $K[x_1, \dots, x_r]_1$, with $m \geq r - \binom{n+1}{2} - n$, that define $I(O_{2,X,\underline{0}}) \subset K[x_1, \dots, x_r]$; let them be:

$$\begin{cases} a_{11}x_1 + \dots + a_{1r}x_r = 0 \\ \vdots \\ a_{m1}x_1 + \dots + a_{mr}x_r = 0 \end{cases} \tag{3.5}$$

where the a_{ij} are determined by the relations:

$$\begin{cases} (a_{i1},\dots,a_{ir})\cdot\left(\frac{\partial f_1}{\partial u_j}(\underline{0}),\dots,\frac{\partial f_r}{\partial u_j}(\underline{0})\right)=0, & \forall i=1,\dots,m;\ \forall j=1,\dots,n;\\ (a_{i1},\dots,a_{ir})\cdot\left(\frac{\partial^2 f_1}{\partial u_{i_1}\partial u_{i_2}}(\underline{0}),\dots,\frac{\partial^2 f_r}{\partial u_{i_1}\partial u_{i_2}}(\underline{0})\right)=0, & \forall i=1,\dots,m;\ \forall i_1,i_2\in\{1,\dots,n\}.\end{cases} \tag{3.6}$$

Now, using Taylor's formula, the system (3.1) can be rewritten:

$$\begin{cases} x_1 &= f_1(u_1,\dots,u_n)=\\ &=\sum_{i=1}^n \frac{\partial f_1}{\partial u_i}(\underline{0})u_i+\frac{1}{2}\sum_{i_1,i_2=1}^n \frac{\partial^2 f_1}{\partial u_{i_1}\partial u_{i_2}}(\underline{0})u_{i_1}u_{i_2}+\frac{1}{3!}\sum_{i_1,i_2,i_3=1}^n \frac{\partial^3 f_1}{\partial u_{i_1}\partial u_{i_2}\partial u_{i_3}}(\underline{0})u_{i_1}u_{i_2}u_{i_3}+\cdots\\ \vdots\\ x_r &= f_r(u_1,\dots,u_n)=\\ &=\sum_{i=1}^n \frac{\partial f_r}{\partial u_i}(\underline{0})u_i+\frac{1}{2}\sum_{i_1,i_2=1}^n \frac{\partial^2 f_r}{\partial u_{i_1}\partial u_{i_2}}(\underline{0})u_{i_1}u_{i_2}+\frac{1}{3!}\sum_{i_1,i_2,i_3=1}^n \frac{\partial^3 f_r}{\partial u_{i_1}\partial u_{i_2}\partial u_{i_3}}(\underline{0})u_{i_1}u_{i_2}u_{i_3}+\cdots \end{cases}. \tag{3.7}$$

Then, by using (3.7), we have that the relations of (3.5), for all $i=1,\dots,m$, are determined as follows:

$$0=a_{i1}x_1+\cdots+a_{ir}x_r=(a_{i1},\dots,a_{ir})\cdot(x_1,\dots,x_r)=$$

$$=(a_{i1},\dots,a_{ir})\cdot\left(\sum_{i=1}^n\frac{\partial f_1}{\partial u_i}(\underline{0})u_i+\frac{1}{2}\sum_{i_1,i_2=1}^n\frac{\partial^2 f_1}{\partial u_{i_1}\partial u_{i_2}}(\underline{0})u_{i_1}u_{i_2}+\frac{1}{3!}\sum_{i_1,i_2,i_3=1}^n\frac{\partial^3 f_1}{\partial u_{i_1}\partial u_{i_2}\partial u_{i_3}}(\underline{0})u_{i_1}u_{i_2}u_{i_3}+\cdots,\right.$$

$$\left.\dots,\sum_{i=1}^n\frac{\partial f_r}{\partial u_i}(\underline{0})u_i+\frac{1}{2}\sum_{i_1,i_2=1}^n\frac{\partial^2 f_r}{\partial u_{i_1}\partial u_{i_2}}(\underline{0})u_{i_1}u_{i_2}+\frac{1}{3!}\sum_{i_1,i_2,i_3=1}^n\frac{\partial^3 f_r}{\partial u_{i_1}\partial u_{i_2}\partial u_{i_3}}(\underline{0})u_{i_1}u_{i_2}u_{i_3}+\cdots\right)=$$

$$=(a_{i_1},\dots,a_{i_r})\cdot\left(\sum_{i=1}^n\frac{\partial f_1}{\partial u_i}(\underline{0})u_i,\dots,\sum_{i=1}^n\frac{\partial f_r}{\partial u_i}(\underline{0})u_i\right)+$$

$$+(a_{i_1},\dots,a_{i_r})\cdot\left(\frac{1}{2}\sum_{i_1,i_2=1}^n\frac{\partial^2 f_1}{\partial u_{i_1}\partial u_{i_2}}(\underline{0})u_{i_1}u_{i_2},\dots,\frac{1}{2}\sum_{i_1,i_2=1}^n\frac{\partial^2 f_r}{\partial u_{i_1}\partial u_{i_2}}(\underline{0})u_{i_1}u_{i_2}\right)+$$

$$+(a_{i_1},\dots,a_{i_r})\cdot\left(\frac{1}{3!}\sum_{i_1,i_2,i_3=1}^n\frac{\partial^3 f_1}{\partial u_{i_1}\partial u_{i_2}\partial u_{i_3}}(\underline{0})u_{i_1}u_{i_2}u_{i_3}+\cdots,\dots,\frac{1}{3!}\sum_{i_1,i_2,i_3=1}^n\frac{\partial^3 f_r}{\partial u_{i_1}\partial u_{i_2}\partial u_{i_3}}(\underline{0})u_{i_1}u_{i_2}u_{i_3}+\cdots\right)$$

Now, by (3.6), we know that $(a_{i_1},\dots,a_{i_r})\cdot\left(\sum_{i=1}^n\frac{\partial f_1}{\partial u_i}(\underline{0})u_i,\dots,\sum_{i=1}^n\frac{\partial f_r}{\partial u_i}(\underline{0})u_i\right)=(a_{i_1},\dots,a_{i_r})\cdot\left(\sum_{i_1,i_2=1}^n\frac{\partial^2 f_1}{\partial u_{i_1}\partial u_{i_2}}(\underline{0})u_{i_1}u_{i_2},\dots,\sum_{i_1,i_2=1}^n\frac{\partial^2 f_r}{\partial u_{i_1}\partial u_{i_2}}(\underline{0})u_{i_1}u_{i_2}\right)=0$ then $a_{i_1}x_1+\cdots+a_{i_r}x_r=$

$$=\frac{1}{3!}(a_{i_1},\dots,a_{i_r})\cdot\left(\sum_{i_1,i_2,i_3=1}^n\frac{\partial^3 f_1}{\partial u_{i_1}\partial u_{i_2}\partial u_{i_3}}(\underline{0})u_{i_1}u_{i_2}u_{i_3}+\cdots,\dots,\sum_{i_1,i_2,i_3=1}^n\frac{\partial^3 f_r}{\partial u_{i_1}\partial u_{i_2}\partial u_{i_3}}(\underline{0})u_{i_1}u_{i_2}u_{i_3}+\cdots\right)$$

for all $i = 1, \dots, m$.

Now, since $K[x_1, \dots, x_r] \supset I(X) = (F_1, \dots, F_s)$ as in (3.2), we want to consider, for all $j = 1, \dots, r$ and for any $u_{i_1}, u_{i_2} \in \{1, \dots, n\}$, the following equality and compute it: $0 = \frac{\partial^2 F_j}{\partial u_{i_1} \partial u_{i_2}} = \frac{\partial}{\partial u_{i_1}} \left(\frac{\partial F_j}{\partial u_{i_2}} \right) =$
$\frac{\partial}{\partial u_{i_1}} \left(\sum_{h=1}^r \frac{\partial F_j}{\partial x_h} \frac{\partial f_h}{\partial u_{i_2}} \right) = \sum_{h=1}^r \frac{\partial}{\partial u_{i_1}} \left(\frac{\partial F_j}{\partial x_h} \frac{\partial f_h}{\partial u_{i_2}} \right) =$
$= \sum_{h=1}^r \left(\left(\left(\frac{\partial}{\partial u_{i_1}} \left(\frac{\partial F_j}{\partial x_h} \right) \right) \frac{\partial f_h}{\partial u_{i_2}} + \frac{\partial F_j}{\partial x_h} \left(\frac{\partial}{\partial u_{i_1}} \left(\frac{\partial f_h}{\partial u_{i_2}} \right) \right) \right) \right) = \sum_{h=1}^r \left(\left(\frac{\partial}{\partial x_h} \left(\frac{\partial F_j}{\partial u_{i_1}} \right) \right) \frac{\partial f_h}{\partial u_{i_2}} + \frac{\partial F_j}{\partial x_h} \frac{\partial^2 f_h}{\partial u_{i_1} \partial u_{i_2}} \right) \right) =$
$= \sum_{h=1}^r \left(\left(\frac{\partial}{\partial x_h} \left(\sum_{l=1}^r \frac{\partial F_j}{\partial x_l} \frac{\partial f_l}{\partial u_{i_1}} \right) \right) \frac{\partial f_h}{\partial u_{i2}} \right) + \sum_{h=1}^r \left(\frac{\partial F_j}{\partial x_h} \frac{\partial^2 f_h}{\partial u_{i_1} \partial u_{i_2}} \right) = \sum_{h=1}^r \left(\sum_{l=1}^r \frac{\partial^2 F_j}{\partial x_h \partial x_l} \frac{\partial f_l}{\partial u_{i_1}} + \frac{\partial F_j}{\partial x_l} \frac{\partial^2 f_l}{\partial x_h \partial u_{i_1}} \right) \frac{\partial f_h}{\partial u_{i_2}} +$
$\sum_{h=1}^r \left(\frac{\partial F_j}{\partial x_h} \frac{\partial^2 f_h}{\partial u_{i_1} \partial u_{i_2}} \right) = \sum_{h=1}^r \left(\sum_{l=1}^r \frac{\partial^2 F_j}{\partial x_h \partial x_l} \frac{\partial f_h}{\partial u_{i_1}} + \frac{\partial F_j}{\partial x_h} \frac{\partial f_l}{\partial u_{i_1}} \right) \frac{\partial f_h}{\partial u_{i_2}} + \sum_{h=1}^r \left(\frac{\partial F_j}{\partial x_h} \frac{\partial^2 f_h}{\partial u_{i_1} \partial u_{i_2}} \right) =$

$$= \sum_{h=1}^r \left(\sum_{l=1}^r \left(\frac{\partial^2 F_j}{\partial x_h \partial x_l} + \frac{\partial F_j}{\partial x_h} \right) \frac{\partial f_l}{\partial u_{i_1}} \right) \frac{\partial f_h}{\partial u_{i_2}} + \sum_{h=1}^r \left(\frac{\partial F_j}{\partial x_h} \frac{\partial^2 f_h}{\partial u_{i_1} \partial u_{i_2}} \right). \tag{3.8}$$

This implies, by (3.6), that

$$\sum_{h=1}^r \left(\sum_{l=1}^r \left(\frac{\partial^2 F_j}{\partial x_h \partial x_l}(\underline{0}) + \frac{\partial F_j}{\partial x_h}(\underline{0}) \right) x_l \right) x_h + \sum_{h=1}^r \left(\frac{\partial F_j}{\partial x_h}(\underline{0}) x_h \right) \in I(O_{2,X,\underline{0}}) \tag{3.9}$$

where $I(O_{2,X,\underline{0}}) = (a_{11}x_1 + \dots + a_{1r}x_r, \dots, a_{m1}x_1 + \dots + a_{mx}x_r)$.

Consider now

$$\operatorname{Spec}\left(\frac{K[x_1, \dots, x_r]}{I(X \cap O_{2,X,\underline{0}})} \right) = \operatorname{Spec}\left(\frac{K[x_1, \dots, x_r]}{I(X) + I(O_{2,X,\underline{0}})} \right) =$$

$$= \operatorname{Spec}\left(\frac{K[x_1, \dots, x_r]}{(F_1, \dots, F_s, \sum_{i=1}^r a_{1i}x_i, \dots, \sum_{i=1}^r a_{mi}x_i)} \right) =$$

$$= \operatorname{Spec}\left(\frac{K[x_1, \dots, x_r]}{(\sum_{i=1}^{d_1} F_{1i}, \dots, \sum_{i=1}^{d_s} F_{si}, \sum_{i=1}^r a_{1i}x_i, \dots, \sum_{i=1}^r a_{mi}x_i)} \right) := A$$

Since $T_{\underline{0}}(X) \subseteq O_{2,X,\underline{0}}$ then

1. $\sum_{h=1}^r \frac{\partial F_j}{\partial x_h}(\underline{0}) x_h \in I(X \cap O_{2,X,\underline{0}})$ then

$$A \simeq \operatorname{Spec}\left(\frac{K[x_1, \dots, x_r]}{(\sum_{i=2}^{d_1} F_{1i}, \dots, \sum_{i=2}^{d_s} F_{si}, \sum_{i=1}^r a_{1i}x_i, \dots, \sum_{i=1}^r a_{mi}x_i)} \right) := B$$

2. together with (3.9), $\sum_{h,l=1}^{r} \frac{\partial^2 F_j}{\partial x_h \partial x_l}(\underline{0}) x_h x_l \in I(X \cap O_{2,X,\underline{0}})$ then

$$B \simeq \operatorname{Spec}\left(\frac{K[x_1,\dots,x_r]}{(\sum_{i=3}^{d_1} F_{1i},\dots,\sum_{i=3}^{d_s} F_{si}, \sum_{i=1}^{r} a_{1i}x_i,\dots,\sum_{i=1}^{r} a_{mi}x_i)}\right) := C$$

because, as in the case of the tangential space, if $F_{j2} = b_{j11}x_1^2 + b_{j12}x_1x_2 + \cdots + b_{jrr}x_r^2$ for all $j = 1,\dots,s$ and some $b_{j,h,l} \in K$, then $\frac{\partial^2 F_j}{\partial x_{i_1}\partial x_{i_2}}(\underline{0}) = b_{ji_1i_2}$, ans so $F_{j,2} = \sum_{i_1,i_2=1}^{r} \frac{\partial^2 F_j}{\partial x_{i_1}\partial x_{i_2}}(\underline{0}) x_{i_1}x_{i_2}$.

But now

$$C \simeq \operatorname{Spec}\left(\frac{K[x_1,\dots x_{r'}]}{(\sum_{i=3}^{d_1} F'_{1i},\dots,\sum_{i=3}^{d_s} F'_{si})}\right)$$

for some $F'_j \in K[x_1,\dots,x_{r'}]$ for all $j = 1,\dots,s$ and $r - \binom{n+2}{2} - n \le r' \le r$ and

$$\operatorname{Spec}\left(\frac{K[x_1,\dots x_{r'}]}{(\sum_{i=3}^{d_1} F'_{1i},\dots,\sum_{i=3}^{d_s} F'_{si})}\right) \supseteq \operatorname{Spec}\left(\frac{K[x_1,\dots,x_{r'}]}{(x_1,\dots,x_{r'})^3}\right).$$

Therefore the intersection between X and $O_{2,X,\underline{0}}$ is at least a 3-fat point.

The k-th osculating space

We can generalize the argument of the tangent space and of the second osculating space to the k-th osculating space.

Proof. of Proposition 3.1.3 .

- From (3.3) we always know that

$$\frac{\partial^k F_j}{\partial u_{i_1}\cdots\partial u_{i_k}} = 0.$$

- If we write it in function of $x_1,\dots,x_r$ we find an expression of the type:

$$\sum_{h_1,\dots,h_k=1}^{r}\left(\frac{\partial^k F_j}{\partial x_{h_1}\cdots\partial x_{h_k}} + \frac{\partial^{k-1} F_j}{\partial x_{h_1}\cdots\partial x_{h_{k-1}}} + \cdots + \frac{\partial F_j}{\partial x_{h_1}}\right)\frac{\partial f_{h_1}}{\partial u_{i_1}}\cdots\frac{\partial f_{h_k}}{\partial u_{i_k}} +$$

$$+ \sum_{h_1,\dots,h_{k-1}=1}^{r}\left(\frac{\partial^{k-1} F_j}{\partial x_{h_1}\cdots\partial x_{h_{k-1}}} + \cdots + \frac{\partial F_j}{\partial x_{h_1}}\right)\left(\frac{\partial^2 f_{h_1}}{\partial u_{i_1}\partial u_{i_2}}\frac{\partial f_{h_1}}{\partial u_{i_3}}\cdots\frac{\partial f_{h_{k-1}}}{\partial u_{i_k}} + \cdots +\right.$$

$$+\frac{\partial^2 f_{h_{k-1}}}{\partial u_{i_{k-1}}\partial u_{i_k}}\frac{\partial f_{h_1}}{\partial u_{i_1}}\cdots\frac{\partial f_{h_{k-2}}}{\partial u_{i_{k-2}}}\Bigg)+$$

$$+\cdots+\sum_{h_1=1}^{r}\frac{\partial F_j}{\partial x_{h_1}}\frac{\partial^k f_{h_1}}{\partial u_{i_1}\cdots\partial u_{i_k}}=0$$

from which we get that the elements of the type:

$$\left(\sum_{i_1,\ldots,i_k=1}^{r}\left(\frac{\partial^k F_j}{\partial x_{i_1}\cdots\partial x_{i_k}}(\underline{0})+\frac{\partial^{k-1}F_j}{\partial x_{i_1}\cdots\partial x_{i_{k-1}}}(\underline{0})+\cdots+\frac{\partial F_j}{\partial x_{i_1}(\underline{0})}\right)x_{i_1}\cdots x_{i_k}\right)+\cdots+$$

$$+\cdots+\left(\sum_{i_1,i_2=1}^{r}\left(\frac{\partial^2 F_j}{\partial x_{i_1}\partial x_{i_2}}(\underline{0})+\frac{\partial F_j}{\partial x_{i_1}}(\underline{0})\right)x_{i_1}x_{i_2}\right)+\sum_{i=1}^{r}\frac{\partial F_j}{\partial x_i}(\underline{0})x_i$$

belong to $I(O_{k,X,\underline{0}})$ by using the perpendicularity relations:

$$(a_{i1},\ldots,a_{ir})\cdot F_{u_{i_1}\cdots u_{i_l}}=0$$

for all $u_{i_1},\ldots,u_{i_l}\in\{u_1,\ldots,u_n\}$, for all $l\leq n$, and where the $a_{i1},\ldots,a_{ir}$ are the coefficients of the defining system of $O_{k,X,\underline{0}}$:

$$\begin{cases} a_{11}x_1+\cdots+a_{1r}x_r=0\\ \vdots\\ a_{m1}x_1+\cdots+a_{mr}x_r=0\end{cases}$$

and $m\geq\sum_i^k\binom{n+i}{i}\geq\dim(O_{k,X,\underline{0}})$.

- Now, if we observe that

$$T_{\underline{0}}(X)\subseteq O_{2,X,\underline{0}}\subseteq O_{3,X,\underline{0}}\subseteq\cdots\subseteq O_{k,X,\underline{0}}$$

we should be able to proove the following chain of isomorphisms:

$$\operatorname{Spec}\left(\frac{K[x_1,\ldots,x_r]}{I(X\cap O_{k,X,\underline{0}})}\right)=\operatorname{Spec}\left(\frac{K[x_1,\ldots,x_r]}{(F_1,\ldots,F_s,\sum_{i=1}^r a_{1i}x_i,\ldots,\sum_{i=1}^r a_{mi}x_i)}\right)\simeq$$

$$\simeq\operatorname{Spec}\left(\frac{K[x_1,\ldots,x_r]}{(\sum_{i=2}^{d_1}F_{1i},\ldots,\sum_{i=2}^{d_s}F_{si},\sum_{i=1}^r a_{1i}x_i,\ldots,\sum_{i=1}^r a_{mi}x_i)}\right)\simeq$$

$$\simeq \operatorname{Spec}\left(\frac{K[x_1,\dots,x_r]}{(\sum_{i=3}^{d_1} F_{1i},\dots,\sum_{i=3}^{d_s} F_{si},\sum_{i=1}^{r} a_{1i}x_i,\dots,\sum_{i=1}^{r} a_{mi}x_i)}\right)\simeq$$

$$\simeq\cdots\simeq \operatorname{Spec}\left(\frac{K[x_1,\dots,x_r]}{(\sum_{i=k+1}^{d_1} F_{1i},\dots,\sum_{i=k+1}^{d_s} F_{si},\sum_{i=1}^{r} a_{1i}x_i,\dots,\sum_{i=1}^{r} a_{mi}x_i)}\right)\simeq$$

$$\simeq \operatorname{Spec}\left(\frac{K[x_1,\dots,x_{r'}]}{(\sum_{i=k+1}^{d_1} F_{1i},\dots,\sum_{i=k+1}^{d_s} F_{si})}\right)\supseteq \operatorname{Spec}\left(\frac{K[x_1,\dots,x_{r'}]}{(x_1,\dots,x_{r'})^{k+1}}\right)$$

with $\sum_{i=1}^{k}\binom{n+i}{i}\leq r'\leq r$.
The statement of Proposition 3.1.3 follows.

□

We can also prove Proposition 3.1.3 in a shorter way.

Proof. Let $U_0\subset\mathbb{C}^n$ be an open neighborhood of $\underline{0}\in\mathbb{C}^n$ in the Euclidian topology; and let $\{u_1,\dots,u_n\}$ be a set of coordinates in $\mathbb{C}^n$.
Now let $F:U_0\to X$ be a local parametrization such that $F(\underline{0})=P$ and

$$F(u_1,\dots,u_n)=(f_1(u_1,\dots,u_n),\dots f_r(u_1,\dots,u_n)).$$

Fix the following notation:

$$F_{u_{i_1}\cdots u_{i_h}}(\underline{0})=\left(\frac{\partial^h f_1}{\partial u_{i_1}\cdots u_{i_h}}(\underline{0}),\dots,\frac{\partial^h f_r}{\partial u_{i_1}\cdots\partial u_{i_h}}(\underline{0})\right)=:(a_{I1},\dots,a_{Ir}) \tag{3.10}$$

if $I=(i_1,\dots,i_h)$ and $i_1,\dots,i_h\in\{1,\dots,n\}$.
The affine k-th osculating space to X at P is spanned by:

$$O_{k,X,P}=<(a_{01},\dots,a_{0r}),\dots,(a_{I1},\dots,a_{Ir}),\dots,(a_{N1},\dots,a_{Nr})>$$

where $I=(i_1,\dots,i_h)$, $h\leq k$ and $N=\{\underbrace{n,\dots,n}_{k}\}$.

Let M be the matrix whose columns are the vectors spanning $O_{k,X,P}$, i.e.:

$$M:=\begin{pmatrix} a_{01} & \dots & a_{I1} & \dots & a_{N1}\\ \vdots & & \vdots & & \vdots\\ a_{0r} & \dots & a_{Ir} & \dots & a_{Nr}\end{pmatrix}$$

and let $m = \mathrm{rk}(M)$. Then there exist $b_{i,j} \in K$, with $i = 1, \dots, m$ and $j = 1, \dots, r$, such that

$$I(O_{k,X,P}) =< b_{11}x_1 + \cdots + b_{1r}x_r, \dots, b_{m1}x_1 + \cdots + b_{mr}x_r >\subset K[x_1, \dots, x_r].$$

Now

$$F^*(b_{i1}x_1 + \cdots + b_{ir}x_r) = b_{i1}f_1(u_1, \dots, u_n) + \cdots + b_{ir}f_r(u_1, \dots, u_n) \tag{3.11}$$

for all $i = 1, \dots, m$. Each f_j can be decomposed via Taylor's polynomial around $P = \underline{0}$, in particular there exist some coefficients $c_I \in K$ such that

$$f_j(u_1, \dots, u_n) = \sum_{|I| \leq k} c_I a_{Ij} u^I + \sum_{|I|=k+1}^{\deg(f_j)} c_I a_{Ij} u^I$$

where the a_{Ij}'s are defined as in (3.10) and $u^I = u_{i_1} \cdots u_{i_h}$ if $I = (i_1, \dots, i_h)$. Hence (3.11) can be rewritten as

$$b_{i1}\left(\sum_{|I| \leq k} c_I a_{I1} u^I\right) + \cdots + b_{ir}\left(\sum_{|I| \leq k} c_I a_{Ir} u^I\right) +$$
$$+ b_{i1}\left(\sum_{|I|=k+1}^{\deg(f_1)} c_I a_{I1} u^I\right) + \cdots + b_{ir}\left(\sum_{|I|=k+1}^{\deg(f_r)} c_I a_{Ir} u^I\right).$$

Now, the first r addends of the above summand are all zero since $b_{i1}x_1 + \cdots + b_{ir}x_r = 0$ for all $(x_1, \dots, x_r) \in O_{k,X,P}$. This means that $b_{i1}x_1 + \cdots + b_{ir}x_r \in \wp^{k+1}$ where $\wp$ is the prime ideal associated to P. Then

$$\mathrm{Spec}\left(\frac{K[x_1, \dots, x_r]}{I(X \cap O_{k,X,P})}\right) \supseteq \mathrm{Spec}\left(\frac{K[x_1, \dots, x_r]}{\wp^{k+1}}\right).$$

□

3.1.3 Dimension of the k-th osculating space

We are interested in discovering the dimension of the k-th osculating space of an n-dimensional projective variety $X \subset \mathbb{P}^r$ at a smooth point $P \in X$.
Since $O_{k,X,P}$ is spanned by all the first k partial derivatives of the polynomials defining a parameterization of X around a smooth point P, it is clear that

$$\dim(O_{k,X,P}) \leq \min\left\{r, \binom{n+k}{k} - 1\right\} := e.$$

It is also clear that if $P \in X$ is a flex point (for a definition of flex point see Definition 4.1.3) and $e = \binom{n+k}{k} - 1$ then $\dim(O_{k,X,P}) < e$ because if $F : U_0 \to X$ is a local parametrization around the point P such that $F(u_1, \ldots, u_n) = (f_1(u_1, \ldots, u_n), \ldots, f_r(u_1, \ldots, u_n))$ and $F_{u_i}(\underline{0})$ and $F_{u_i u_j}(\underline{0})$ are defined as in Definition 3.1.1 then the vectors $F_{u_i}(\underline{0})$, $F_{u_i u_j}(\underline{0})$ are not independent for all $i, j = 1, \ldots, n$.

Example: Suppose that $\max_{j=1,\ldots,r}\{\deg(f_j)\} = m$ then $\dim(O_{k,X,P}) < \binom{n+k}{k} - 1$ for all $k > m$; hence for the integers $k > m$ such that $e = \binom{n+k}{k} - 1$ it always happens that $\dim(O_{k,X,P}) < e$.

Example: Consider a projective variety $X \subset \mathbb{P}^r$ having around $[\underline{0}] \in X$ the following parameterization:

$$\begin{array}{rcl} \mathbb{C}^n \times \mathbb{C}^m & \to & \mathbb{C}^N \\ (u_1, \ldots, u_n; v_1, \ldots, v_m) & \mapsto & (g_1(u_1, \ldots, u_n), \ldots, g_{\binom{n}{k}}(u_1, \ldots, u_n); \\ & & f_1(v_1, \ldots, v_m), \ldots, f_{r-\binom{n}{k}}(v_1, \ldots, v_m)) \end{array}$$

such that

$$\begin{aligned} g_1(u_1, \ldots, u_1) &= u_1^k, \\ g_2(u_1, \ldots, u_1) &= u_1^{k-1} u_2, \\ &\vdots \\ g_{\binom{n}{k}}(u_1, \ldots, u_k) &= u_n^k. \end{aligned}$$

Then:

- $\frac{\partial f_i(v_1,\ldots,v_m)}{\partial u_j} = 0$ for $i = 1, \ldots, r - \binom{n}{k}$ and $j = 1, \ldots, n$;
- $\frac{\partial^{k+1} g_i(u_1,\ldots,u_n)}{\partial u_{i_1} \cdots \partial u_{i_{k+1}}} = 0$ for $i = 1, \ldots, \binom{n}{k}$ and $i_1, \ldots, i_k \in \left\{1, \ldots, \binom{n}{k}\right\}$.

Therefore, if $\binom{n+m+k}{k} - 1 \leq r$ and $h \geq k+1$, then $\dim(O_{h,X,P}) \leq \binom{n+m+h}{h} - 1 - \sum_{l=k+1}^{h} \binom{n}{l}$.

Example: Let C be a non degenerate projective curve. Let

$$\begin{array}{rcl} v : \mathbb{C} & \to & \mathbb{P}^n \\ t & \mapsto & (f_0(t), \ldots, f_n(t)) \end{array}$$

be a local parameterization of C such that $v(t_0) = P \in C$. We use the following notation: $v^{(i)}(t) := \frac{\partial^i v}{\partial^i t}$. The k-th osculating space to C at P is

$$O_{k,C,P} = v(t_0) + < v^{(1)}(t_0), \ldots, v^{(k)}(t_0) > .$$

It has the expected dimension if and only if

$$v(t_0) \wedge v^{(1)}(t_0) \wedge \cdots \wedge v^{(k)}(t_0) \neq 0.$$

Suppose in fact that $v(t_0) \wedge v^{(1)}(t_0) \wedge \cdots \wedge v^{(k)}(t_0) = 0$ and that $w(t_0) := v(t_0) \wedge v^{(1)}(t_0) \wedge \cdots \wedge v^{(k-1)}(t_0) \neq 0$, these imply that there exist $\alpha_i \in \mathbb{C}$, for $i = 0, \ldots, k-1$, such that:

$$v^{(k)}(t_0) = \sum_{i=0}^{k-1} \alpha_i v^{(i)}(t_0). \tag{3.12}$$

Moreover

$$\begin{aligned} w^{(1)}(t_0) = \; & v^{(1)}(t_0) \wedge v^{(1)}(t_0) \wedge \cdots \wedge v^{(k-1)}(t_0) + \\ & + v(t_0) \wedge v^{(2)}(t_0) \wedge v^{(2)}(t_0) \wedge \cdots \wedge v^{(k-1)}(t_0) + \\ & \vdots \\ & + v(t_0) \wedge \cdots \wedge v^{(k-1)}(t_0) \wedge v^{(k-1)}(t_0) + \\ & + v(t_0) \wedge \cdots \wedge v^{(k-2)}(t_0) \wedge v^{(k)}(t_0) \\ = \; & v(t_0) \wedge \cdots \wedge v^{(k-2)}(t_0) \wedge v^{(k)}(t_0). \end{aligned}$$

So, by (3.12), we have

$$w^{(1)}(t_0) = \alpha_{k-1} v(t_0) \wedge \cdots \wedge v^{(k-1)}(t_0) = \alpha_{k-1} w(t_0).$$

Hence there exist $c = (c_1, \ldots, c_n) \in \mathbb{C}^n$ such that

$$w(t_0) = c \cdot e^{\alpha_{k-1} t_0} = [c_1, \ldots, c_n] \in \mathbb{P}^n.$$

Therefore for a generic $t \in \mathbb{C}$ the vectors $v(t), v^{(1)}(t), \ldots, v^{(k-1)}(t)$ span the same $\mathbb{P}^{k-1}$, then $v(t) \in \mathbb{P}^{k-1}$ for a generic t, so the curve $C \subset \mathbb{P}^{k-1}$ that is a contradiction since C is not degenerate in $\mathbb{P}^n$.

3.2 The k-th osculating varieties to Veronese varieties

Definition 3.2.1. *Let $X \subset \mathbb{P}^N$ be a variety and let $X_0 \subset X$ be the dense set of the smooth points where $O_{k,X,P}$ has maximal dimension. The k-th osculating variety to X is defined as:*

$$O_{k,X} = \overline{\bigcup_{P \in X_0} O_{k,X,P}}.$$

We are interested in the study of the k-th osculating variety to the Veronese variety $\nu_d(\mathbb{P}^n)$. We set $O_{k,n,d} := O_{k,\nu_d(\mathbb{P}^n)}$.

For all this section we will assume that $N := \binom{n+d}{d} - 1$.

Let us assume (and from now on this assumption will be implicit) that $d \geq k$. If $L \in S_1$ is a linear form, we can write a point $P \in \nu_d(\mathbb{P}^n)$ as $P = [L^d]$. It is easy to see that the k-th osculating space to $\nu_d(\mathbb{P}^n)$ at a point $P = [L^d]$ is

$$O_{k,\nu_d(\mathbb{P}^n),P} = \{[M] \in \mathbb{P}(S_d) \mid M = L^{d-k}F, \text{ where } F \in S_k\}. \tag{3.13}$$

Notice that $O_{k,\nu_d(\mathbb{P}^n),P}$ has maximal dimension $\dim(S_k) - 1 = \binom{k+n}{n} - 1$ for all $P \in \nu_d(\mathbb{P}^n)$. This can be seen in the following way: the fat point $(k+1)P$ on $\nu_d(\mathbb{P}^n)$ gives independent conditions to the hyperplanes of $\mathbb{P}^N$, since it gives independent conditions to the forms of degree d in $\mathbb{P}^n$.
Hence, $O_{k,n,d} = \bigcup_{P \in \nu_d(\mathbb{P}^n)} O_{k,\nu_d(\mathbb{P}^n),P}$.
As we have already noticed, for $k = 0$ the equality (3.13) gives $O_{k,\nu_d(\mathbb{P}^n),P} = \{P\} = \{[L^d]\}$, and for $k = 1$ it becomes $O_{1,\nu_d(\mathbb{P}^n),P} = T_P(\nu_d(\mathbb{P}^n)) = \{[M] \in \mathbb{P}(S_d) \mid M = L^{d-1}F, \text{ where } F \in S_1\}$.
In general, we have:

$$O_{k,n,d} = \{[M] \in \mathbb{P}(S_d) \mid M = L^{d-k}F, \text{ where } L \in S_1, \text{ and } F \in S_k\}.$$

In the following we also need to know the tangent space $T_Q(O_{k,n,d})$ of $O_{k,n,d}$ at the generic point $Q = [L^{d-k}F]$ with $L \in S_1$ and $F \in S_k$; one has that the affine cone over $T_Q(O_{k,n,d})$ is

$$W = W(L, F) =< L^{d-k}R_k, L^{d-k-1}FR_1 > . \tag{3.14}$$

Lemma 3.2.2. *The dimension of $O_{k,n,d}$ is always the expected one, that is:*

$$\dim(O_{k,n,d}) = \min\left\{N,\ n + \binom{k+n}{n} - 1\right\}$$

Proof. By (3.14), the dimension of $O_{k,n,d}$ is $\dim(W(L,F)) - 1$, for a generic choice of $L \in S_1$ and $F \in S_k$, so that we can assume that L does not divide F. When $\mathbb{P}(W) \neq \mathbb{P}^N$, we have $\dim(W) = \dim(< L^{d-k}S_k >) + \dim(< L^{d-k-1}FS_1 >) - \dim(< L^{d-k}S_k > \cap < L^{d-k-1}FS_1 >) = \binom{k+n}{n} + (n+1) - 1 = \binom{k+n}{n} + n$, since there is only the obvious relation between LS_k and FS_1, namely $LF - FL = 0$. □

3.3 The $(s-1)$-secant variety to $O_{k,n,d}$

What we present in this section for $n > 2$ is also described in the joint work [**BCGI**].

From the analysis of the previous section it is now obvious that the $(s-1)$-secant variety to the k-th osculating variety to the Veronese $\nu_d(\mathbb{P}^n)$ is

$$\mathrm{Sec}_{s-1}(O_{k,n,d}) = \{[M] \in \mathbb{P}(S_d) \mid M = L_1^{d-k}F_1 + \cdots + L_s^{d-k}F_s, \text{ where } L_i \in S_1 \text{ and } F_i \in S_k, \text{ for } i = 1, \ldots, s\}.$$

Hence if we are interested in answering to the question:

> "which is the minimum integer s such that the form $M = L_1^{d-k}F_1 + \cdots + L_s^{d-k}F_s$, where $L_i \in S_1$ and $F_i \in S_k$, for $i = 1, \ldots, s$, is canonical",

we have to answer to:

> "which is the minimum integer s such that $\mathrm{Sec}_{s-1}(O_{k,n,d})$ is equal to $\mathbb{P}^N$?".

In this chapter we will study the dimension of $\mathrm{Sec}_{s-1}(O_{k,n,d})$.

Notice that, since $d \geq k$, one has $\dim(O_{k,n,d}) = N$ if and only if $\binom{d+n}{n} \leq n + \binom{k+n}{n}$, hence for all such k, n, d and for any s we have $\dim(\mathrm{Sec}_{s-1}(O_{k,n,d})) = \mathrm{expdim}(\mathrm{Sec}_{s-1}(O_{k,n,d})) = N$.
So we have to study this problem when $\binom{d+n}{n} > n + \binom{k+n}{n}$ and $s \geq 2$; it is easy to check that whenever $n \geq 2$ this condition is equivalent to $d \geq k+1$; on the other hand the case $n = 1$ (osculating varieties of rational normal curves) can be easily described (we will prove that $\mathrm{Sec}_{s-1}(O_{k,1,d})$ have always the expected dimension), thus the question becomes:

> "For all k, n, d such that $d \geq k+1$, $n \geq 2$, describe all s for which
>
> $$\dim(\mathrm{Sec}_{s-1}(O_{k,n,d})) < \min\left\{N, s\left(n + \binom{k+n}{n} - 1\right) + s - 1\right\}".$$

Remark: Terracini's Lemma 2.6.1 says that $\dim(\mathrm{Sec}_{s-1}(O_{k,n,d})) = N - h^0(\mathcal{I}_X \otimes \mathcal{O}_{\mathbb{P}^N}(1))$, where X is a generic union of 2-fat points on $O_{k,n,d}$; we are not able to handle directly the study of $h^0(\mathcal{I}_X \otimes \mathcal{O}_{\mathbb{P}^N}(1))$, nevertheless, Terracini's Lemma says that the tangent space to $\mathrm{Sec}_{s-1}(O_{k,n,d})$ at a generic point of $< P_1, \ldots, P_s >$, with $P_i \in O_{k,n,d}$ for $i = 1, \ldots, s$, is the span of the tangent spaces of $O_{k,n,d}$ at each P_i; i.e. if $T_{P_i}(O_{k,n,d}) = \mathbb{P}(W_i)$, then

$$\dim(\mathrm{Sec}_{s-1}(O_{k,n,d})) = \dim(< T_{P_1}(O_{k,n,d}), \ldots, T_{P_s}(O_{k,n,d}) >) = \dim(< W_1, \ldots, W_s >) - 1.$$

We want to prove, via Macaulay's theory of "inverse systems", that there exists a 0-dimensional projective scheme $Z = Z(k,n)$, that we will analyze further, such that for a single W_i,

$$\dim(W_i) = N + 1 - h^0(\mathbb{P}^n, \mathcal{I}_Z(d))$$

and, if $Y = Y(k,n,s)$ is a generic union in $\mathbb{P}^n$ of s 0-dimensional schemes isomorphic to Z, then

$$\dim(< W_1, ..., W_s >) = N + 1 - h^0(\mathbb{P}^n, \mathcal{I}_Y(d))$$

Hence,

$$\dim(\mathrm{Sec}_{s-1}(O_{k,n,d})) = \dim(< W_1, ..., W_s >) - 1 = N - h^0(\mathbb{P}^n, \mathcal{I}_Y(d)).$$

Notation: If $W \subset S_d = K[x_0, \ldots, x_n]_d$, we indicate with $W^\perp \subset R_d = K[y_0, \ldots, y_n]_d$ the orthogonal to W with respect to the Inverse System perfect pairing, i.e. $(W^\perp)^{-1} = W$.

Lemma 3.3.1. *There exists a 0-dimensional projective scheme $Z(k,n,d) \in \mathbb{P}^n$ such that the degree d part of the inverse system of its defining ideal is equal to the affine cone over the tangent space to $O_{k,n,d}$ at a generic point $Q \in O_{k,n,d}$. Moreover, if $O \in \mathbb{P}^n$ is the support of $Z(k,n,d)$, then for all k, n and $d \geq k+2$, we have:*

$$(k+1)O \subset Z(k,n,d) \subset (k+2)O.$$

Proof. Let $W =< L^{d-k}S_k,\ L^{d-k-1}FS_1 >\subset S_d$ be the affine cone over $T_Q(O_{k,n,d})$ at a generic point $Q = [L^{d-k}F]$, with $L \in S_1$ and $F \in S_k$. Without loss of generality we can choose $L = x_0$, so that $W =< x_0^{d-k-1}x_0S_k, x_0^{d-k-1}FS_1 >$, hence $< x_0^{d-k}S_k >\subset W \subset< x_0^{d-k-1}S_{k+1} >$. So for any (k,n,d),

$$< x_0^{d-k-1}S_{k+1} >^\perp \subset W^\perp \subset< x_0^{d-k}S_k >^\perp . \tag{3.15}$$

Now, denoting by $\wp$ the ideal $(x_1, \ldots, x_n)$, we have:

$$(x_0^{d-t}S_t)^\perp =< \{x_0^{i_0} \cdots x_n^{i_n} \mid \Sigma_j i_j = d, i_0 \leq d-t-1\} >=$$
$$=< (\wp^d)_d,\ x_0(\wp^{d-1})_{d-1}, \ldots, x_0^{d-t-1}(\wp^{t+1})_{t+1} >= (\wp^{t+1})_d.$$

Let us view everything in (3.15) as the degree d part of an homogeneous ideal; we get:

$$(\wp^{k+2})_d \subset (I(Z(k,n,d)))_d \subset (\wp^{k+1})_d.$$

Let $(x_1, \ldots, x_n)$ be local coordinates in $\mathbb{P}^n$ around the point $O = (1, 0, \ldots, 0)$; the above inclusions give, in terms of 0-dimensional schemes in $\mathbb{P}^n$:

$$(k+1)O \subset Z(k,n,d) \subset (k+2)O.$$

□

Lemma 3.3.2. *For any k, n, d with $d \geq k+2$, the length of $Z = Z(k,n,d)$ defined in Lemma 3.3.1 is:*

$$l(Z) = \dim(W) = \left(\binom{k+n}{n} + n\right).$$

Proof. We have seen that $Z(k,n,d) \subset (k+2)O$, with $O \in \mathbb{P}^n$. Setting $X := (k+2)O$, the condition $d \geq k+2$ then gives $\binom{d+n}{n} \geq l(X) = \binom{k+1+n}{n} \geq l(Z)$.
We have $W \neq S_d$ by assumption, since $d \geq k+2$ implies $\binom{d+n}{n} \geq \binom{k+2+n}{n} = \binom{k+n}{n} + \binom{k+n}{n-1} + \binom{k+1+n}{n-1} \geq \binom{k+n}{n} + n$.
Hence, $\dim(I_d) = \dim(W^\perp) = \binom{d+n}{n} - \dim(W)$, hence if we prove that $\dim(I_d) = \binom{d+n}{n} - l(Z)$, i.e. Z imposes indipendent conditions to the forms of degree d, thesis follows.
One $(k+2)$-fat point always imposes independent conditions to the forms of degree d, and since $\binom{d+n}{n} \geq l(X)$, then $h^1(\mathcal{I}_X(d)) = 0$. The cohomology of the exact sequence:

$$0 \to \mathcal{I}_X(d) \to \mathcal{I}_Z(d) \to \mathcal{I}_{Z,X}(d) \simeq \mathcal{O}_D \to 0$$

where D is a 0-dimensional scheme of length $l(X) - l(Z)$ then gives $h^1(\mathcal{I}_Z(d)) = 0$. □

Now we have seen that our problem can be translated into a problem of studying certain schemes $Z(k,n,d) \subset \mathbb{P}^n$; we want to check that actually these schemes are the same for all $d \geq k+2$, say $Z(k,n,d) = Z(k,n)$.

Lemma 3.3.3. *For any k, n and $d \geq k+2$, we have $Z(k,n,d) = Z(k,n,k+2)$. Henceforth we will denote $Z(k,n) = Z(k,n,d)$, for all $d \geq k+2$.*

Proof. By the previous lemmata we already know that $Z(k,n,d)$ and $Z(k,n,k+2)$ have the same support and the same length, hence it is enough to show that $Z(k,n,d) \subset Z(k,n,k+2)$ (as schemes) in order to conclude. This will be done if we check that $I(Z(k,n,k+2))_d \subset I(Z(k,n,d))_d$; in fact, since both ideals are generated in degrees $\leq d$, this will imply that $I(Z(k,n,k+2))_j \subset I(Z(k,n,d))_j$, for all $j \geq d$, hence the inclusion will hold also between the two saturations, implying $Z(k,n,d) \subset Z(k,n,k+2)$.

Let $f \in I(Z(k,n,k+2))_d$, then $f = h_1 g_1 + \cdots + h_r g_r$, where $h_j \in S_{d-k-2}$ and $g_j \in I(Z(k,n,k+2))_{k+2}$; since $I(Z(k,n,d))_d$ is the perpendicular (via apolarity duality) to $W =< L^{d-k}S_k, L^{d-k-1}FS_1 >$, it is enough to check that $h_j g_j \in W^\perp$, $j = 1, \ldots, r$. Without loss of generality we can assume $L = x_0$; hence, since $g_j \in < L^2 S_k, LFS_1 >^\perp$, $g_j = x_0 g' + g''$, with $g', g'' \in K[x_1, \ldots, x_n]$ and $g' \in (FS_1)^\perp$. It will be enough to prove $x_0^{i_0} \cdots x_n^{i_n} g_j = x_0^{i_0+1} \cdots x_n^{i_n} g' + x_0^{i_0} \cdots x_n^{i_n} g'' \in W^\perp$, for all $i_0, \ldots, i_n$ such that $i_0 + \cdots + i_n = d-k-2$. It is clear that $x_0^{i_0} \cdots x_n^{i_n} g'' \in W^\perp$, since $i_0 \leq d-k-2$; on the other hand, $x_0^{i_0+1} \cdots x_n^{i_n} g' \in (x_0^{d-k} S_k)^\perp$ again by looking at the degree of x_0, while $x_0^{i_0+1} \cdots x_n^{i_n} g' \in (x_0^{d-k-1} FS_1)^\perp$ since $g' \in (FS_1)^\perp$. □

Remark: From the lemmata above it follows that in order to study the dimension of $\mathrm{Sec}_{s-1}(O_{k,n,d})$, for all $d \geq k+2$, we only need to study the postulation of unions of schemes $Z(k,n)$. For $d = k+1$, we will work directly on W, see Proposition 3.3.9
What we got is a sort of "generalized Terracini" for osculating varieties to Veronesean, since the formula $\dim(\mathrm{Sec}_{s-1}(O_{k,n,d})) = N - h^0(\mathcal{I}_Y(d))$ reduces to the one in Corollary 2.6.2 for $k = 0$. Instead of studying 2-fat points on $O_{k,n,d}$, we can study the schemes $Y \subset \mathbb{P}^n$.

Definition 3.3.4. *Let $Y \subset \mathbb{P}^n$ be a 0-dimensional scheme; we say that Y is Regular in degree d, $d \geq 0$, if the restriction map $\rho : H^0(\mathcal{O}_{\mathbb{P}^n}(d)) \to H^0(\mathcal{O}_Y(d))$ has maximal rank, i.e. if*

$$h^0(\mathcal{I}_Y(d)) \cdot h^1(\mathcal{I}_Y(d)) = 0.$$

We set $\exp(h^0(\mathcal{I}_Y(d))) := \max\{0, \binom{d+n}{n} - l(Y)\}$*; hence to say that Y is regular in degree d amounts to saying that $h^0(\mathcal{I}_Y(d)) = \exp(h^0(\mathcal{I}_Y(d)))$.*

Since we always have $h^0(\mathcal{I}_Y(d)) \geq \exp(h^0(\mathcal{I}_Y(d)))$, we write

$$h^0(\mathcal{I}_Y(d)) = \exp(h^0(\mathcal{I}_Y(d))) + \delta,$$

where $\delta = \delta(Y,d)$; hence whenever $\binom{d+n}{n} - l(Y) \geq 0$, we have $\delta = h^1(\mathcal{I}_Y(d))$, while if $\binom{d+n}{n} - l(Y) \leq 0$, $\delta = \binom{d+n}{n} - l(Y) + h^1(\mathcal{I}_Y(d))$; in any case, by setting $\exp\, h^1(\mathcal{I}_Y(d)) := \max\left\{0, l(Y) - \binom{d+n}{n}\right\}$, we get: $h^1(\mathcal{I}_Y(d)) = \exp(h^1(\mathcal{I}_Y(d))) + \delta$.

Remark: For any k, n, d such that $d \geq k+1$, let $Z = Z(k,n)$ be the scheme defined and studied in Lemmas 3.3.1 and 3.3.3, let $Y = Y(k,n,s) \subset \mathbb{P}^n$ be the generic union of s 0-dimensional schemes isomorphic to $Z(k,n)$ and $\delta = \delta(Y,d)$. Then

$$\dim(\mathrm{Sec}_{s-1}(O_{k,n,d})) = \mathrm{expdim}(\mathrm{Sec}_{s-1}(O_{k,n,d})) - \delta.$$

In particular, $\dim(\mathrm{Sec}_{s-1}(O_{k,n,d})) = \mathrm{expdim}(\mathrm{Sec}_{s-1}(O_{k,n,d}))$ if and only if:

- $h^0(\mathcal{I}_Y(d)) = 0$, when $\binom{d+n}{n} \leq s\binom{k+n}{n} + sn$;
- $h^0(\mathcal{I}_Y(d)) = N+1-l(Y) = \binom{d+n}{n} - s\binom{k+n}{n} - sn$ (i.e. $h^1(\mathcal{I}_Y(d)) = 0$), when $\binom{d+n}{n} \geq s\binom{k+n}{n} + sn$.

Example: In the case of $n = 1$ every $\mathrm{Sec}_{s-1}(O_{k,1,d})$, with $d \geq k+2$, has the expected dimension; in fact here $Z(k,1) = (k+2)O$, and the scheme $Y = \{s\ (k+2)\text{-fat points}\} \subset \mathbb{P}^1$ is regular in any degree d. Notice that for $d = k+1$ we trivially have $O_{k,1,k+1} = \mathbb{P}^N$.

The case of $n = 2$ will be treat in the next section.

Example: If $k = 1$ we have already observed at the beginning of this chapter that $O_{1,n,d} = T(\nu_d(\mathbb{P}^n))$, i.e. the tangential variety to the Veronese variety. In [**CGG2**] it is shown that $Z(1,n)$ is a "$(2,3)-$scheme" (i.e. the intersection in $\mathbb{P}^n$ of a 3-fat point with a double line); this is easy to see, e.g. by choosing coordinates so that $L = x_0$ and $F = x_1$.
The postulation of generic unions of such schemes in $\mathbb{P}^n$, and hence the defectivity of $\mathrm{Sec}_{s-1}(O_{1,n,d})$, has been studied. Moreover, a conjecture regarding all defective cases is stated there:

Conjecture 3.3.5. ([**CGG2**]) *The variety* $\mathrm{Sec}_{s-1}(O_{1,n,d})$ *is not defective, except in the following cases:*

1. *for $d = 2$ and $n \geq 2s$, $s \geq 2$;*

2. *for $d = 3$ and $n = s = 2, 3, 4$.*

In [**CGG2**] the conjecture is proved for $s \leq 5$ (any d, n), for $s \geq \frac{1}{3}\binom{n+2}{2} + 1$ (any d, n); for $d = 2$ (any s, n), for $d \geq 3$ and $n \geq s+1$, for $d \geq 4$ and $s = n$. In [**Ba**], the conjecture is proved for $n = 2, 3$ (any s, d).

The following lemma describes what can be deduced about the postulation of the scheme Y from information on fat points:

Lemma 3.3.6. *Let $P_1, \ldots, P_s$ be generic points in $\mathbb{P}^n$, and set $X := (k+1)P_1 \cup \cdots \cup (k+1)P_s$, $T := (k+2)P_1 \cup \cdots \cup (k+2)P_s$. Now let Z_i be a 0-dimensional scheme supported on P_i , $(k+1)P_i \subset Z_i \subset (k+2)P_i$, with $l(Z_i) = l((k+1)P_i) + n$ for each $i = 1, \ldots, s$, , and set $Y := Z_1 \cup \cdots \cup Z_s$. Then:*

1. *Y is regular in degree d if one of the following (a) or(b) holds:*

   (a) *$h^1(\mathcal{I}_T(d)) = 0$, (hence $\binom{d+n}{n} \geq s\binom{k+n+1}{n}$);*

   (b) *$h^0(\mathcal{I}_X(d)) = 0$, (hence $\binom{d+n}{n} \leq s\binom{k+n}{n}$).*

2. *Y is not regular in degree d, with defectivity δ, if one of the following (a) or (b) holds:*

   (a) *$h^1(\mathcal{I}_X(d)) > \exp$, $h^1(\mathcal{I}_Y(d)) = \max\{0, l(Y) - \binom{d+n}{n}\}$; in this case $\delta \geq h^1(\mathcal{I}_X(d))) - \exp(h^1(\mathcal{I}_Y(d))$.*

   (b) *$h^0(\mathcal{I}_T(d)) > \exp(h^0(\mathcal{I}_Y(d))) = \max\{0, \binom{d+n}{n} - l(Y)\}$; in this case $\delta \geq h^0(\mathcal{I}_T(d)) - \exp(h^0(\mathcal{I}_Y(d)))$.*

Proof. The statement follows by considering the cohomology of the exact sequences:

$$0 \to \mathcal{I}_T(d) \to \mathcal{I}_Y(d) \to \mathcal{I}_{Y,T}(d) \to 0$$

and

$$0 \to \mathcal{I}_Y(d) \to \mathcal{I}_X(d) \to \mathcal{I}_{X,Y}(d) \to 0$$

where we have: $h^1(\mathcal{I}_{Y,T}(d)) = h^1(\mathcal{I}_{X,Y}(d)) = 0$ since those two sheaves are supported on a 0-dimensional scheme. □

Lemma 3.3.7. *Let $s \geq n+2$ and $d < k+1+2\left(\frac{k+1}{n}\right)$. Then $\mathrm{Sec}_{s-1}(O_{k,n,d})$ is not defective and $\mathrm{Sec}_{s-1}(O_{k,n,d}) = \mathbb{P}^N$.*

Proof. Let $Y \subset \mathbb{P}^n$ be as in Lemma 3.3.6; we have to prove that $h^0(\mathcal{I}_Y(d)) = 0$ in our hypotheses. Let $\{P_1, \ldots, P_s\}$ be the support of Y; we can always choose a rational normal curve $C \subset \mathbb{P}^n$ containing $n+2$ of the P_i's . For any hypersurface F given by a section of $\mathcal{I}_Y(d)$, since $nd < (k+1)(n+2)$, by Bezout we get $C \subset F$. But we can always find a rational normal curve containing $n+3$ points in $\mathbb{P}^n$, so this would imply that any $P \in \mathbb{P}^n$ is on F, i.e. $\mathcal{I}_Y(d) = 0$. □

Lemma 3.3.8. *Assume $s = n+1$; if $d \leq k+1+\frac{k+2}{n}$, then $\mathrm{Sec}_{s-1}(O_{k,n,d}) = \mathbb{P}^N$.*

Proof. Since $d \geq k+1$, we can set $d = k+j$ with $j > 0$; let $W_i =< L_i^j S_k,\ L_i^{j-1} F_i S_1 >$ with $F_i \in S_k$ for $i = 1, \ldots, s$; since $s = n+1$, without loss of generality we can assume that $L_1 = x_0, \ldots, L_{n+1} = x_n$.
Hence $W_1 + \cdots + W_s$ contains $U := x_0^j S_k + \cdots + x_n^j S_k$; now in U the missing monomials are $x_0^{i_0} \cdots x_n^{i_n}$ with $i_l \leq j-1$ for each $l = 0, \ldots, n$, and $d = \deg(x_0^{i_0} \cdots x_n^{i_n}) \leq (n+1)(j-1)$. Hence if $d \geq (n+1)(j-1)$, i.e. $d < k+1+\frac{k+1}{n}$, we get $U = S_d$.
If $d = (n+1)(j-1)$ the only missing monomial in U is $x_0^{j-1} \cdots x_n^{j-1}$, hence it is enough to choose one of the F_i's in a proper way to get $W_1 + \cdots + W_s = S_d$.
If $d = (n+1)(j-1)-1$, i.e. $d = k+1+\frac{k+2}{n}$, the $n+1$ missing monomials in U are $x_0^{j-1} \cdots x_i^{j-2} \cdots x_n^{j-1}$ with $i = 0, \ldots, n$ and again it is possible to choose the F_i's so that $W_1 + \cdots + W_s = S_d$. □

3.3.1 The case of $O_{k,n,k+1}$

The description for $k=1$ given in [**CGG2**], together with following proposition, describes this case completely.

Proposition 3.3.9. *If $s \geq 2$, $k \geq 2$ and $d = k+1$, consider the secant variety $\mathrm{Sec}_{s-1}(O_{k,n,d}) \subset \mathbb{P}^N$; then:*

1. *if $s \leq n-1$ and its expected dimension is $s\binom{k+n}{n} + sn - 1$, then $\mathrm{Sec}_{s-1}(O_{k,n,k+1})$ is defective with defect*
$$\delta = s^2 - s + s\binom{k+n}{n} + \binom{n-s+d}{d} - N;$$

2. *if $s \leq n-1$ and the expected dimension is $N = \binom{d+n}{n} - 1$ then*
   (a) *$\mathrm{Sec}_{s-1}(O_{d-1,n,d})$ is defective with defect $\delta = \binom{n-s+d}{d} - s(n-s+1)$ if $s < \frac{1}{d}\binom{n-s+d}{d-1}$;*
   (b) *$\mathrm{Sec}_{s-1}(O_{d-1,n,d}) = \mathbb{P}^N$ if $s \geq \frac{1}{d}\binom{n-s+d}{d-1}$;*

3. *if $s \geq n$ then $\mathrm{Sec}_{s-1}(O_{d-1,n,d}) = \mathbb{P}^N$.*

Proof. 1. We have that $W = W_1 + \cdots + W_s =< x_0S_k, \ldots, x_{s-1}S_k; F_1S_1, \ldots, F_sS_1 >$ in S_d. We can suppose that the F_i's, $i = 1, \ldots, s$ are generic in $K[x_s, \ldots, x_n]_{d-1} := S'_{d-1}$, and we have that $\frac{S_d}{W} \simeq \frac{S'_d}{(F_1,\ldots,F_s)_d}$. Then, since $(F_1, \ldots, F_s)_d =< F_1S_1, \ldots, F_sS_1 >$ and the F_i's are generic, $\dim(F_1, \ldots, F_s)_d = \min\left\{\binom{n-s+d}{d}, s(n-s+1)\right\}$.

From this, and from our hypothesis about the expected dimension, we immediately get that $\dim(W) = N - \binom{n-s+d}{d} + s(n-s+1)$, and hence that the defectivity is $\delta = s^2 - s + s\binom{k+n}{n} + \binom{n-s+d}{d} - N$.

2. If $s\binom{n+d-1}{n} + ns \geq \binom{n+d}{n}$ we expect that $\mathrm{Sec}_{s-1}(O_{d-1,n,d}) = \mathbb{P}^N$. Proceeding as in the previous case, in order to compute $\dim(W)$ we can actually just consider the vector space $< F_1S_1, \ldots, F_sS_1 >$; whose dimension is $\min\left\{\binom{n-s+d}{d}, s(n-s+1)\right\}$; so we get that

   (a) if $s(n-s+1) < \binom{n-s+d}{d}$, then $\mathrm{Sec}_{s-1}(O_{d-1,n,d})$ is defective. This happens if and only if $s < \frac{1}{d}\binom{n-s+d}{d-1}$, in this case the defect is $\delta = \binom{n-s+d}{d} - s(n-s+1)$.

   (b) if $s(n-s+1) \geq \binom{n-s+d}{d}$, then $\mathrm{Sec}_{s-1}(O_{d-1,n,d}) = \mathbb{P}^N$ (for example this is always true for $d \geq n$);

3. It suffices to prove that $\mathrm{Sec}_{s-1}(O_{d-1,n,d}) = \mathbb{P}^N$ for $s = n$.

   If $s = n$ and $d = k+1$, then the subspace $W_1 + \cdots + W_s$ can be written as follows: $< x_0S_k, F_1S_1, \ldots, x_{n-1}S_k, F_nS_1 >$; it turns out to be equal to $< x_0S_k, \ldots, x_{n-1}S_k, x_n^{k+1} >= S_{k+1}$, so $\mathrm{Sec}_{n-1}(O_{d-1,n,d}) = \mathbb{P}^N$.

□

Example: Let us consider the secant varieties of the 4th-osculating variety $O_{4,6,5} \subset \mathbb{P}^{461}$. We begin with $\mathrm{Sec}_1(O_{4,6,5})$; we are in case 1. of Proposition 3.3.9, and we expect that $\dim(\mathrm{Sec}_1(O_{4,6,5})) = 431$, but we get that the defectivity is $\delta = 86$ so that $\dim(\mathrm{Sec}_1(O_{4,6,5})) = 345$.

When $s = 3, 4$ we are in case 2. of Proposition 3.3.9, and $\delta = 44$ for $\mathrm{Sec}_2(O_{4,6,5})$, while $\delta = 9$ for $\mathrm{Sec}_3(O_{4,6,5})$. Eventually, $\mathrm{Sec}_4(O_{4,6,5})) = \mathbb{P}^{461}$

So, even if we expect that $\mathrm{Sec}_2(O_{4,6,5})$ should fill up $\mathbb{P}^N$, even the 3-secant variety does not.

In terms of forms we get that neither we can write a generic $f \in (K[x_0, \ldots, x_6])_5$ as $f = L_1F_1 + L_2F_2 + L_3F_3$ with $L_i \in S_1$ and $F_i \in S_4$ (as we expect), nor as $f = L_1F_1 + \cdots + L_4F_4$, but we need five addenda.

3.3.2 Some examples for $d = k+2$ and $d = k+3$

The case of $d = k+2$

- Let us consider first the Veronese surface $\nu_{k+2}(\mathbb{P}^2)$.

  Corollary 3.3.10. *Assume $d = k+2$ and $n = 2$. Then, $\mathrm{Sec}_{s-1}(O_{k,2,k+2})$ is not defective for $s \geq 3$ and $k \geq 1$, and $\mathrm{Sec}_{s-1}(O_{k,2,k+2})$ is defective for $s = 2$ and $k \geq 1$.*

  Proof. By Lemma 3.3.7 and Lemma 3.3.8, $\mathrm{Sec}_{s-1}(O_{k,2,k+2})$ is not defective for $s \geq 3$ and $d \geq 3$, i.e. $k \geq 2$; the case $k = 1$ is already known by [**Ba**].
  For $s = 2$ and $k \geq 1$, let $Y = Y(k, 2) \subset \mathbb{P}^2$ be the 0-dimensional scheme defined in 3.3.6; it is easy to check that $\exp(h^0(\mathcal{I}_Y(d))) = \exp(h^0(\mathcal{I}_T(d))) = 0$, where T denotes the generic union of two $(k+2)$-fat points in $\mathbb{P}^2$. Since T is not regular in degree $d = k+2$ for any $k \geq 1$, we conclude by Lemma 3.3.6 case 2.(b) that $\mathrm{Sec}_{s-1}(O_{k,n,k+2})$ is defective with defectivity grater or equal than $h^0(\mathcal{I}_T(d)) = 1$ (the only section is given by the $(k+2)$-ple line through the two points). □

- Let us now consider the case of $\nu_{k+2}(\mathbb{P}^3)$.

  Corollary 3.3.11. *Assume $d = k+2$ and $n = 3$. Then, $\mathrm{Sec}_{s-1}(O_{k,3,k+2}) = \mathbb{P}^N$ for $s \geq n+1 = 4$ and $k \geq 1$, while $\mathrm{Sec}_{s-1}(O_{k,3,k+2})$ is defective for $s \leq 3$.*

Proof. The case $s \leq 3$ will be treated in Proposition 3.3.15.
If $s = 4$ and $k = 1$, $\mathrm{Sec}_3(O_{1,3,3}) = \mathbb{P}^N$ by [**CGG2**], (4.6). If $s = 4$ and $k = 2$, we have $\mathrm{Sec}_3(O_{2,3,4}) = \mathbb{P}^N$ by Lemma 3.3.8.
If $s \geq 5$ and $k \geq 1$, or $s = 4$ and $k \geq 3$, the thesis follows by Lemmata 3.3.7 and 3.3.8, respectively. □

- As last case we consider $\nu_{k+2}(\mathbb{P}^4)$.

  Corollary 3.3.12. *Assume $d = k+2$ and $n = 4$. Then, $\mathrm{Sec}_{s-1}(O_{k,4,k+2}) = \mathbb{P}^N$ for $s \geq 5$ and $k \geq 1$, while $\mathrm{Sec}_{s-1}(O_{k,4,k+2})$ is defective for $s \leq 4$.*

  Proof. The case $s \leq 4$ will be given by Proposition 3.3.15.
  If $s \geq 5$ and $k = 1$, $\mathrm{Sec}_{s-1}(O_{1,4,3}) = \mathbb{P}^N$ by [**CGG2**], (4.6) and (4.5). If $s = 5$ and $k = 2, 3$, we have $\mathrm{Sec}_4(O_{k,4,k+2}) = \mathbb{P}^N$ by Lemma 3.3.8.
  If $s \geq n+2 = 6$ and $k \geq 2$, or $s = 5$ and $k \geq 4$, thesis follows by Lemmata 3.3.7 and 3.3.8, respectively. □

The case of $d = k+3$

For the Veronese surface we can prove the following:

Corollary 3.3.13. *Assume $d = k+3$ and $n = 2$. Then:*

1. *for $s = 2$ and $k = 1, 2$: $\dim(\mathrm{Sec}_1(O_{k,2,k+3})) = s\binom{k+2}{2} + 2s - 1$ (the expected one);*
2. *for $s = 2$ and $k \geq 3$: $\mathrm{Sec}_1(O_{k,2,k+3})$ is defective;*
3. *for $s \geq 3$ and $k \geq 1$: $\mathrm{Sec}_{s-1}(O_{k,2,k+3}) = \mathbb{P}^N$.*

Proof. If $s \geq n+2 = 4$ and $k \geq 2$, or $s = 3$ and $k \geq 4$, the thesis follows by Lemmata 3.3.7 and 3.3.8, respectively.
If $s \geq 3$ and $k = 1$, $\mathrm{Sec}_{s-1}(O_{1,2,k+3}) = \mathbb{P}^N$ by [**CGG2**], (4.5).
If $s = 3$ and $k = 2, 3$, we have $\mathrm{Sec}_1(O_{k,2,k+3}) = \mathbb{P}^N$ by Lemma 3.3.8.
If $s = 2$ and $k = 1$, or $s = 2$ and $k = 2$, $\mathrm{Sec}_1(O_{k,2,k+3}) \neq \mathbb{P}^N$ is not defective by [**CGG2**], (4.6) and [**BF1**], Theorem 1, respectively.
If $s = 2$ and $k \geq 3$, then, in the notations of Lemma 3.3.6, we have :
for $k = 3, 4$ $\exp(h^1(\mathcal{I}_X(d))) = \exp(h^1(\mathcal{I}_Y(d))) = 0$, and the union X of 2 $(k+1)$-fat points is not regular in degree $d = k+3$;
for $k \geq 5$ $\exp(h^0(\mathcal{I}_Y(d))) = \exp(h^0(\mathcal{I}_T(d))) = 0$, and the union T of 2 $(k+2)$-fat points is not regular in degree $d = k+3$; so we conclude by 3.3.6, cases 2.(a) and 2.(b). □

3.3.3 Partial results for $s \leq n+1$

If we want to study the $(s-1)$-secant variety to $O_{k,n,d}$ and we know that $s \leq n+1$ we can deeply use the Inverse System theory because we can always choose a particular tangent space to $\mathrm{Sec}_{s-1}(O_{k,n,d})$ in such a way we can be sure that it is not a restrictive hypothesis.

Proposition 3.3.14. *If $s \leq n+1$, $d \geq 2k+1$ and $k \geq 2$ then $\mathrm{Sec}_{s-1}(O_{k,n,d})$ is regular.*

Proof. We have to study the dimension of the vector space $W_1+\cdots+W_s =< L_1^{d-k}S_k, L_1^{d-k-1}F_1S_1, \ldots, L_s^{d-k}S_k, L_s^{d-k-1}F_sS_1 >$, where $L_1, \ldots, L_s$ are generic in S_1 and $F_1, \ldots, F_s$ are generic in S_k. Since $s \leq n+1$, without loss of generality we may suppose $L_i = x_{i-1}$ for $i = 1, \ldots, s$. Since $d \geq 2k+1$, for $\beta = d-k \geq 3$, the vector space $W_1+\cdots+W_s$ can be written as $< x_0^{\beta}S_k, x_0^{\beta-1}F_1S_1, \ldots, x_{s-1}^{\beta}S_k, x_{s-1}^{\beta-1}F_sS_1 >$. If we show that for a particular choice of $F_1, \ldots, F_s \in S_k$ the dimension of $W_1 + \cdots + W_s = \mathrm{expdim}(\mathrm{Sec}_{s-1}(O_{k,n,d}))+1$ we can conclude by semi-continuity that $\mathrm{Sec}_{s-1}(O_{k,n,d})$ has the expected dimension. Let us consider the case $F_i = x_ix_{i+1}\widetilde{F}_i$ for $i = 1, \ldots, s-2$, $F_{s-1} = x_{s-1}x_0\widetilde{F}_{s-1}$ and $F_s = x_0x_1\widetilde{F}_s$, where the $\widetilde{F}_j$'s are generic forms in S_{k-2}, $j = 1, \ldots, n+1$. Let $< x_i^{\beta}S_k >=: A_i$ and $< x_i^{\beta-1}F_{i+1}S_1 >=: A'_i$, $i = 0, \ldots, s-1$; then we get $A'_i =< x_i^{\beta-1}x_{i+1}x_{i+2}\widetilde{F}_{i+1}S_1 >$, $i = 0, \ldots, s-3$; $A'_{s-2} =< x_{s-2}^{\beta-1}x_{s-1}x_0\widetilde{F}_{s-1}S_1 >$ and $A'_{s-1} =< x_{s-1}^{\beta-1}x_0x_1\widetilde{F}_sS_1 >$. Now $W_1 + \cdots + W_s = \sum_{j=0}^{s-1} A_j + \sum_{j=0}^{s-1} A'_j$. We can easily notice that $A'_i \cap \left(\sum_{j=0}^{s-1} A_j + \sum_{j=0,j\neq i}^{s-1} A'_j\right) = A_i \cap \left(\sum_{j=0,j\neq i}^{s-1} A_j + \sum_{j=0}^{s-1} A'_j\right) = A_i \cap A'_i =< x_i^{\beta}S_k > \cap < x_i^{\beta-1}x_{i+1}x_{i+2}\widetilde{F}_{i+1}S_1 >=< x_i^{\beta}x_{i+1}x_{i+2}\widetilde{F}_{i+1} >$ for any fixed $i = 0, \ldots, s-3$ (analogously if $i = s-2, s-1$). So we have found exactly s relations and we can conclude that $\dim(W_1 + \cdots + W_s) = \dim\left(\sum_{j=0}^{s-1} A_j\right) + \dim\left(\sum_{j=0}^{s-1} A'_j\right) - s = s\binom{k+n}{n} + s(n+1) - s$, which is exactly the expected dimension. □

Proposition 3.3.15. *If $s \leq n$ and $k+2 \leq d \leq 2k$ then $\mathrm{Sec}_{s-1}(O_{k,n,d})$ is defective with defect δ such that:*

1. *$\delta \geq \binom{n-s+d}{d}$ if the expected dimension is $\binom{d+n}{n} - 1$;*
2. *$\delta \geq \binom{s}{2}\binom{2k-d+n}{n}$ if the expected dimension is $s\binom{k+n}{n} + sn - 1$.*

Proof. Let $\beta := d-k \geq 2$; we can rewrite the vector space $W_1 + \cdots + W_s$ as follows:
$< x_0^{\beta}S_k, x_0^{\beta-1}F_1S_1, \ldots, x_{s-1}^{\beta}S_k, x_{s-1}^{\beta-1}F_sS_1 >$.

1. We can observe that $K[x_s, \ldots, x_n]_d \cap (W_1+\cdots+W_s) = \{0\}$, so if we expect that $\mathrm{Sec}_{s-1}(O_{k,n,d}) = \mathbb{P}^N$ we get a defect $\delta \geq \binom{n-s+d}{d}$.

2. Suppose now that $s\left[\binom{k+n}{n}+n\right] < \binom{d+n}{n}$. If $\mathrm{Sec}_{s-1}(O_{k,n,d})$ were to have the expected dimension we would not be able to find more relations among the W_i's other than $x_i^\beta F_{i+1} \in < x_i^\beta S_k > \cap < x_i^{\beta-1} F_{i+1} S_1 >$, for $i = 0, \dots, s-1$ (as it happens in Proposition 3.3.14. But it is easy to see that $x_i^\beta x_j^\beta F \in < x_i^\beta S_k > \cap < x_j^\beta S_k >$ with $i \neq j$ and $F \in S_{k-\beta}$. We have exactly $\binom{s}{2}$ such terms for any choice of $F \in S_{k-\beta}$. We can also suppose that the $F_i \in S_k$ that appear in $W_1 + \cdots + W_s$ are different from $x_j^\beta F$ for any $F \in S_{k-\beta}$ and $j = 0, \dots, s-1$, because $F_1, \dots, F_s$ are generic forms of S_k. Then we can be sure that the form $x_i^\beta x_j^\beta F$ belonging to $< x_i^\beta S_k > \cap < x_j^\beta S_k >$ is not one of the $x_i^\beta F_{i+1}$ that belongs to $< x_i^\beta S_k > \cap < x_i^{\beta-1} F_{i+1} S_1 >$. Now $\dim(S_{k-\beta}) = \binom{k-\beta+n}{n}$ so we can find $\binom{s}{2}\binom{k-\beta+n}{n}$ independent forms that give defectivity. Hence in case $s\left[\binom{k+n}{n}+n\right] < \binom{d+n}{n}$ we have $\dim(\mathrm{Sec}_{s-1}(O_{k,n,d})) \leq \mathrm{expdim}(\mathrm{Sec}_{s-1}(O_{k,n,d})) - \binom{s}{2}\binom{k-\beta+n}{n} = \mathrm{expdim}(\mathrm{Sec}_{s-1}(O_{k,n,d})) - \binom{s}{2}\binom{2k-d+n}{n}$.

□

Proposition 3.3.16. *If $s = n+1$, $k+2 \leq d \leq 2k$ and $\mathrm{expdim}(\mathrm{Sec}_n(O_{k,n,d})) = (n+1)\left(\binom{k+n}{n}+n\right)-1$ then $\mathrm{Sec}_n(O_{k,n,d})$ is defective with defect $\delta \geq \binom{n+1}{2}\binom{2k-d+n}{n}$.*

Proof. The proof of this fact is the same as case 2. of the previous proposition. □

Proposition 3.3.17. *If $s = n+1$, $n \geq \frac{k+2}{d-k-2}$, $k+2 < d \leq 2k$ and $\mathrm{expdim}(\mathrm{Sec}_n(O_{k,n,d})) = N$ then $\mathrm{Sec}_n(O_{k,n,d})$ is defective with defect $\delta \geq \binom{(n+1)(d-k-1)-(d+1)}{n}$.*

Proof. If $k+2 < d \leq 2k$, then $2 < \beta := d-k \leq k$ and we have to study the dimension of $W_1 + \cdots + W_{n+1} =< x_0^\beta S_k, x_0^{\beta-1} F_1 S_1, \dots, x_n^\beta S_k, x_n^{\beta-1} F_{n+1} S_1 >$. It is easy to see that a monomial of the form $f = x_0^{\beta_0} \cdots x_n^{\beta_n}$ with $\sum_{i=0}^n \beta_i = d$ and $0 \leq \beta_i \leq \beta - 2$ for all $i \in \{0, \dots, n\}$ is a form of degree d which does not belong to $W_1 + \cdots + W_{n+1}$. In fact f can be written as $x_0^{d-(\gamma_0+k+2)} \cdots x_n^{d-(\gamma_n+k+2)}$ with $\sum_{i=0}^n \gamma_i = nd - (n+1)(k+2)$ and $\gamma_i \geq 0$ for all $i = 0, \dots, n$ and these forms are exactly $\binom{n+(n+1)(d-k-2)-d}{n} = \binom{(n+1)(d-k-1)-(d+1)}{n}$. In order for these forms to exist, one needs that $(n+1)(d-k-2) - d \geq 0$, i.e. that $n \geq \frac{k+2}{d-k-2}$. This is sufficient to show that if we expect that $\mathrm{Sec}_n(O_{k,n,d}) = \mathbb{P}^N$, and if $n \geq \frac{k+2}{d-k-2}$ and $k+2 < d \leq 2k$, then $\mathrm{Sec}_n(O_{k,n,d})$ is defective.

Let us notice that what we just saw is not sufficient to say that the defect δ is exactly equal to $\binom{(n+1)(d-k-1)-(d+1)}{n}$, because in $S_d \setminus < W_1 + \cdots W_{n+1} >$ we can find also monomials like $x_0^{\beta_0} \cdots x_n^{\beta_n}$ with $\sum_{i=0}^n \beta_i = d$, at least one $\beta_i = \beta - 1$ and each of the others $\beta_j \leq \beta - 2$. Hence $\delta \geq \binom{(n+1)(d-k-1)-(d+1)}{n}$. □

All the results on defectivity lead us to formulate the following:

Conjecture 3.3.18. *The variety* $\mathrm{Sec}_{s-1}(O_{k,n,d})$ *is defective only if* Y *is as in case 2. (a) or 2. (b) of Lemma 3.3.6.*

The conjecture amounts to say that the defectivity of Y can only occur if defectivity of the fat points schemes X or T imposes it.

Remark: In many examples the defectivity of Y is exactly the one imposed by X or by T (i.e. the inequalities on δ in Lemma 3.3.6 are equalities), but this is not always the case: for example if we consider the variety $\mathrm{Sec}_1(O_{4,5,6})$ (see the example after Proposition 3.3.9), here we get that the corresponding scheme Y has defectivity 86 in degree 5. Here we have that X is given by two 5-fat points in $\mathbb{P}^6$, and it is easy to check that $h^0(\mathcal{I}_X(5)) = 126$ (all 5-tics through X can be viewed as cones over a 5-tic of a $\mathbb{P}^4$), so that its defectivity is 84. Hence, even if Y is "forced" to be defective by X, its defectivity is bigger, i.e. Y should impose to 5-tics 12 conditions more than X, but it imposes only ten conditions more.

It is easy to find similar behavior if $d = k + 1$, for instance for $n = 8$, $s = 3$, $d = k + 1 = 2$ or $n = 10$, $s = 3$, $d = k + 1 = 2$.

3.4 The secant varieties to the osculating varieties to the Veronese surface

In this section we want to study the particular case of $\mathrm{Sec}_{s-1}(O_{k,2,d})$. Since for all this section we will work with $n = 2$, we write $O_{k,n}$ instead of $O_{k,2,n}$.

What we are going to present here is in part contained in the joint work [**BC**]: in that note we proved the Conjecture 3.3.18 for cases $n = 2$ and $s = 3, 4, 5, 6, 9$ (with some omitted details); here we want to give all the detailed proofs and to show that Conjecture 3.3.18 holds also for $n = 2$ and $s = 7, 8$.

In [**Ba**] and [**BF1**] the authors study the $(s-1)$-secant varieties of $O_{k,n}$ for $k = 1, 2$ and they prove the following results:

Proposition 3.4.1. *For* $k = 1$*, the* $(s-1)$*-secant variety of the tangential variety to* $\nu_d(\mathbb{P}^2)$ *has the expected dimension, unless* $s = 2$ *and* $d = 3$*.*

Proposition 3.4.2. *For* $k = 2$*, the* $(s-1)$*-secant variety of* 2*-osculating variety to* $\nu_d(\mathbb{P}^2)$ *has the expected dimension, unless* $s = 2$ *and* $d = 4$*.*

Remark: In general it is a hard problem to determine the postulation for a union of m-fat points. There is a conjecture for the postulation of a generic union $X \subset \mathbb{P}^2$ of s m-fat points (e.g. see

[**Harb**]): for $s \geq 10$, the conjecture says that X is regular in any degree d. This has been proved for $m \leq 20$ in [**CCMO**], and, when s is a square, by L.Evain in [**Ev**]. For $s \leq 9$ all the defective cases are known (e.g., see [**CCMO**] or [**Harb**]), more precisely, for any $m \in \mathbb{N}$ and $s \leq 9$ the cases in which $X \subset \mathbb{P}^2$ is not regular are:

1. $s = 2$, and $m \leq d \leq 2m - 2$;
2. $s = 3$, and $\frac{3m}{2} \leq d \leq 2m - 2$;
3. $s = 5$, and $2m \leq d \leq \frac{5m-2}{2}$;
4. $s = 6$, and $\frac{12m}{5} \leq d \leq \frac{5m-2}{2}$;
5. $s = 7$, and $\frac{21m}{8} \leq d \leq \frac{8m-2}{3}$;
6. $s = 8$, and $\frac{48m}{17} \leq d \leq \frac{17m-2}{6}$.

Notation: Only for this section, since we are studying the case $n = 2$, we indicate with S the coordinate ring $K[x, y, z]$.

Proposition 3.4.3. *For $d = k + 1$ and $s \geq 2$, we have $\mathrm{Sec}_{s-1}(O_{k,d}) = \mathbb{P}^N$.*

Proof. It is an easy consequence of Proposition 3.3.9, point 3. Since for $d = k + 1$ we have that $\mathrm{Sec}_1(O_{k,d}) = \mathbb{P}^N$ then the statement holds for $s \geq 2$. □

Notation: Let $P = [L^{d-k}F]$ be a generic point of $O_{k,d}$ with $L \in S_1$ and $F \in S_k$, and let $T_P(O_{k,d})$ be the tangent space of $O_{k,d}$ at P. The affine cone over $T_P(O_{k,d})$ is

$$W =< L^{d-k}S_k, L^{d-k-1}FS_1 > .$$

Terracini's Lemma says that the tangent space of $\mathrm{Sec}_{s-1}(O_{k,d})$ at a generic point of $< P_1, \ldots, P_s >$ for $P_1, \ldots, P_s \in O_{k,d}$, is the span of the tangent spaces to $O_{k,d}$ at $P_i = [L_i^{d-k}F_i]$ with $L_i \in S_1$ and $F_i \in S_k$ for $1 \leq i \leq s$. If $T_{O_{k,d},P_i} = \mathbb{P}(W_i) = \mathbb{P}(< L_i^{d-k}S_k, L_i^{d-k-1}F_iS_1 >)$, then

$$\dim(\mathrm{Sec}_{s-1}(O_{k,d})) = \dim(< T_{P_1}(O_{k,d}), \ldots, T_{P_s}(O_{k,d}) >) = \dim(< W_1, \ldots, W_s >) - 1. \tag{3.16}$$

With an abuse of notation we consider $W_i^{\perp} \subset S_d$, for all $1 \leq i \leq s$. It generates an ideal in S defining a scheme $Z_i(k, d) \subset \mathbb{P}^2$. Let Y be a generic union of s schemes

$$Z_i(k, d) \subset \mathbb{P}^2, \tag{3.17}$$

for $1 \leq i \leq s$. Since $\dim(< W_1, \ldots, W_s >) - 1 = N - \dim[< W_1, \ldots, W_s >]^{\perp} = N - \dim(W_1^{\perp} \cap \cdots \cap W_s^{\perp}) = N - h^0(\mathbb{P}^2, \mathcal{I}_Y(d))$, we have:

$$\dim(\mathrm{Sec}_{s-1}(O_{k,d})) = N - h^0(\mathbb{P}^2, \mathcal{I}_Y(d)) = H(Y,d) - 1 \tag{3.18}$$

where $H(Y,d)$ is the Hilbert function of Y in degree d.

For $d \geq k+2$, the schemes $Z_i(k,d)$ are zero-dimensional, and do not depend on d, in fact we can rephrase the Lemmata 3.3.1, 3.3.2, 3.3.3 as follows:

Lemma 3.4.4. *Let $Z(k,d) = Z_i(k,d)$ be one such scheme with support at P. For $d \geq k+2$, we have:*

1. $(k+1)P \subset Z(k,d) \subset (k+2)P$;
2. *the length of $Z(k,d)$ is $l(Z) = \binom{k+2}{2} + 2$;*
3. $Z(k,d) = Z(k,k+2)$.

Henceforth for $d \geq k+2$ we will denote $Z(k,d)$ by $Z(k)$, or Z, if k is obvious by the context.

From (3.18) and the lemma above it follows that for $d \geq k+2$ in order to study the dimension of $\mathrm{Sec}_{s-1}(O_{k,d})$, we only need to study the postulation of unions of generic schemes $Z(k)$.

Remark: Let $d \geq k+2$. Recall that $Z(k)$ is defined by the ideal generated by $W^\perp \subset S_d$, where $W =< L^{d-k}S_k, L^{d-k-1}FS_1 >$, with $L \in S_1$ and $F \in S_k$. Now we choose the scheme $Z(k)$: set $L = x$ and $F = y^k$; we get

$$W =< x^{d-k}S_k, x^{d-k-1}y^kS_1 >$$

hence

$$\begin{aligned} W^\perp = \; & < x^{d-k-1}y^{k-1}z^2, \ldots, x^{d-k-1}yz^k, x^{d-k-1}z^{k+1}, x^{d-k-2}y^{k+2}, x^{d-k-2}y^{k+1}z, \ldots, \\ & x^{d-k-2}yz^{k+1}, x^{d-k-2}z^{k+2}, x^{d-k-3}y^{k+1}, x^{d-k-3}y^kz, \ldots, x^{d-k-3}yz^k, x^{d-k-3}z^{k+1}, \ldots, \\ & xy^{d-1}, xy^{d-2}z, \ldots, xyz^{d-2}, xz^{d-1}, y^d, y^{d-1}z, \ldots, yz^{d-1}, z^d > . \end{aligned}$$

Let I be the ideal generated by $W^\perp$. By a direct computation, it is easy to show that the saturation of I is the ideal

$$(I)^{sat} = (y,z)^{k+1} \cap \big((y,z)^{k+2} + (z^2)\big) \tag{3.19}$$

that defines a scheme supported at a point of $\mathbb{P}^2$, whose structure is given by the union of its k-th infinitesimal neighbourhood, with the intersection of its $(k+1)$-th infinitesimal neighbourhood with a double line.

Notation: We fix, as in the previous section, the following notation:

- let $P_1, \dots, P_s$ be s generic points in $\mathbb{P}^2$;
- let X be the union of s generic $(k+1)$-fat points in $\mathbb{P}^2$, with support in $P_1, \dots, P_s$;
- let T be the union of s generic $(k+2)$-fat points in $\mathbb{P}^2$, with support in $P_1, \dots, P_s$;
- let Z_i be a 0-dimensional scheme in $\mathbb{P}^2$, as defined in (3.17), with support in P_i;
- let $Y = Z_1 + \cdots + Z_s$;
- denote by "$(k+1, k+2)P$" a 0-dimensional scheme whose defining ideal is $\wp^{k+1} \cap (\wp^{k+2} + l^2)$ where $\wp$ is the homogeneous ideal in $S = K[x, y, z]$ of a point $P \in \mathbb{P}^2$, and l is the ideal of a generic line through P; we call $(k+1, k+2)P$ a "$(k+1, k+2)$-*point*";
- let $\mathcal{Z}_i$ be a $(k+1, k+2)$-point with support in P_i. By (3.19), the scheme $\mathcal{Z}_i$ is a specialization of the scheme Z_i;
- let $\mathcal{Y} = \mathcal{Z}_1 + \cdots + \mathcal{Z}_s$ (so $\mathcal{Y}$ is a specialization of the scheme Y). We have
$$\deg(\mathcal{Y}) = \deg(Y) = s\left(\binom{k+2}{2} + 2\right) = \deg(X) + 2s;$$
- if $\mathcal{C} \subset \mathbb{P}^2$ is a curve, and Z is a zero-dimensional scheme, the scheme Z' defined by the ideal $(I_Z : I_{\mathcal{C}})$ is called the residual of Z with respect to $\mathcal{C}$, and it is denoted by $Res_{\mathcal{C}} Z$.

In the following lemma we determine the subscheme of a $(k+1, k+2)$-point with support in P, residual to a curve $\mathcal{C}$.

Lemma 3.4.5. *Let $\mathcal{Z}$ be a $(k+1, k+2)$-point, with support in P with defining ideal $\wp^{k+1} \cap (\wp^{k+2} + l^2)$, where $\wp$ is the ideal of P, and $l = (L)$ is the ideal of a generic line through P. Let $\mathcal{C} \subset \mathbb{P}^2$ be a curve having at P a singularity of multiplicity m, and having L as tangent direction with multiplicity t. Then $Res_{\mathcal{C}}(\mathcal{Z})$ is defined by the ideal*
$$I_{Res_{\mathcal{C}}(\mathcal{Z})} = \wp^{\max\{k+1-m;0\}} \cap \left(\wp^{\max\{k+2-m;0\}} + l^{\max\{2-t;0\}}\right).$$
The residual $Res_{\mathcal{C}}(\mathcal{Z})$ is a fat point or a $(k+1-m, k+2-m)$-point, except for $m < k+1$ and $t = 1$, more precisely:
$$Res_{\mathcal{C}}(\mathcal{Z}) = \begin{cases} 0P & \text{for } m \geq k+2, \text{ or } m = k+1 \text{ and } t \geq 2; \\ 1P & \text{for } m = k+1 \text{ and } t \leq 1; \\ 2P & \text{for } m = k \text{ and } t = 0; \\ (k+1-m)P & \text{for } m < k+1 \text{ and } t \geq 2; \\ (k+1-m, k+2-m)P & \text{for } m < k \text{ and } t = 0. \end{cases}$$

Proof. Without loss of generality, we assume that $\wp = (x, y)$, $L = x$, and, by abuse of notation, that x, y are affine coordinates.

Let $x^t f_1 + f_2 = 0$ be an equation defining the curve $\mathcal{C}$, where f_1 is a homogeneous polynomial of degree $m - t$, $f_1 \notin (x)$, and $f_2 \in (x, y)^{m+1}$. We have to prove that

$$((x,y)^{k+1} \cap ((x,y)^{k+2} + (x^2))) : (x^t f_1 + f_2) = (x,y)^{\max\{k+1-m;0\}} \cap \left((x,y)^{\max\{k+2-m;0\}} + (x^{\max\{2-t;0\}})\right). \tag{3.20}$$

This is obvious for $m \geq k+2$, and for $m = k+1, t \geq 2$, since in these cases $Res_{\mathcal{C}}(\mathcal{Z})$ is supported on the emptyset.

Let $m = k+1$, $t \leq 1$. In this case the equality (3.20) becomes

$$((x,y)^{k+1} \cap ((x,y)^{k+2} + (x^2))) : (x^t f_1 + f_2) = (x,y).$$

"$\subseteq$" : To prove this inclusion, let $g = a + h$, $a \in K$, $h \in (x, y)$. If $g \cdot (x^t f_1 + f_2) = (a+h)(x^t f_1 + f_2) \in ((x,y)^{k+2} + (x^2))$, since $f_2 \in (x,y)^{m+1}$, $hx^t f_1 \in (x,y)^{m+1}$ and $m + 1 = k + 2$, it follows that $ax^t f_1 \in ((x,y)^{k+2} + (x^2))$. But f_1 is a homogeneous polynomial of degree $m - t$, $f_1 \notin (x)$, $t \leq 1$, so it easily follows that $a = 0$, and we get $g \in (x, y)$. The reverse inclusion is obvious.

Since $I_{Res_{\mathcal{C}}(\mathcal{Z})} = (x, y)$, we have $Res_{\mathcal{C}}(\mathcal{Z}) = 1P$

Now, let $m < k+1$, $t \geq 2$. In this case we have to prove that:

$$((x,y)^{k+1} \cap ((x,y)^{k+2} + (x^2))) : (x^t f_1 + f_2) = (x,y)^{k+1-m}.$$

If $g \cdot (x^r f_1 + f_2) \in (x,y)^{k+1}$, it immediately follows that $g \in (x,y)^{k+1-m}$, and the reverse inclusion is obvious. Moreover, since $I_{Res_{\mathcal{C}}(\mathcal{Z})} = (x,y)^{k+1-m}$, we have that $Res_{\mathcal{C}}(\mathcal{Z}) = (k+1-m)P$.

Let $m < k+1$, $t \leq 1$. Now we have to prove that

$$((x,y)^{k+1} \cap ((x,y)^{k+2} + (x^2))) : (x^t f_1 + f_2) = (x,y)^{k+1-m} \cap ((x,y)^{k+2-m} + (x^{2-t})).$$

"$\subseteq$" : As in the previous case, if $g \cdot (x^t f_1 + f_2) \in (x,y)^{k+1}$, it follows that $g \in (x,y)^{k+1-m}$, so we can write

$$g = xg_1 + ay^{k+1-m} + g_2,$$

where $g_1 \in (x,y)^{k-m}$ is homogeneous of degree $k - m$, $g_2 \in (x,y)^{k+2-m}$, $a \in K$. In order to prove that

$$g \cdot (x^t f_1 + f_2) = (xg_1 + ay^{k+1-m} + g_2)(x^t f_1 + f_2) \in ((x,y)^{k+2} + (x^2)))$$

since $g_2 x^t f_1$, and $f_2 \in (x,y)^{k+2}$, it suffices that

$$x^{t+1} g_1 f_1 + ax^t y^{k+1-m} f_1 \in ((x,y)^{k+2} + (x^2)).$$

Since $x^{t+1} g_1 f_1 + ax^t y^{k+1-m} f_1$ is homogeneous of degree $k+1$, and $f_1 \notin (x)$, we get that $x^{t+1} g_1 + ax^t y^{k+1-m} \in (x^2)$. For $t = 1$, this implies $a = 0$, so $g \in ((x,y)^{k+2-m} + (x))$. For $t = 0$ this implies $a = 0$, and $g_1 \in (x)$, so $g \in ((x,y)^{k+2-m} + (x^2))$.

"$\supseteq$" : This inclusion is obvious.
So we have proved that, for $m \leq k$ and $t \leq 1$:

$$I_{Res_{\mathcal{C}}(\mathcal{Z})} = \begin{cases} (x,y)^{k+1-m} \cap ((x,y)^{k+2-m} + (x)) = (x,y)^{k+1-m} \cap (x, y^{k+2-m}) & \text{for } m \leq k \text{ and } t = 1, \\ (x,y) \cap ((x,y)^2 + (x^2)) = (x,y)^2 & \text{for } m = k \text{ and } t = 0, \\ (x,y)^{k+1-m} \cap ((x,y)^{k+2-m} + (x^2)) & \text{for } m < k \text{ and } t, = 0 \end{cases}$$

hence for $m = k$ and $t = 0$ we have $Res_{\mathcal{C}}(\mathcal{Z}) = 2P$, for $m < k$ and $t = 0$ we have $Res_{\mathcal{C}}(\mathcal{Z}) = (k+1-m, k+2-m)P$, while for $m \leq k$ and $t = 1$, $Res_{\mathcal{C}}(\mathcal{Z})$ is the union of the fat point $(k+1-m)P$ with the intersection of the line $\{x = 0\}$ with the fat point $(k+2-m)P$. $\square$

We wish to notice that the expected dimension for $\mathrm{Sec}_{s-1}(O_{k,d})$ is

$$\mathrm{expdim}(\mathrm{Sec}_{s-1}(O_{k,d})) = \min\{sn + s - 1,\ N\},$$

where $n = \dim(O_{k,d}) = \min\left\{\binom{k+2}{2} + 1, \binom{d+2}{2} - 1\right\} = \min\left\{\binom{k+2}{2} + 1, N\right\} = \min\left\{\frac{\deg(Y)}{s} - 1,\ N\right\}$.
Hence it easily follows that

$$\mathrm{expdim}(\mathrm{Sec}_{s-1}(O_{k,d})) = \min\{\deg(Y), N+1\} - 1 = \exp(H(Y,d)) - 1$$

where $\exp(H(Y,d))$ is the expected value for the Hilbert function $H(Y,d)$ of Y in degree d.
In next Lemmata we show that the postulation of Y is strictly related with the postulation of the specialized scheme $\mathcal{Y}$, and of the scheme of fat points X.

Lemma 3.4.6. *If the Hilbert function of the specialized scheme $\mathcal{Y}$ in degree d is*

$$H(\mathcal{Y}, d) = \min\{H(X,d) + 2s, N+1\},$$

then

$$H(Y,d) = \min\{H(X,d) + 2s, N+1\}.$$

Proof. It follows from the obvious inequalities: $H(\mathcal{Y},d) \leq H(Y,d) \leq \min\ \{H(X,d)+2s, N+1\}$. $\square$

Lemma 3.4.7. *Let $s > 2$. Then:*

1. *for $k = 1$, $\mathcal{Y} = Y = (2,3)P_1 + \cdots + (2,3)P_s$, and $H(\mathcal{Y},d) = \min\{\deg(Y), N+1\}$;*
2. *for $k = 2$, $\mathcal{Y} = (3,4)P_1 + \cdots + (3,4)P_s$, and $H(\mathcal{Y},d) = \min\{\deg(Y), N+1\}$.*

Proof. 1. If $d = 2$ see [**CGG2**], Proposition 3.3; for $d = 3$ see [**CGG2**], Proposition 4.5; for $d \geq 4$ see [**Ba**], Theorem 1.

2. follows from [**BF1**] Theorems 1 and 2.

□

Lemma 3.4.8. *1. If $H(\mathcal{Y}, d_0) = H(X, d_0) + 2s$, then for every $d \geq d_0$ we have*

$$H(\mathcal{Y}, d) = H(X, d) + 2s;$$

2. if $(I_{\mathcal{Y}})_{d_0} = (0)$, then for every $d \leq d_0$ we have $(I_{\mathcal{Y}})_d = (0)$.

Proof. 1. Since $X \subset \mathcal{Y}$ and $H(\mathcal{Y}, d_0) = H(X, d_0) + 2s$, then it easily follows that $\dim (I_X/I_{\mathcal{Y}})_{d_0} = 2s$. Therefore there are $2s$ forms $f_1, \ldots, f_{2s} \in (I_X)_{d_0}$ linearly independent module $(I_{\mathcal{Y}})_{d_0}$. Let $\{l = 0\}$ be a line not through any of the points $P_1, \ldots, P_s$. The forms $f_1 l^{d-d_0}, \ldots, f_{2s} l^{d-d_0} \in (I_X)_d$ are linearly independent module $(I_{\mathcal{Y}})_d$, hence $\dim (I_X/I_{\mathcal{Y}})_d \geq 2s$, so we have $H(\mathcal{Y}, d) \geq H(X, d) + 2s$. Since obviously $H(\mathcal{Y}, d) \leq H(X, d) + 2s$, then the conclusion follows.

2. Obvious.

□

Now we will study the postulation of $\mathcal{Y}$ for each $s = 3, \ldots, 9$ separately, but first we wish to mention the case $s = 2$.

Proposition 3.4.9. *For $s = 2$ we have:*

$$H(\mathcal{Y}, d) = \begin{cases} \text{for } k = 1: & \begin{cases} N+1 & \text{if } d \leq 2; \\ H(T, d) = 9 < \exp(H(\mathcal{Y}, d)) & \text{if } d = 3; \\ H(X, d) + 4 = \deg(Y) & \text{if } d \geq 4; \end{cases} \\ \text{for } k = 2: & \begin{cases} N+1 & \text{if } d \leq 3; \\ H(T, d) = 14 < \exp(H(\mathcal{Y}, d)) & \text{if } d = 4; \\ H(X, d) + 4 = \deg(Y) & \text{if } d \geq 5; \end{cases} \\ \text{for } k \geq 3: & \begin{cases} N+1 & \text{if } d \leq k+1; \\ H(T, d) < \exp(H(\mathcal{Y}, d)) & \text{if } d = k+2; \\ H(X, d) + 4 < \exp(H(\mathcal{Y}, d)) & \text{if } k+3 \leq d \leq 2k; \\ H(X, d) + 4 = \deg(Y) & \text{if } d \geq 2k+1. \end{cases} \end{cases}$$

Proof. The case $d \leq k+1$ follows from Lemma 3.4.8, 2., and Proposition 3.3.9, 3.

For $d = k+2$ observe that the line L through P_1 and P_2 is a component of multiplicity at least $2(k+1) - d = k$ for the curves defined by the forms both of $(I_{\mathcal{Y}})_d$ and of $(I_T)_d$. Since $Res_{kL}\mathcal{Y} = Res_{kL}T = 2P_1 + 2P_2$ (see Lemma 3.4.5), we get

$$\dim(I_{\mathcal{Y}})_{k+2} = \dim(I_T)_{k+2} = \dim(I_{2P_1+2P_2})_2 = 1$$

(the only curve is the $(k+2)$-uple line through the two points)). Thus $H(\mathcal{Y}, d) = H(T, d)$. Moreover, since T is not regular in degree $k+2$, we get $H(\mathcal{Y}, d) < \exp(H(\mathcal{Y}, d))$ (see Corollary 3.3.10).

For $k = 1, 2$ and $d \geq k+3$, see Corollary 3.3.13. For $k \geq 3$, and $d \geq 2k+1$ see Proposition 3.3.14.

Now let $k \geq 3$, and $k+3 \leq d \leq 2k$. For $d = k+3$ the line L through P_1 and P_2 is a component of multiplicity at least $\nu = 2(k+1) - d = k-1$ for the curves defined by the forms of both $(I_{\mathcal{Y}})_d$, and $(I_X)_d$, hence from the case $k = 1$, $d = 4$, we get

$$\dim(I_{\mathcal{Y}})_{k+3} = \dim(I_{\mathcal{Y}'})_{k+3-(k-1)} = \dim(I_{\mathcal{Y}'})_4 = 15 - 10 = 5,$$

$$\dim(I_X)_{k+3} = \dim(I_{X'})_4 = 9,$$

where $\mathcal{Y}' = Res_{\nu L}\mathcal{Y} = (2,3)P_1 + (2,3)P_2$ (see Lemma 3.4.5), and $X' = Res_{\nu L}X = 2P_1 + 2P_2$.

It follows that $H(\mathcal{Y}, k+3) = H(X, k+3) + 4$. Hence by Lemma 3.4.8 1., for every $d \geq k+3$ we have

$$H(\mathcal{Y}, d) = H(X, d) + 4.$$

Since two $(k+1)$-fat points impose independent conditions to curves of degree d if and only if $d \geq 2k+1$ (see the first Remark in the present section), then, for $k+3 \leq d \leq 2k$, we have $H(X, d) < \deg(X)$, thus

$$H(\mathcal{Y}, d) = H(X, d) + 4 < \deg(X) + 4 = \deg(Y).$$

Moreover, since for $d = k+3$, $\dim(I_{\mathcal{Y}})_{k+3} = 5$, then for $d \geq k+3$, $\dim(I_{\mathcal{Y}})_d$ is positive, that is $H(\mathcal{Y}, d) < \binom{d+2}{2}$. It follows that $k+3 \leq d \leq 2k$, then $H(\mathcal{Y}, d) < \min\left\{\deg(Y), \binom{d+2}{2}\right\} \exp(H(\mathcal{Y}, d))$.

(For $k \geq 3$, and $k+3 \leq d \leq 2k$, see also Proposition 3.3.15). □

Proposition 3.4.10. *For $s = 3$ we have:*

1.

$$H(\mathcal{Y}, d) = \begin{cases} N+1 & \text{if } d \leq \left\lceil \frac{3(k+1)}{2} \right\rceil; \\ H(X, d) + 6 < \deg(Y) & \text{if } \left\lceil \frac{3(k+1)}{2} \right\rceil + 1 \leq d \leq 2k; \\ H(X, d) + 6 = \deg(Y) & \text{if } d \geq \max\left\{ \left\lceil \frac{3(k+1)}{2} \right\rceil + 1;\ 2k+1 \right\}. \end{cases}$$

2.

$$H(\mathcal{Y},d) < \exp(H(\mathcal{Y},d)) \quad \textit{iff} \quad \begin{cases} \left\lceil \frac{3(k+1)}{2} \right\rceil + 2 \leq d \leq 2k & \textit{if } k+1 \textit{ is even;} \\ \\ \left\lceil \frac{3(k+1)}{2} \right\rceil + 1 \leq d \leq 2k & \textit{if } k+1 \textit{ is odd.} \end{cases}$$

Proof. 1. In case $d \leq \left\lceil \frac{3(k+1)}{2} \right\rceil$, it suffices to prove that $(I_{\mathcal{Y}})_d = (0)$ for $d = \left\lceil \frac{3(k+1)}{2} \right\rceil$.

Let $\mathcal{C}$ be the curve formed by the three lines P_1P_2, P_1P_3, P_2P_3. For $d = \left\lceil \frac{3(k+1)}{2} \right\rceil$, the curve $\mathcal{C}$ is a fixed component, of multiplicity at least

$$\nu = 2(k+1) - d = \begin{cases} \frac{k+1}{2} & \text{if } k+1 \text{ is even,} \\ \frac{k}{2} & \text{if } k+1 \text{ is odd,} \end{cases}$$

for the curves defined by the forms of $(I_{\mathcal{Y}})_d$, so we have (see Lemma 3.4.5)

$$\dim(I_{\mathcal{Y}})_d = \dim(I_{\mathcal{Y}'})_{d-3\nu}$$

where

$$\mathcal{Y}' = Res_{\nu\mathcal{C}}\mathcal{Y} = \begin{cases} P_1 + P_2 + P_3 & \text{if } k+1 \text{ is even,} \\ 2P_1 + 2P_2 + 2P_3 & \text{if } k+1 \text{ is odd;} \end{cases}$$

$$d - 3\nu = \begin{cases} 0 & \text{if } k+1 \text{ is even,} \\ 2 & \text{if } k+1 \text{ is odd.} \end{cases}$$

It immediately follows that $(I_{\mathcal{Y}})_d = (0)$.

Now let $d \geq \left\lceil \frac{3(k+1)}{2} \right\rceil + 1$. In order to prove that $H(\mathcal{Y},d) = H(X,d) + 6$, by Lemma 3.4.8 it suffices to prove that $H(\mathcal{Y},d) = H(X,d) + 6$ for $d = \left\lceil \frac{3(k+1)}{2} \right\rceil + 1$.

Let $d = \left\lceil \frac{3(k+1)}{2} \right\rceil + 1$. The curve $\mathcal{C}$ is a fixed component, with multiplicity at least

$$\nu = 2(k+1) - d = \begin{cases} \frac{k-1}{2} & \text{if } k+1 \text{ is even,} \\ \frac{k-2}{2} & \text{if } k+1 \text{ is odd,} \end{cases}$$

for the curves defined by the forms of both $(I_{\mathcal{Y}})_d$ and $(I_X)_d$, then we have

$$\dim(I_{\mathcal{Y}})_d = \dim(I_{\mathcal{Y}'})_{d-3\nu},$$

$$\dim(I_X)_d = \dim(I_{X'})_{d-3\nu}$$

where (see Lemma 3.4.5)

$$d-3\nu=\begin{cases}4 & \text{if } k+1 \text{ is even}\\ 6 & \text{if } k+1 \text{ is odd,}\end{cases}$$

$$\mathcal{Y}'=Res_{\nu\mathcal{C}}\mathcal{Y}=\begin{cases}(2,3)P_1+(2,3)P_2+(2,3)P_3 & \text{if } k+1 \text{ is even,}\\ (3,4)P_1+(3,4)P_2+(3,4)P_3 & \text{if } k+1 \text{ is odd,}\end{cases}$$

$$X'=\begin{cases}2P_1+2P_2+2P_3 & \text{if } k+1 \text{ is even,}\\ 3P_1+3P_2+3P_3 & \text{if } k+1 \text{ is odd.}\end{cases}$$

Since it is well known that $\dim(I_{2P_1+2P_2+2P_3})_4=6$ and $\dim(I_{3P_1+3P_2+3P_3})_6=10$, we have

$$\dim(I_{X'})_{d-3\nu}=\begin{cases}6 & \text{if } k+1 \text{ is even,}\\ 10 & \text{if } k+1 \text{ is odd,}\end{cases}$$

moreover, by Lemma 3.4.7 we get that

$$\dim(I_{\mathcal{Y}'})_{d-3\nu}=\begin{cases}0 & \text{if } k+1 \text{ is even,}\\ 4 & \text{if } k+1 \text{ is odd.}\end{cases}$$

It follows that $\dim(I_X)_d-\dim(I_{\mathcal{Y}})_d=6$, hence $H(\mathcal{Y},d)-H(X,d)=6$.

Since three $(k+1)$-fat points impose independent conditions to curves of degree d if and only if $d\geq 2k+1$ (see the first Remark of this section), then for $\left\lceil\frac{3(k+1)}{2}\right\rceil+1\leq d\leq 2k$ we have $H(X,d)<\deg(X)$, while if $d\geq\max\left\{\left\lceil\frac{3(k+1)}{2}\right\rceil+1;\ 2k+1\right\}$, then $H(X,d)=\deg(X)$. Since $\deg(Y)=\deg(X)+6$ we get:

$$H(\mathcal{Y},d)=\begin{cases}H(X,d)+6<\deg(Y) & \text{if } \left\lceil\frac{3(k+1)}{2}\right\rceil+1\leq d\leq 2k,\\ H(X,d)+6=\deg(Y) & \text{if } d\geq\max\left\{\left\lceil\frac{3(k+1)}{2}\right\rceil+1;2k+1\right\}.\end{cases}$$

2. For $d\leq\left\lceil\frac{3(k+1)}{2}\right\rceil$, or $d\geq\max\left\{\left\lceil\frac{3(k+1)}{2}\right\rceil+1;\ 2k+1\right\}$, from 1. we have $H(\mathcal{Y},d)=\exp(H(\mathcal{Y},d))$.

   If $k+1$ is even and $d=\left\lceil\frac{3(k+1)}{2}\right\rceil+1$, then $\dim(I_{\mathcal{Y}})_d=0$, hence $H(\mathcal{Y},d)=\binom{d+2}{2}$, the expected one.

   If $k+1$ is even and $d=\left\lceil\frac{3(k+1)}{2}\right\rceil+2$, from 1., since $\dim(I_X)_{d-1}=6$ implies $\dim(I_X)_d>6$, we have:

$$\dim(I_{\mathcal{Y}})_d=\binom{d+2}{2}-H(\mathcal{Y},d)=\binom{d+2}{2}-H(X,d)-6=\dim(I_X)_d-6>0.$$

Hence if $k+1$ is even, $d \geq \left\lceil \frac{3(k+1)}{2} \right\rceil + 2$, and so also for $d \geq \left\lceil \frac{3(k+1)}{2} \right\rceil + 2$ we have $\dim(I_{\mathcal{Y}})_d > 0$, that is $H(\mathcal{Y}, d) < \binom{d+2}{2}$. Since, by 1., if $\left\lceil \frac{3(k+1)}{2} \right\rceil + 1 \leq d \leq 2k$, then $H(\mathcal{Y}, d) < \deg(Y)$, it follows that $\left\lceil \frac{3(k+1)}{2} \right\rceil + 2 \leq d \leq 2k$ we have $H(\mathcal{Y}, d) < \min\left\{\deg(Y), \binom{d+2}{2}\right\} = \exp(H(\mathcal{Y}, d))$

If $k+1$ is odd and $d \geq \left\lceil \frac{3(k+1)}{2} \right\rceil + 1$, from the proof of $i)$ we get $\dim(I_{\mathcal{Y}})_d > 0$, hence $H(\mathcal{Y}, d) < \binom{d+2}{2}$.

Moreover, by 1., if $\left\lceil \frac{3(k+1)}{2} \right\rceil + 1 \leq d \leq 2k$, then $H(\mathcal{Y}, d) < \deg(Y)$, and the conclusion immediately follows.

□

Proposition 3.4.11. *For $s = 4$ we have:*

$$H(\mathcal{Y}, d) = \begin{cases} \textit{for } k \leq 6: & \begin{cases} N+1 & \textit{if } d \leq 2k+2, \\ H(X,d) + 8 = \deg(Y) & \textit{if } d \geq 2k+3, \end{cases} \\ \textit{for } \ k \geq 6: & \begin{cases} N+1 & \textit{if } d \leq 2k+1, \\ H(X,d) + 8 = \deg(Y) & \textit{if } d \geq 2k+2. \end{cases} \end{cases}$$

Proof. If $d \leq 2k+1$, by Bezout Theorem, each element of $(I_{\mathcal{Y}})_d$ is divisible by every form defining an irreducible conic through $P_1, \ldots, P_4$, hence $(I_{\mathcal{Y}})_d = (0)$.

Let $d = 2k+2$. Recall that the ideal of the scheme $\mathcal{Z}_i$ is $\wp_i^{k+1} \cap (\wp_i^{k+2} + l_i^2)$, where l_i defines a generic line L_i through P_i $(1 \leq i \leq 4)$ such that $\deg(\mathcal{Y} \cap L_i) = k+2$. Let $\mathcal{C}_i$ be the conic through $P_1, \ldots, P_4$, tangent in P_i to L_i. For the genericity of the L_i's, the conics $\mathcal{C}_1, \ldots, \mathcal{C}_4$ are distinct. Bezout's Theorem implies that each conic $\mathcal{C}_i$ is a component of each curve defined by the forms of $(I_{\mathcal{Y}})_d$. By Lemma 3.4.5 we can determine $I_{Res_{\mathcal{C}_1+\cdots+\mathcal{C}_4}\mathcal{Y}}$, and it is an easy computation that the intersection multiplicities of the curves defined by the forms of $(I_{Res_{\mathcal{C}_1+\cdots+\mathcal{C}_4}\mathcal{Y}})_{d-8}$ with a conic $\mathcal{C}_i$, is bigger than $2(d-8)$. Hence by Bezout's Theorem we get that each conic $\mathcal{C}_i$ is a component with multiplicity at least 2 of each curve defined by the forms of $(I_{\mathcal{Y}})_d$. So these curves have a component of degree 16. It follows that, if $(I_{\mathcal{Y}})_d \neq (0)$, then $d \geq 16$, that is $k \geq 7$. Thus, for $k \leq 6$, we have $(I_{\mathcal{Y}})_d = (0)$, hence $H(\mathcal{Y}, d) = N+1$. Observe that for $k = 6$, we have $N+1 = H(X,d) + 8 = \deg(Y)$, in fact in this case $d = 2k+2 = 14$, $N+1 = \binom{16}{2} = 120$, and, since four 7-fat points impose independent conditions to curves of degree 14 (see the first Remark of this section), then $H(X,d) = 112$. If $k \geq 7$ we have

$$\dim(I_{\mathcal{Y}})_{2k+2} = \dim(I_{\mathcal{Y}'})_{2k+2-16},$$

where $\mathcal{Y}' = Res_{2\mathcal{C}_1+\cdots+2\mathcal{C}_4}\mathcal{Y} = (k-7)P_1 + \cdots + (k-7)P_4$ is a scheme of four $(k-7)$-fat points (see Lemma 3.4.5). Since $\mathcal{Y}'$ imposes independent conditions to curves of degree $2k-14$ (see the first Remark of this section), then $H(\mathcal{Y}, 2k+2) = \binom{2k+4}{2} - \dim(I_{\mathcal{Y}})_{2k+2} = \binom{2k+4}{2} - \dim(I_{\mathcal{Y}'})_{2k-14} = \binom{2k+4}{2} - \binom{2k-12}{2} + 4\binom{k-6}{2} = 4\binom{k+2}{2} + 8 = H(X, 2k+2) + 8 = \deg(Y)$.

Now let $d \geq 2k+3$. It suffices to prove that $H(\mathcal{Y}, 2k+3) = H(X, 2k+3) + 8 = \deg(Y)$ (see Lemma 3.4.8 point 1.), hence let $d = 2k+3$. By induction on k. For $k = 1$ see Lemma 3.4.7. Let $k \geq 2$. Let $\mathcal{C}$ be an irreducible conic through $P_1, \ldots, P_4$, and let Q_1, Q_2, Q_3 be three points on $\mathcal{C}$. Let $\widetilde{\mathcal{Y}} = \mathcal{Y} + Q_1 + Q_2 + Q_3$. By Bezout's Theorem, the conic $\mathcal{C}$ is a fixed component for the curves of degree $2k+3$ through $\widetilde{\mathcal{Y}}$, then

$$\dim(I_{\widetilde{\mathcal{Y}}})_{2k+3} = \dim(I_{\widetilde{\mathcal{Y}}'})_{2k+1} = \binom{2k+3}{2} - H(\widetilde{\mathcal{Y}}', 2k+1),$$

where $\widetilde{\mathcal{Y}}' = Res_{\mathcal{C}}\widetilde{\mathcal{Y}} = Res_{\mathcal{C}}\mathcal{Y} = \sum_{i=1}^{4}(k, k+1)P_i$ (see Lemma 3.4.5). By the inductive hypothesis we have that $H(\widetilde{\mathcal{Y}}', 2k+1) = \deg(\widetilde{\mathcal{Y}}') = 4\binom{k+1}{2} + 8$, hence

$$H(\widetilde{\mathcal{Y}}, 2k+3) = \binom{2k+5}{2} - \binom{2k+3}{2} + 4\binom{k+1}{2} + 8 = \deg(\mathcal{Y}) + 3 = \deg(\widetilde{\mathcal{Y}}).$$

Hence $\widetilde{\mathcal{Y}}$ imposes independent conditions to curves of degree $2k+3$. Since $\mathcal{Y} \subset \widetilde{\mathcal{Y}}$, also $\mathcal{Y}$ imposes independent conditions to curves of degree $2k+3$, that is $H(\mathcal{Y}, 2k+3) = \deg(\mathcal{Y})$. □

Proposition 3.4.12. *For $s = 5$ we have:*

$$H(\mathcal{Y}, d) = \begin{cases} N+1 & \text{if } d \leq 2k+3, \\ H(X,d) + 10 < \exp(H(\mathcal{Y}, d)) & \text{if } 2k+4 \leq d \leq \left\lfloor \frac{5(k+1)}{2} \right\rfloor - 1, \\ H(X,d) + 10 = \deg(Y) & \text{if } d \geq \max\left\{2k+4; \left\lfloor \frac{5(k+1)}{2} \right\rfloor\right\}. \end{cases}$$

Proof. Let $d \leq 2k+3$. If we prove that $(I_{\mathcal{Y}})_d = (0)$ for $d = 2k+3$ we are done. So let $d = 2k+3$. For $k = 1$ see Lemma 3.4.7.

Let $k \geq 2$. Any curve defined by a nonzero element of $(I_X)_d$ has the conic $\mathcal{C}$ through $P_1, \ldots, P_5$ as a component of multiplicity at least $5(k+1) - 2d = k-1$, where X is the fat point subscheme of 5 points of multiplicity $k+1$, hence the same is true for $\mathcal{Y}$ in place of X, since $X \subset \mathcal{Y}$, so we have:

$$\dim(I_{\mathcal{Y}})_{2k+3} = \dim(I_{\mathcal{Y}'})_{2k+3-2(k-1)} = \dim(I_{\mathcal{Y}'})_5,$$

where, by Lemma 3.4.5, $\mathcal{Y}' = Res_{(k-1)\mathcal{C}}\mathcal{Y} = (2,3)P_1 + \cdots + (2,3)P_5$. Since, by Lemma 3.4.7 point 1., $\dim(I_{\mathcal{Y}'})_5 = 0$, then the conclusion follows.

Now let $d \geq 2k+4$. We have to prove that

$$H(\mathcal{Y}, d) = H(X, d) + 10.$$

By Lemma 3.4.8, it is sufficient to prove that $H(\mathcal{Y}, d) = H(X, d) + 10$ for $d = 2k+4$, so let $d = 2k+4$. For $k = 1, 2$ see Lemma 3.4.7. If $k = 3$ (hence $d = 10$), let Q be a point on the conic $\mathcal{C}$ through $P_1, \ldots, P_5$. The scheme $\mathcal{Y} + Q$ imposes independent conditions to the curves of degree 10. In fact, since the conic $\mathcal{C}$ is a fixed locus for $(I_{\mathcal{Y}+Q})_{10}$, from the case $k = 2$ we get:

$$\dim(I_{\mathcal{Y}+Q})_{10} = \dim(I_{\mathcal{Y}'})_8 = \binom{8+2}{2} - 5(8) = 5 = \binom{10+2}{2} - 5(12) - 1 = \binom{10+2}{2} - \deg(\mathcal{Y}+Q),$$

where $\mathcal{Y}' = Res_{\mathcal{C}}(\mathcal{Y} + Q) = (3,4)P_1 + \cdots + (3,4)P_5$ (see Lemma 3.4.5). Since $\mathcal{Y} + Q$ imposes independent conditions to curves of degree 10, then also $\mathcal{Y}$ and X do the same. It follows that

$$H(\mathcal{Y}, 10) = \deg(\mathcal{Y}) = \deg(X) + 10 = H(X, 10) + 10.$$

For $k \geq 4$, since $\mathcal{C}$ is a fixed component with multiplicity at least $(k-3)$ for curves defined both by $(I_{\mathcal{Y}})_{2k+4}$ and by $(I_X)_{2k+4}$, it follows that

$$\dim(I_{\mathcal{Y}})_{2k+4} = \dim(I_{\mathcal{Y}'})_{2k+4-2(k-3)} = \dim(I_{\mathcal{Y}'})_{10},$$

$$\dim(I_X)_{2k+4} = \dim(I_{X'})_{10},$$

where (see Lemma 3.4.5)

$$\mathcal{Y}' = Res_{(k-3)\mathcal{C}}\mathcal{Y} = (4,5)P_1 + \cdots + (4,5)P_5,$$

$$X' = Res_{(k-3)\mathcal{C}}4P_1 + \cdots + 4P_5.$$

From the case $k = 3$ it follows that

$$\dim(I_{\mathcal{Y}})_{2k+4} = 6,$$

$$\dim(I_X)_{2k+4} = 16,$$

hence $H(\mathcal{Y}, d) = H(X, d) + 10$.

So we have proved that for $d \geq 2k+4$

$$H(\mathcal{Y}, d) = H(X, d) + 10$$

Now, since $\dim(I_{\mathcal{Y}})_{2k+4}$ is positive, then $H(\mathcal{Y},d) < \binom{d+2}{2}$ for any $d \geq 2k+4$. Moreover, since five generic $(k+1)$-fat points impose independent conditions to curves of degree d if and only if $d \geq \left\lfloor \frac{5(k+1)}{2} \right\rfloor$ (see the first Remark of this section), then for $2k+4 \leq d \leq \left\lfloor \frac{5(k+1)}{2} \right\rfloor - 1$, we have $H(X,d) < \deg(X)$, hence

$$H(\mathcal{Y},d) = H(X,d)+10 < \min\left\{\deg(X)+10, \binom{d+2}{2}\right\} = \min\left\{\deg(Y), \binom{d+2}{2}\right\} = \exp(H(\mathcal{Y},d)).$$

If $d \geq \max\left\{2k+4; \left\lfloor \frac{5(k+1)}{2} \right\rfloor\right\}$, then $H(X,d) = \deg(X)$, so $H(\mathcal{Y},d) = \deg(Y)$. □

Proposition 3.4.13. *For $s = 6$ we have:*

$$H(\mathcal{Y},d) = \begin{cases} \textit{for } k=1: & \begin{cases} N+1 & \textit{if } d \leq 6, \\ H(X,d)+12 = \deg(Y) & \textit{if } d \geq 7, \end{cases} \\ \textit{for } k=2: & \begin{cases} N+1 & \textit{if } d \leq 8, \\ H(X,d)+12 = \deg(Y) & \textit{if } d \geq 9, \end{cases} \\ \begin{array}{l}\textit{for } k \geq 3 \\ k \equiv 2 \ (mod\, 5):\end{array} & \begin{cases} N+1 & \textit{if } d \leq \left\lceil \frac{12(k+1)}{5} \right\rceil - 1, \\ H(X,d)+12 < \exp(H(\mathcal{Y},d)) & \textit{if } \left\lceil \frac{12(k+1)}{5} \right\rceil \leq d \leq \left\lfloor \frac{5(k+1)}{2} \right\rfloor - 1, \\ H(X,d)+12 = \deg(Y) & \textit{if } d \geq \max\left\{\left\lceil \frac{12(k+1)}{5} \right\rceil; \left\lfloor \frac{5(k+1)}{2} \right\rfloor\right\}, \end{cases} \\ \begin{array}{l}\textit{for } k \geq 3 \\ k \not\equiv 2 (mod\, 5):\end{array} & \begin{cases} N+1 & \textit{if } d \leq \left\lceil \frac{12(k+1)}{5} \right\rceil, \\ H(X,d)+12 < \exp(H(\mathcal{Y},d)) & \textit{if } \left\lceil \frac{12(k+1)}{5} \right\rceil + 1 \leq d \leq \left\lfloor \frac{5(k+1)}{2} \right\rfloor - 1, \\ H(X,d)+12 = \deg(Y) & \textit{if } d \geq \max\left\{\left\lceil \frac{12(k+1)}{5} \right\rceil + 1; \left\lfloor \frac{5(k+1)}{2} \right\rfloor\right\}. \end{cases} \end{cases}$$

Proof. We start by proving four particular cases, that we need later in the proof.

Lemma 3.4.14. *We have:*

1. $\dim(I_{(8,9)P_1+\cdots+(8,9)P_6})_{20} = 3$;
2. $\dim(I_{(6,7)P_1+\cdots+(6,7)P_6})_{15} = 0$;
3. $\dim(I_{(5,6)P_1+\cdots+(5,6)P_6})_{13} = 3$;
4. $\dim(I_{(4,5)P_1+\cdots+(4,5)P_6})_{11} = 6$.

Proof. We prove 1. by specializing the scheme $\mathcal{Y}$. The proofs of 2., 3. and 4. are done by using [**CoCoA**].

1. Let $Q \in \mathbb{P}^2$ be a generic point, and let $\mathcal{F} = \{F = 0\}$ be a rational integral curve of degree 5 passing through Q, and having at each P_i, $1 \leq i \leq 6$, an ordinary singularity of multiplicity 2, (so $F \in (I_{2P_1+\cdots+2P_6})_5$), and let $\{\widetilde{l_i} = 0\}$ be one of the two distinct lines contained in the tangent space $T_{\mathcal{F},P_i}$ to $\mathcal{F}$ at the point P_i.

   Recall that the defining ideal of $\mathcal{Y} = (8,9)P_1 + \cdots + (8,9)P_6$ is

   $$I_{\mathcal{Y}} = (\wp_1^8 \cap (\wp_1^9 + l_1^2)) \cap \cdots \cap (\wp_6^8 \cap (\wp_6^9 + l_6^2)).$$

   Specialize the scheme $\mathcal{Y}$ putting $l_i = \widetilde{l_i}$ for $i = 1,2,3,4$, and let $\mathcal{Y}^*$ be such specialization of $\mathcal{Y}$. Since the expected dimension of $(I_{\mathcal{Y}})_{20}$ is $\binom{20+2}{2} - \deg(\mathcal{Y}) = 231 - 228 = 3$, then if we prove that $\dim(I_{\mathcal{Y}^*})_{20} = 3$, we are done.

   It is easy to see that the curves defined by the forms of $(I_{\mathcal{Y}^*+Q})_{20}$ have the quintic $\mathcal{F}$ as fixed component with multiplicity 2, hence

   $$\dim(I_{\mathcal{Y}^*+Q})_{20} = \dim(I_{\mathcal{W}})_{10}$$

   where $\mathcal{W} = Res_{2\mathcal{F}}(\mathcal{Y}^* + Q) = 4P_1 + 4P_2 + 4P_3 + 4P_4 + (4,5)P_5 + (4,5)P_6$. Now let $\mathcal{W}^*$ be a specialization of $\mathcal{W}$ obtained by putting $l_i = \widetilde{l_i}$ for $i = 5,6$. Since the quintic $\mathcal{F}$ is as fixed component with multiplicity 2 for $(I_{\mathcal{W}^*+Q})_{10}$, and since $Res_{2\mathcal{F}}(\mathcal{W}^* + Q) = \emptyset$ (see Lemma 3.4.5) we have

   $$\dim(I_{\mathcal{W}^*+Q})_{10} = \dim(I_{Res_{2\mathcal{F}}(\mathcal{W}^*+Q)})_0 = 1.$$

   Thus for the specialized scheme $\mathcal{W}^*$ we have $\dim(I_{\mathcal{W}^*})_{10} = 2 = \binom{10+2}{2} - \deg\mathcal{W}^*$. Then $\mathcal{W}^*$, and so also $\mathcal{W}$, imposes independent conditions to curves of degree 10. It follows that $\dim(I_{\mathcal{W}})_{10} = 2$. So $\dim(I_{\mathcal{Y}^*+Q})_{20} = 2$, hence $\dim(I_{\mathcal{Y}^*})_{20} = 3$, and we are done.

2. By using [**CoCoA**] we verified that $H(\mathcal{Y}, 15) = N + 1 = 136$.

3. By using [**CoCoA**] we verified that $H(\mathcal{Y}, 13) = H(X, 13) + 12 = 90 + 12 = 102$.

4. By using [**CoCoA**] we verified that $H(\mathcal{Y}, 11) = H(X, 11) + 12 = 60 + 12 = 72$.

□

Now let $k + 1 = 5q + r$, $(0 \leq r \leq 4)$. Thus $k \equiv 2 \pmod 5$ iff $r = 3$.
For $k = 1, 2$ see Lemma 3.4.7.

Let $k \geq 3$. Let $\mathcal{C}_i$ be the conic through $P_1, \ldots, \widehat{P_i}, \ldots, P_6$, $(i = 1, \ldots, 6)$, and let $\mathcal{C} = \sum_{i=1}^{6} \mathcal{C}_i$. Observe that if $2d < 5(k+1)$, then the curves defined by the forms of $(I_{\mathcal{Y}})_d$, and by the forms of $(I_X)_d$ have the six conics $\mathcal{C}_i$ as fixed components with multiplicity at least $\nu = 5(k+1) - 2d$.

Then

$$\dim(I_{\mathcal{Y}})_d = \dim(I_{\mathcal{Y}'})_{d-12\nu},$$

$$\dim(I_X)_d = \dim(I_{X'})_{d-12\nu},$$

where

$$\mathcal{Y}' = Res_{\nu\mathcal{C}}\mathcal{Y} = (k+1-5\nu, k+2-5\nu)P_1 + \cdots + (k+1-5\nu, k+2-5\nu)P_6,$$

$$X' = Res_{\nu\mathcal{C}}X = (k+1-5\nu)P_1 + \cdots + (k+1-5\nu)P_6.$$

We split the proof in four cases.

1. $k \equiv 2 \pmod 5$, and $d \leq \left\lceil \frac{12(k+1)}{5} \right\rceil - 1 = 12q + 7$.

   In this case it suffices to prove that $(I_{\mathcal{Y}})_d = (0)$ for $d = 12q + 7$. Since $2d = 2(12q+7) < 5(k+1) = 5(5q+3)$, then the curves defined by the forms of $(I_{\mathcal{Y}})_d$ should have a fixed locus of degree $12\nu = 12q + 12$, and this is impossible, since $d = 12q + 7$. It follows that $(I_{\mathcal{Y}})_d = (0)$.

2. $k \equiv 2 \pmod 5$, and $d \geq \left\lceil \frac{12(k+1)}{5} \right\rceil = 12q + 8$.

   First we will prove that

   $$H(\mathcal{Y}, d) = H(X, d) + 12.$$

   By Lemma 3.4.8, it suffices to prove that $H(\mathcal{Y}, d) = H(X, d) + 12$, for $d = 12q + 8$. Since $k \geq 3$, and $k + 1 = 5q + 3$, then we have $q \geq 1$. Let $q = 1$, so $d = 20$, $k + 1 = 8$, $\mathcal{Y} = (8,9)P_1 + \cdots + (8,9)P_6$, and $X = 8P_1 + \cdots + 8P_6$. Since $\dim(I_{(8,9)P_1 + \cdots + (8,9)P_6})_{20} = 3$ (see Lemma 3.4.14 point 1.), and six 8-fat points impose independent conditions to curves of degree 20 (see the first Remark of this section), we have $\dim(I_X)_{20} = 15$. It follows that $H(\mathcal{Y}, d) = H(X, d) + 12$. If $q > 1$, then $\nu\mathcal{C} = \sum_{i=1}^{6} \nu\mathcal{C}_i$ is a fixed locus for $(I_{\mathcal{Y}})_d$ and $(I_X)_d$. Since $\nu = 5(k+1) - 2d = 5(5q+3) - 2(12q+8) = q - 1$, we have $d - 12\nu = 12q + 8 - 12(q-1) = 20$, and $k + 1 - 5\nu = 5q + 3 - 5(q-1) = 8$. So

   $$\dim(I_{\mathcal{Y}})_d = \dim(I_{\mathcal{Y}'})_{20} = 3,$$

   $$\dim(I_X)_d = \dim(I_{X'})_{20} = 15,$$

   where $\mathcal{Y}' = Res_{\nu\mathcal{C}}\mathcal{Y} = (8,9)P_1 + \cdots + (8,9)P_6$, $X' = Res_{\nu\mathcal{C}}X = 8P_1 + \cdots + 8P_6$. Hence, we easily get that $H(\mathcal{Y}, d) = H(X, d) + 12$.

So we have proved that $H(\mathcal{Y}, d) = H(X, d) + 12$.

Now, since for $d = 12q + 8$, $\dim(I_{\mathcal{Y}})_d$ is positive (and in fact it is equal to $\dim(I_{\mathcal{Y}'})_{20} = 3$), then $H(\mathcal{Y}, d) < \binom{d+2}{2}$ for any $d \geq 12q + 8$.

Since six generic $(k+1)$-fat points impose independent conditions to curves of degree d if and only if $d \geq \left\lfloor \frac{5(k+1)}{2} \right\rfloor$ (see the first Remark of this section), then for $12q+8 \leq d \leq \left\lfloor \frac{5(k+1)}{2} \right\rfloor - 1$ we have $H(X, d) < \deg(X)$, hence

$$H(\mathcal{Y}, d) = H(X, d) + 12 < \min\left\{\deg(X) + 12, \binom{d+2}{2}\right\} =$$

$$= \min\left\{\deg(Y), \binom{d+2}{2}\right\} = \exp(H(\mathcal{Y}, d)).$$

While for $d \geq \max\left\{12q + 8; \left\lfloor \frac{5(k+1)}{2} \right\rfloor\right\}$, we have $H(X, d) = \deg(X)$, so $H(\mathcal{Y}, d) = H(X, d) + 12 = \deg(X) + 12 = \deg(Y)$. If $12q+8 \leq d \leq \left\lfloor \frac{5(k+1)}{2} \right\rfloor - 1$, we have $H(X, d) < \deg(X)$, hence $H(\mathcal{Y}, d) < \deg(Y)$. Moreover, since for $d = 12q + 8$, $\dim(I_{\mathcal{Y}})_d$ is positive (as shown above, it is equal to $\dim(I_{\mathcal{Y}'})_{20}$), then $H(\mathcal{Y}, d) < \binom{d+2}{2}$ for any $d \geq 12q + 8$.

That is enough to finish the proof of this case.

3. $k \not\equiv 2 \pmod 5$, and $d \leq \left\lceil \frac{12(k+1)}{5} \right\rceil$.

   By Lemma 3.4.8 we have only to prove that $H(\mathcal{Y}, d) = N + 1$ for $d = \left\lceil \frac{12(k+1)}{5} \right\rceil = 12q + \left\lceil \frac{12r}{5} \right\rceil$. Since $k \geq 3$, we have $k + 1 = 5q + r \geq 4$, hence $q \geq \frac{4-r}{5}$. As above, let $\nu = 5(k + 1) - 2d$, $\mathcal{Y}' = Res_{\nu\mathcal{C}}\mathcal{Y}$, and let $d' = d - 12\nu$. We have:

r	$k+1$	d	ν	$\mathcal{Y}'$	d'
0	$5q$	$12q$	$q > 0$	$P_1 + \cdots + P_6$	0
1	$5q+1$	$12q+3$	$q - 1 \geq 0$	$(6,7)P_1 + \cdots + (6,7)P_6$	15
2	$5q+2$	$12q+5$	$q > 0$	$(2,3)P_1 + \cdots + (2,3)P_6$	5
4	$5q+4$	$12q+10$	$q \geq 0$	$(4,5)P_1 + \cdots + (4,5)P_6$	10

   Since for $\nu = 0$, we have $\mathcal{Y}' = \mathcal{Y}$ and $d' = d$, then for every $\nu \geq 0$ we have:

$$\dim(I_{\mathcal{Y}})_d = \dim(I_{\mathcal{Y}'})_{d'}.$$

Now we will prove that $\dim(I_{\mathcal{Y}'})_{d'} = 0$.

For $r = 0$ it is obvious. For $r = 2$ see Lemma 3.4.7. For $r = 1$ by Lemma 3.4.14 point 2., we have $\dim(I_{(6,7)P_1+\cdots+(6,7)P_6})_{15} = 0$. For $r = 4$, let $\mathcal{F} = \{F = 0\}$ be a rational integral curve of degree 5 having at each P_i $(1 \leq i \leq 6)$ an ordinary singularity of multiplicity 2, $(F \in (I_{2P_1+\cdots+2P_6})_5)$. If there exists a form $G \neq 0$, $G \in (I_{(4,5)P_1+\cdots+(4,5)P_6})_{10}$, then $FG \neq 0$ and $FG \in (I_{(6,7)P_1+\cdots+(6,7)P_6})_{15}$, but this is impossible by the previous case $r = 1$.

4. $k \not\equiv 2 \pmod 5$, and $d \geq \left\lceil \frac{12(k+1)}{5} \right\rceil + 1$.

First we will to prove that

$$H(\mathcal{Y}, d) = H(X, d) + 12.$$

By Lemma 3.4.8, it suffices to prove that $H(\mathcal{Y}, d) = H(X, d) + 12$ for $d = \left\lceil \frac{12(k+1)}{5} \right\rceil + 1 = 12q + \left\lceil \frac{12r}{5} \right\rceil + 1$.

As usual, let $\nu = 5(k+1) - 2d$, $\mathcal{Y}' = Res_{\nu\mathcal{C}}\mathcal{Y}$, $X' = Res_{\nu\mathcal{C}}X$, and $d' = d - 12\nu$. We have:

r	$k+1$	d	ν	$k+1-5\nu$	$\mathcal{Y}'$	X'	d'
0	$5q$	$12q+1$	$q-2$	10	$\sum_{i=1}^{6}(10,11)P_i$	$\sum_{i=1}^{6} 10P_i$	25
1	$5q+1$	$12q+4$	$q-3$	16	$\sum_{i=1}^{6}(16,17)P_i$	$\sum_{i=1}^{6} 16P_i$	40
2	$5q+2$	$12q+6$	$q-2$	12	$\sum_{i=1}^{6}(12,13)P_i$	$\sum_{i=1}^{6} 12P_i$	30
4	$5q+4$	$12q+11$	$q-2$	14	$\sum_{i=1}^{6}(14,15)P_i$	$\sum_{i=1}^{6} 14P_i$	35

Since for $\nu = 0$, we have $\mathcal{Y}' = \mathcal{Y}$, $X' = X$, and $d' = d$, then for every $\nu \geq 0$ we have:

$$\dim(I_{\mathcal{Y}})_d = \dim(I_{\mathcal{Y}'})_{d'},$$

$$\dim(I_X)_d = \dim(I_{X'})_{d'}.$$

It follows that

$$H(\mathcal{Y}, d) - H(X, d) = H(\mathcal{Y}', d') - H(X', d').$$

Hence in case $\nu \geq 0$ we have only to prove that:

(a) $H(\sum_{i=1}^{6}(10,11)P_i,\ 25) = H(\sum_{i=1}^{6} 10P_i,\ 25) + 12$;
(b) $H(\sum_{i=1}^{6}(12,13)P_i,\ 30) = H(\sum_{i=1}^{6} 12P_i,\ 30) + 12$;
(c) $H(\sum_{i=1}^{6}(14,15)P_i,\ 35) = H(\sum_{i=1}^{6} 14P_i,\ 35) + 12$;
(d) $H(\sum_{i=1}^{6}(16,17)P_i,\ 40) = H(\sum_{i=1}^{6} 16P_i,\ 40) + 12$;

Now we need the following lemma:

Lemma 3.4.15. *Let:*

$$\mathcal{Y} = (m, m+1)P_1 + \cdots + (m, m+1)P_6,$$
$$\widetilde{\mathcal{Y}} = (m+2, m+3)P_1 + \cdots + (m+2, m+3)P_6,$$
$$\widetilde{X} = (m+2)P_1 + \cdots + (m+2)P_6.$$

If the integer $\eta = 5(d+5) - 12(m+2) + 1 \geq 0$, *and* $H(\mathcal{Y}, d) = \deg(\mathcal{Y})$, *then*

1. $H(\widetilde{\mathcal{Y}}, d+5) = \deg(\widetilde{\mathcal{Y}})$, $H(\widetilde{X}, d+5) = \deg(\widetilde{X})$;
2. $H(\widetilde{\mathcal{Y}}, d+5) = H(\widetilde{X}, d+5) + 12$.

Proof. 1. Let $\mathcal{F}$ be (as above) a rational curve of degree 5 having at each P_i $(1 \leq i \leq 6)$, an ordinary singularity of multiplicity 2. Let $Q_1, \ldots, Q_\eta \in \mathcal{F}$ be generic points. Since $5(d+5) < 6(2(m+2)) + \eta$, by Bezout Theorem $\mathcal{F}$ is a fixed component for the curves defined by the forms of $\left(I_{\widetilde{\mathcal{Y}}+Q_1+\cdots+Q_\eta}\right)_{d+5}$. It follows that

$$\dim\left(I_{\widetilde{\mathcal{Y}}+Q_1+\cdots+Q_\eta}\right)_{d+5} = \dim\left(I_{\mathcal{Y}}\right)_d.$$

Since $\binom{d+5+2}{2} - \deg(\widetilde{\mathcal{Y}} + Q_1 + \cdots + Q_\eta) = \frac{1}{2}(d+7)(d+6) - (\deg(\mathcal{Y}) + 6(m+2) + 6(m+1) + \eta) = \binom{d+2}{2} - \deg(\mathcal{Y}) = \binom{d+2}{2} - H(\mathcal{Y}, d) = \dim(I_{\mathcal{Y}})_d$, we have

$$\dim\left(I_{\widetilde{\mathcal{Y}}+Q_1+\cdots+Q_\eta}\right)_{d+5} = \binom{d+5+2}{2} - \deg\left(\widetilde{\mathcal{Y}} + Q_1 + \cdots + Q_\eta\right)$$

hence $H(\widetilde{\mathcal{Y}} + Q_1 + \cdots + Q_\eta, d+5) = \deg(\widetilde{\mathcal{Y}} + Q_1 + \cdots + Q_\eta)$.

Since obviously $\widetilde{X} \subset \widetilde{\mathcal{Y}} \subset \widetilde{\mathcal{Y}} + Q_1 + \cdots + Q_\eta$, it follows that $H(\widetilde{\mathcal{Y}}, d+5) = \deg(\widetilde{\mathcal{Y}})$, and $H(\widetilde{X}, d+5) = \deg(\widetilde{X})$.

2. Obvious.

□

By *Case 2)* we know that $H\left(\sum_{i=1}^6 (8,9)P_i,\ 20\right) = H\left(\sum_{i=1}^6 8P_i,\ 20\right) + 12 = \deg\left(\sum_{i=1}^6 (8,9)P_i\right)$, so by Lemma 3.4.15 point 2. we have (a): $H\left(\sum_{i=1}^6 (10,11)P_i, 25\right) = H\left(\sum_{i=1}^6 10P_i,\ 25\right) + 12$.

Moreover, by Lemma 3.4.15 point 1., $H\left(\sum_{i=1}^6 (10,11)P_i,\ 25\right) = \deg\left(\sum_{i=1}^6 (10,11)P_i\right)$, hence by Lemma 3.4.15 point 2. we get (b): $H\left(\sum_{i=1}^6 (12,13)P_i, 30\right) = H\left(\sum_{i=1}^6 12P_i,\ 30\right) + 12$.

Analogously, by Lemma 3.4.15, we have that $(b) \Rightarrow (c) \Rightarrow (d)$, so, for $\nu \geq 0$, we have proved that $H(\mathcal{Y}, d) = H(X, d) + 12$.

Now let $\nu < 0$. In this case, since $k + 1 = 5q + r \geq 3$, we are left with the folloving cases:

r	q	$k+1$	Y	X	d
0	1	5	$\sum_{i=1}^{6}(5,6)P_i$	$\sum_{i=1}^{6}5P_i$	13
1	1	6	$\sum_{i=1}^{6}(6,7)P_i$	$\sum_{i=1}^{6}6P_i$	16
1	2	11	$\sum_{i=1}^{6}(11,12)P_i$	$\sum_{i=1}^{6}11P_i$	28
2	1	7	$\sum_{i=1}^{6}(7,8)P_i$	$\sum_{i=1}^{6}7P_i$	18
4	0	4	$\sum_{i=1}^{6}(4,5)P_i$	$\sum_{i=1}^{6}4P_i$	11
4	1	9	$\sum_{i=1}^{6}(9,10)P_i$	$\sum_{i=1}^{6}9P_i$	23

hence we have to prove that:
(e): $H\left(\sum_{i=1}^{6}(5,6)P_i,\ 13\right) = H\left(\sum_{i=1}^{6}5P_i,\ 13\right) + 12$;
(f): $H\left(\sum_{i=1}^{6}(6,7)P_i,\ 16\right) = H\left(\sum_{i=1}^{6}6P_i,\ 16\right) + 12$;
(g): $H\left(\sum_{i=1}^{6}(11,12)P_i,\ 28\right) = H\left(\sum_{i=1}^{6}11P_i,\ 28\right) + 12$;
(h): $H\left(\sum_{i=1}^{6}(7,8)P_i,\ 18\right) = H\left(\sum_{i=1}^{6}7P_i,\ 18\right) + 12$;
(i): $H\left(\sum_{i=1}^{6}(4,5)P_i,\ 11\right) = H\left(\sum_{i=1}^{6}4P_i,\ 11\right) + 12$;
(l): $H\left(\sum_{i=1}^{6}(9,10)P_i,\ 23\right) = H\left(\sum_{i=1}^{6}9P_i,\ 23\right) + 12$.

By Lemma 3.4.14 points 3. and 4., it easily follows that (e) and (i) hold, moreover by Lemma 3.4.15 we have that $(e) \Rightarrow (h) \Rightarrow (l) \Rightarrow (g)$, and $(i) \Rightarrow (f)$, so we have proved that $H(\mathcal{Y}, d) = H(X, d) + 12$ also for $\nu < 0$.

Now, for $d = \left\lceil \frac{12(k+1)}{5} \right\rceil + 1$, as shown above, we have:
for $\nu \geq 0$:

$$\dim(I_{\mathcal{Y}})_d = \dim(I_{\mathcal{Y}'})_{d'} = \binom{d'+2}{2} - \deg(X') - 12 = \begin{cases} \binom{25+2}{2} - 6\binom{10+1}{2} - 12 = 9 & \text{for } r = 0, \\ \binom{40+2}{2} - 6\binom{16+1}{2} - 12 = 33 & \text{for } r = 1, \\ \binom{30+2}{2} - 6\binom{12+1}{2} - 12 = 16 & \text{for } r = 2, \\ \binom{35+2}{2} - 6\binom{14+1}{2} - 12 = 24 & \text{for } r = 4; \end{cases}$$

for $\nu < 0$:

$$\dim(I_{\mathcal{Y}})_d = \begin{cases} \binom{13+2}{2} - 6\binom{5+1}{2} - 12 = 3 & \text{for } r = 0,\ q = 1, \\ \binom{16+2}{2} - 6\binom{6+1}{2} - 12 = 15 & \text{for } r = 1,\ q = 1, \\ \binom{28+2}{2} - 6\binom{11+1}{2} - 12 = 27 & \text{for } r = 1,\ q = 2, \\ \binom{18+2}{2} - 6\binom{7+1}{2} - 12 = 10 & \text{for } r = 2,\ q = 1, \\ \binom{11+2}{2} - 6\binom{4+1}{2} - 12 = 6 & \text{for } r = 4,\ q = 0, \\ \binom{23+2}{2} - 6\binom{9+1}{2} - 12 = 18 & \text{for } r = 4,\ q = 1; \end{cases}$$

hence $\dim(I_{\mathcal{Y}})_d$ is positive, and this implies that $H(\mathcal{Y}, d) < \binom{d+2}{2}$ for any $d \geq \left\lceil \frac{12(k+1)}{5} \right\rceil + 1$. Moreover, since six generic $(k+1)$-fat points impose independent conditions to curves of degree d if and only if $d \geq \left\lfloor \frac{5(k+1)}{2} \right\rfloor$ (see the first Remark of this section), then for $\left\lceil \frac{12(k+1)}{2} \right\rceil + 1 \leq d \leq \left\lfloor \frac{5(k+1)}{2} \right\rfloor - 1$, we have $H(X, d) < \deg(X)$, hence

$$H(\mathcal{Y}, d) = H(X, d) + 12 < \min\left\{\deg(X) + 12, \binom{d+2}{2}\right\} = \min\left\{\deg(Y), \binom{d+2}{2}\right\} = \exp(H(\mathcal{Y}, d)).$$

While for $d \geq \max\left\{\left\lceil \frac{12(k+1)}{5} \right\rceil + 1; \left\lfloor \frac{5(k+1)}{2} \right\rfloor\right\}$, we have $H(X, d) = \deg(X)$, so $H(\mathcal{Y}, d) = H(X, d) + 12 = \deg(X) + 12 = \deg(Y)$.

If $\left\lceil \frac{12(k+1)}{5} \right\rceil + 1 \leq d \leq \left\lfloor \frac{5(k+1)}{2} \right\rfloor - 1$, we have $H(X, d) = \deg(X)$, so $H(\mathcal{Y}, d) = H(X, d) + 12 = \deg(X) + 12 = \deg(Y)$. $\square$

For the study of cases $s = 6, 7$ we need to introduce here a Lemma that uses Cremona transformations.

Lemma 3.4.16. *Let $\Pi_1 \simeq \Pi_2 \simeq \mathbb{P}^2$ be two copies of $\mathbb{P}^2$ with coordinate rings $K[x, y, z]$ and $K[X, Y, Z]$ respectively. Let $\varphi : \Pi_1 - - > \Pi_2$ be the Cremona transformation $\varphi(x, y, z) = (yz, xz, xy) = (X, Y, Z)$. Let $I \subset K[x, y, z]$ be the ideal of a $(m, m+1)$-point with support $P = (0, 0, 1)$, $\wp = (x - y, y)$ be its representative ideal and $l = \{x - y = 0\}$ the line such that $I = \wp^{k+1} \cap (\wp^{k+2} + l^2)$. Then $< I_d > \simeq < J_{2d-m} >$ where*

$$J = (X, Y)^d \cap (X, Z)^{d-m} \cap (Y, Z)^{d-m} \cap ((Y - X)^2, Z).$$

Proof. By hypothesis I is the ideal $I = (x-y,y)^m \cap ((x-y,y)^{m+1},(x-y)^2) = ((x-y)^m,(x-y)^{m-1}y,(x-y)^{m-2}y^2,\dots,(x-y)^2y^{m-2},(x-y)y^m,y^{m+1})$.
Let $f_d(x,y,z) \in I_d$ then $f_d(x,y,z) = (\sum_{i=0}^{m-2} a_i(x-y)^{m-i}y^i)z^{d-m} + \sum_{i=m+1}^{d} g_i z^{d-i}$ where $g_i \in (K[x,y])_i$ and $a_i \in K$ for all $i = 1,\dots,d$.
Obviously $\varphi(f_d(x,y,z)) =: F_{2d}(X,Y,Z) \in (K[X,Y,Z])_{2d}$ and
$F_{2d}(X,Y,Z) = (XY)^{d-m}Z^m\left(\sum_{i=0}^{m-2} a_i(Y-X)^{m-i}X^i\right)+\sum_{i=m+1}^{d}(XY)^{d-i}Z^iG_i$ where $G_i \in (K[X,Y])_i$ for all $i = m+1,\dots,d$.
Now F_{2d} is the total transforme of f_d but we are looking for the strict transforme $\widetilde{F}_{2d-m}$ defined by $F_{2d} = Z^m\widetilde{F}_{2d-m}$; then $\widetilde{F}_{2d-m}(X,Y,Z) = (XY)^{d-m}\left(\sum_{i=0}^{m-2} a_i(Y-X)^{m-i}X^i\right)+\sum_{i=m+1}^{d}(XY)^{d-i}Z^{i-m}G_i$.
Now the $\widetilde{F}_{2d-m}(X,Y,Z)$, as $a_i \in K$ and $G_i \in (K[X,Y])_i$ vary, give the part of degree $2d-m$ of the ideal $J \subset K[X,Y,Z]$ where $J = (X,Y)^d \cap (X,Z)^{d-m} \cap (Y,Z)^{d-m} \cap ((Y-X)^2,Z)$. It is easy to see that $\widetilde{F}_{2d-m} \in J_{2d-m}$.

The other inclusion, i.e. that all $h(X,Y,Z) \in J_{2d-m}$ can be written as $\widetilde{F}_{2d-m}$, can be computationally verified. □

Remark: Let P_1,P_2,P_3 be three generic points of $\mathbb{P}^2$. Consider the Cremona transformation that acts as an isomorphism on $\mathbb{P}^2 \smallsetminus \{P_1,P_2,P_3\}$. Therefore, if I and J are as in the lemma above and the point P of the lemma is one of the P_i for $i = 1,\dots,s$, then

$$H(K[x,y,z]/I,d) = H(K[X,Y,Z]/J,2d-m).$$

Proposition 3.4.17. *For $s = 7$ we have:*

$$H(\mathcal{Y},d) = \begin{cases} \textit{for } k=1: & \begin{cases} N+1 & \textit{if } d \le 6, \\ H(X,d)+14 = \deg(Y) & \textit{if } d \ge 7, \end{cases} \\ \textit{for } k=2: & \begin{cases} N+1 & \textit{if } d \le 9, \\ H(X,d)+14 = \deg(Y) & \textit{if } d \ge 9, \end{cases} \\ \begin{array}{l}\textit{for } k \ge 3 \\ k \equiv 1,4 \ (\textit{mod } 8):\end{array} & \begin{cases} N+1 & \textit{if } d \le \left\lceil \frac{21(k+1)}{8} \right\rceil - 1, \\ H(X,d)+14 < \deg(Y) & \textit{if } \left\lceil \frac{21(k+1)}{8} \right\rceil \le d \le \left\lfloor \frac{8k}{3} \right\rfloor + 2, \\ H(X,d)+14 = \deg(Y) & \textit{if } d \ge \max\left\{\left\lceil \frac{21(k+1)}{8} \right\rceil, \left\lfloor \frac{8k}{3} \right\rfloor + 3\right\}, \end{cases} \\ \begin{array}{l}\textit{for } k \ge 3 \\ k \not\equiv 1,4 \ (\textit{mod } 8):\end{array} & \begin{cases} N+1 & \textit{if } d \le \left\lceil \frac{21(k+1)}{8} \right\rceil, \\ H(X,d)+14 < \deg(Y) & \textit{if } \left\lceil \frac{21(k+1)}{8} \right\rceil + 1 \le d \le \left\lfloor \frac{8k}{3} \right\rfloor + 2, \\ H(X,d)+14 = \deg(Y) & \textit{if } d \ge \max\left\{\left\lceil \frac{21(k+1)}{8} + 1 \right\rceil, \left\lfloor \frac{8k}{3} \right\rfloor + 3\right\}. \end{cases} \end{cases}$$

Proof. For $k = 1, 2$ the statement is known by [**Ba**] and [**BF1**].

For $k \geq 3$ we prove first that $(I_{\mathcal{Y}})_d = (0)$ if

$$d = \begin{cases} \left\lceil \frac{21(k+1)}{8} \right\rceil & \text{and } k \not\equiv 1, 4 \pmod 8 \text{ or} \\ \\ \left\lceil \frac{21(k+1)}{8} \right\rceil - 1 & \text{and } k \equiv 1, 4 \pmod 8 \end{cases}$$

so that $(I_{\mathcal{Y}})_d = (0)$ for all d less or equal to that value.

Consider the seven cubics C_i, for $i = 1, \ldots, 7$, through $P_1, \ldots, 2P_i, \ldots, P_7$. If $8(k+1) > 3d$, then the seven cubics C_i are fixed components for $(I_{\mathcal{Y}})_d$ with multiplicity $\nu = 8(k+1) - 3d$, therefore

$$\dim (I_{\mathcal{Y}})_d = \dim \left(I_{\mathcal{Y} - \sum_{i=1}^{7} \nu C_i} \right)_{d - 21\nu} \tag{3.21}$$

where $\mathcal{Y} - \sum_{i=1}^{7} \nu C_i$ is the union of seven 0-dimensional schemes of the type $(k+1-8\nu, k+2-8\nu)$.

Let

$$k + 1 = 8q + r.$$

In the following table we summarize the cases we need to study in order to compute

$$\dim \left(I_{\mathcal{Y} - \sum_{i=1}^{7} \nu C_i} \right)_{d - 21\nu}.$$

r	d	ν	$\mathcal{Y} - \sum_{i=1}^{7} \nu C_i$	$d - 21\nu$	
0	$21q$	q	$\sum_{i=1}^{7} P_i$	0	
1	$21q + 3$	$q - 1$	$\sum_{i=1}^{7} (9, 10) P_i$	24	(*)
2	$21q + 5$	$q + 1$	$\sum_{i=1}^{7} (-6, -5) P_i$	-16	
3	$21q + 8$	q	$\sum_{i=1}^{7} (3, 4) P_i$	8	(*)
4	$21q + 11$	$q - 1$	$\sum_{i=1}^{7} (12, 13) P_i$	32	(*)
5	$21q + 13$	$q + 1$	$\sum_{i=1}^{7} (-3, -2) P_i$	-8	
6	$21q + 16$	q	$\sum_{i=1}^{7} (6, 7) P_i$	16	(*)
7	$21q + 19$	$q - 1$	$\sum_{i=1}^{7} (15, 16) P_i$	40	(*)

Since some case has to be excluded from this procedure, in the table above there are some cases excluded:

r	q	d	$\mathcal{Y}$	
1	1	24	$\sum_{i=1}^{7}(9,10)P_i$	(*)
3	0	8	$\sum_{i=1}^{7}(3,4)P_i$	
4	0	11	$\sum_{i=1}^{7}(4,5)P_i$	(*)
4	1	32	$\sum_{i=1}^{7}(12,13)P_i$	(*)
6	0	16	$\sum_{i=1}^{7}(6,7)P_i$	(*)
7	0	19	$\sum_{i=1}^{7}(7,8)P_i$	(*)
7	1	40	$\sum_{i=1}^{7}(15,16)P_i$	(*)

In the cases where $d-21\nu = 0, -16, -8$ we clearly have $\left(I_{\mathcal{Y}-\sum_{i=1}^{7}\nu C_i}\right)_{d-21\nu} = (0)$. The case of seven $(3,4)$-points is from [**Ba**]. For all the remaining cases (that are those we have marked with a "(*)") we will use Lemma 3.4.16.

Let us start with the scheme $(k+1,k+2)P_1 \cup \cdots \cup (k+1,k+2)P_7$. Consider the Cremona transformation

$$\varphi_{1,2,3} \text{ that acts as an isomorphism on } \mathbb{P}^2 \smallsetminus \{P_1, P_2, P_3\}. \tag{3.22}$$

The image of $\left(I_{(k+1,k+2)P_1+\cdots+(k+1,k+2)P_7}\right)_d$ via $\varphi_{1,2,3}$ is the degree $(2d-3k-3)$ part of the ideal representing the following scheme:

$$\sum_{i=1}^{3}(d-2k-2)P_i' + \sum_{i=1}^{6} R_i + \sum_{i=1}^{3}(k+1,k+2)Q_i$$

where $P_i' = \varphi_{1,2,3}(< P_1, \hat{P}_i, P_2 >)$ for $i = 1,2,3$; $Q_i = \varphi_{1,2,3}(P_{i+3})$ for $i = 1,2,3$; and R_1, R_2 are two simple points on the line $< P_1', P_2' >$, R_3, R_4 are two simple points on the line $< P_1', P_3' >$, and R_5, R_6 are two simple points on the line $< P_2', P_3' >$.
From now on we will use an abuse of notation and we will keep calling, after the Cremona transformation, a point P_i instead of P_i'.

Consider now the following successive Cremona transformations:

$$\begin{aligned}
&\varphi_{4,5,6}, \text{ that acts as isomorphism on } \mathbb{P}^2 \smallsetminus \{P_4, P_5, P_6\},\\
&\varphi_{1,2,7}, \text{ that acts as isomorphism on } \mathbb{P}^2 \smallsetminus \{P_1, P_2, P_7\},\\
&\varphi_{3,4,5}, \text{ that acts as isomorphism on } \mathbb{P}^2 \smallsetminus \{P_3, P_4, P_5\},\\
&\varphi_{3,6,7}, \text{ that acts as isomorphism on } \mathbb{P}^2 \smallsetminus \{P_3, P_6, P_7\}.
\end{aligned} \tag{3.23}$$

At the end we have to study the Hilbert function in degree $8d-21(k+1)$ of the scheme:

$$\mathcal{Y}' = \sum_{i=1}^{7}(3d-8k-8)P_i + \sum_{i=1}^{14} R_i \tag{3.24}$$

where P_i are seven generic points of $\mathbb{P}^2$ and $R_i \in \mathbb{P}^2$ are 14 points such that:

$$\begin{gathered} R_1, R_2 \text{ belong to a conic through } P_1, P_2; \\ R_3, R_4 \text{ belong to another different conic through } P_1, P_2; \\ R_5, R_6 \text{ belong to a conic through } P_1, P_3; \\ R_7, R_8 \text{ belong to a conic through } P_2, P_3; \\ R_9, R_{10} \text{ belong to a conic through } P_4, P_5; \\ R_{11}, R_{12} \text{ belong to a conic through } P_4, P_6; \\ R_{13}, R_{14} \text{ belong to a conic through } P_5, P_6. \end{gathered} \tag{3.25}$$

Let φ be the composition of the Cremona transformations defined in (3.22) and in (3.23):

$$\varphi := \varphi_{3,6,7} \circ \varphi_{3,4,5} \circ \varphi_{1,2,7} \circ \varphi_{4,5,6} \circ \varphi_{1,2,3}. \tag{3.26}$$

The action of φ on $(I_{\mathcal{Y}})_d$ in the cases "(*)" of the last table, gives the degree $(8d - 21k - 21)$ part of the ideal $I_{\mathcal{Y}'}$ described in (3.24), in particular they become the following:

1. $(I_{\mathcal{Y}'})_{8d-21k-21} = \left(I_{\sum_{i=1}^{14} R_i}\right)_3 = (0)$, if $r = 1$ and $q > 1$;
2. $(I_{\mathcal{Y}'})_{8d-21k-21} = \left(I_{\sum_{i=1}^{14} R_i}\right)_1 = (0)$, if $r = 3$ and $q > 0$;
3. $(I_{\mathcal{Y}'})_{8d-21k-21} = \left(I_{\sum_{i=1}^{14} R_i}\right)_4$, if $r = 4$ and $q > 1$;
4. $(I_{\mathcal{Y}'})_{8d-21k-21} = \left(I_{\sum_{i=1}^{14} R_i}\right)_2 = (0)$, if $r = 6$ and $q > 0$;
5. $(I_{\mathcal{Y}'})_{8d-21k-21} = \left(I_{\sum_{i=1}^{14} R_i}\right)_5$, if $r = 7$ and $q \geq 0$;
6. $(I_{\mathcal{Y}'})_{8d-21k-21} = \left(I_{\sum_{i=1}^{7} P_i'' + \sum_{i=1}^{14} R_i}\right)_4 = (0)$, if $r = 4$ and $q = 0$.

For the cases 3. and 5. the idea to use Cremona transformations is not useful. Those cases, before transformations, were $\left(I_{\sum_{i=1}^{7}(4,5)P_i}\right)_{11}$ and $\left(I_{\sum_{i=1}^{7}(7,8)P_i}\right)_{19}$ respectively. For the first one we used the help of [**CoCoA**] system that immediately gives that $\left(I_{\sum_{i=1}^{7}(4,5)P_i}\right)_{11} = (0)$. For the second case we study the residual scheme obtained by cutting with some particular conics.
Let $\wp_i^7 \cap (\wp_i^8 + l_i^2)$ be the ideal of the scheme $(7,8)P_i$, let $C_{a,b,c,d,e}^{i,j,h,k,l} \subset \mathbb{P}^2$ be the conics through

P_a, P_b, P_c, P_d, P_e for $a, b, c, d, e \in \{1, \dots, 7\}$ such that the tangent space to $C^{i,j,h,k,l}_{a,b,c,d,e}$ at P_α is l_α if $\alpha \in \{i, j, k, l\} \subset \{1, \dots, 7\}$ (i.e. we are specializing the scheme $(7, 8)P_\alpha$ in such a way that this can be possible). The curve $C = C^{1,2,3,4}_{1,2,3,4,5} \cup C^{1,2,3,4,5}_{1,2,3,4,5} \cup C^{5,6}_{3,4,5,6,7} \cup C^{6,7}_{1,4,5,6,7} \cup C^{7}_{1,2,4,6,7} \cup C_{1,2,3,6,7} \cup C_{2,3,5,6,7}$ is a fixed component for $\left(I_{\sum_{i=1}^7 (7,8)P_i}\right)_{19}$, hence $\left(I_{\sum_{i=1}^7 (7,8)P_i}\right)_{19} = \left(I_{Res_C \sum_{i=1}^7 (7,8)P_i}\right)_5$. Then Lemma 3.4.5 assures that

$$\left(I_{\sum_{i=1}^7 (7,8)P_i}\right)_{19} = \left(I_{Res_C \sum_{i=1}^7 (7,8)P_i}\right)_5 = \left(I_{\sum_{i=1}^7 2P_i}\right)_5 = (0).$$

This conclude the proof for

$$d \leq \begin{cases} \left\lceil \frac{21(k+1)}{8} \right\rceil \text{ and } k \not\equiv 1, 4 \pmod 8 \text{ or} \\ \\ \left\lceil \frac{21(k+1)}{8} \right\rceil - 1 \text{ and } k \equiv 1, 4 \pmod 8 \end{cases}.$$

Now for

$$d \geq \begin{cases} \left\lceil \frac{21(k+1)}{8} \right\rceil + 1 \text{ and } k \not\equiv 1, 4 \pmod 8 \text{ or} \\ \\ \left\lceil \frac{21(k+1)}{8} \right\rceil \text{ and } k \equiv 1, 4 \pmod 8 \end{cases} \tag{3.27}$$

it is sufficient to prove that $H(\mathcal{Y}, d) = H(X, d) + 14$ for $d = \left\lceil \frac{21(k+1)}{8} \right\rceil + 1$ if $k \not\equiv 1, 4 \pmod 8$ or $d = \left\lceil \frac{21(k+1)}{8} \right\rceil$ if $k \equiv 1, 4 \pmod 8$ by Lemma 3.4.8.

The criterion we use in order to reduce the number of cases to check is the same of the previous discussion. In the table that follows we summarize the passages:

- we start with $(I_{\mathcal{Y}})_d$ and we write $k + 1 = 8q + r$,
- we use the relation (3.21), and we compute the scheme $\mathcal{Y} - \sum_{i=1}^7 \nu C_i$ and the degree $d - 21\nu$,
- we apply to $\left(I_{\mathcal{Y} - \sum_{i=1}^7 \nu C_i}\right)_{d-21\nu}$ the map φ defined in (3.26) and we write the "result" in the last column of the table: the notation will be very concise in order to make the material better readable, we will write $L(\alpha, \beta)$ to indicate $\left(I_{\sum_{i=1}^7 \beta P_i + \sum_{i=1}^{14} R_i}\right)_\alpha$.

r	d	ν	$\mathcal{Y} - \sum_{i=1}^{7} \nu C_i$	$d - 21\nu$	$L(\alpha, \beta)$
0	$21q+1$	$q-3$	$\sum_{i=1}^{7}(24,25)P_i$	64	$L(8,0)$
1	$21q+4$	$q-4$	$\sum_{i=1}^{7}(33,34)P_i$	88	$L(11,0)$
2	$21q+6$	$q-2$	$\sum_{i=1}^{7}(18,19)P_i$	48	$L(6,0)$
3	$21q+9$	$q-3$	$\sum_{i=1}^{7}(27,28)P_i$	72	$L(9,0)$
4	$21q+12$	$q-4$	$\sum_{i=1}^{7}(36,37)P_i$	96	$L(12,0)$
5	$21q+14$	$q-2$	$\sum_{i=1}^{7}(21,22)P_i$	56	$L(7,0)$
6	$21q+17$	$q-3$	$\sum_{i=1}^{7}(30,31)P_i$	80	$L(10,0)$
7	$21q+20$	$q-4$	$\sum_{i=1}^{7}(39,40)P_i$	104	$L(13,0)$

Since in the table above ν has to be positive, we have some cases that do not appear in that table, we enumerate them in the following tables (the last column of the tables below describe, with the same notation of the table above, the schemes we obtain after having applied to $(I_{\mathcal{Y}})_d$ the composition of Cremona transformations φ defined in (3.26)):

$r=0$	q	d	$\mathcal{Y}$	$L(\alpha, \beta)$
	1	22	$\sum_{i=1}^{7}(8,9)P_i$	$L(8,2)$
	2	43	$\sum_{i=1}^{7}(16,17)P_i$	$L(8,1)$
	3	64	$\sum_{i=1}^{7}(24,25)P_i$	$L(8,0)$

$r=1$	q	d	$\mathcal{Y}$	$L(\alpha, \beta)$
	1	25	$\sum_{i=1}^{7}(9,10)P_i$	$L(11,3)$
	2	45	$\sum_{i=1}^{7}(17,18)P_i$	$L(11,2)$
	3	67	$\sum_{i=1}^{7}(25,26)P_i$	$L(11,1)$
	4	88	$\sum_{i=1}^{7}(33,34)P_i$	$L(11,0)$

$r=2$	q	d	$\mathcal{Y}$	$L(\alpha, \beta)$
	0	6	$\sum_{i=1}^{7}(2,3)P_i$	solved in [**CGG2**]
	1	27	$\sum_{i=1}^{7}(10,11)P_i$	$L(6,1)$
	2	48	$\sum_{i=1}^{7}(18,19)P_i$	$L(6,0)$

$r=3$	q	d	$\mathcal{Y}$	$L(\alpha, \beta)$
	0	9	$\sum_{i=1}^{7}(3,4)P_i$	solved in [**Ba**]
	1	30	$\sum_{i=1}^{7}(11,12)P_i$	$L(9,2)$
	2	51	$\sum_{i=1}^{7}(19,20)P_i$	$L(9,1)$
	3	72	$\sum_{i=1}^{7}(27,28)P_i$	$L(9,0)$

$r=4$	q	d	$\mathcal{Y}$	$L(\alpha,\beta)$
	0	12	$\sum_{i=1}^{7}(4,5)P_i$	$L(12,4)$
	1	33	$\sum_{i=1}^{7}(12,13)P_i$	$L(12,3)$
	2	54	$\sum_{i=1}^{7}(20,21)P_i$	$L(12,2)$
	3	75	$\sum_{i=1}^{7}(28,29)P_i$	$L(12,1)$
	4	96	$\sum_{i=1}^{7}(36,37)P_i$	$L(12,0)$

$r=5$	q	d	$\mathcal{Y}$	$L(\alpha,\beta)$
	0	14	$\sum_{i=1}^{7}(5,6)P_i$	$L(7,2)$
	1	35	$\sum_{i=1}^{7}(13,14)P_i$	$L(7,1)$
	2	56	$\sum_{i=1}^{7}(21,22)P_i$	$L(7,0)$

$r=6$	q	d	$\mathcal{Y}$	$L(\alpha,\beta)$
	0	17	$\sum_{i=1}^{7}(6,7)P_i$	L(10,3)
	1	38	$\sum_{i=1}^{7}(14,15)P_i$	$L(10,2)$
	2	59	$\sum_{i=1}^{7}(22,23)P_i$	$L(10,1)$
	3	80	$\sum_{i=1}^{7}(30,31)P_i$	$L(10,0)$

$r=7$	q	d	$\mathcal{Y}$	$L(\alpha,\beta)$
	0	20	$\sum_{i=1}^{7}(7,8)P_i$	$L(13,4)$
	1	41	$\sum_{i=1}^{7}(15,16)P_i$	$L(13,3)$
	2	62	$\sum_{i=1}^{7}(23,24)P_i$	$L(13,2)$
	3	83	$\sum_{i=1}^{7}(31,32)P_i$	$L(13,1)$
	4	104	$\sum_{i=1}^{7}(39,40)P_i$	$L(13,0)$

The use of Cremona transformations allows us to study the degree α part of the ideals representing only five schemes: $L(\alpha,4)$, $L(\alpha,3)$, $L(\alpha,2)$, $L(\alpha,1)$ and $L(\alpha,0)$; Lemma 3.4.8 allows us to compute the regularity of the Hilbert functions of those schemes only for the lowest values of α. Hence we have to study only the following cases:

1. $\left(I_{\sum_{i=0}^{7}(10,11)P_i}\right)_{27}$,
2. $\left(I_{\sum_{i=0}^{7}(18,19)P_i}\right)_{48}$,
3. $\left(I_{\sum_{i=0}^{7}(4,5)P_i}\right)_{12}$,

4. $\left(I_{\sum_{i=0}^{7}(5,6)P_i}\right)_{14}$,

5. $\left(I_{\sum_{i=0}^{7}(6,7)P_i}\right)_{17}$.

The cases 3., 4. and 5. were verified by using [**CoCoA**] since the computational complexity was not too hight. For the cases 1. and 2. we study the residual schemes obtained by cutting with some particular curves. We write here only the curve we use in order to compute the residual. We will specialize the schemes $(k+1,k+2)P_1,\dots,(k+1,k+2)P_7$ in such a way that the curves we are going to describe do exist. If the dimension of the degree d part of the ideal representing the specialized scheme is the expected one, then, by semi-continuity, the Hilbert function of the not specialized scheme in degree d is the expected one.

1. Let $R_1,\dots,R_7\in\mathbb{P}^2$ be seven points chosen on the cubics we will enumerate below and let $\mathcal{Y}'=\mathcal{Y}+\sum_{i=1}^{7}R_i=\sum_{i=1}^{7}(10,11)P_i+\sum_{i=1}^{7}R_i$. Since $\exp(H(\mathcal{Y},27))=H(X,27)+2\cdot 7=359$ and $N+1=\binom{27+2}{2}=406$, if we prove that $(I_{\mathcal{Y}'})_{27}=(0)$ we will have that $\dim(I_{\mathcal{Y}})_{27}=7$ and $H(\mathcal{Y},27)=\exp(H(\mathcal{Y},27))$.

   Let

   - $C_i^{j,k}$ be the cubic passing through $P_1,\dots,P_7$, having in P_i a double point and such that $T_{P_j}(C_i^{j,k})=l_j$, $T_{P_k}(C_i^{j,k})=l_k$ (where l_j and l_k are the lines appearing in the definition of the schemes $(k+1,k+2)P_j$ and $(k+1,k+2)P_k$);
   - $C_{\underline{m},i}^{j,k}$ be the cubic passing through $P_1,\dots,P_7,R_m$, having in P_i a double point and such that $T_{P_j}(C_i^{j,k})=l_j$, $T_{P_k}(C_i^{j,k})=l_k$;
   - $C_{\underline{m},i}^{j}$ be the cubic passing through $P_1,\dots,P_7,R_m$ having in P_i a double point and such that $T_{P_j}(C_{\underline{l}i}^{j})=l_j$;
   - $C_{\underline{5,6},7}^{5}$ be the cubic passing through $P_1,\dots,P_7,R_5,R_6$, having in P_7 a double point and such that $T_{P_5}(C_{\underline{5,6},7}^{5})=l_5$;
   - C be the cubic passing through P_3,P_4,P_6,R_7 and having in P_3,P_4,P_6 three double points.

   For the first case we use the degree 27 curve

   $$Q=C_1^{2,3}\cup C_{\underline{1},1}^{2,3}\cup C_2^{1,4}\cup C_{\underline{2},2}^{1,4}\cup C_5^{6,7}\cup C_{\underline{3},5}^{6,7}\cup C_{\underline{4},7}^{5}\cup C_{\underline{5,6},7}^{5}\cup C$$

   that is a fixed component for the curves defined by $(I_{\mathcal{Y}'})_{27}$. By using Lemma 3.4.5 we have that $Res_Q\mathcal{Y}'=\emptyset$ then $\dim((I_{\mathcal{Y}'})_{27})=0$ and we are done.

2. Let $R_1, \dots, R_{14} \in \mathbb{P}^2$ be points on the cubics and conic we will enumerate below and let $\mathcal{Y}' = \mathcal{Y} + \sum_{i=1}^{14} R_i = \sum_{i=1}^{7}(18,19)P_i + \sum_{i=1}^{14} R_i$. Since $\exp(H(\mathcal{Y},48)) = H(X,48) + 2\cdot 7 = 1211$ and $N+1 = \binom{48+2}{2} = 1225$, if we prove that $(I_{\mathcal{Y}'})_{48} = (0)$ we will have that $\dim (I_{\mathcal{Y}})_{48} = 14$ and $H(\mathcal{Y},48) = \exp(H(\mathcal{Y},48))$.

   Let

   - C_i^j be the cubic passing through $P_1, \dots, P_7$, having in P_i a double point and such that $T_{P_j}(C_i^{j,k}) = l_j$ (where l_j is the line appearing in the definition of the scheme $(k+1, k+2)P_j$);
   - $C_{\underline{m},i}^j$ be the cubic passing through $P_1, \dots, P_7, R_m$, having in P_i a double point and such that $T_{P_j}(C_i^{j,k}) = l_j$;
   - $C_{\underline{8,9},7}$ be the cubic passing through $P_1, \dots, P_7, R_8, R_9$ and having in P_7 a double point;
   - $C_{\underline{10,11,12},7}$ be the cubic passing through $P_1, \dots, P_7, R_{10}, R_{11}, R_{12}$ and having in P_7 a double point;
   - C be the conic passing through $P_1, \dots, P_5, R_{13}, R_{14}$;
   - r be the line through P_6, P_7.

   For the this case we use the degree 48 curve

   $$Q = C_1^2 \cup C_{\underline{1},1}^2 \cup C_2^1 \cup C_{\underline{2},2}^1 \cup C_3^4 \cup C_{\underline{3},3}^4 \cup C_4^3 \cup C_{\underline{4},4}^3 \cup C_5^6 \cup C_{\underline{5},5}^6 \cup C_6^5 \cup C_{\underline{6},6}^5 \cup C_{\underline{7},7} \cup C_{\underline{8,9},7} \cup C_{\underline{10,11,12},7} \cup C \cup r$$

   that is a fixed component for the curves defined by $(I_{\mathcal{Y}'})_{48}$. By using Lemma 3.4.5 we have that $Res_Q \mathcal{Y}' = \emptyset$ then $\dim((I_{\mathcal{Y}'})_{48}) = 0$ and we are done.

We have finally proved that $H(\mathcal{Y},d) = H(X,d) + 14$ if we are in the case (3.27). By the first remark of this section we have that $H(X,d) < \exp(H(X,d))$ if $\frac{21(k+1)}{8} \leq d \leq \frac{8(k+1)-2}{3}$, thus, for the same values of d, we have that $H(\mathcal{Y},d) < \deg(\mathcal{Y})$ and this ends the proof. $\square$

Proposition 3.4.18. *For $s = 8$ we have:*

$$H(\mathcal{Y}, d) = \begin{cases} k = 1: & \begin{cases} N+1 & \text{if } d \leq 7, \\ H(X,d) + 16 = \deg(Y) & \text{if } d \geq 8, \end{cases} \\ k = 2: & \begin{cases} N+1 & \text{if } d \leq 9, \\ H(X,d) + 16 = \deg(Y) & \text{if } d \geq 10, \end{cases} \\ \left.\begin{array}{l} 3 \leq k \leq 7 \\ k = 12 \\ k \equiv 0,5,6,11 \ (mod\ 17) \end{array}\right\} & \begin{cases} N+1 & \text{if } d \leq \left\lceil \frac{48(k+1)}{17} \right\rceil \\ H(X,d) + 16 < \deg(Y) & \text{if } \left\lceil \frac{48(k+1)}{17} \right\rceil + 1 \leq d \leq \left\lfloor \frac{17k+15}{6} \right\rfloor, \\ H(X,d) + 16 = \deg(Y) & \text{if } d \geq max\left\{ \left\lceil \frac{48(k+1)}{17} \right\rceil, \left\lfloor \frac{17k+15}{6} \right\rfloor \right\} + 1, \end{cases} \\ \left.\begin{array}{l} k \geq 8 \\ k \neq 12 \\ k \not\equiv 0,5,6,11 \ (mod\ 17) \end{array}\right\} & \begin{cases} N+1 & \text{if } d \leq \left\lceil \frac{48(k+1)}{17} \right\rceil - 1, \\ H(X,d) + 16 < \deg(Y) & \text{if } \left\lceil \frac{48(k+1)}{17} \right\rceil \leq d \leq \left\lfloor \frac{17k+15}{6} \right\rfloor, \\ H(X,d) + 16 = \deg(Y) & \text{if } d \geq max\left\{ \left\lceil \frac{48(k+1)}{17} \right\rceil, \left\lfloor \frac{17k+15}{6} \right\rfloor \right. + 1 \end{cases} \end{cases}$$

Proof. For $k = 1, 2$ see [**Ba**] and [**BF1**].

For $k \geq 3$ consider $P_1, \ldots, P_8 \in \mathbb{P}^2$ eight generic points. Let S_i be the curve of degree 6 that is double at seven generic points $P_1, \ldots, \hat{P}_i, \ldots, P_8$ and triple at P_i. If $17(k+1) > 6d$ then the eight sestics S_i are fixed components with multiplicity $\nu = 17(k+1) - 6d$, then $\dim((I_{\mathcal{Y}})_d) = \dim\left(\left(I_{\mathcal{Y} - \sum_{i=1}^{8} \nu S_i}\right)_{(d-48\nu)}\right)$. The scheme

$$\mathcal{Y}' = \mathcal{Y} - \sum_{i=1}^{8} \nu S_i \tag{3.28}$$

is the union of $(k+1-17\nu, k+2-17\nu)$-points.

Fix the notation:

$\varphi_{i,j,k}$ is the Cremona Transformation that acts as an isomorphism on $\mathbb{P}^2 \setminus \{P_i, P_j, P_k\}$.

Let now φ be the composition of the following Cremona transformations:

$$\varphi := \varphi_{3,6,8} \circ \varphi_{3,4,5} \circ \varphi_{1,2,8} \circ \varphi_{5,6,7} \circ \varphi_{2,3,4} \circ \varphi_{1,7,8} \circ \varphi_{4,5,6} \circ \varphi_{1,2,3}. \tag{3.29}$$

Suppose to apply φ to $\left(I_{\sum_{i=1}^{8}(k+1,k+2)P_i}\right)_d$ (with an abuse of notation we keep calling P_i the points after the transformation):

$$\varphi\left(\left(I_{\sum_{i=1}^{8}(k+1,k+2)P_i}\right)_d\right) = \left(I_{\sum_{i=1}^{8}(6d-17k-17)P_i + \sum_{i=1}^{16} Q_i}\right)_{(17d-48k-48)} \tag{3.30}$$

where:
Q_1, Q_2 belong to a conic through P_1, P_2,
Q_3, Q_4 belong to a conic through P_1, P_3,
Q_5, Q_6 belong to a conic through P_2, P_3,
Q_7, Q_8 belong to a conic through P_1, P_7,
Q_9, Q_{10} belong to a conic through P_1, P_8,
Q_{11}, Q_{12} belong to a conic through P_4, P_5,
Q_{13}, Q_{14} belong to a conic through P_4, P_6,
Q_{15}, Q_{16} belong to a conic through P_5, P_6.

Let us first consider $d = \left\lceil \frac{48(k+1)}{17} \right\rceil$ if $k \leq 7$, $k = 12$, $k \equiv 0, 5, 6, 11 \pmod{17}$, and $d = \left\lceil \frac{48(k+1)}{17} \right\rceil - 1$ if $k \geq 8$, $k \neq 12$, $k \not\equiv 0, 5, 6, 11 \pmod{17}$. If we prove that $H(X, d) = N + 1$ for such a d, we will be done for any smaller d.

In the following table we summarize what happens if we apply the map φ defined in (3.29) to the degree $(d - 48\nu)$ part of the ideal representing the scheme $\mathcal{Y}'$ obtained in (3.28). When in the last column we write

$$L(d', m') \tag{3.31}$$

we mean that we have, after having applied the map φ to $(I_{\mathcal{Y}'})_{(d-48\nu)}$, a scheme of type (3.30) with $d' = 17d - 48k - 48$ and $m' = 6d - 17k - 17$. The values $r, q \in \mathbb{N}$ are defined by

$$k + 1 = 17q + r. \tag{3.32}$$

r	d	ν	$\mathcal{Y} - \sum_{i=1}^{8} \nu S_i$	$d - 48\nu$	$L(d', m')$
0	$48q$	q	$\sum_{i=1}^{8} P_i$	0	/
1	$48q+3$	$q-1$	$\sum_{i=1}^{8}(18,19)P_i$	51	$L(3,0)$
2	$48q+5$	$q+4$	$\sum_{i=1}^{8}(-66,-65)P_i$	-187	/
3	$48q+8$	$q+3$	$\sum_{i=1}^{8}(-48,-47)P_i$	-136	/
4	$48q+11$	$q+2$	$\sum_{i=1}^{8}(-30,-29)P_i$	-85	/
5	$48q+14$	$q+1$	$\sum_{i=1}^{8}(-12,-11)P_i$	-34	/
6	$48q+17$	q	$\sum_{i=1}^{8}(6,7)P_i$	17	$L(1,0)$
7	$48q+20$	$q-1$	$\sum_{i=1}^{8}(24,25)P_i$	68	$L(4,0)$
8	$48q+22$	$q+4$	$\sum_{i=1}^{8}(-60,-59)P_i$	-170	/
9	$48q+25$	$q+3$	$\sum_{i=1}^{8}(-42,-41)P_i$	-119	/
10	$48q+28$	$q+2$	$\sum_{i=1}^{8}(-24,-23)P_i$	-68	/
11	$48q+31$	$q+1$	$\sum_{i=1}^{8}(-6,-5)P_i$	-17	/
12	$48q+34$	q	$\sum_{i=1}^{8}(12,13)P_i$	34	$L(2,0)$
13	$48q+36$	$q+5$	$\sum_{i=1}^{8}(-72,-71)P_i$	-204	/
14	$48q+39$	$q+4$	$\sum_{i=1}^{8}(-54,-53)P_i$	-153	/
15	$48q+42$	$q+3$	$\sum_{i=1}^{8}(-36,-35)P_i$	-102	/
16	$48q+45$	$q+2$	$\sum_{i=1}^{8}(-18,-17)P_i$	-51	/

Since in the table above ν has to be positive, we are not considering all the cases where $\nu \leq 0$; we enumerate them in the following table (now the notation $L(\alpha, \beta)$ denote the degree $\alpha = (17d - 48k - 48)$ part of the ideal representing - as in (3.30) - the scheme $\sum_{i=1}^{8} \beta P_i + \sum_{i=1}^{16} Q_i$, with $\beta = 6d - 17k - 17$, obtained by applying the map φ, defined in (3.29), to $(I_{\mathcal{Y}})_d$):

r	q	$\mathcal{Y}$	d	$L(\alpha, \beta)$
1	0	$\sum_{i=1}^{8}(1,2)$	3	Solved in [**CGG2**]
1	1	$\sum_{i=1}^{8}(18,19)$	51	$L(3,0)$
6	0	$\sum_{i=1}^{8}(6,7)$	17	$L(1,0)$
7	0	$\sum_{i=1}^{8}(7,8)$	20	$L(4,1)$
7	1	$\sum_{i=1}^{8}(24,25)$	68	$L(4,0)$
12	0	$\sum_{i=1}^{8}(12,13)$	34	$L(2,0)$
13	0	$\sum_{i=1}^{8}(13,14)$	37	$L(5,1)$

The use of Cremona transformations allows us to study the degree α part of the ideals representing only two schemes: $L(\alpha, 0)$ and $L(\alpha, 1)$; Lemma 3.4.8 allows us to check only the two following cases (the ones with higher values of α):

1. $\left(I_{\sum_{i=1}^{8}(13,14)P_i}\right)_{37}$ that corresponds to $L(5,1) = \left(I_{\sum_{i=1}^{8}P_i+\sum_{i=1}^{16}Q_i}\right)_5$, then $\dim\left(\left(I_{\sum_{i=1}^{8}(13,14)P_i}\right)_{37}\right) = \dim\left(\left(I_{\sum_{i=1}^{8}P_i+\sum_{i=1}^{16}Q_i}\right)_5\right) = 0$ as expected;

2. $\left(I_{\sum_{i=1}^{8}(24,25)}\right)_{68}$ that corresponds to $L(4,0) = \left(I_{\sum_{i=1}^{16}Q_i}\right)_4$, then $\dim\left(\left(I_{\sum_{i=1}^{8}(24,25)P_i}\right)_{68}\right) = \dim\left(\left(I_{\sum_{i=1}^{16}Q_i}\right)_4\right) = 0$ as expected.

This ends the proof for the cases $d \leq \left\lceil \frac{48(k+1)}{17} \right\rceil$ if $k \leq 7$, $k = 12$, $k \equiv 0,5,6,11 \pmod{17}$, and $d \leq \left\lceil \frac{48(k+1)}{17} \right\rceil - 1$ if $k \geq 8$, $k \neq 12$, $k \not\equiv 0,5,6,11 \pmod{17}$.

Consider now the remaining cases.

With the notation (3.28), (3.29), (3.30), (3.31) and (3.32) we construct the following tables as we did in the previous case.

r	d	ν	$\mathcal{Y} - \sum_{i=1}^{8}\nu S_i$	$d - 48\nu$	$L(d',m')$
0	$48q+1$	$q-6$	$\sum_{i=1}^{8}(102,103)P_i$	289	$L(17,0)$
1	$48q+4$	$q-7$	$\sum_{i=1}^{8}(120,121)P_i$	340	$L(20,0)$
2	$48q+6$	$q-2$	$\sum_{i=1}^{8}(36,37)P_i$	102	$L(6,0)$
3	$48q+9$	$q-3$	$\sum_{i=1}^{8}(54,55)P_i$	153	$L(9,0)$
4	$48q+12$	$q-4$	$\sum_{i=1}^{8}(72,73)P_i$	204	$L(12,0)$
5	$48q+15$	$q-5$	$\sum_{i=1}^{8}(90,91)P_i$	255	$L(15,0)$
6	$48q+18$	$q-6$	$\sum_{i=1}^{8}(108,109)P_i$	306	$L(18,0)$
7	$48q+21$	$q-7$	$\sum_{i=1}^{8}(126,127)P_i$	357	$L(21,0)$
8	$48q+83$	$q-2$	$\sum_{i=1}^{8}(42,43)P_i$	119	$L(7,0)$
9	$48q+26$	$q-3$	$\sum_{i=1}^{8}(60,61)P_i$	170	$L(10,0)$
10	$48q+29$	$q-4$	$\sum_{i=1}^{8}(78,79)P_i$	221	$L(13,0)$
11	$48q+32$	$q-5$	$\sum_{i=1}^{8}(96,97)P_i$	272	$L(16,0)$
12	$48q+35$	$q-6$	$\sum_{i=1}^{8}(114,115)P_i$	323	$L(19,0)$
13	$48q+37$	$q-1$	$\sum_{i=1}^{8}(30,31)P_i$	85	$L(22,0)$
14	$48q+40$	$q-2$	$\sum_{i=1}^{8}(48,49)P_i$	136	$L(5,0)$
15	$48q+43$	$q-3$	$\sum_{i=1}^{8}(66,67)P_i$	187	$L(11,0)$
16	$48q+46$	$q-4$	$\sum_{i=1}^{8}(84,85)P_i$	238	$L(14,0)$

Since in the table above ν has to be positive, we are not considering all the cases where $\nu \leq 0$ that we enumerate in the following tables:

$r=0$	q	$\sum_{i=1}^{8}(k+1,k+1)P_i$	d	$L(\alpha,\beta)$
	1	$\sum_{i=1}^{8}(17,18)P_i$	49	$L(17,5)$
	2	$\sum_{i=1}^{8}(34,35)P_i$	97	$L(17,4)$
	3	$\sum_{i=1}^{8}(51,52)P_i$	145	$L(17,3)$
	4	$\sum_{i=1}^{8}(68,69)P_i$	193	$L(17,2)$
	5	$\sum_{i=1}^{8}(85,86)P_i$	241	$L(17,1)$
	6	$\sum_{i=1}^{8}(102,103)P_i$	289	$L(17,0)$

$r=1$	q	$\sum_{i=1}^{8}(k+1,k+1)P_i$	d	$L(\alpha,\beta)$
	1	$\sum_{i=1}^{8}(18,19)P_i$	52	$L(20,6)$
	2	$\sum_{i=1}^{8}(35,36)P_i$	100	$L(20,5)$
	3	$\sum_{i=1}^{8}(52,53)P_i$	148	$L(20,4)$
	4	$\sum_{i=1}^{8}(69,70)P_i$	196	$L(20,3)$
	5	$\sum_{i=1}^{8}(86,87)P_i$	244	$L(20,2)$
	6	$\sum_{i=1}^{8}(103,104)P_i$	292	$L(20,1)$
	7	$\sum_{i=1}^{8}(120,121)P_i$	340	$L(20,0)$

$r=2$	q	$\sum_{i=1}^{8}(k+1,k+1)P_i$	d	$L(\alpha,\beta)$
	1	$\sum_{i=1}^{8}(19,20)P_i$	54	$L(6,1)$
	2	$\sum_{i=1}^{8}(36,37)P_i$	102	$L(6,0)$

$r=3$	q	$\sum_{i=1}^{8}(k+1,k+1)P_i$	d	$L(\alpha,\beta)$
	1	$\sum_{i=1}^{8}(20,21)P_i$	57	$L(9,2)$
	2	$\sum_{i=1}^{8}(37,38)P_i$	105	$L(9,1)$
	3	$\sum_{i=1}^{8}(54,55)P_i$	153	$L(9,0)$

$r=4$	q	$\sum_{i=1}^{8}(k+1,k+1)P_i$	d	$L(\alpha,\beta)$
	0	$\sum_{i=1}^{8}(4,5)P_i$	13	$L(29,10)$
	1	$\sum_{i=1}^{8}(21,22)P_i$	60	$L(12,3)$
	2	$\sum_{i=1}^{8}(38,39)P_i$	108	$L(12,2)$
	3	$\sum_{i=1}^{8}(55,56)P_i$	156	$L(12,1)$
	4	$\sum_{i=1}^{8}(72,73)P_i$	204	$L(12,0)$

$r=5$	q	$\sum_{i=1}^{8}(k+1,k+1)P_i$	d	$L(\alpha,\beta)$
	0	$\sum_{i=1}^{8}(5,6)P_i$	16	$L(32,11)$
	1	$\sum_{i=1}^{8}(22,23)P_i$	63	$L(15,4)$
	2	$\sum_{i=1}^{8}(39,40)P_i$	111	$L(15,3)$
	3	$\sum_{i=1}^{8}(56,57)P_i$	159	$L(25,2)$
	4	$\sum_{i=1}^{8}(73,74)P_i$	207	$L(15,1)$
	5	$\sum_{i=1}^{8}(90,91)P_i$	255	$L(15,0)$

$r=6$	q	$\sum_{i=1}^{8}(k+1,k+1)P_i$	d	$L(\alpha,\beta)$
	0	$\sum_{i=1}^{8}(6,7)P_i$	18	$L(18,6)$
	1	$\sum_{i=1}^{8}(23,24)P_i$	66	$L(18,5)$
	2	$\sum_{i=1}^{8}(40,41)P_i$	114	$L(18,4)$
	3	$\sum_{i=1}^{8}(57,58)P_i$	162	$L(18,3)$
	4	$\sum_{i=1}^{8}(74,75)P_i$	206	$L(18,2)$
	5	$\sum_{i=1}^{8}(91,92)P_i$	258	$L(18,1)$
	6	$\sum_{i=1}^{8}(108,109)P_i$	306	$L(18,0)$

$r=7$	q	$\sum_{i=1}^{8}(k+1,k+1)P_i$	d	$L(\alpha,\beta)$
	0	$\sum_{i=1}^{8}(7,8)P_i$	21	$L(21,7)$
	1	$\sum_{i=1}^{8}(24,25)P_i$	69	$L(21,6)$
	2	$\sum_{i=1}^{8}(41,42)P_i$	117	$L(21,5)$
	3	$\sum_{i=1}^{8}(58,59)P_i$	165	$L(21,4)$
	4	$\sum_{i=1}^{8}(75,76)P_i$	213	$L(21,3)$
	5	$\sum_{i=1}^{8}(91,93)P_i$	261	$L(21,2)$
	6	$\sum_{i=1}^{8}(109,110)P_i$	309	$L(21,1)$
	7	$\sum_{i=1}^{8}(126,127)P_i$	357	$L(21,0)$

$r=8$	q	$\sum_{i=1}^{8}(k+1,k+1)P_i$	d	$L(\alpha,\beta)$
	0	$\sum_{i=1}^{8}(8,9)P_i$	24	$L(24,8)$
	1	$\sum_{i=1}^{8}(25,26)P_i$	71	$L(7,1)$
	2	$\sum_{i=1}^{8}(42,43)P_i$	116	$L(7,0)$

$r=9$	q	$\sum_{i=1}^{8}(k+1,k+1)P_i$	d	$L(\alpha,\beta)$
	0	$\sum_{i=1}^{8}(9,10)P_i$	26	$L(10,3)$
	1	$\sum_{i=1}^{8}(26,27)P_i$	74	$L(10,2)$
	2	$\sum_{i=1}^{8}(43,44)P_i$	122	$L(10,1)$
	3	$\sum_{i=1}^{8}(60,61)P_i$	170	$L(10,0)$

$r=10$	q	$\sum_{i=1}^{8}(k+1,k+1)P_i$	d	$L(\alpha,\beta)$
	0	$\sum_{i=1}^{8}(10,11)P_i$	29	$L(13,4)$
	1	$\sum_{i=1}^{8}(27,28)P_i$	77	$L(13,3)$
	2	$\sum_{i=1}^{8}(44,45)P_i$	125	$L(13,2)$
	3	$\sum_{i=1}^{8}(61,62)P_i$	173	$L(13,1)$
	4	$\sum_{i=1}^{8}(78,79)P_i$	221	$L(13,0)$

$r=11$	q	$\sum_{i=1}^{8}(k+1,k+1)P_i$	d	$L(\alpha,\beta)$
	0	$\sum_{i=1}^{8}(11,12)P_i$	32	$L(16,5)$
	1	$\sum_{i=1}^{8}(28,29)P_i$	80	$L(16,4)$
	2	$\sum_{i=1}^{8}(45,46)P_i$	128	$L(16,3)$
	3	$\sum_{i=1}^{8}(62,63)P_i$	176	$L(16,2)$
	4	$\sum_{i=1}^{8}(79,80)P_i$	224	$L(16,1)$
	5	$\sum_{i=1}^{8}(96,97)P_i$	272	$L(16,0)$

$r=12$	q	$\sum_{i=1}^{8}(k+1,k+1)P_i$	d	$L(\alpha,\beta)$
	0	$\sum_{i=1}^{8}(12,13)P_i$	35	$L(19,6)$
	1	$\sum_{i=1}^{8}(29,30)P_i$	83	$L(19,5)$
	2	$\sum_{i=1}^{8}(46,47)P_i$	131	$L(19,4)$
	3	$\sum_{i=1}^{8}(63,64)P_i$	179	$L(19,3)$
	4	$\sum_{i=1}^{8}(80,81)P_i$	227	$L(19,2)$
	5	$\sum_{i=1}^{8}(97,98)P_i$	275	$L(19,1)$
	6	$\sum_{i=1}^{8}(114,115)P_i$	323	$L(19,0)$

$r=13$	q	$\sum_{i=1}^{8}(k+1,k+1)P_i$	d	$L(\alpha,\beta)$
	0	$\sum_{i=1}^{8}(13,14)P_i$	38	$L(22,7)$
	1	$\sum_{i=1}^{8}(30,31)P_i$	85	$L(22,0)$

$r=14$	q	$\sum_{i=1}^{8}(k+1,k+1)P_i$	d	$L(\alpha,\beta)$
	0	$\sum_{i=1}^{8}(14,15)P_i$	40	$L(8,2)$
	1	$\sum_{i=1}^{8}(31,32)P_i$	88	$L(8,1)$
	2	$\sum_{i=1}^{8}(48,49)P_i$	136	$L(8,0)$

$r=15$	q	$\sum_{i=1}^{8}(k+1,k+1)P_i$	d	$L(\alpha,\beta)$
	0	$\sum_{i=1}^{8}(15,16)P_i$	43	$L(11,3)$
	1	$\sum_{i=1}^{8}(32,33)P_i$	91	$L(11,2)$
	2	$\sum_{i=1}^{8}(49,50)P_i$	139	$L(11,1)$
	3	$\sum_{i=1}^{8}(66,67)P_i$	187	$L(11,0)$

$r=16$	q	$\sum_{i=1}^{8}(k+1,k+1)P_i$	d	$L(\alpha,\beta)$
	0	$\sum_{i=1}^{8}(16,17)P_i$	46	$L(14,4)$
	1	$\sum_{i=1}^{8}(33,34)P_i$	94	$L(14,3)$
	2	$\sum_{i=1}^{8}(50,51)P_i$	142	$L(14,2)$
	3	$\sum_{i=1}^{8}(67,68)P_i$	190	$L(14,1)$
	4	$\sum_{i=1}^{8}(84,85)P_i$	238	$L(14,0)$

The use of Cremona transformations allows us to study only the degree α part of the ideals representing 11 schemes: $L(\alpha,0)$, $L(\alpha,1)$, $L(\alpha,2)$, $L(\alpha,3)$, $L(\alpha,4)$, $L(\alpha,5)$, $L(\alpha,6)$, $L(\alpha,7)$, $L(\alpha,8)$, $L(\alpha,10)$ and $L(\alpha,11)$. Lemma 3.4.8 allows us to verify only the following cases (those for lower values of α), we checked them by direct computations:

1. $\left(I_{\sum_{i=1}^{8}(19,20)P_i}\right)_{54}$ that corresponds to $L(6,1)=\left(I_{\sum_{i=1}^{8}P_i+\sum_{i=1}^{16}Q_i}\right)_6$, then $\dim\left(\left(I_{\sum_{i=1}^{8}(19,20)P_i}\right)_{54}\right)=\dim\left(\left(I_{\sum_{i=1}^{8}P_i+\sum_{i=1}^{16}Q_i}\right)_6\right)=4$ as expected;

2. $\left(I_{\sum_{i=1}^{8}(36,37)P_i}\right)_{102}$ that corresponds to $L(6,0)=\left(I_{\sum_{i=1}^{16}Q_i}\right)_6$ then $\dim\left(\left(I_{\sum_{i=1}^{8}(36,37)P_i}\right)_{102}\right)=\dim\left(\left(I_{\sum_{i=1}^{16}Q_i}\right)_6\right)=12$ as expected;

3. $\left(I_{\sum_{i=1}^{8}(4,5)P_i}\right)_{13}$ that corresponds to $L(29,10)=\left(I_{\sum_{i=1}^{8}10P_i+\sum_{i=1}^{16}Q_i}\right)_{29}$ then $\dim\left(\left(I_{\sum_{i=1}^{8}(4,5)P_i}\right)_{13}\right)=\dim\left(\left(I_{\sum_{i=1}^{8}10P_i+\sum_{i=1}^{16}Q_i}\right)_{29}\right)=9$ as expected;

4. $\left(I_{\sum_{i=1}^{8}(5,6)P_i}\right)_{16}$ that corresponds to $L(32,11) = \left(I_{\sum_{i=1}^{8}11P_i+\sum_{i=1}^{16}Q_i}\right)_{32}$ then $\dim\left(\left(I_{\sum_{i=1}^{8}(5,6)P_i}\right)_{16}\right)$ $\dim\left(\left(I_{\sum_{i=1}^{8}11P_i+\sum_{i=1}^{16}Q_i}\right)_{32}\right) = 17$ as expected;

5. $\left(I_{\sum_{i=1}^{8}(6,7)P_i}\right)_{18}$ that corresponds to $L(18,6) = \left(I_{\sum_{i=1}^{8}6P_i+\sum_{i=1}^{16}Q_i}\right)_{18}$ then $\dim\left(\left(I_{\sum_{i=1}^{8}(6,7)P_i}\right)_{18}\right) =$ $\dim\left(\left(I_{\sum_{i=1}^{8}6P_i+\sum_{i=1}^{16}Q_i}\right)_{18}\right) = 6$ as expected;

6. $\left(I_{\sum_{i=1}^{8}(7,8)P_i}\right)_{21}$ that corresponds to $L(21,7)) = \left(I_{\sum_{i=1}^{8}7P_i+\sum_{i=1}^{16}Q_i}\right)_{21}$ then $\dim\left(\left(I_{\sum_{i=1}^{8}(7,8)P_i}\right)_{21}\right) =$ $\dim\left(\left(I_{\sum_{i=1}^{8}7P_i+\sum_{i=1}^{16}Q_i}\right)_{21}\right) = 13$ as expected;

7. $\left(I_{\sum_{i=1}^{8}(8,9)P_i}\right)_{24}$ that corresponds to $L(24,8) = \left(I_{\sum_{i=1}^{8}8P_i+\sum_{i=1}^{16}Q_i}\right)_{24}$ then $\dim\left(\left(I_{\sum_{i=1}^{8}(8,9)P_i}\right)_{24}\right) =$ $\dim\left(\left(I_{\sum_{i=1}^{8}8P_i+\sum_{i=1}^{16}Q_i}\right)_{24}\right) = 21$ as expected;

8. $\left(I_{\sum_{i=1}^{8}(9,10)P_i}\right)_{26}$ that corresponds to $L(10,3) = \left(I_{\sum_{i=1}^{8}3P_i+\sum_{i=1}^{16}Q_i}\right)_{10}$ then $\dim\left(\left(I_{\sum_{i=1}^{8}(9,10)P_i}\right)_{26}\right)$ $\dim\left(\left(I_{\sum_{i=1}^{8}3P_i+\sum_{i=1}^{16}Q_i}\right)_{10}\right) = 2$ as expected;

9. $\left(I_{\sum_{i=1}^{8}(10,11)P_i}\right)_{29}$ that corresponds to $L(13,4) = \left(I_{\sum_{i=1}^{8}4P_i+\sum_{i=1}^{16}Q_i}\right)_{13}$ then $\dim\left(\left(I_{\sum_{i=1}^{8}(10,11)P_i}\right)_{29}\right)$ $\dim\left(\left(I_{\sum_{i=1}^{8}4P_i+\sum_{i=1}^{16}Q_i}\right)_{13}\right) = 9$ as expected;

10. $\left(I_{\sum_{i=1}^{8}(11,12)P_i}\right)_{32}$ that corresponds to $L(16,5) = \left(I_{\sum_{i=1}^{8}5P_i+\sum_{i=1}^{16}Q_i}\right)_{16}$ then $\dim\left(\left(I_{\sum_{i=1}^{8}(11,12)P_i}\right)_{32}\right)$ $\dim\left(\left(I_{\sum_{i=1}^{8}5P_i+\sum_{i=1}^{16}Q_i}\right)_{16}\right) = 17$ as expected;

11. $\left(I_{\sum_{i=1}^{8}(14,15)P_i}\right)_{40}$ that corresponds to $L(8,2) = \left(I_{\sum_{i=1}^{8}2P_i+\sum_{i=1}^{16}Q_i}\right)_{8}$ then $\dim\left(\left(I_{\sum_{i=1}^{8}(14,15)P_i}\right)_{40}\right)$ $\dim\left(\left(I_{\sum_{i=1}^{8}2P_i+\sum_{i=1}^{16}Q_i}\right)_{8}\right) = 5$ as expected.

We have finally proved that $H(\mathcal{Y},d) = H(X,d)+16$ for $d \geq \left\lceil \frac{48(k+1)}{17} \right\rceil + 1$ and $3 \leq k \leq 7$ and $k = 12$ and $k \equiv 0,5,6,11 \pmod{17}$, and for $d \geq \left\lceil \frac{48(k+1)}{17} \right\rceil$ and $k \geq 8$ and $k \neq 12$ and $k \not\equiv 0,5,6,11 \pmod{17}$. Now the statement of the proposition follows from the first remark of this section that assure that $H(X,d) < \exp(H(X,d))$ if $\frac{48(k+1)}{17} \leq d \leq \frac{17(k+1)}{6}$. $\square$

Proposition 3.4.19. *For $s = 9$ we have:*

$$H(\mathcal{Y}, d) = \begin{cases} k = 1: & \begin{cases} N+1 & \text{if } d \leq 8, \\ H(X,d) + 18 = \deg(Y) & \text{if } d \geq 8, \end{cases} \\ k = 2: & \begin{cases} N+1 & \text{if } d \leq 10, \\ H(X,d) + 18 = \deg(Y) & \text{if } d \geq 11, \end{cases} \\ k = 3: & \begin{cases} N+1 & \text{if } d \leq 13, \\ H(X,d) + 18 = \deg(Y) & \text{if } d \geq 14, \end{cases} \\ k \geq 4: & \begin{cases} N+1 & \text{if } d \leq 3k+3, \\ H(X,d) + 18 = \deg(Y) & \text{if } d \geq 3k+4. \end{cases} \end{cases}$$

Proof. For $k = 1, 2$ the statement is known by [**Ba**] and [**BF1**].

Let $k = 3$, so $\mathcal{Y} = (4,5)P_1 + \cdots + (4,5)P_9$. For $d = 13$, by [**CoCoA**], or by specializing the scheme $\mathcal{Y}$ it is easy to check that $\dim(I_{\mathcal{Y}})_{13} = 0$, hence for $d \leq 13$ the conclusion follows from Lemma 3.4.8.

Now let C be the unique (smooth) cubic curve passing through the support of $\mathcal{Y}$, i.e., through $P_1, \ldots, P_9$. Consider the following exact sequence, where $\mathcal{Y}' = Res_C\mathcal{Y}$:

$$0 \to \mathcal{I}_{\mathcal{Y}'}(d-3) \to \mathcal{I}_{\mathcal{Y}}(d) \to \mathcal{I}_{\mathcal{Y}\cap C, C}(d) \to 0.$$

We have that $\mathcal{I}_{\mathcal{Y}\cap C,C}(d) = \mathcal{O}_C(dH - \mathcal{Y} \cap C)$, where H is a line section of C, and $\deg(\mathcal{O}_C(dH - \mathcal{Y} \cap C)) = 3d - 9(k+1)$.

Let $d = 14$. Since $k = 3$, we have $\deg(\mathcal{O}_C(dH - \mathcal{Y} \cap C)) = 14 \cdot 3 - 4 \cdot 9 = 6$. It follows that $h^1(\mathcal{O}_C(dH - \mathcal{Y} \cap C)) = 0$. Since $\mathcal{Y}' = (3,4)P_1 + \cdots + (3,4)P_9$, from the case $k = 2$ we get $h^1(\mathcal{I}_{\mathcal{Y}'}(d-3)) = h^1(\mathcal{I}_{\mathcal{Y}'}(11)) = 0$. So by the exact sequence above it follows that $h^1(\mathcal{I}_{(4,5)P_1+\cdots+(4,5)P_9}(14)) = 0$, which implies $H(\mathcal{Y}, 14) = \deg Y$. For $d > 14$ the conclusion follows from Lemma 3.4.8.

Let $k \geq 4$.

Now we proceed by induction on k. For $k = 4$, we have $\mathcal{Y} = (5,6)P_1 + \cdots + (5,6)P_9$, and $3k+4 = 16$. By [**CoCoA**], or by specializing the scheme $\mathcal{Y}$ it is easy to check that $\dim(I_{\mathcal{Y}})_{16} = 0$. So, since $N + 1 = \binom{16+2}{2} = 9 \cdot 17 = \deg(\mathcal{Y})$, it follows that $H(\mathcal{Y}, 16) = N + 1 = \deg(\mathcal{Y})$. Hence, by Lemma 3.4.8 it follows that for $d \leq 16$ we have $H(\mathcal{Y}, d) = N+1$, while for $d \geq 16$ we have $H(\mathcal{Y}, d) = \deg(\mathcal{Y})$.

Let $k > 4$. We have:

$$\mathcal{Y} = (k+1, k+2)P_1 + \cdots + (k+1, k+2)P_9$$

$$\mathcal{Y}' = (k, k+1)P_1 + \cdots + (k, k+1)P_9.$$

Since obviously if $d \geq 3k+4$, then $d-3 \geq 3(k-1)+4$, and if $d \leq 3k+3$, then $d-3 \leq 3(k-1)+3$,by induction hypothesis we have $H(\mathcal{Y}', d-3) = N'+1$ for $d-3 \leq 3(k-1)+3$, ($N' = \binom{d-3+2}{2}$), and $H(\mathcal{Y}', d-3) = \deg(\mathcal{Y}')$ for $d-3 \geq 3(k-1)+4$. That is:

$$h^0(\mathcal{I}_{Y'}(d-3)) = 0 \quad \text{for} \quad d-3 \leq 3(k-1)+3,$$

$$h^1(\mathcal{I}_{Y'}(d-3)) = 0 \quad \text{for} \quad d-3 \geq 3(k-1)+4.$$

Moreover, since $\deg(\mathcal{O}_C(dH - \mathcal{Y} \cap C)) = 3d - 9(k+1) \leq 0$ for $d \leq 3k+3$, and $\deg(\mathcal{O}_C(dH - \mathcal{Y} \cap C)) = 3d - 9(k+1) \geq 3$ for $d \geq 3k+4$, we have:

$$h^0(\mathcal{I}_{\mathcal{Y} \cap C, C}(d)) = 0 \quad \text{for} \quad d \leq 3k+3,$$

$$h^1(\mathcal{I}_{\mathcal{Y} \cap C, C}(d)) = 0 \quad \text{for} \quad d \geq 3k+4.$$

So whenever $d \leq 3k+3$, we get $h^0(\mathcal{I}_{Y'}(d-3)) = h^0(\mathcal{I}_{\mathcal{Y} \cap C, C}(d)) = 0$, which by the exact sequence above implies $h^0(\mathcal{I}_Y(d)) = 0$.

When $d \geq 3k+4$, we get $h^1(\mathcal{I}_{Y'}(d-3)) = h^1(\mathcal{I}_{\mathcal{Y} \cap C, C}(d)) = 0$, so by the exact sequence above we have $h^1(\mathcal{I}_Y(d)) = 0$, and we are done. □

With all these partial results we have actually proved the main results of this section:

Theorem 3.4.20. *For $s \leq 9$, then*

$$\dim(\mathrm{Sec}_{s-1}(O_{k,d})) = \min\left\{H(X,d) + 2s,\ \binom{d+2}{2}\right\} - 1$$

except when $s = 2$, $d = k+2$ where $\dim(\mathrm{Sec}_1(O_{k,k+2})) = H(T,d) - 1 = \binom{d+2}{2} - 2 = N - 1$.

Proof. For $s = 1$, since $H(X,d) = \min\left\{\binom{k+2}{2}, \binom{d+2}{2}\right\}$, then the result follows from (3.16).

For $s = 2$ and $d = k+2$, since $H(\mathcal{Y}, d) = H(T, d)$ (see Proposition 3.4.9), by the obvious inequalities $H(\mathcal{Y}, d) \leq H(Y, d) \leq H(T, d)$ we get $H(Y, d) = H(\mathcal{Y}, d) = H(T, d)$ and the conclusion follows from (3.18).

In the other cases by Lemma 3.4.6 and Propositions from 3.4.9 to 3.4.19 we have

$$H(Y,d) = H(\mathcal{Y},d) = \min\{H(X,d) + 2s, N+1\},$$

hence from (3.18) we get the conclusion. □

Corollary 3.4.21. *Let $\delta = \min\{\deg(Y) - 1,\ N\} - \dim(\mathrm{Sec}_{s-1}(O_{k,d}))$ be the defect of $\mathrm{Sec}_{s-1}(O_{k,d})$. If $s \leq 9$, then $\mathrm{Sec}_{s-1}(O_{k,d})$ is defective only in the following cases:*

1. $s=2$, $d=k+2$; *with defect* $\delta=1$.

2. $s=2$, $k\geq 3$, $k+3\leq d\leq 2k$; *with defect* $\delta=\min\left\{\binom{2(k+1)-d}{2};(d-k)^2-4\right\}$.

3. $s=3$, $k\geq 7$, k *odd* , $\left\lceil\frac{3(k+1)}{2}\right\rceil+2\leq d\leq 2k$; *with defect* $\delta=\min\left\{3\binom{2(k+1)-d}{2};\binom{2d-3k-1}{2}-6\right\}$.

4. $s=3$, $k\geq 6$, k *even*, $\left\lceil\frac{3(k+1)}{2}\right\rceil+1\leq d\leq 2k$; *with defect* $\delta=\min\left\{3\binom{2(k+1)-d}{2};\binom{2d-3k-1}{2}-6\right\}$.

5. $s=5$, $k\geq 5$, $2k+4\leq d\leq\left\lfloor\frac{5(k+1)}{2}\right\rfloor-1$; *with defect* $\delta=\min\left\{\binom{5(k+1)-2d}{2};5\binom{d-2k-1}{2}-9\right\}$.

6. $s=6$, $k\equiv 2\ (mod\,5)$, $k\geq 17$, $\left\lceil\frac{12(k+1)}{5}\right\rceil\leq d\leq\left\lfloor\frac{5(k+1)}{2}\right\rfloor-1$, *with defect* $\delta=\min\left\{6\binom{5(k+1)-2d}{2};\binom{5d-12k-10}{2}-12\right\}$.

7. $s=6$, $k\not\equiv 2\ (mod\,5)$, $k\geq\begin{cases}19 & \text{if } k \text{ odd}\\ 24 & \text{if } k \text{ even}\end{cases}$, $\left\lceil\frac{12(k+1)}{5}\right\rceil+1\leq d\leq\left\lfloor\frac{5(k+1)}{2}\right\rfloor-1$, *with defect* $\delta=\min\left\{6\binom{5(k+1)-2d}{2};\binom{5d-12k-10}{2}-12\right\}$.

8. $s=7$, $k\equiv 1,4\ (mod\,8)$, $k\geq\begin{cases}33 & \text{if } k\equiv 1\ (mod\,8)\\ 36 & \text{if } k\equiv 4\ (mod\,8)\end{cases}$, $\left\lceil\frac{21(k+1)}{8}\right\rceil\leq d\leq\left\lfloor\frac{8k}{3}\right\rfloor+2$, *with defect* $\delta=\min\left\{7\binom{8(k+1)-3d}{2},\binom{8d-21k-19}{2}-14\right\}$.

9. $s=7$, $k\not\equiv 1,4\ (mod\,8)$, $k\geq\begin{cases}39 & \text{if } k\equiv 0\ (mod\,3)\\ 43 & \text{if } k\equiv 1\ (mod\,3)\\ 47 & \text{if } k\equiv 2\ (mod\,3)\end{cases}$, $\left\lceil\frac{21(k+1)}{8}\right\rceil+1\leq d\leq\left\lfloor\frac{8k}{3}\right\rfloor+2$, *with defect* $\delta=\min\left\{7\binom{8(k+1)-3d}{2},\binom{8d-21k-19}{2}-14\right\}$.

10. $s=8$, $k\equiv 0,5,6,11\ (mod\,17)$, $k\geq\begin{cases}153 & \text{if } k\equiv 0\ (mod\,17)\\ 141 & \text{if } k\equiv 5\ (mod\,17)\\ 159 & \text{if } k\equiv 6\ (mod\,17)\\ 147 & \text{if } k\equiv 11\ (mod\,17)\end{cases}$, $\left\lceil\frac{48(k+1)}{17}\right\rceil+1\leq d\leq\left\lfloor\frac{17k+15}{6}\right\rfloor$, *with defect* $\delta=\min\left\{8\binom{17(k+1)-6d}{2},\binom{17d-48k-46}{2}-16\right\}$.

11. $s = 8$, $k \not\equiv 0, 5, 6, 11 \ (mod\, 17)$, $k \geq \begin{cases} 36 & if\ k \equiv 0 \ (mod\, 6) \\ 67 & if\ k \equiv 1 \ (mod\, 6) \\ 50 & if\ k \equiv 2 \ (mod\, 6) \\ 33 & if\ k \equiv 3 \ (mod\, 6) \\ 118 & if\ k \equiv 4 \ (mod\, 6) \\ 101 & if\ k \equiv 5 \ (mod\, 6) \end{cases}$, $\left\lceil \frac{48(k+1)}{17} \right\rceil \leq d \leq \left\lfloor \frac{17k+15}{6} \right\rfloor$ *with defect* $\delta = \min\left\{8\binom{17(k+1)-6d}{2}, \binom{17d-48k-46}{2} - 16\right\}$.

Proof. First we observe that:

- $k + 3 \leq 2k$ implies $k \geq 3$;
- if k is odd and $\left\lceil \frac{3(k+1)}{2} \right\rceil + 2 \leq 2k$, then $3(k+1) + 4 \leq 4k$, that is $k \geq 7$;
- while if k is even and $\left\lceil \frac{3(k+1)}{2} \right\rceil + 1 \leq 2k$, then $k \geq 6$;
- from $2k + 4 \leq \left\lfloor \frac{5(k+1)}{2} \right\rfloor - 1$ we get $k \geq 5$;
- for $k \equiv 2 \pmod 5$, it is easy to compute that $\left\lceil \frac{12(k+1)}{5} \right\rceil \leq \left\lfloor \frac{5(k+1)}{2} \right\rfloor - 1$ implies $k \geq 17$;
- while for $k \not\equiv 2 \pmod 5$, if $\left\lceil \frac{12(k+1)}{5} \right\rceil + 1 \leq \left\lfloor \frac{5(k+1)}{2} \right\rfloor - 1$, then $k \geq \begin{cases} 19 & \text{if } k \text{ odd} \\ 24 & \text{if } k \text{ even} \end{cases}$;
- for $k \equiv 1, 4 \pmod 8$, if $\left\lceil \frac{21(k+1)}{8} \right\rceil \leq d \leq \left\lfloor \frac{8k}{3} \right\rfloor + 2$, then $k \geq \begin{cases} 33 & \text{if } k \equiv 1 \pmod 8 \\ 36 & \text{if } k \equiv 4 \pmod 8 \end{cases}$;
- while for $k \not\equiv 1, 4 \pmod 8$, if $\left\lceil \frac{21(k+1)}{8} \right\rceil + 1 \leq d \leq \left\lfloor \frac{8k}{3} \right\rfloor + 2$, then $k \geq \begin{cases} 39 & \text{if } k \equiv 0 \pmod 3 \\ 43 & \text{if } k \equiv 1 \pmod 3 \\ 47 & \text{if } k \equiv 2 \pmod 3 \end{cases}$;
- for $k \equiv 0, 5, 6, 11 \pmod{17}$, if $\left\lceil \frac{48(k+1)}{17} \right\rceil + 1 \leq d \leq \left\lfloor \frac{17k+15}{6} \right\rfloor$, then $k \geq \begin{cases} 153 & \text{if } k \equiv 0 \pmod{17} \\ 141 & \text{if } k \equiv 5 \pmod{17} \\ 159 & \text{if } k \equiv 6 \pmod{17} \\ 147 & \text{if } k \equiv 11 \pmod{17} \end{cases}$

- while for $k \not\equiv 0, 5, 6, 11 \pmod{17}$, if $\left\lceil \frac{48(k+1)}{17} \right\rceil \leq d \leq \left\lfloor \frac{17k+15}{6} \right\rfloor$,

  then $k \geq \begin{cases} 36 & \text{if } k \equiv 0 \pmod 6 \\ 67 & \text{if } k \equiv 1 \pmod 6 \\ 50 & \text{if } k \equiv 2 \pmod 6 \\ 33 & \text{if } k \equiv 3 \pmod 6 \\ 118 & \text{if } k \equiv 4 \pmod 6 \\ 101 & \text{if } k \equiv 5 \pmod 6 \end{cases}$.

From what we have seen above, by (3.16), and Propositions form 3.4.9 to 3.4.19, we get that $\mathrm{Sec}_{s-1}(O_{k,d})$ is defective only in the cases from 1. to 11., and, except for $s = 2$ and $d = k + 2$, we know that $H(Y, d) = H(X, d) + 2s$.

For $s = 2$ and $d = k + 2$, since $\dim(\mathrm{Sec}_1(O_{k,k+2})) = N - 1$, while the expected dimension is N, we have $\delta = 1$.
In the other cases we have:

$$\delta = \min\{\deg(Y) - 1,\ N\} - \dim(\mathrm{Sec}_{s-1}(O_{k,d})) = \min\{\deg(Y) - 1,\ N\} - H(Y, d) + 1$$

$$= \min\{\deg(Y) - H(X, d) - 2s,\ N + 1 - H(X, d) - 2s\} = \min\{\deg(X) - H(X, d),\ \dim(I_X)_d - 2s\}.$$

For $s = 2$, $k \geq 3$ and $k + 2 \leq d \leq 2k$, computing the dimension of $(I_X)_d$ by removing the line P_1P_2 $(2(k + 1) - d)$ times, we get:

$$\dim(I_X)_d = \dim(I_{X'})_{2(d-k-1)} = \binom{2(d-k-1)+2}{2} - 2\binom{d-k}{2} = (d-k)^2,$$

where $X' = (d - k - 1)P_1 + (d - k - 1)P_2$, hence

$$\deg(X) - H(X, d) = 2\binom{k+2}{2} - \binom{d+2}{2} + (d-k)^2 = \binom{2(k+1)-d}{2},$$

$$\delta = \min\left\{\binom{2(k+1)-d}{2}; (d-k)^2 - 4\right\}.$$

In cases 3. and 4., computing the dimension of $(I_X)_d$ by cutting off the three lines P_1P_2, P_1P_3, P_2P_3, $2(k + 1) - d$ times each, we have:

$$\dim(I_X)_d = \dim(I_{X'})_{d-3(2k+2-d)} = \dim(I_{X'})_{2(2d-3k-3)} =$$

$$= \binom{2(2d-3k-3)+2}{2} - 3\binom{2d-3k-2}{2} = \binom{2d-3k-1}{2},$$

where $X' = \sum_{i=1}^{3}(k+1-2(2k+2-d))P_i = \sum_{i=1}^{3}(2d-3k-3)P_i$, and from here we easily get:

$$\deg(X) - H(X,d) = 3\binom{k+2}{2} - \binom{d+2}{2} + \binom{2d-3k-1}{2} = 3\binom{2(k+1)-d}{2},$$

$$\delta = \min\left\{3\binom{2(k+1)-d}{2}; \binom{2d-3k-1}{2} - 6\right\}.$$

For $s = 5$ computing the dimension of $(I_X)_d$ by cutting off the three lines P_1P_2, P_1P_3, P_2P_3, $2(k+1)-d$ times each, we have:

$$\dim(I_X)_d = \dim(I_{X'})_{d-2(5k+5-2d)} = \dim(I_{X'})_{5(d-2k-2)} =$$

$$= \binom{5(d-2k-2)+2}{2} - 5\binom{2d-4k-3}{2} = 5\binom{d-2k-1}{2} + 1,$$

where $X' = \sum_{i=1}^{5}(k+1-(5k+5-2d))P_i = \sum_{i=1}^{5}(2d-4k-4)P_i$, and from here we get:

$$\deg(X) - H(X,d) = 5\binom{k+2}{2} - \binom{d+2}{2} + 5\binom{d-2k-1}{2} + 1 = \binom{5(k+1)-2d}{2},$$

$$\delta = \min\left\{\binom{5(k+1)-2d}{2}; 5\binom{d-2k-1}{2} - 9\right\}.$$

For $s = 6$, calculating the dimension of $(I_X)_d$ by removing every conic C_i (see the proof of Proposition 3.4.13) $(5(k+1)-2d)$ times, we get

$$\dim(I_X)_d = \dim(I_{X'})_{d-12(5k+5-2d)} = \dim(I_{X'})_{25d-60k-60} =$$

$$= \binom{25d-60k-60+2}{2} - 6\binom{10d-24k-24+1}{2} = \binom{5d-12k-10}{2},$$

where $X' = \sum_{i=1}^{6}(k+1-5(5k+5-2d))P_i = \sum_{i=1}^{6}(10d-24k-24)P_i$, and from here we get:

$$\deg(X) - H(X,d) = 6\binom{k+2}{2} - \binom{d+2}{2} + \binom{5d-12k-10}{2} = 6\binom{5(k+1)-2d}{2},$$

$$\delta = \min\left\{6\binom{5(k+1)-2d}{2}; \binom{5d-12k-10}{2} - 12\right\}.$$

For $s = 7$, computing the dimension of $(I_X)_d$ by cutting off the fixed locus (that is the union of the seven cubics through $P_1, \ldots, 2P_i, \ldots, P_7$ with multiplicity $8(k+1)-3d$) we get:

$$\dim(I_X)_d = \dim(I_{X'})_{d-21(8k+8-3d)} = \dim(I_{X'})_{64d-168k-168} =$$

$$= \binom{64d-168k-168+2}{2} - 7\binom{24d-63k-63+1}{2} = \binom{8d-21k-19}{2}$$

where $X' = \sum_{i=1}^{7}(k+1-8(8(k+1)-3d))P_i = \sum_{i=1}^{7}(24d-63k-63)$. Then

$$\deg(X) - H(X,d) = 7\binom{k+2}{2} - \binom{d+2}{2} + \binom{8d-21k-19}{2} = 7\binom{8(k+1)-3d}{2}$$

and

$$\delta = \min\left\{7\binom{8(k+1)-3d}{2}, \binom{8d-21k-19}{2} - 14\right\}.$$

Finally, for $s = 8$, computing the dimension of $(I_X)_d$ by removing the fixed locus (that is the union of eight curves of degree 6 that are triple in one point and double in the other seven remaining points) we get:

$$\dim(I_X)_d = \dim(I_{X'})_{d-48(17(k+1)-6d)} = \dim(I_{X'})_{289d-816k-816} =$$

$$= \binom{289d-816k-816+2}{2} - 8\binom{102d-288k-288+1}{2} = \binom{17d-48k-46}{2}$$

where $X' = \sum_{i=1}^{8}(k+1-17(17(k+1)-6d))P_i = \sum_{i=1}^{8}(102d-288k-288)P_i$. Then

$$\deg(X) - H(X,d) = 8\binom{k+2}{2} - \binom{d+2}{2} + \binom{17d-48k-46}{2} = 8\binom{17(k+1)-6d}{2}$$

and

$$\delta = \min\left\{8\binom{17(k+1)-6d}{2}, \binom{17d-48k-46}{2} - 16\right\}.$$

□

E.Ballico and C.Fontanari in [**BF2**] give partial results about the regularity of $\mathrm{Sec}_{s-1}(O_{k,d})$ for $2 \leq s \leq 8$. Our Corollary 3.4.21, for $s \leq 9$, improves the results of [**BF2**] and gives a complete classification of all the defective cases.

Remark: We wish to notice that there are no defective cases for $s = 4$ or $s = 9$.

In case $s = 2$, $d = k+2$ defectivity is forced by the defectivity of T, in fact, since $Y \subset T$ implies that $H(Y, k+2) \leq H(T, k+2)$, and since $H(T, k+2) = N < \exp(H(Y, k+2)) = N+1$, it follows that $H(Y, k+2) < \exp(H(Y, k+2))$. In the other cases defectivity of $\mathrm{Sec}_{s-1}(O_{k,d})$ is forced by the defectivity of X.

Remark: At the light of this last Remark and the results of L.Evain in [**Ev**], we would like to conjecture that if s is a square, then $\mathrm{Sec}_{s-1}(O_{k,d})$ is regular in any degree d.

Anyway by the results of L.Evain, and 3.3.6, we easily deduce a partial result about the regularity of $\mathrm{Sec}_{s-1}(O_{k,d})$:

Corollary 3.4.22. *If s is a square, and $N+1 \leq \deg(X)$ or $N+1 \geq \deg(T)$, then $\dim(\mathrm{Sec}_{s-1}(O_{k,d}))$ is as expected.*

In fact if s is a square, by [**Ev**] we know that X and T have maximal Hilbert function. Hence if $N+1 \leq \deg(X)$, then $\dim(I_X)_d = 0$, and if $N+1 \geq \deg(T)$, then $H(T,d) = \deg(T)$. Since $X \subset Y \subset T$, it follows that if $\dim(I_X)_d = 0$, then $H(Y,d) = N+1$, and if $H(T,d) = \deg(T)$, then $H(Y,d) = \deg(Y)$, and now the conclusion follows from the first Remark of this section.

Chapter 4

Secant varieties to the Split varieties

In this chapter we study the second question presented in Section 2.6.1:

> "which is the least integer s such that the following form is canonical:
>
> $$L_1^{(1)} \cdots L_d^{(1)} + \cdots + L_1^{(s)} \cdots L_d^{(s)}?"$$

where $L_i^{(j)} \in K[x_0, \ldots, x_n]_1 \; \forall \; i = 1, \ldots, d$ and $j = 1, \ldots, s$.

We have already observed that this problem is equivalent to the following:

> "which is the minimum integer s such that the $(s-1)$-secant variety to the Split variety $\mathrm{Split}_d(\mathbb{P}^n)$ fills up $\mathbb{P}^{\binom{n+d}{d}-1}$?"

where the Split variety $\mathrm{Split}_d(\mathbb{P}^n)$ is the variety that parameterizes forms of degree d that split into product of d linear forms of $S = K[x_0, \ldots, x_n]$. We have defined it as the image of the map (2.15).

The dimension of $\mathrm{Split}_d(\mathbb{P}^n)$ is nd, hence the expected dimension of its $(s-1)$-secant variety is:

$$\mathrm{expdim}(\mathrm{Sec}_{s-1}(\mathrm{Split}_d(\mathbb{P}^n))) = \min\left\{\binom{n+d}{d} - 1, snd + s - 1\right\}.$$

4.1 What we can do with Inverse Systems

In the section 2.6.1 we have shown why the Eherenborg conjecture 2.6.4 (see [**Eh**]) does not work. The counterexample we produced was:

"the typical rank of the Grassmannian $\mathbb{G}(3,6)$ is 4, but the typical rank of $\mathrm{Split}_4(\mathbb{P}^3)$ is 3."

The typical rank of $\mathbb{G}(3,6)$ is well known (see for example [**CGG4**]), but for the typical rank of $\mathrm{Split}_4(\mathbb{P}^3)$ we made computations with [**CoCoA**]. The method we used for these computations uses Inverse Systems.
Consider the map

$$\begin{aligned} \phi : \underbrace{S_1 \times \cdots \times S_1}_{d} &\rightarrow S_d, \\ (L_1, \ldots, L_d) &\mapsto L_1 \cdots L_d. \end{aligned}$$

If $A_1, \ldots, A_d \in S_1$, then $\lim_{t\to 0} \frac{d}{dt}(\phi(L_1+tA_1, \ldots, L_d+tA_d)) = \sum_{i=1}^{d} L_1 \cdots L_{i-1} A_i L_{i+1} \cdots L_d$. Therefore the affine cone over the tangent space $T_P(\mathrm{Split}_d(\mathbb{P}^n))$ at a regular point $P = [L_1 \cdots L_d] \in \mathrm{Split}_d(\mathbb{P}^n)$ is spanned by:

$$\hat{T}_P(\mathrm{Split}_d(\mathbb{P}^n)) =< S_1 L_2 \cdots L_d, \ldots, S_1 L_1 \cdots L_{i-1} L_{i+1} \cdots L_d, \ldots, S_1 L_1 \cdots L_{d-1} > . \tag{4.1}$$

By using Terracini's lemma we can write the affine cone W over the tangent space to $\mathrm{Sec}_{s-1}(\mathrm{Split}_d(\mathbb{P}^n))$ at a regular point Q: let $P_1, \ldots, P_s \in \mathrm{Split}_d(\mathbb{P}^n)$ such that $P_i = [L_1^{(i)} \cdots L_d^{(i)}] \in \mathbb{P}(S_d)$, let $Q \in < P_1, \ldots . P_s >$, then

$$\begin{aligned} W =< & S_1 L_2^{(1)} \cdots L_d^{(1)}, \ldots, S_1 L_1^{(1)} \cdots L_{i-1}^{(1)} L_{i+1}^{(1)} \cdots L_d^{(1)}, \ldots, S_1 L_1^{(1)} \cdots L_{d-1}^{(1)}; \ldots \\ & \ldots; S_1 L_2^{(s)} \cdots L_d^{(s)}, \ldots, S_1 L_1^{(s)} \cdots L_{i-1}^{(s)} L_{i+1}^{(s)} \cdots L_d^{(s)}, \ldots, S_1 L_1^{(s)} \cdots L_{d-1}^{(s)} > . \end{aligned} \tag{4.2}$$

What can be done in the case $\mathrm{Split}_4(\mathbb{P}^3)$ is:

- choose twelve forms $L_1^{(1)}, \ldots, L_4^{(1)}, L_1^{(2)}, \ldots, L_4^{(2)}, L_1^{(3)}, \ldots, L_4^{(3)} \in K[x_0, x_1, x_2, x_3]_1$,
- explicitly write down the particular tangent space to $\mathrm{Sec}_2(\mathrm{Split}_4(\mathbb{P}^3))$ that we obtain with this (particular) choice of linear forms,
- make computation and find out that the dimension is actually the expected one (by using [**CoCoA**]).

It is clear that this easy check works if the particular tangent space we choose, via the choice of the linear forms, has the expected dimension (that is what happens for $\mathrm{Sec}_2(\mathrm{Split}_4(\mathbb{P}^3))$). This method works only if we have to verify few particular examples and if we find that $\mathrm{Sec}_{s-1}(\mathrm{Split}_d(\mathbb{P}^n))$ has the expected dimension.

Remark: If $d = 2$ the variety $\mathrm{Split}_2(\mathbb{P}^n)$ parameterizes forms of the type: $L_1 L_2 \in S_2$ and this means that $\mathrm{Split}_2(\mathbb{P}^n)$ is nothing else that the tangential variety to the double Veronese variety $T(\nu_2(\mathbb{P}^n))$ defined in the previous chapter. This case was already studied in [**CGG2**] (see Proposition 3.3): the authors proved the following proposition:

Proposition 4.1.1. *For all $s \geq 2$*

1. *if $n < 2s$, then $\mathrm{Sec}_{s-1}(T(\nu_2(\mathbb{P}^n))) = \mathbb{P}^{\binom{n+2}{2}-1}$ as expected;*
2. *if $n \geq 2s$, then $\dim(\mathrm{Sec}_{s-1}(T(\nu_2(\mathbb{P}^n)))) = s(2n+1) - 2(s^2 - s) - 1$, i.e.:*
   - *if $\binom{n+2}{2} \geq s(2n+1)$, then $\dim(\mathrm{Sec}_{s-1}(T(\nu_2(\mathbb{P}^n)))) = 2(s^2-s) < \mathrm{expdim}(\mathrm{Sec}_{s-1}(T(\nu_2(\mathbb{P}^n))))$,*
   - *if $\binom{n+2}{2} \leq s(2n+1)$, then $\dim(\mathrm{Sec}_{s-1}(T(\nu_2(\mathbb{P}^n)))) = \binom{n-2s+2}{2} < \mathrm{expdim}(\mathrm{Sec}_{s-1}(T(\nu_2(\mathbb{P}^n))))$.*

We want to point out here that the defective cases found in proposition 4.1.1 are the only known defective ones for secant varieties to the Split varieties.

Remark: Since K is an algebraically closed field it is obvious that if $L_1, L_2 \in S_1 = K[x_0, \ldots, x_n]_1$, then there always exist $L', L'' \in S_1$ such that $L_1^2 + L_2^2 = L' \cdot L''$. In terms of varieties parameterizing forms this means that $\mathrm{Sec}_1(\nu_2(\mathbb{P}^n)) = \mathrm{Split}_2(\mathbb{P}^n)$. Therefore

$$\mathrm{Sec}_1(\nu_2(\mathbb{P}^n)) = \mathrm{Split}_2(\mathbb{P}^n) = T(\nu_2(\mathbb{P}^n)).$$

This implies that such secant variety is defective, since its dimension is $2n = \dim(T(\nu_2(\mathbb{P}^n)))$ instead that $2n + 1$.

In general for any d such that $sd \leq n + 1$, since we can choose coordinate so that $L_i = x_i \in S_1$, we can compute the ideal $I \subset R = K[y_0, \ldots, y_d]$ such that $(I^{-1})_d = \hat{T}_Q(\mathrm{Sec}_{s-1}(\mathrm{Split}_d(\mathbb{P}^n)))$.
We present here the case of $d = 3$ in order to show the complexity of the computational problem.

4.1.1 The case of $\mathrm{Split}_3(\mathbb{P}^n)$

If $d = 3$ then the affine cone over the tangent space to $\mathrm{Split}_3(\mathbb{P}^n)$ at a point $P = [L_1 L_2 L_3] \in \mathbb{P}(S_3)$ is

$$\hat{T}_P(\mathrm{Split}_3(\mathbb{P}^n)) =< S_1 L_2 L_3, S_1 L_1 L_3, S_1 L_1 L_2 > .$$

Now if $3 \leq n + 1$ then we can suppose, without loss of generality, that $L_i = x_{i-1}$, for $i = 1, 2, 3$. With those assumptions it is not difficult to verify that the ideal $I_{0,1,2} \subset R = K[y_0, \ldots, y_n]$ such that $(I_{0,1,2}^{-1})_3 = \hat{T}_P(\mathrm{Split}_3(\mathbb{P}^n))$ is

$$I_{0,1,2} = (y_0^3, y_1^3, y_2^3) + (y_0^2, y_1^2, y_2^2)(y_3, \ldots, y_n) + (y_0, y_1, y_2)(y_3, \ldots, y_n)^2 + (y_3, \ldots, y_n)^3. \tag{4.3}$$

(The choice of the name "$I_{0,1,2}$" is motivated by the assumption "$L_i = x_{i-1}$, for $i = 1, 2, 3$".)
The projective scheme associated to I has dimension -1, so its support is the empty set. This fact can be verified by observing that

$$\sqrt{I_{0,1,2}} = (y_0, ..., y_n).$$

Consider now the $(s-1)$-secant variety to $\mathrm{Split}_3(\mathbb{P}^n)$. Let $P_i = [L_1^{(i)} L_2^{(i)} L_3^{(i)}] \in \mathrm{Split}_3(\mathbb{P}^n)$, for $i = 1, \ldots, s$, then the affine tangent space W to $\mathrm{Sec}_{s-1}(\mathrm{Split}_d(\mathbb{P}^n))$ at $Q \in < P_1, \ldots, P_s >$ is (by (4.2), for $d = 3$):

$$W =< S_1 L_2^{(1)} L_3^{(1)}, S_1 L_1^{(1)} L_3^{(1)}, S_1 L_1^{(1)} L_2^{(1)}; \ldots; S_1 L_2^{(s)} L_3^{(s)}, S_1 L_1^{(s)} L_3^{(s)}, S_1 L_1^{(s)} L_2^{(s)} > .$$

Now, if $3s \leq n+1$ we can choose $L_1^{(1)} = x_0, \ldots, L_3^{(s)} = x_{3s-1}$; therefore the ideal $I \subset R = K[y_0, \ldots, y_n]$ such that the degree 3 part of its inverse system is W, can be obtained as the intersection of s ideals of the type (4.3):

$$I = I_{0,1,2} \cap \cdots \cap I_{3s-3,3s-2,3s-1}.$$

It is not difficult to verify that

$$I = (y_0^3, \ldots, y_{3s-1}^3) + \sum_{\substack{i \equiv_3 0 \\ i=0,\ldots,3s-3}} (y_i^2, y_{i+1}^2, y_{i+2}^2)(y_0, \ldots, y_{i-1}, \hat{y}_i, \hat{y}_{i+1}, \hat{y}_{i+2}, y_{i+3}, \ldots, y_n) +$$

$$+ (y_0, \ldots, y_{3s-1})(y_{3s}, \ldots, y_n)^2 +$$

$$\sum_{\substack{i,j,k \equiv_3 0 \\ i \neq j, i \neq k, j \neq k; \\ i,j,k=0,\ldots,3s-3}} (y_i, y_{i+1}, y_{i+2})(y_j, y_{j+1}, y_{j+2})(y_k, y_{k+1}, y_{k+2}) +$$

$$+ \left(\sum_{\substack{i,j \equiv_3 0 \\ i \neq j=0,\ldots,3s-3}} (y_i, y_{i+1}, y_{i+2})(y_j, y_{j+1}, y_{j+2}) \right) (y_{3s}, \ldots, y_n) +$$

$$+ (y_{3s}, \ldots, y_n)^3 + \sum_{\substack{i,j \equiv_3 0 \\ i \neq j=0,\ldots,3s-3}} (y_i y_{i+1}, y_i y_{i+2}, y_{i+1} y_{i+2})(y_j y_{j+1}, y_j y_{j+2}, y_{j+1} y_{j+2}),$$

where $i \equiv_3 0$ means that there exists $m \in \mathbb{Z}$ such that $i = 3m$ and we write $\hat{y}_i$ when the term y_i does not appear.

The Hilbert function $H(R/I, 3)$ of I can be easily computed; it turns out to be:

$$\binom{n+3}{3} - \left(3s + 3s(n-2) + 3s\binom{n-3s+2}{2} + 27\binom{s}{3} + 9(n-3s+1)\binom{s}{2} + \binom{n-3s+3}{3} \right) =$$

$$= 3n + s - 1 = \mathrm{expdim}(\mathrm{Sec}_{s-1}(\mathrm{Split}_3(\mathbb{P}^n))).$$

Since $\dim(\mathrm{Sec}_{s-1}(\mathrm{Split}_3(\mathbb{P}^n))) = H(R/I, 3) - 1$, we have proved the following Proposition:

Proposition 4.1.2. *If $3s \leq n+1$ then $\mathrm{Sec}_{s-1}(\mathrm{Split}_3(\mathbb{P}^n))$ has the expected dimension.*

The general case $\mathrm{Sec}_{s-1}(\mathrm{Split}_d(\mathbb{P}^n))$ for $sd \leq n+1$ can be treated in an analogous way for $d \geq 3$:

- the affine cone over the tangent space to $\mathrm{Split}_d(\mathbb{P}^n)$ at a point $P = [L_1 \cdots L_d] \in \mathrm{Split}_d(\mathbb{P}^n)$ is computed in (4.1);
- we can suppose that $L_i = x_{i-1}$, for $i \geq 1$, since $sd \leq n+1$;
- the ideal $I_{0,\ldots,d-1} \subset R = K[y_0, \ldots, y_n]$ such that $(I^{-1}_{0,\ldots,d-1})_d = W$ defines a scheme whose support is the empty set and it is of the form:

$$
\begin{aligned}
I_{0,\ldots,d-1} = \; & (y_d, \ldots, y_n)^d + \\
& +(y_d, \ldots, y_n)^{d-1}(y_0, \ldots, y_{d-1}) + \\
& +(y_d, \ldots, y_n)^{d-2}(y_0, \ldots, y_{d-1})^2 + \\
& \vdots \\
& +(y_d, \ldots, y_n)^2(y_0, \ldots, y_{d-1})^{d-2} + \\
& \quad +(y_d, \ldots, y_n) \quad \big[(y_0^{d-1}, \ldots, y_{d-1}^{d-1}) + \\
& \qquad\qquad + \textstyle\sum_{i=0}^{d-1}(y_i^{d-2})(y_0, \ldots, \hat{y}_i, \ldots, y_{d-1}) + \\
& \qquad\qquad + \textstyle\sum_{i=0}^{d-1}(y_i^{d-3})(y_0, \ldots, \hat{y}_i, \ldots, y_{d-1})^2 + \\
& \qquad\qquad \vdots \\
& \qquad\qquad + \textstyle\sum_{i=0}^{d-1}(y_i^2)(y_0, \ldots, \hat{y}_i, \ldots, y_{d-1})^{d-3}\big] + \\
& +(y_0^d, \ldots, y_{d-1}^d) + \\
& + \textstyle\sum_{i=0}^{d-1}(y_i^{d-1})(y_0, \ldots, \hat{y}_i, \ldots, y_{d-1}) + \\
& + \textstyle\sum_{i=0}^{d-1}(y_i^{d-2})(y_0, \ldots, \hat{y}_i, \ldots, y_{d-1})^2 + \\
& \vdots \\
& + \textstyle\sum_{i=0}^{d-1}(y_i^3)(y_0, \ldots, \hat{y}_i, \ldots, y_{d-1})^{d-3} + \\
& \quad + \textstyle\sum_{i=0}^{d-1}(y_i)^2 \quad \big[(y_0^{d-2}, \ldots, \hat{y}_i^{d-2}, \ldots, y_{d-1}^{d-2}) + \\
& \qquad\qquad + \textstyle\sum_{j\neq i;\, j=0}^{d-1}(y_i)^{d-3}(y_0, \ldots, \hat{y}_i, \ldots, \hat{y}_j, \ldots, y_{d-1}) + \\
& \qquad\qquad + \textstyle\sum_{j\neq i;\, j=0}^{d-1}(y_i)^{d-4}(y_0, \ldots, \hat{y}_i, \ldots, \hat{y}_j, \ldots, y_{d-1})^2 + \\
& \qquad\qquad \vdots \\
& \qquad\qquad + \textstyle\sum_{j\neq i;\, j=0}^{d-1}(y_i)^2(y_0, \ldots, \hat{y}_i, \ldots, \hat{y}_j, \ldots, y_{d-1})^{d-4}\big];
\end{aligned}
$$

- let $W = \hat{T}_Q(\mathrm{Sec}_{s-1}(\mathrm{Split}_d(\mathbb{P}^n)))$ defined as in (4.2); the ideal $I \subset R$ such that $(I^{-1})_d = W$, is obtained as the intersection of s ideals of the type $I_{0,\ldots,d-1}$:

$$I = I_{0,\ldots,d-1} \cap \cdots \cap I_{(s-1)d,\ldots,s(d-1)};$$

At this point it would be possible to compute $H(I,d)$ but we prefer to perform a different and more efficient kind of approach to establish the regularity or the defectivity of $\mathrm{Sec}_{s-1}(\mathrm{Split}_d(\mathbb{P}^n))$ in many cases. We will present it in the section 4.2; before doing that we want to offer motivation for the study of $\mathrm{Split}_3(\mathbb{P}^n)$ which is of particular interest in fact this variety can be characterized in the following way.

A characterization of $\mathrm{Split}_3(\mathbb{P}^n)$

Let $X \subset \mathbb{P}^r$ be an irreducible hypersurface. Let $P \in X$ be a simple point of X and let $A(P)$ be the set of lines such that their intersection with X in P has multiplicity at least 3. We have the following:

- If $A(P) \equiv T_P(X)$ then P is a flex point, otherwise
- the set $A(P)$ is an $(r-2)$-dimensional quadric cone, with a double point in P and contained in $T_P(X)$; $A(P)$ is called the *Asymptotic Cone to X in P*.

Definition 4.1.3. *If $1 \leq k < r-1$, then a simple point P of an irreducible hypersurface $X \subset \mathbb{P}^r$ is said to be a k-Parabolic Point for X if the vertex of the asymptotic cone $A(P)$ is a k-dimensional linear subspace of $T_P(X)$. The point P is an $(r-1)$-parabolic point of X if P is a flex point.*

If $f = 0$ is an equation of X, one can check that $P \in X$ is a k-parabolic point for X if and only if

$$\mathrm{rk}\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{i,j=0,\ldots,r} \leq r-k+1.$$

Proposition 4.1.4. *The variety $\mathrm{Split}_d(\mathbb{P}^n)$ is contained in the projectivization of*

$$\{p \in S_d \,:\, p \text{ divides all the } 3\times 3 \text{ minors of } Hess(p)\}.$$

Proof. Let $[p] \in \mathrm{Split}_d(\mathbb{P}^n)$, i.e. p represents a polynomial of degree d that can be written as a product of d linear forms: $p = L_1 \cdots L_d$; hence p represents a hypersurface $H \subset \mathbb{P}^n$ which is the union of d hyperplanes of $\mathbb{P}^n$, so each point of H is a flex point, i.e. an $(n-1)$-parabolic point (see [**Il**]). If p is without multiple components then this last condition is equivalent to the fact that the polynomial p divides all the 3×3 minors of $Hess(p)$ (see [**Se**]). If p has multiple components we consider the dense open subset of $\mathrm{Split}_d(\mathbb{P}^n)$ contained in the projectivization of the algebraic set of the forms dividing their Hessian. Now since $\mathrm{Split}_d(\mathbb{P}^n)$ is an irreducible variety, we can conclude, by continuity, that $\mathrm{Split}_d(\mathbb{P}^n)$ is contained in the projectivization of the set $\{p \in S_d \,:\, p$ divides all the 3×3 minors of $Hess(p)\}$. $\square$

Corollary 4.1.5. *If $d = 3$ the variety $\mathrm{Split}_3(\mathbb{P}^n) \subset \mathbb{P}(S_3)$ is the locus of all cubics of S_3 which divide all the 3×3 minors of their Hessian.*

Proof. One inclusion is a direct consequence of previous Proposition. For the other inclusion it is sufficient to observe that if $p \in S_3$ is such that p divides all the 3×3 minors of its Hessian then:

- if p has not multiple components, then, by [**Se**], p divides all the 3×3 minors of its Hessian if and only if the hypersurface $H \subset \mathbb{P}(S_3)$ of degree 3 represented by p is made only by flexes points;
- if p has multiple components then p can only be a product of three linear forms.

Both the conclusions above are equivalent to the fact that H is the union of three hyperplanes, therefore $[p] \in \mathrm{Split}_3(\mathbb{P}^n)$. □

4.2 Another approach

In this section we show a different way to approach the study of the dimension of secant varieties of the Split varieties which will turn out to be more efficient.

In (2.15) we have given the definition of the Split variety as the image of the map

$$\begin{array}{rcl} \phi : \underbrace{\mathbb{P}(S_1) \times \cdots \times \mathbb{P}(S_1)}_{d} & \to & \mathbb{P}(S_d) \\ ([L_1], \ldots, [L_d]) & \mapsto & [L_1 \cdots L_d] \end{array}$$

where $S = K[x_0, \ldots, x_n]$. Let us work in more details. Let $A_1, \ldots, A_d$ be vector spaces of dimension $n+1$; consider the space $\mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_d)$. On each $\mathbb{P}(A_j)$, for $j = 1, \ldots, d$, we consider the coordinate ring $S^{(j)} = K[x_0^{(j)}, \ldots, x_n^{(j)}]$. On $\mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_d)$ we consider the ring $S := K[x_0^{(1)}, \ldots, x_n^{(1)}; \ldots; x_0^{(d)}, \ldots, x_n^{(d)}]$ of multi-homogeneous coordinates, then $\mathrm{Split}_d(\mathbb{P}^n)$ is the image of the map

$$\begin{array}{rcl} \phi : \mathbb{P}(S_1^{(1)}) \times \cdots \times \mathbb{P}(S_1^{(d)}) & \to & \mathbb{P}(S_d) \\ ([x_0^{(1)}, \ldots, x_n^{(1)}], \ldots, [x_0^{(d)}, \ldots, x_n^{(d)}]) & \mapsto & [x_0^{(1)} \cdots x_0^{(d)}, \\ & & \sum_{i=1}^d x_0^{(1)} \cdots x_0^{(i-1)} x_1^{(i)} x_0^{(i+1)} \cdots x_0^{(d)}, \\ & & \vdots \\ & & x_n^{(1)} \cdots x_n^{(d)}] \end{array}$$

i.e. $\mathrm{Split}_d(\mathbb{P}^n)$ parameterizes the multi-degree $(1,\dots,1)$ forms of S that are symmetric.

Notation: If a point $Q \in \mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_d)$ we will write:

$$Q = (Q^{(1)}, \dots, Q^{(d)})$$

with $Q^{(j)} \in \mathbb{P}(A_j)$, for $j = 1, \dots, d$.

This characterization of the Split variety together with Terracini's Lemma 2.6.1, Corollary 2.6.2 and Proposition 2.6.3 allow us to say that $\mathrm{Sec}_{s-1}(\mathrm{Split}_d(\mathbb{P}^n))$ has the expected dimension if and only if s double points of $\mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_d)$ impose independent conditions to the multi-degree $(1,\dots,1)$ symmetric forms of S.

Lemma 4.2.1. *Let R be a generic point of $\mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_d)$ with defining multi-homogeneous ideal $I_R \subset S$. Then a 2-fat point $2R$ imposes independent conditions to the symmetric multi-degree $(1,\dots,1)$-forms of S.*

Proof. A double point R in $\mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_d)$ imposes $dn+1$ conditions to forms of S hence, in order to prove the lemma, it is sufficient to find $dn+1$ symmetric forms in S of multi-degree $(1,\dots,1)$ such that nd of them generate I_R, the other one does not vanish at R and such that all those $dn+1$ forms must be independent module the ideal $(I_R)^2$. Let $R \in \mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_d)$, $R = (R^{(1)}, \dots, R^{(d)})$. Let us view the points $R^{(1)}, \dots, R^{(d)}$ in a same projective space $\mathbb{P}^n$ with coordinate ring $\tilde{S} = K[y_0, \dots, y_n]$; i.e. consider for $j = 1, \dots, d$ the maps:

$$\begin{array}{rccc} \phi_j : & \mathbb{P}(A_j) & \to & \mathbb{P}^n \\ & (x_0^{(j)}, \dots, x_n^{(j)}) & \mapsto & (y_0, \dots, y_n) \end{array}, \tag{4.4}$$

we indicate $\phi_j(R^{(j)})$ with $Q^{(j)}$. Let $P_1, \dots, P_n$ be generic points of $\mathbb{P}^n$. Consider the linear forms $f_1^{(1)}, \dots, f_n^{(1)}; f_1^{(2)}, \dots, f_n^{(2)}; \dots; f_1^{(d)}, \dots, f_n^{(d)}$ of $\tilde{S}$ that define the following hyperplanes $\pi_i^{(j)}$ of $\mathbb{P}^n$ for $j = 1, \dots, d$ and $i = 1, \dots, n$:

$$\begin{array}{rcl} f_1^{(1)}(y_0, \dots, y_n) = 0 & \longleftrightarrow & \pi_1^{(1)} =< Q^{(1)}, \widehat{P}_1, P_2, \dots, P_n > \\ f_2^{(1)}(y_0, \dots, y_n) = 0 & \longleftrightarrow & \pi_2^{(1)} =< Q^{(1)}, P_1, \widehat{P}_2, \dots, P_n > \\ & \vdots & \\ f_n^{(1)}(y_0, \dots, y_n) = 0 & \longleftrightarrow & \pi_n^{(1)} =< Q^{(1)}, P_1, \dots, P_{n-1}, \widehat{P}_n > \end{array}$$

$$
\begin{aligned}
f_1^{(2)}(y_0,\dots,y_n)=0 \quad &\longleftrightarrow \quad \pi_1^{(2)} =< Q^{(2)}, \widehat{P}_1, P_2, \dots, P_n > \\
&\vdots \\
f_n^{(2)}(y_0,\dots,y_n)=0 \quad &\longleftrightarrow \quad \pi_n^{(2)} =< Q^{(2)}, P_1, \dots, P_{n-1}, \widehat{P}_n > \\
\\
f_1^{(3)}(y_0,\dots,y_n)=0 \quad &\longleftrightarrow \quad \pi_1^{(3)} =< Q^{(3)}, \widehat{P}_1, P_2, \dots, P_n > \\
&\vdots \\
&\vdots \\
f_n^{(d)}(y_0,\dots,y_n)=0 \quad &\longleftrightarrow \quad \pi_n^{(d)} =< Q^{(d)}, P_1, \dots, P_{n-1}, \widehat{P}_n > .
\end{aligned}
\tag{4.5}
$$

I.e. the $f_i^{(j)} \in S$ are linear forms such that $f_i^{(j)}|_{\pi_i^{(j)}} \equiv 0$ where

$$\pi_i^{(j)} :=< Q^{(j)}, P_1, \dots, \widehat{P}_i, \dots, P_n >$$

is the space spanned by $Q^{(j)}, P_1, \dots, P_{i-1}, P_{i+1}, \dots, P_n$, for all $i = 1, \dots, n$ and for all $j = 1, \dots, d$. Now consider the linear form $g \in \tilde{S}$ which defines the hyperplane $\pi :=< P_1, \dots, P_n >$. Let us define the following $nd + 1$ symmetric $(1, \dots, 1)$-forms in S:

$$F_i^{(j)} = f_i^{(j)}(x_0^{(1)}, \dots, x_n^{(1)}) \cdot f_i^{(j)}(x_0^{(2)}, \dots, x_n^{(2)}) \cdots f_i^{(j)}(x_0^{(d)}, \dots, x_n^{(d)}), \tag{4.6}$$

with $i = 1, \dots, n$ and $j = 1, \dots, d$; and

$$G = g(x_0^{(1)}, \dots, x_n^{(1)}) \cdot g(x_0^{(2)}, \dots, x_n^{(2)}) \cdots g(x_0^{(d)}, \dots, x_n^{(d)}). \tag{4.7}$$

By construction the $F_i^{(j)}$ are nd symmetric multi-degree $(1, \dots, 1)$-forms in S that generate the ideal $I_Q \subset S$, moreover G is a form of the same type that does not vanish on Q. In order to check that those forms are independent module the ideal $(I_Q)^2$ it is sufficient to verify that the tangent spaces to the hypersurfaces individuated by the forms above generate the ideal I_Q. Also this last fact follows from our construction. □

Proposition 4.2.2. *If $d > 2$ and $d(s-1) \leq n$, then*

$$\dim(\mathrm{Sec}_{s-1}(\mathrm{Split}_d(\mathbb{P}^n))) = \mathrm{expdim}(\mathrm{Sec}_{s-1}(\mathrm{Split}_d(\mathbb{P}^n))).$$

Proof. As we have already recalled, the statement of the proposition can be equivalently reformulated in the following way:

> "if $d > 2$ and $d(s-1) \leq n$ then s double points of $\mathbb{P}(A_1)\times\cdots\times\mathbb{P}(A_d)$ impose independent conditions to the multi-degree $(1,\ldots,1)$ symmetric forms of S."

Let $R, T_1, \ldots, T_{s-1}$ be s generic points of $\mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_d)$. If we want to prove that they impose independent conditions to the $(1,\ldots,1)$ symmetric forms of S, it is sufficient to find $dn+1$ symmetric $(1,\ldots,1)$ forms of S such that:

- dn of them vanish with multiplicity at least 2 on $T_1,\ldots,T_{s-1}$ and with multiplicity 1 on R,
- one of them does not vanish on R,
- all the $dn+1$ are independent in $S/(I_R)^2$.

We can apply the same construction used to prove the previous Lemma.
Let $\mathbb{P}^n$ be the projective space with coordinate ring $K[y_0,\ldots,y_n]$.
Since $n \geq d(s-1)$ we can choose $d(s-1)$ $P_1,\ldots,P_{d(s-1)} \in \mathbb{P}^n$ (since $n \geq d(s-1)$, they will be linearly independent). Let $\phi_j : \mathbb{P}(A_j) \to \mathbb{P}^n$ defined for all $j = 1,\ldots,d$ as in (4.4). We can choose $T_i \equiv (T_i^{(1)},\ldots,T_i^{(d)})$, for $i = 1,\ldots,s-1$, in such a way that

$$\phi_j(T_i^{(j)}) = P_{d(i-1)+j} \tag{4.8}$$

for $i = 1,\ldots,d$. In that way $T_1,\ldots,T_{s-1}$ are $s-1$ generic points of $\mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_d)$.
Choose a generic point $R \equiv (R^{(1)},\ldots,R^{(d)})$ and let $\phi_j(R^{(j)}) =: Q^{(j)} \in \mathbb{P}^n$.
We can construct $f_i^{(j)}, g \in K[y_0,\ldots,y_n]$ such that if

$$\pi_i^{(j)} =< Q^{(j)}, P_1,\ldots,\hat{P}_i,\ldots,P_{d(s-1)} >\subset \mathbb{P}^n$$

for $i = 1,\ldots,d(s-1)$, $j = 1,\ldots,d$, and

$$\pi =< P_1,\ldots,P_{d(s-1)} >\subset \mathbb{P}^n$$

then $f_i^{(j)}|_{\pi_i^{(j)}} \equiv 0$ for $i = 1,\ldots,d(s-1)$, $j = 1,\ldots,d$, and $g|_\pi \equiv 0$.
Now we can construct $F_i^{(j)} \in S = K[x_0^{(1)},\ldots,x_n^{(1)};\ldots;x_0^{(d)},\ldots,x_n^{((d))}]$, for $i = 1,\ldots,n$ and $j = 1,\ldots,d$, as in (4.6), and $G \in S$ as in (4.7). They will turn out to be $dn+1$ symmetric $(1,\ldots,1)$ forms of S such that $F_1^{(1)},\ldots,F_{dn}^{(d)}$ vanish with multiplicity at least 2 on $T_1,\ldots,T_{s-1}$ and simply on R; the form G does not vanish on R and they all are independent module $(I_R)^2$. □

Example: We consider the case of $n = 7$, $d = 3$ and $s = 3$. We want to see that $\mathrm{Sec}_2(\mathrm{Split}_3(\mathbb{P}^7))$ has the expected dimension. In terms of number of conditions imposed by fat points to forms of $S = K[x_0^{(1)}, \ldots, x_7^{(1)}; x_0^{(2)}, \ldots, x_7^{(2)}; x_0^{(3)}, \ldots, x_7^{(3)}]$, it means that three double points of $\mathbb{P}(A_1) \times \mathbb{P}(A_2) \times \mathbb{P}(A_3)$, with $\dim(\mathbb{P}(A_1)) = \dim(\mathbb{P}(A_2)) = \dim(\mathbb{P}(A_3)) = 7$, impose independent conditions to the $(1, 1, 1)$ symmetric forms of S.

Consider the points $R, T_1, T_2 \in \mathbb{P}(A_1) \times \mathbb{P}(A_2) \times \mathbb{P}(A_3)$ such that:

- if ϕ_j is defined as in (4.4), then $R \equiv (R^{(1)}, R^{(2)}, R^{(3)}) \in \mathbb{P}(A_1) \times \mathbb{P}(A_2) \times \mathbb{P}(A_3)$ is such that:
  - $\phi_1(R^{(1)}) := Q^{(1)} = (1, 0, \ldots, 0) \in \mathbb{P}^7 = \phi_1(\mathbb{P}(A_1))$,
  - $\phi_2(R^{(2)}) := Q^{(2)} = (1, \alpha_1, \ldots, \alpha_7) \in \mathbb{P}^7 = \phi_2(\mathbb{P}(A_2))$, with $\alpha_1, \ldots, \alpha_7 \in K$,
  - $\phi_3(R^{(3)}) := Q^{(3)} = (1, \beta_1, \ldots, \beta_7) \in \mathbb{P}^7 = \phi_3(\mathbb{P}(A_3))$, with $\beta_1, \ldots, \beta_7 \in K$;
- $T_1 \equiv (T_1^{(1)}, T_1^{(2)}, T_1^{(3)}) \in \mathbb{P}(A_1) \times \mathbb{P}(A_2) \times \mathbb{P}(A_3)$ is such that
  - $\phi_1(T_1^{(1)}) := P_1 = (0, 1, 0, \ldots, 0) \in \mathbb{P}^7 = \phi_1(\mathbb{P}(A_1))$,
  - $\phi_2(T_1^{(2)}) := P_2 = (0, 0, 1, 0, \ldots, 0) \in \mathbb{P}^7 = \phi_2(\mathbb{P}(A_2))$,
  - $\phi_3(T_1^{(3)}) := P_3 = (0, 0, 0, 1, 0, 0, 0, 0) \in \mathbb{P}^7 = \phi_3(\mathbb{P}(A_3))$;
- $T_2 \equiv (T_2^{(1)}, T_2^{(2)}, T_2^{(3)}) \in \mathbb{P}(A_1) \times \mathbb{P}(A_2) \times \mathbb{P}(A_3)$ is such that
  - $\phi_1(T_2^{(1)}) := P_4 = (0, 0, 0, 0, 1, 0, 0, 0) \in \mathbb{P}^7 = \phi_1(\mathbb{P}(A_1))$,
  - $\phi_2(T_2^{(2)}) := P_5 = (0, 0, 0, 0, 0, 1, 0, 0) \in \mathbb{P}^7 = \phi_2(\mathbb{P}(A_2))$,
  - $\phi_3(T_2^{(3)}) := P_6 = (0, 0, 0, 0, 0, 0, 1, 0) \in \mathbb{P}^7 = \phi_3(\mathbb{P}(A_3))$;

If $\tilde{S} = K[y_0, \ldots, y_7]$ is the coordinate ring over $\mathbb{P}^7 = \phi_j(A_j)$ for $j = 1, 2, 3$, the forms of (4.5)

are the following:

$$Q^{(1)}: \begin{cases} f_1^{(1)}(y_0,\dots,y_7) = y_1 \\ f_2^{(1)}(y_0,\dots,y_7) = y_2 \\ \vdots \\ f_7^{(1)}(y_0,\dots,y_7) = y_7 \end{cases}$$

$$Q^{(2)}: \begin{cases} f_1^{(2)}(y_0,\dots,y_7) = y_1 - \alpha_1 y_0 \\ f_2^{(2)}(y_0,\dots,y_7) = y_2 - \alpha_2 y_0 \\ \vdots \\ f_7^{(2)}(y_0,\dots,y_7) = y_7 - \alpha_7 y_0 \end{cases}$$

$$Q^{(3)}: \begin{cases} f_1^{(3)}(y_0,\dots,y_7) = y_1 - \beta_1 y_0 \\ \vdots \\ f_7^{(3)}(y_0,\dots,y_7) = y_7 - \beta_7 y_0 \end{cases}$$

$$g(y_0,\dots,y_7) = y_0.$$

Consider now the corresponding multi-degree $(1,1,1)$ symmetric forms in S constructed as in (4.6) e (4.7) we get the following 22 forms $F_i^{(j)}, G \in S = K[x_0^{(1)},\dots,x_7^{(1)};x_0^{(2)},\dots,x_7^{(2)};x_0^{(3)},\dots,x_7^{(3)}]$ for $j = 1,2,3$ and $i = 1,\dots,7$:

$$\begin{aligned} F_1^{(1)}(\underline{x}) &= x_1^{(1)}x_1^{(2)}x_1^{(3)} \\ F_2^{(1)}(\underline{x}) &= x_2^{(1)}x_2^{(2)}x_2^{(3)} \\ &\vdots \\ F_7^{(1)}(\underline{x}) &= x_7^{(1)}x_7^{(2)}x_7^{(3)} \end{aligned}$$

$$\begin{aligned} F_1^{(2)}(\underline{x}) &= (x_1^{(1)} - \alpha_1 x_0^{(1)})(x_1^{(2)} - \alpha_1 x_0^{(2)})(x_1^{(3)} - \alpha_1 x_0^{(3)}) \\ &\vdots \\ F_7^{(2)}(\underline{x}) &= (x_7^{(1)} - \alpha_7 x_0^{(1)})(x_7^{(2)} - \alpha_7 x_0^{(2)})(x_7^{(3)} - \alpha_7 x_0^{(3)}) \end{aligned} \tag{4.9}$$

$$\begin{aligned} F_1^{(3)}(\underline{x}) &= (x_1^{(1)} - \beta_1 x_0^{(1)})(x_1^{(2)} - \beta_1 x_0^{(2)})(x_1^{(3)} - \beta_1 x_0^{(3)}) \\ &\vdots \\ F_7^{(3)}(\underline{x}) &= (x_7^{(1)} - \beta_7 x_0^{(1)})(x_7^{(2)} - \beta_7 x_0^{(2)})(x_7^{(3)} - \beta_7 x_0^{(3)}) \end{aligned}$$

$$G(\underline{x}) = x_0^{(1)} x_0^{(2)} x_0^{(3)}.$$

It is evident that the $F_i^{(j)}$, with $i = 1, \ldots, 7$ and $j = 1, 2, 3$, are the 21 symmetric forms of multidegree $(1, 1, 1)$ of S that generate the ideal $I_R \subset S$ and that they are double at T_1 and at T_2. Moreover G is double in T_1 and T_2 (actually, G has multiplicity 3 at T_i, it is sufficient that they are at least double points) but it does not vanish in R.

Remark: The forms $F_i^{(j)}$ of the previous example are double in T_1, T_2 but in general, the $F_i^{(j)}$'s, with $i = 1, \ldots, n$ and $j = 1, \ldots, d$, defined in (4.6), vanish up to order $(d-1)$ at $T_1, \ldots, T_{s-1}$ chosen as in the previous example. Moreover, the proof works because in the hypothesis we have that $d > 2$ and this allows to construct the $F_i^{(j)}$, vanishing to the order at least two $(d - 1 \geq 2)$ at $T_1, \ldots, T_{s-1}$, for $i = 1, \ldots, n$, $j = 1, \ldots, d$.

Now we have to verify that the 22 forms defined in (4.9) are independent module the ideal $(I_R)^2$. In order to do that we consider the following construction (as it is done in [**CGG3**], see Theorem 1.1): let f be the map

$$f : \mathbb{P}(A_1) \times \mathbb{P}(A_2) \times \mathbb{P}(A_3) \longrightarrow \mathbb{A}^{21}$$

with $\dim(\mathbb{P}(A_i)) = 7$ for $i = 1, 2, 3$, defined on the open affine set $\{x_0^{(1)} x_0^{(2)} x_0^{(3)} \neq 0\}$ such that:

$$f((x_0^{(1)}, \ldots, x_7^{(1)}), (x_0^{(2)}, \ldots x_7^{(2)}), (x_0^{(3)}, \ldots x_7^{(3)})) = \left(1; \frac{x_1^{(1)}}{x_0^{(1)}}, \ldots, \frac{x_7^{(1)}}{x_0^{(1)}}; \frac{x_1^{(2)}}{x_0^{(2)}}, \ldots, \frac{x_7^{(2)}}{x_0^{(2)}}; \frac{x_1^{(3)}}{x_0^{(3)}}, \ldots, \frac{x_7^{(3)}}{x_0^{(3)}}\right).$$

If $Z \subset \mathbb{P}(A_1) \times \mathbb{P}^(A_2) \times \mathbb{P}^(A_3)$ is a 0-dimensional scheme contained in the affine chart $\{x_0^{(1)} x_0^{(2)} x_0^{(3)} \neq 0\}$ then $Z \simeq f(Z)$.
Consider the image of R via f, i.e. $f(R) = (1; 0, 0, 0, 0, 0, 0; \alpha_1, \ldots, \alpha_7; \beta_1, \ldots, \beta_7)$ which, with a slight abuse of notation, we will still indicate with R. With the same kind of notation, the affinization of the $F_i^{(j)}$'s, with $i = 1, \ldots, 7$ and $j = 1, 2, 3$, and of G are:

$$F_1^{(1)}(\underline{x}) = x_1^{(1)} x_1^{(2)} x_1^{(3)}$$

$$\vdots$$

$$F_7^{(1)}(\underline{x}) = x_7^{(1)} x_7^{(2)} x_7^{(3)}$$

$$F_1^{(2)}(\underline{x}) = (x_1^{(1)} - \alpha_1)(x_1^{(2)} - \alpha_1)(x_1^{(3)} - \alpha_1)$$

$$\vdots$$

$$F_7^{(2)}(\underline{x}) = (x_7^{(1)} - \alpha_7)(x_7^{(2)} - \alpha_7)(x_7^{(3)} - \alpha_7)$$

$$F_1^{(3)}(\underline{x}) = (x_1^{(1)} - \beta_1)(x_1^{(2)} - \beta_1)(x_1^{(3)} - \beta_1)$$
$$F_7^{(3)}(\underline{x}) = (x_7^{(1)} - \beta_7)(x_7^{(2)} - \beta_7)(x_7^{(3)} - \beta_7)$$

$$G(\underline{x}) = 1$$

where $(\underline{x}) := (x_0^{(1)}, \dots, x_7^{(1)}; x_0^{(2)}, \dots x_7^{(2)}; x_0^{(3)}, \dots x_7^{(3)})$. We consider, only for simplicity, the translation of $\mathbb{A}^{21}$ that sends R to the origin:

$$\begin{cases} X_1^{(1)} = x_1^{(1)} \\ \vdots \\ X_7^{(1)} = x_7^{(1)} \\ X_1^{(2)} = x_1^{(2)} - \alpha_1 \\ \vdots \\ X_7^{(2)} = x_7^{(2)} - \alpha_7 \\ X_1^{(3)} = x_1^{(3)} - \beta_1 \\ \vdots \\ X_7^{(3)} = x_7^{(3)} - \beta_7. \end{cases}$$

After this translation the $F_i^{(j)}$ and G become the following $\widetilde{F}_i^{(j)}$ and $\widetilde{G}$ respectively:

$$\widetilde{F}_1^{(1)}(\underline{X}) = X_1^{(1)}(X_1^{(2)} + \alpha_1)(X_1^{(3)} + \beta_1)$$

$$\vdots$$

$$\widetilde{F}_7^{(1)}(\underline{X}) = X_7^{(1)}(X_7^{(2)} + \alpha_7)(X_7^{(3)} + \beta_7)$$

$$\widetilde{F}_1^{(2)}(\underline{X}) = X_1^{(2)}(X_1^{(1)} + \alpha_1)(X_1^{(3)} + \beta_1 - \alpha_1)$$

$$\vdots$$

$$\widetilde{F}_7^{(2)}(\underline{X}) = X_7^{(2)}(X_7^{(1)} + \alpha_7)(X_7^{(3)} + \beta_7 - \alpha_7)$$

$$\widetilde{F}_1^{(3)}(\underline{X}) = X_1^{(3)}(X_1^{(1)} + \beta_1)(X_1^{(2)} + \alpha_1 - \beta_1)$$

$$\vdots$$

$$\widetilde{F}_7^{(3)}(\underline{X}) = X_7^{(3)}(X_7^{(1)} + \beta_7)(X_7^{(2)} + \alpha_7 - \beta_7)$$

$$\widetilde{G}(\underline{X}) = 1$$

where, as above, $(\underline{X}) := (X_1^{(1)}, \dots, X_7^{(1)}; \dots; X_1^{(3)}, \dots, X_7^{(3)})$. The form $\widetilde{G}$ does not vanish at the origin; the tangent planes to the $\widetilde{F}_i^{(j)}$ have equations: $X_i^{(j)} = 0$ respectively for all $i = 1, \dots, 7$ and $j = 1, 2, 3$.
Now it is clear that those forms generate I_O and so they are independent module the ideal $(I_O)^2$.

As we have already observed, the forms $F_i^{(j)}$ and G defined in (4.6) and (4.7) are symmetric forms of S of multi-degree $(1, \dots, 1)$ that vanish with multiplicity $(d-1)$ on $T_1, \dots, T_{s-1}$ chosen as in (4.8).

We have already observed that if we want to prove that s double fat points $R, T_1, \dots, T_{s-1} \in \mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_d)$ impose independent conditions to the multi-degree $(1, \dots, 1)$ symmetric forms of S, it is sufficient to find $dn + 1$ of these forms such that:

- dn of them vanish with multiplicity at least 2 on $T_1, \dots, T_{s-1}$ and with multiplicity 1 on R,
- one of them does not vanish on R,
- all the $dn + 1$ are independent in $S/(I_R)^2$.

Therefore the $T_1, \dots, T_{s-1}$ can be chosen in a less restrictive way with respect to what we have done in the last proposition: it is not necessary that the $F_i^{(j)}$ and G are zero on them up to the degree $d-1$, it is sufficient that they are double in those points. Hence we can improve Proposition 4.2.2.

Proposition 4.2.3. *If $d > 2$ and $n \geq 3(s-1)$, then*

$$\dim(\mathrm{Sec}_{s-1}(\mathrm{Split}_d(\mathbb{P}^n))) = \mathrm{expdim}(\mathrm{Sec}_{s-1}(\mathrm{Split}_d(\mathbb{P}^n))).$$

Proof. The proof of this proposition is very similar to that of the previous one; it is sufficient to choose s generic points of $\mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_d)$: R and $T_i \equiv (T_i^{(1)}, \ldots, T_i^{(d)}) \in \mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_d)$ with $i = 1, \ldots, s-1$ such that:

$$T_i^{(1)} := P_{3(i-1)+1} \in \mathbb{P}(A_1),$$
$$T_i^{(2)} := P_{3(i-1)+2} \in \mathbb{P}(A_2),$$
$$T_i^{(3)} := P_{3(i-1)+3} \in \mathbb{P}(A_3)$$

where P_j, for $j = 1, \ldots, 3(s-1)$, is the j-th coordinate point of $\mathbb{P}^n$. Now the proof works exactly as the proof of Proposition 4.2.2 with the only difference that $F_i^{(j)}$ defined in (4.6) are only double at $T_1, \ldots, T_{s-1}$ and G defined in (4.7) vanishes with multiplicity 3, and then this is sufficient to prove that the $F_i^{(j)}$'s and G are independent in $S/(I_R)^2$. □

4.3 Linear subspaces of $\mathrm{Split}_d(\mathbb{P}^n)$

Our study of the Split varieties will now aim to understand the structure of the linear subspaces of $\mathrm{Split}_d(\mathbb{P}^n)$.

Let us recall Bertini's Theorem:

Theorem 4.3.1. (**Bertini**) *If $X \subset \mathbb{P}^n$ is a complex projective variety, Σ a linear system on X without fixed components and such that all $D \in \Sigma$ are reducible, then the rational map $\rho : X \overset{|D|}{\dashrightarrow} Y \subset \mathbb{P}^r$ factorizes through a curve C:*

$$\begin{tikzcd} \rho: \; X \arrow[rr, dashed, "|D|"] \arrow[dr, dashed, "\tau"'] & & Y \\ & C \arrow[ur, "\sigma"'] & \end{tikzcd}$$

where $\tau : X \dashrightarrow C$ has connected fibers, and $\sigma : C \to Y$ is a finite morphism. The composition $\rho = \sigma \circ \tau$ is called the Stein factorization.

Proposition 4.3.2. *Let $[M_0], \ldots, [M_r]$ be linearly independent in $\mathrm{Split}_d(\mathbb{P}^n)$. Let $L_1 \cdots L_s$ be their greatest common divisor and, for $i = 0, \ldots, r$ write*

$$M_i := L_1 \cdots L_s L_{i,s+1} \cdots L_{i,d}.$$

Then, the r-dimensional span $V_r \subset \mathbb{P}^{\binom{n+d}{d-1}}$ of $M_0, \ldots, M_r$ is contained in $\mathrm{Split}_d(\mathbb{P}^n)$ if and only if one of the following cases occur:

1. *$s \leq d-2$ and there exist $L', L'' \in S_1$ such that all $L_{i,j} \in K[L', L'']$. In this case, $V_r \subset \mathbb{P}(K[L', L'']_{d-s}) = \mathbb{P}^{d-s}$.*

2. *$s \geq d-1$. In this case, $V_r \subset \mathbb{P}(S_1) = \mathbb{P}^n$.*

Proof. Since $V_r = \mathbb{P}^r \subset \mathrm{Split}_d(\mathbb{P}^n) \subset \mathbb{P}(S_d)$ and the elements of $\mathrm{Split}_d(\mathbb{P}^n)$ represent forms of degree d obtained as product of d linear forms, V_r is the span of $r+1$ of them

$$V_r =< L_{0,1} \cdots L_{0,d}, \ldots, L_{r,1} \cdots L_{r,d} > .$$

An element $D \in V_r$ is such that:

$$D := L_1 \cdots L_d = \alpha_0 L_{0,1} \cdots L_{0,d} + \cdots + \alpha_r L_{r,1} \cdots L_{r,d}$$

for some $\alpha_0, \ldots, \alpha_r \in K$.
Therefore we are interested in studying linear systems $\Sigma = |L_{0,1} \cdots L_{0,d}, \ldots, L_{r,1} \cdots L_{r,d}|$ on $\mathbb{P}^n$ whose elements are all of the form $L_1 \cdots L_d$.

1. (a) If $s = 0$ we can apply Bertini's Theorem because the hypothesis assure us that Σ is without fixed components. Therefore there exist three maps $\rho : \mathbb{P}^n \to \Gamma \subset \mathbb{P}^r$, $\tau : \mathbb{P}^n \to \mathcal{C}$ and $\sigma : \mathcal{C} \to \Gamma$ such that $\mathcal{C}$ is a curve, ρ is the rational map given by Σ, τ has connected fibers, σ is a finite morphism and $\rho = \sigma \circ \tau$. If $P \in \Gamma$, the pre-image $\sigma^{-1}(P)$ is a set of d points on $\mathcal{C}$. The curve $\mathcal{C}$ is a $\mathbb{P}^1$ because τ is linear. Therefore $\rho^{-1}(P) = \tau^{-1}(\sigma^{-1}(P))$ is a set of d hyperplanes of $\mathbb{P}^n$. This fact implies also that the d fibers of τ meet in the same $\mathbb{P}^{n-2}$. Therefore V_r is contained in a $\mathbb{P}^d$ that is obtained as $\mathbb{P}(K[L', L'']_d)$ with $L', L'' \in S_1$.

   (b) If $0 \leq s \leq d-2$ then $D \in V_r$ is of the form:

   $$D = \alpha_0 M_0 + \cdots + \alpha_r M_r = L_1 \cdots L_s \cdot (\alpha_0 L_{0,s+1} \cdots L_{0,d} + \cdots + \alpha_r L_{r,s+1} \cdots L_{r,d}).$$

   We can apply Bertini's Theorem to a system Σ' whose elements are all of the form $D' = \alpha_0 L_{0,s+1} \cdots L_{0,d} + \cdots + \alpha_r L_{r,s+1} \cdots L_{r,d}$. If ρ, τ, σ are defined as in the previous case (the map ρ now is the the rational map given by Σ' and τ, σ are then well defined) we observe now that $length(\{\sigma^{-1}(P)\}) = d - l$ and then $\rho^{-1}(P)$ is a set of $d-s$ hyperplanes of $\mathbb{P}^n$ meeting in the same $\mathbb{P}^{n-2}$ then $D = L_1 \cdots L_s \cdot F$ where $F \in K[L', L'']_{d-s}$.

2. The last case is obvious.

□

4.4 $\mathrm{Split}_d(\mathbb{P}^n)$ and $\mathbb{G}(n-1, n+d-1)$

The variety $\mathrm{Split}_d(\mathbb{P}^n)$ has dimension nd and parameterizes all forms of degree d that are decomposable as products of d linear forms. From a geometric point of view $\mathrm{Split}_d(\mathbb{P}^n)$ parameterizes the unions of d hyperplanes.

Example: If $d = 2$ an element of $\mathrm{Split}_2(\mathbb{P}^n)$ represents a quadric of $\mathbb{P}^n$ which is the union of two hyperplanes; therefore, if M_Q is the symmetric matrix of $M_n(K)$ representing a quadric $Q \in S_2$, then:

$$\mathrm{Split}_2(\mathbb{P}^n) = \{[Q] \in \mathbb{P}(S_2) \mid \mathrm{rk}(M_Q) \leq 2\}. \tag{4.10}$$

We need to recall that the *Grassmannian* $\mathbb{G}(k, n)$ is the projective variety which parameterizes the k-spaces in $\mathbb{P}^N$. Grassmannians can be viewed in a projective space by looking at their *Plücker embedding*. For this, we will use the Plücker coordinates, but in a way that is dual to the standard one (i.e. describing the k-spaces as intersection of hyperplanes rather than as spanned by points).

Let $\Lambda \subset \mathbb{P}^N$ be the space $H_1 \cap \cdots \cap H_{N-k} \subset \mathbb{P}^N$ where H_i is the hyperplane $u_{i,0}x_0 + \cdots + u_{i,N}x_N = 0$. For each $i_0 < \cdots < i_k$ we define $p_{i_0 \cdots i_k}$ to be the determinant

$$p_{i_0 \cdots i_k} := \begin{vmatrix} u_{1,i_0} & \cdots & u_{1,i_k} \\ \vdots & & \vdots \\ u_{N-k,i_0} & \cdots & u_{N-k,i_k} \end{vmatrix}.$$

The *Plücker embedding* is defined as follows:

$$\begin{array}{rcl} p : \mathbb{G}(k,n) & \hookrightarrow & \mathbb{P}^{\binom{n+1}{k+1}-1} \\ \Lambda & \mapsto & \{\{p_{i_0 \cdots i_k}\} \mid 0 \leq i_0 < \cdots < i_k \leq n\} \end{array} \tag{4.11}$$

Example: If $d = 2$, it easy to find the equations for $\mathbb{G}(n-1, n+1)$. Let Λ be an $(n-1)$-space of $\mathbb{P}^{n+1}$, then Λ is defined by the intersection of two independent hyperplanes $H_1, H_2 \subset \mathbb{P}^{n+1}$. Let their equations be $u_{1,0}x_0 + \cdots + u_{1,n+1}x_{n+1} = 0$ and $u_{2,0}x_0 + \cdots + u_{2,n+1}x_{n+1} = 0$ respectively. If p is the map defined in (4.11), the image $p(\Lambda)$ in $\mathbb{P}^{\frac{n^2+3n}{2}}$ is

$$\begin{array}{llll} (p_{0,1}, & p_{0,2}, & \ldots, & p_{0,n+1}, \\ & p_{1,2}, & \ldots, & p_{1,n+1}, \\ & & \ddots & \\ & & & p_{n,n+1}) \end{array}$$

where $p_{i,j} = \begin{vmatrix} u_{1,i} & u_{1,j} \\ u_{2,i} & u_{2,j} \end{vmatrix}$ for $i, j = 0, \dots, n+1$.
Let $\pi_0, \dots, \pi_{n+1} \subset \mathbb{P}^{n+1}$ be the $n+2$ hyperplanes defined by the following equations respectively:

$$0x_0 + p_{0,1}x_1 + p_{0,2}x_2 + \cdots + p_{0,n+1}x_{n+1} = 0,$$

$$-p_{0,1}x_0 + 0x_1 + p_{1,2}x_2 + \cdots + p_{1,n+1}x_{n+1} = 0,$$

$$\vdots$$

$$-p_{0,n+1}x_0 - \cdots - p_{n,n+1}x_n + 0x_{n+1} = 0.$$

If we consider the intersections $\Lambda_{i,j} = \pi_i \cap \pi_j$ for all $i \neq j$, and the image $p(\Lambda_{i,j})$, we obtain that the coordinates of $p(\Lambda_{i,j})$ are the 2×2 minors of the following skew symmetric matrix:

$$M_{n+1} := \begin{pmatrix} 0 & p_{0,1} & \cdots & p_{0,n+1} \\ -p_{0,1} & 0 & \cdots & p_{1,n+1} \\ \vdots & \ddots & \ddots & \vdots \\ -p_{0,n+1} & \cdots & -p_{n,n+1} & 0 \end{pmatrix}. \tag{4.12}$$

If the rank of this matrix is two, then $\Lambda_{0,1}, \dots, \Lambda_{n,n+1}$ are the same codimension 2 subspace of $\mathbb{P}^{n+1}$.

Then a rank 2 skew symmetric $(n+2) \times (n+2)$ matrix describes an element $\Lambda \in \mathbb{G}(n-1, n+1)$. Vice versa if an $(n-1)$-subspace of $\mathbb{P}^{n+1}$ is given by the intersection of $n+2$ hyperplanes $\pi_0, \dots, \pi_{n+1}$ as above then $\mathrm{rk}(M_{n+1}) = 2$.
Now it is not difficult to believe that imposing $\mathrm{rk}(M_{n+1}) = 2$ is equivalent to finding the equations of $\mathbb{G}(n-1, n+1)$ in $\mathbb{P}^{\frac{n^2+3n}{2}}$.

We like to observe that we have already noticed that the elements of $\mathrm{Split}_2(\mathbb{P}^n)$ are represented by the rank at most 2 quadrics of $\mathbb{P}^n$. We will see that this is not only a coincidence.

Remark: Observe that, when $N = n+d-1$ and $k = n-1$ we obtain that $\mathbb{G}(n-1, n+d-1)$ has dimension nd and it is contained in $\mathbb{P}^{\binom{n+d}{d}-1}$, exactly as it happens for $\mathrm{Split}_d(\mathbb{P}^n)$.

The remark above induced Ehrenborg (in [**Eh**]) to state a conjecture (see 2.6.4) in terms of Secant varieties of $\mathrm{Split}_d(\mathbb{P}^n)$ and of $\mathbb{G}(n-1, n+d-1)$. In Section 2.6.1 we have already shown a counterexample to this conjecture but we have also checked that it easily works when $d = 2$. The Ehrenborg conjecture and the many cases where it works suggest the interest in the study of the intersection between $\mathrm{Split}_d(\mathbb{P}^n)$ and the Grassmannian $\mathbb{G}(n-1, n+d-1)$.
The first idea to study this intersection comes from a way to embed the Veronese variety $\nu_d(\mathbb{P}^n)$ into the Grassmannian $\mathbb{G}(n-1, n+d-1)$ (see for instance [**AP**]).

Proposition 4.4.1. *Identify S_1 with $K[t_0, t_1]_n$ by assigning to $u_0x_0 + \cdots + u_nx_n \in S_1$ the homogeneous form $u_0t_0^n + u_1t_0^{n-1}t_1 + \cdots + u_nt_1^n \in K[t_0, t_1]$. Then $\mathbb{P}(S_1)$ is naturally identified with the set of schemes of length n on $\mathbb{P}^1$. The identifications are the following:*

1. *The map $\phi : \mathbb{P}(S_1) \to \mathbb{G}(n-1, n+d-1)$ that assigns to any scheme of length n on $\mathbb{P}^1$ its span as a scheme in $\nu_{n+d-1}(\mathbb{P}^1)$ is defined in coordinates as $\phi((u_0, \ldots, u_n))$ = intersection of the hyperplanes:*
$$\begin{cases} u_0x_0 + \cdots + u_nx_n = 0 \\ u_0x_1 + \cdots + u_nx_{n+1} = 0 \\ \vdots \\ u_0x_{d-1} + \cdots + u_nx_{n+d-1} = 0 \end{cases}.$$

2. *There is a linear change of coordinates in the Plücker space $\mathbb{P}^{\binom{n+d}{d}-1}$ such that the image of ϕ (which is the set of $(n-1)$-spaces of $\mathbb{P}^{n+d-1}$ that are n-secant to the rational normal curve) is the Veronese variety. This yields a canonical identification of $\mathbb{P}^{\binom{n+d}{d}-1}$ with $\mathbb{P}(S_d)$.*

By using the idea of this proposition that shows how one can embed the Veronese variety inside a Grassmannian, we can prove the following theorem which will be of some interest in order to discover a set that is contained in the intersection between $\mathrm{Split}_d(\mathbb{P}^n)$ and $\mathbb{G}(n-1, n+d-1)$. An interesting fact will be that this set, in the case of $d = 2$, will be exactly the intersection $\mathrm{Split}_d(\mathbb{P}^n) \cap \mathbb{G}(n-1, n+d-1)$.

Consider the embedding

$$\phi : \mathbb{P}^n \overset{\nu_d}{\hookrightarrow} \nu_d(\mathbb{P}^n) \overset{\mu_d}{\hookrightarrow} \mathbb{G}(n-1, n+d-1) \tag{4.13}$$

which sends a point $(u_0, \ldots, u_n) \in \mathbb{P}^n$ into the $(n-1)$-space obtained as the intersection of hyperplanes:

$$\begin{cases} u_0x_0 + \cdots + u_nx_n = 0 \\ u_0x_1 + \cdots + u_nx_{n+1} = 0 \\ \vdots \\ u_0x_{d-1} + \cdots + u_nx_{n+d-1} = 0 \end{cases}.$$

If we think at ν_d as the dual embedding we can think at an element of $\nu_d(\mathbb{P}^n)$ as the d-th power of a linear form $L \in K[x_0, \ldots, x_n]$, but Proposition 4.4.1 lets us to interpret an element of $\nu_d(\mathbb{P}^n)$ as an $(n-1)$-space of $\mathbb{P}^{n+d-1}$ that is n-secant to the rational normal curve $\nu_{n+d-1}(\mathbb{P}^1)$, or better,

consider the following composition of maps:

$$\begin{array}{ccccc} & & \mathbb{P}^1 & & \\ & & \downarrow & & \nu_{n+d-1} \\ & & \nu_{n+d-1}(\mathbb{P}^1) & & \\ & & \downarrow & & \eta_{n+d-1} \\ \phi : \mathbb{P}^n \overset{\nu_d}{\to} & \nu_d(\mathbb{P}^n) & \overset{\mu_d}{\to} & \mathbb{G}(n-1, n+d-1) & \end{array}$$

then Proposition 4.4.1 says that

$$\mu_d(\nu_d(\mathbb{P}^n)) \cap \mathbb{G}(n-1, n+d-1) = \{\Lambda \in \mathbb{G}(n-1, n+d-1) \mid length\{\Lambda \cap \eta_{n+d-1}(\nu_{n+d-1}(\mathbb{P}^1))\} \geq n\}. \tag{4.14}$$

Now let $P_1, \ldots, P_{n-1}, P$ be points of $\nu_{n+d-1}(\mathbb{P}^1)$. By Proposition 4.4.1 the span of $< P_1, \ldots, P_{n-1}, P >$ is represented by a point in $\mu_d(\nu_d(\mathbb{P}^n)) \cap \mathbb{G}(n-1, n+d-1)$: i.e. $\Lambda_P := \eta_{n+d-1}(< P_1, \ldots, P_{n-1}, P >) \in \mu_d(\nu_d(\mathbb{P}^n)) \cap \mathbb{G}(n-1, n+d-1)$, hence there exists a linear form $u_0x_0 + \cdots + u_nx_n \in K[x_0, \ldots, x_n]_1 = (\mathbb{P}^n)^*$ such that

$$\mu_d((u_0x_0 + \cdots + u_nx_n)^d) = \Lambda_P.$$

Consider now the following embedding:

$$\nu_n : \mathbb{P}^1 \hookrightarrow \mathbb{P}^n \simeq < \nu_n(\mathbb{P}^1) >,$$

whose dual is

$$(\mathbb{P}^1)^* = \mathbb{P}(K[t_0, t_1]_1) \overset{\nu_n}{\hookrightarrow} < \mathbb{P}(K[t_0, t_1]_n) > = (\mathbb{P}^n)^*. \tag{4.15}$$

By this identification we can view a linear form $u_0x_0 + \cdots + u_nx_n \in (\mathbb{P}^n)^*$ as a polynomial of degree n in the variable t_0, t_1:

$$\begin{array}{rcl} \mathbb{P}(K[x_0, \ldots, x_n]_1) = (\mathbb{P}^n)^* & \simeq & \nu_n(\mathbb{P}^1) = < \mathbb{P}(K[t_0, t_1]_n) > \\ [u_0x_0 + \cdots + u_nx_n] & \leftrightarrow & [u_0t_0^n + u_1t_0^{n-1}t_1 + \cdots + u_nt_1^n] \end{array}. \tag{4.16}$$

Therefore

$$\begin{array}{rcl} \mathbb{G}(n-1, n+d-1) \cap \mu_d(\nu_d(\mathbb{P}^n)) & \leftrightarrow & \mathbb{P}(K[t_0, t_1]_n) \\ \eta_{n+d-1}(< P_1, \ldots, P_{n-1}, P >) & \leftrightarrow & [u_0t_0^n + u_1t_0^{n-1}t_1 + \cdots + u_nt_1^n] \end{array}. \tag{4.17}$$

By the same reason

$$\begin{array}{rcl} \mathbb{G}(n-2, n+d-2) \cap \mu_d(\nu_d(\mathbb{P}^{n-1})) & \leftrightarrow & \mathbb{P}(K[t_0, t_1])_{n-1} \\ \eta_{n+d-2}(< P_1, \ldots, P_{n-1} >) & \leftrightarrow & [v_0t_0^{n-1} + v_1t_0^{n-2}t_1 + \cdots + v_{n-1}t_1^{n-1}] \end{array}. \tag{4.18}$$

Theorem 4.4.2. *Let* $a_0t_0^{n-1} + a_1t_0^{n-2}t_1 + \cdots + a_{n-1}t_1^{n-1} \in K[t_0, t_1]_{n-1}$, *let* $P_1, \ldots, P_{n-1}$ *be* $n-1$ *points on* $\nu_{n+d-1}(\mathbb{P}^1)$ *corresponding to the solutions of* $a_0t_0^{n-1} + a_1t_0^{n-2}t_1 + \cdots + a_{n-1}t_1^{n-1} = 0$ *and set* $L_1 = \sum_{i=0}^{n-1} a_ix_i \in S_1$ *and* $L_2 = \sum_{i=0}^{n-1} a_ix_{i+1} \in S_1$. *Then the locus* $\{\Lambda \in \mathbb{G}(n-1, n+d-1) \mid \eta_{n+d-1}(P_1), \ldots, \eta_{n+d-1}(P_{n-1}) \in \Lambda\}$ *parameterizes the forms of* $K[L_1, L_2]_d$.

Proof. If $P \in \nu_{n+d-1}(\mathbb{P}^1) =< K[t_0, t_1]_{n+d-1} >$, then there exists $(e_0t_0 + e_1t_1) \in K[t_0, t_1]$ such that:

$$P \leftrightarrow \left[(e_0t_0 + e_1t_1)^{n+d-1}\right] \in \mathbb{P}(K[t_0, t_1]_{n+d-1})$$

for some $e_0, e_1 \in K$. For (4.17) there exist $u_0, \ldots, u_n \in K$ such that $\eta_{n+d-1}(< P_1, \ldots, P_{n-1}, P >)$ corresponds to $[u_0t_0^n + u_1t_0^{n-1}t_1 + \cdots + u_nt_1^n]$. Now, the considerations (4.16), (4.17) and (4.18) above imply that if $P_1, \ldots, P_{n-1}$ are roots of $a_0t_0^{n-1} + a_1t_0^{n-2}t_1 + \cdots + a_{n-1}t_1^{n-1} \in K[t_0, t_1]_{n-1}$ then $u_0t_0^n + u_1t_0^{n-1}t_1 + \cdots + u_nt_1^n$ has to factorizes as $(e_0t_0 + e_1t_1)(a_0t_0^{n-1} + a_1t_0^{n-2}t_1 + \cdots + a_{n-1}t_1^{n-1}) = e_0(a_0t_0^n + a_1t_0^{n-1}t_1 + \cdots + a_{n-1}t_0t_1^{n-1}) + e_1(a_0t_0^{n-1}t_1 + a_1t_0^{n-2}t_1^2 + \cdots + a_{n-1}t_1^n) = e_0(a_0x_0 + a_1x_1 + \cdots + a_{n-1}x_{n-1}) + e_1(a_0x_1 + \cdots + a_nx_n) = e_0L_1 + e_1L_2$. So we have shown that there is a $1:1$ correspondence between the $\mathbb{P}^d = \{\Lambda \in \mathbb{G}(n-1, n+d-1) \mid \eta_{n+d-1}(P_1), \ldots, \eta_{n+d-1}(P_n) \in \Lambda\}$ and $\mathbb{P}(K[L_1, L_2]_d)$ obtained by the following construction:

$$\begin{array}{rcccl} \mathbb{P}^d = \{\Lambda \in \mathbb{G}(n-1, n+d-1) \mid \eta_{n+d-1}(P_1), \ldots, \eta_{n+d-1}(P_n) \in \Lambda\} & \hookrightarrow & \mathbb{P}(K[x_1, \ldots, x_n]_d) & \to & \mathbb{P}(K[L_1, L_2]_d) \\ \Lambda_P =< P_1, \ldots, P_{n-1}, P > & \mapsto & (u_0x_0 + \cdots + u_nx_n)^d & \mapsto & (e_0L_1 + e_1L_2)^d \end{array}$$

where $P_1, \ldots, P_{n-1}, P$ and L_1, L_2 are defined as above. □

Corollary 4.4.3. *The locus of* $(n-1)$*-linear spaces which are* $(n-1)-$*secant to* $\eta_{n+d-1}(\nu_{n+d-1}(\mathbb{P}^1))$ *is contained in* $\mathrm{Split}_d(\mathbb{P}^n) \cap \mathbb{G}(n-1, n+d-1)$.

Proof. It is a consequence of the previous theorem and of the fact that if L_1 and L_2 are two linear forms of S_1 then $\mathbb{P}(K[L_1, L_2]_d) \subset \mathrm{Split}_d(\mathbb{P}^n)$. □

Definition 4.4.4. *Let* $f, g \in K[x_1, \ldots, x_n]$ *of positive degree in* x_1*:*

$$f = a_0x_1^l + \cdots + a_l, \quad a_0 \neq 0,$$

$$g = b_0x_1^m + \cdots + b_m, \quad b_0 \neq 0,$$

where $a_i, b_i \in K[x_2, \ldots, x_n]$. *We define the Resultant of* f *and* g *with respect to* x_1 *to be the determinant*

$$Res(f, g, x_1) = \det \begin{pmatrix} \underbrace{\begin{matrix} a_0 & & & \\ a_1 & a_0 & & \\ & a_1 & \ddots & \\ \vdots & & \ddots & a_0 \\ & \vdots & & a_1 \\ a_l & & & \\ & a_l & & \vdots \\ & & \ddots & \\ & & & a_l \end{matrix}}_{m \text{ columns}} & \underbrace{\begin{matrix} b_0 & & & \\ b_1 & b_0 & & \\ & b_1 & \ddots & \\ \vdots & & \ddots & b_0 \\ & \vdots & & b_1 \\ b_m & & & \\ & b_m & & \vdots \\ & & \ddots & \\ & & & b_m \end{matrix}}_{l \text{ columns}} \end{pmatrix}$$

where the empty spaces are filled by zeros.

Lemma 4.4.5. *Let* $f, g \in K[x_1, \ldots, x_n]$ *have positive degree in* x_1. *Then* $Res(f, g, x_1) = 0$ *if and only if* f *and* g *have a common factor in* $K[x_1, \ldots, x_n]$ *which has positive degree in* x_1.

Proof. For a proof see for example [**CLO**]. □

Consider now a generalization of the Resultant.

Lemma 4.4.6. *Let* $f = a_0 x_1^d + \cdots + a_d$, $g = b_0 x_1^d + \cdots + b_d \in K[x_1, \ldots, x_n]$ *of degree* d *in the variable* x_1, *where* $a_i, b_i \in K[x_2, \ldots, x_n]$ *and* $a_0, b_0 \neq 0$. *The two polynomials* f *and* g *have a common factor of degree* $d - r$ *in* x_1 *if and only if the rank of the following* $(r + d + 1) \times (2r + 2)$ *matrix is strictly less then* $2r + 2$:

$$\begin{pmatrix} a_0 & & & b_0 & & \\ \vdots & \ddots & & \vdots & \ddots & \\ a_r & \cdots & a_0 & b_r & \cdots & b_0 \\ a_{r+1} & & \vdots & b_{r+1} & & \vdots \\ \vdots & & \vdots & \vdots & & \vdots \\ a_d & & \vdots & b_d & & \vdots \\ & \ddots & \vdots & & \ddots & \vdots \\ & & a_d & & & b_d \end{pmatrix} \tag{4.19}$$

Proof. The two polynomials a and b have a common factor $e \in K[x_1, \ldots, x_n]$ of degree $d-r$ in the variable x_1 if and only if there exist $c, d \in K[x_1, \ldots, x_n]$ of degree r in x_1 such that $a = e \cdot d$ and $b = e \cdot c$. This is equivalent to

$$a \cdot c + b \cdot d = 0. \tag{4.20}$$

Now, by taking as variables the coefficients of c and d, the equation (4.20) becomes a homogeneous linear system, and saying that it admits solution is equivalent to ask that the matrix (4.19) has rank at most $2r+1$. □

Proposition 4.4.7. *The intersection between the Grassmannian* $\mathbb{G}(n-1, n+1)$ *and* $\mathrm{Split}_2(\mathbb{P}^n)$ *in* $\mathbb{P}^{\frac{n^2+3n}{2}}$ *is the locus* $\{\Lambda \in \mathbb{G}(n-1, n+1) \mid length\{\Lambda \cap \eta_{n+1}(\nu_{n+1}(\mathbb{P}^1))\} \geq n-1\}$.

Proof. The inclusion $\{\Lambda \in \mathbb{G}(n-1, n+1) \mid length\{\Lambda \cap \eta_{n+1}(\nu_{n+1}(\mathbb{P}^1))\} \geq n-1\} \subset \mathrm{Split}_2(\mathbb{P}^n) \cap \mathbb{G}(n-1, n+1)$ is a consequence of Theorem 4.4.2. Let us look at the other inclusion.

As we have recalled above $\mathbb{G}(n-1, n+1) \simeq \mathbb{G}(1, n+1)$. Moreover, if M_{n+1} is defined as in (4.12), we have shown, in that example, that the equations of $\mathbb{G}(1, n+1)$ are obtained by imposing that $\mathrm{rk}(M_{n+1}) = 2$.
Let $\{p_{i,j}\}_{0 \leq i \leq n, 1 \leq j \leq n+1}$ be the Plücker coordinates for $\mathbb{G}(1, n+1)$. For example, if L is the line of $\mathbb{P}^{n+1}$ spanned by the points $(u_0, \ldots, u_n, 0), (0, u_0, \ldots, u_n) \in \mathbb{P}^{n+1}$, then

$$p_{i,j} = \begin{vmatrix} u_i & u_j \\ u_{i-1} & u_{j-1} \end{vmatrix}$$

with the assumption that $u_{i-1} = 0$ if $i = 0$ and $u_j = 0$ if $j = n+1$.

Let us consider the Veronese variety $\nu_2(\mathbb{P}^n)$, embedded into $\mathbb{P}^N$ with $N = \binom{n+2}{2} - 1$ as follows:

$$\begin{array}{rcl} \nu_2(\mathbb{P}^n) & \hookrightarrow & \mathbb{P}^N \\ (u_0, \ldots, u_n) & \mapsto & (p_{0,1}, \ldots, p_{n,n+1}). \end{array}$$

The points $(\alpha_{0,0}, \ldots, \alpha_{n,n})$ of $\mathbb{P}^N$ are in $(1:1)$-correspondence with quadrics $\sum_{i,j=0}^{n} \alpha_{i,j} x_i x_j$ of S_2 and the relation between $\alpha_{i,j}$ and $p_{i,j}$ is via the u_i:

$$\begin{cases} \alpha_{i,i} = u_i^2, & \text{if } i = j; \\ \alpha_{i,j} = 2u_i u_j, & \text{if } i \neq j. \end{cases}$$

The quadric of $\mathbb{P}^n$ can be represented by the symmetric matrix $A_n = (\alpha_{i,j})_{0 \leq i,j \leq n}$ where $\alpha_{i,j} =$

$\begin{cases} p_{i,j+1} + \alpha_{i-1,j+1}, & \text{if } i \leq j; \\ \alpha_{j,i}, & \text{if } i > j. \end{cases}$ with $\alpha_{i-1,j+1} = 0$ if $i = 0$ or $j = n$. Therefore:

$$A_n = \begin{pmatrix} p_{0,1} & p_{0,2} & p_{0,3} & \cdots & p_{0,n+1} \\ p_{0,2} & p_{1,2} + p_{0,3} & p_{1,3} + p_{0,4} & \cdots & p_{1,n+1} \\ p_{0,3} & p_{1,3} + p_{0,4} & p_{2,3} + p_{1,4} + p_{0,5} & \cdots & p_{2,n+1} \\ \vdots & \vdots & & & \vdots \\ p_{0,n+1} & p_{1,n+1} & \cdots & \cdots & p_{n,n+1} \end{pmatrix}. \tag{4.21}$$

With this description it turns out that imposing the vanishing of all 3×3 minors of A_n is equivalent to describe $\mathrm{Split}_2(\mathbb{P}^n)$ set theoretically.
This condition is equivalent to asking that there exist $a_0, \ldots, a_n, b_0, \ldots, b_n \in K$ such that

$$A_n = \begin{pmatrix} a_0 & b_0 \\ \vdots & \vdots \\ a_n & b_n \end{pmatrix} \begin{pmatrix} a_0 & \cdots & a_n \\ b_0 & \cdots & b_n \end{pmatrix}.$$

We can rewrite the matrix M_{n+1} defined in (4.12) as $M_{n+1} = (m_{i,j})_{0 \leq i,j \leq n}$ by using these $a_i, b_j \in K$. The matrix M_{n+1} is skew symmetric and

$$m_{i,j} = \alpha_{i,j-1} - \alpha_{i-1,j} = a_{i-1}a_{j-2} + b_{i-1}b_{j-2} - (a_{i-2}a_{j-1} - b_{i-2}b_{j-1}) =$$

$$= \begin{pmatrix} -a_{i-2} & a_{i-1} & -b_{i-2} & b_{i-1} \end{pmatrix} \begin{pmatrix} a_{j-1} \\ a_{j-2} \\ b_{j-1} \\ b_{j-2} \end{pmatrix}.$$

Now one can observe that M_{n+1} can be obtained as follows:

$$M_{n+1} = \begin{pmatrix} a_0 & 0 & b_0 & 0 \\ a_1 & a_0 & b_1 & b_0 \\ \vdots & \vdots & \vdots & \vdots \\ a_n & a_{n-1} & b_n & b_{n-1} \\ 0 & a_n & 0 & b_n \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} a_0 & a_1 & \cdots & a_n & 0 \\ 0 & a_0 & \cdots & a_{n-1} & a_n \\ b_0 & b_1 & \cdots & b_n & 0 \\ 0 & b_0 & \cdots & b_{n-1} & b_n \end{pmatrix} =: C^T G C.$$

This means that if $\mathrm{rk}(A_n) = 2$ then $\mathrm{rk}(M_{n+1}) \leq 4$.

Now we want to prove the inclusion $\mathbb{G}(n-1, n+1) \cap \mathrm{Split}_2(\mathbb{P}^n) \subseteq \{\Lambda \in \mathbb{G}(n-1, n+1) \mid length\{\Lambda \cap \eta_{n+1}(\nu_{n+1}(\mathbb{P}^1))\} \geq n-1\}$.

We have to consider the condition "$\mathrm{rk}(M_{n+1}) = 2$", which, since M_{n+1} is skew symmetric, is equivalent at "$\mathrm{rk}(M_{n+1}) \leq 3$". Now $\mathrm{rk}(M_{n+1}) \leq 3$ iff $\mathrm{rk}(C) \leq 3$ that is equivalent, by Lemma

4.4.6, to say that the two polynomials $a := a_0t_0^n + a_1t_0^{n-1}t_1 + \cdots + a_nt_1^n \in K[t_0, t_1]_n$ and $b := b_0t_0^n + b_1t_0^{n-1}t_1 + \cdots + b_nt_1^n \in K[t_0, t_1]_n$ have a common factor of degree $n-1$. This implies exactly that the elements of $\mathbb{G}(n-1, n+1) \cap \mathrm{Split}_2(\mathbb{P}^n)$ are $(n-1)$-spaces that intersect $\nu_{n+1}(\mathbb{P}^1)$ at least $n-1$ times. □

Remark: We have already observed that $\mathrm{Sec}_1(\nu_2(\mathbb{P}^n)) = \mathrm{Split}_2(\mathbb{P}^n)$. In (4.10) we have characterized $\mathrm{Split}_2(\mathbb{P}^n)$ via the symmetric $n \times n$ matrices of rank at most 2. This is not by chance because the elements of $\nu_2(\mathbb{P}^n)$ represent the symmetric $n \times n$ matrices of rank 1, hence $\mathrm{Sec}_1(\nu_2(\mathbb{P}^n))$ has to parameterizes the quadrics of $\mathbb{P}^n$ whose representative $n \times n$ matrices are symmetric and have at most rank 2. In fact a generic element of $\mathrm{Sec}_{r-1}(\nu_2(\mathbb{P}^n))$ is obtained as a linear combination of r elements of $\nu_2(\mathbb{P}^n)$ and the linear combination of r symmetric matrices of rank 1 is a symmetric matrix of rank less or equal than r.

Lemma 4.4.8. *If A_n and M_{n+1} are defined as in (4.21) and (4.12) respectively, and if*

$$T_{n+1} := \begin{pmatrix} t_0^{n+1} \\ t_0^n t_1 \\ \vdots \\ t_1^{n+1} \end{pmatrix}, \tag{4.22}$$

then $\mathrm{rk}(A_n) \leq r$ *if and only if the system* $M_{n+1} \cdot T_{n+1} = 0$ *admits at least* $n - r + 1$ *solutions in* $\mathbb{P}^1$*, counted with multiplicity.*

Proof. For the easiest implication ("$\Leftarrow$") we will show that solving the system

$$M_{n+1} \cdot T_{n+1} = 0 \tag{4.23}$$

is equivalent to solve

$$A_n \cdot T_n = 0 \tag{4.24}$$

then, since $M_{n+1} \cdot T_{n+1} = 0$ admits $n + 1 - r$ solutions if and only if the polynomials appearing in $M_{n+1} \cdot T_{n+1}$ have a degree $n + 1 - r$ common factor, then also the entries of $A_n \cdot T_n$ have a common factor of the same degree, and this implies that $\mathrm{rk}(A_n) = r$.

Let $M_{n+1(i)}$ and $A_{n(i)}$ be the i-th rows of M_{n+1} and A_n respectively. The first row of the system (4.23) can be written as $[0, A_{n(1)}] \cdot T_{n+1} = 0$ where $[0, A_{n(1)}]$ is a row whose first element is zero and the others are the same of $A_{n(1)}$. Now $[0, A_{n(1)}] \cdot T_{n+1} = 0$ is equivalent to $[0, A_{n(1)}] \cdot \begin{pmatrix} 0 \\ t_0^n t_1 \\ \vdots \\ t_1^{n+1} \end{pmatrix} = 0$

that is clearly equivalent to $t_1 A_{n(1)} \cdot T_n = 0$. This meas that, if $t_1 \neq 0$, then $M_{n+1(1)} \cdot T_{n+1} = 0$ iff $A_{n(1)} \cdot T_n = 0$.
For $i > 1$ we will show that

$$\big([A_{n(i-1)}, 0] + M_{n+1(i)}\big) \cdot T_{n+1} = [0, A_{n(i)}] \cdot T_{n+1}$$

from which $M_{n+1(i)} \cdot T_{n+1} = 0$ if and only if $A_{n(i)} \cdot T_n = 0$ that will give the first implication.
Let $a_{i,j}$, $a'_{i,j}$ and $m_{i,j}$ be the j-th elements of $A_{n(i)}$, $[A_{n(i)}, 0]$ and $M_{n+1(i)}$ respectively. Now we need only to make an easy calculation to prove that

$$a'_{i-1,j} + m_{i,j} = \begin{cases} 0, & \text{if } j = 1; \\ a_{i,j-1}, & \text{if } j > 1; \end{cases}$$

where $(a_{i,j})_{1 \leq i,j \leq n+1} = A_n$:

$$\begin{array}{ll}
\text{if } j = 1: & a'_{i-1,1} + m_{i,1} = a_{i-1,1} + m_{i,1} = p_{0,i-1} - p_{0,i-1} = 0; \\
\text{if } 1 < j < i: & a'_{i-1,j} + m_{i,j} = a_{i-1,j} + m_{i,j} = p_{j-1,i-1} + a_{i,j-1} - p_{j-1,i-1} = a_{i,j-1}; \\
\text{if } j = i: & a'_{i-1,j} + m_{i,j} = a_{i-1,i} + m_{i,i} = a_{i-1,i} + 0 = a_{i,i-1} = a_{i,j-1}; \\
\text{if } i < j < n+2: & a'_{i-1,j} + m_{i,j} = a_{i-1,j} + m_{i,j} = a_{i-1,j} + p_{i-1,j-1} = a_{i,j-1}; \\
\text{if } j = n+2: & a'_{i-1,j} + m_{i,j} = 0 + m_{i,j} = p_{i-1,n+1} = a_{i,n+1}.
\end{array}$$

The other implication ("$\mathrm{rk}(A_n) = r \Rightarrow M_{n+1} \cdot T_{n+1} = 0$ admits $n - r + 1$ solutions") is more computational.

First we observe that if $i \leq j$, then:

$$a_{i,j} = \sum_{k=0}^{\min\{i-1, n-j+1\}} m_{i-k,j+k+1}.$$

Moreover, since $\mathrm{rk}(M_{n+1}) = 2$ and M_{n+1} is a skew symmetric matrix, there exist $\alpha_0, \ldots, \alpha_{n+1}$ and $\beta_0, \ldots, \beta_{n+1} \in K$ such that

$$M_{n+1} = \begin{pmatrix} \alpha_0 & \beta_0 \\ \vdots & \vdots \\ \alpha_{n+1} & \beta_{n+1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \alpha_0 & \cdots & \alpha_{n+1} \\ \beta_0 & \cdots & \beta_{n+1} \end{pmatrix}.$$

Therefore $a_{i,j} = \alpha_{i-1}\beta_j - \alpha_j\beta_{i-1} + a_{i-1,j+1} = \sum_{k=1}^{\min\{i,n-(j-1)\}} \alpha_{i-k}\beta_{j+k-1} - \sum_{k=1}^{\min\{i,n-(j-1)\}} \alpha_{j+(k-1)}\beta_{i-k}$ if $i \leq j$.

Let us define the following matrices:

$$E_r := \begin{pmatrix} \alpha_0 & \cdots & & \alpha_r & \alpha_{r+1} & \cdots & \alpha_{n+1} & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots & \vdots & & & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & & & & \ddots & 0 \\ 0 & \cdots & 0 & \alpha_0 & \alpha_1 & \cdots & \cdots & \cdots & \cdots & \alpha_{n+1} \\ \beta_0 & \cdots & & \beta_r & \beta_{r+1} & \cdots & \beta_{n+1} & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots & \vdots & & & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & & & & \ddots & 0 \\ 0 & \cdots & 0 & \beta_0 & \beta_1 & \cdots & \cdots & \cdots & \cdots & \beta_{n+1} \end{pmatrix} \in M_{2r+2,r+n+2}(K),$$

$$H_r := \begin{pmatrix} 0 & & \cdots & & 0 & 1 \\ & & & 0 & 1 & 0 \\ \vdots & & 0 & 1 & 0 & \\ & 0 & -1 & 0 & & \vdots \\ 0 & -1 & 0 & & & \\ -1 & 0 & & \cdots & & 0 \end{pmatrix} \in M_{2r+2}(K).$$

One can observe (we omit the computations because they are too tedious) that the product $E_r^T H_r E_r$ (which we write by blocks) is the sum of the following $(n+r+2) \times (n+r+2)$ matrices:

$$E_r^T H_r E_r = \left(\begin{array}{c|c} 0 & 0 \\ \hline -A_n & 0 \end{array}\right) + \left(\begin{array}{c|c} 0 & A_n \\ \hline 0 & 0 \end{array}\right).$$

Now, since the rank of A_n is at most r by hypothesis, the rank of $E_r^T H_r E_r$ has to be at most $2r$; this fact is equivalent to "$\mathrm{rk}(E_r^T H_r E_r) \leq 2r+1$" because $E_r^T H_r E_r$ is skew symmetric, then also $\mathrm{rk}(E_r) \leq 2r+1$. This last condition is equivalent, by Lemma 4.4.6, to say that $a := \alpha_0 t_0^{n+1} + \alpha_1 t_0^n t_1 + \cdots + \alpha_{n+1} t_1^{n+1} \in K[t_0, t_1]_{n+1}$ and $b := \beta_0 t_0^{n+1} + \beta_1 t_0^n t_1 + \cdots + \beta_{n+1} t_1^{n+1} \in K[t_0, t_1]_{n+1}$ have a common factor of degree $n-r+1$. $\square$

By last Remark in Section 4.1 we know that $\mathrm{Split}_2(\mathbb{P}^n) = \mathrm{Sec}_1(\nu_2(\mathbb{P}^n))$, then Proposition 4.4.7 cas be rephrased as

> "The intersection between the Grassmannian $\mathbb{G}(n-1, n+1)$ and $\mathrm{Sec}_1(\nu_2(\mathbb{P}^n))$ in $\mathbb{P}^{\frac{n^2+3n}{2}}$ is the locus $\{\Lambda \in \mathbb{G}(n-1, n+1) \mid length\{\Lambda \cap \eta_{n+1}(\nu_{n+1}(\mathbb{P}^1))\} \geq n-1\}$. "

We can now generalize Proposition 4.4.7 to $\mathrm{Sec}_{r-1}(\nu_2(\mathbb{P}^n))$.

Proposition 4.4.9. *The intersection between the Grassmannian* $\mathbb{G}(n-1, n+1)$ *and the variety* $\mathrm{Sec}_{r-1}(\nu_2(\mathbb{P}^n))$ *(which corresponds to the locus* $Q_r := \{A \in M_{n+1}(K)$ *s.t.* A *is symmetric and* $\mathrm{rk}(A) \leq r\}$*) corresponds to the set of all* $(n-1)$*-spaces of* $\mathbb{P}^{n+1}$ *that are* $(n-r+1)$*-secant to the rational normal curve* $\nu_{n+1}(\mathbb{P}^1)$ *embedded into* $\mathbb{G}(n-1, n+1)$ *via the map* η_{n+1} *defined in (4.14).*

Proof. Let us identify only for this proof, with an abuse of notation, an element of $\mathbb{G}(n-1, n+1)$ with a skew symmetric $(n+1) \times (n+1)$ matrix M_{n+1} defined in (4.12). By the previous lemma, the locus $\{A \in M_{n+1}(K) \mid \mathrm{rk}(A) \leq r \text{ and } A = A^T\}$ corresponds to the subset of $\mathbb{G}(n-1, n+1)$ of the skew symmetric $(n+1) \times (n+1)$ matrices M_{n+1} such that the system $M_{n+1} \cdot T_{n+1} = 0$, where T_{n+1} is defined in (4.22), admits at most $n-r+1$ solutions. Such an M_{n+1} describes an $(n-1)$-space of $\mathbb{P}^{n+1}$ that is $(n-r+1)$- secant to the embedding of $\nu_{n+1}(\mathbb{P}^1)$ into $\mathbb{G}(n-1, n+1)$ via η_{n+1}. $\square$

Corollary 4.4.10. *The intersection between* $\mathrm{Sec}_{s-1}(\mathrm{Split}_2(\mathbb{P}^n))$ *and* $\mathbb{G}(n-1, n+1)$ *is set-theoretically the locus* $\{\Lambda \in \mathbb{G}(n-1, n+1) \mid \Lambda$ *is* $(n-2s+1)-$ *secant to* $\eta_{n+1}(\nu_{n+1}(\mathbb{P}^1))\}$.

Proof. This is a consequence of the previous proposition and of the observation that, since $\mathrm{Split}_2(\mathbb{P}^n) = \{A \in M_{n+1}(K) \text{ s.t. } A \text{ is symmetric and } \mathrm{rk}(A) = 2\}$ and the elements of $\mathrm{Split}_2(\mathbb{P}^n)$ are of the form $[L_1 \cdot L_2]$ with $L_1, L_2 \in S_1$, then $\mathrm{Sec}_{s-1}(\mathrm{Split}_2(\mathbb{P}^n)) = \{[L_1L_2 + \cdots + L_{2s-1}L_{2s}] \in \mathbb{P}(S_2) \mid L_i \in S_1 \text{ for } i = 1, \ldots, 2s\}$ is the set of all symmetric matrices of $M_{n+1}(K)$ of rank at most $2s$. $\square$

4.4.1 A conjecture

Conjecture 4.4.11. *The intersection between* $\mathrm{Split}_d(\mathbb{P}^n)$ *and* $\mathbb{G}(n-1, n+d-1)$ *is the locus* $\{\Lambda \in \mathbb{G}(n-1, n+d-1) \mid \mathit{length}\{\Lambda \cap \eta_{n+d-1}(\nu_{n+d-1}(\mathbb{P}^1))\} \geq n-1\}$.

As we have seen, the conjecture is proved for $d = 2$.
In the case of $d = 3$ we have computed with [**Macaulay**] the example of $n = 2$ and it turns out that in fact:

$$\mathbb{G}(1,4) \cap \mathrm{Split}_3(\mathbb{P}^2) = \{l \in \mathbb{G}(1,4) \mid \mathit{length}\{l \cap \eta_4(\nu_4(\mathbb{P}^1))\} \geq 1\}.$$

We will only give a hint that suggests at least one inclusion (see Proposition 4.4.14).

The embedding μ_d defined in (4.13) can be generalized to the domain $K[x_0, \ldots, x_n]_d$. With this generalization there can exist degree d forms, different from d-th power of linear forms, whose image via μ_d are elements of $\mathbb{G}(n-1, n+d-1)$.

Proposition 4.4.12. *Let* L_1, L_2 *be two linear forms of* $S_1 = K[x_0, \ldots, x_n]_1$. *If there exists* $M \in K[L_1, L_2]_3$ *such that* $\mu_3(M) \in \mathbb{G}(n-1, n-2)$ *then* $\mu_3(K[L_1, L_2]_3)$ *is completely contained in* $\mathbb{G}(n-1, n+2)$.

Proof. Consider the cubic $C = (\alpha L_1 + \beta L_2)^3 \in K[L_1, L_2]_3$. Since it is the third power of a linear form $\alpha L_1 + \beta L_2 \in K[x_0, \ldots, x_n]_1$, C can be interpreted as an element of $\nu_3(\mathbb{P}^n)$, then $\mu_3(C) \in \mathbb{G}(n-1, n+2)$ for all $\alpha, \beta \in K$. Therefore the image of the whole twisted cubic $\mu_3(\nu_3(\mathbb{P}^1)) = \mu_3((\alpha L_1 + \beta L_2)^3) \in \mu_3(K[L_1, L_2]_3)$ for $\alpha, \beta \in K$ is contained in $\mathbb{G}(n-1, n+2)$. Hence we have that both $\mu_3(M)$ and the image of the twisted cubic $\mu_3(\nu_3(\mathbb{P}^1))$ are in $\mathbb{G}(n-1, n+2)$, then it is possible to find a line completely contained in $\mathbb{G}(n-1, n+2)$ passing through $\mu_3(M)$ and bi-secant to $\mu_3(\nu_3(\mathbb{P}^1))$. But the Grassmannian is generated by quadrics, then the span of $\mu_3(\nu_3(\mathbb{P}^1))$ is completely contained in $\mathbb{G}(n-1, n+2)$; i.e. $\mu_3(K[L_1, L_2]_3) \subset \mathbb{G}(n-1, n+2)$. □

Lemma 4.4.13. *Let $A, B \in \mu_d(\nu_d(\mathbb{P}^n))$. If there exists a point $C \in \mu_d(\mathrm{Sec}_1(\nu_d(\mathbb{P}^n))) \cap \mathbb{G}(n-1, n+d-1)$ such that $C \in < A, B > \setminus \mu_d(\nu_d(\mathbb{P}^n))$, then $< A, B > \subset \mathbb{G}(n-1, n+d-1)$.*

Proof. The set of the three points $\{A, B, C\}$ is contained in the intersection $< A, B > \cap \mathbb{G}(n-1, n+d-1)$. Since the Grassmannian is an intersection of quadrics, it cannot exist a point $D \in < A, B >$ but $D \notin \mathbb{G}(n-1, n+d-1)$ then $< A, B > \subset \mathbb{G}(n-1, n+d-1)$. □

Proposition 4.4.14. *The intersection between $\mu_d(\mathrm{Sec}_1(\nu_d(\mathbb{P}^n)))$ and $\mathbb{G}(n-1, n+d-1)$ is contained in $\{\Lambda \in \mathbb{G}(n-1, n+d-1) \mid length\{\Lambda \cap \eta_{n+d-1}(\nu_{n+d-1}(\mathbb{P}^1))\} \geq n-1\}$.*

Proof. Let us take a point $A \in \mu_d(Sec_1(\nu_d(\mathbb{P}^n)) \cap \mathbb{G}(n-1, n+d-1)) \setminus \nu_d(\mathbb{P}^n)$, then there exist $\pi_1, \pi_2 \in \nu_d(\mathbb{P}^n)$ such that $A \in \mu_d(< \pi_1, \pi_2 >)$. Since $\mu_d(\nu_d(\mathbb{P}^n))$ is the locus of the $(n-1)$-spaces of $\mathbb{P}^{n+d-1}$ that are n-secant to $\eta_{n+d-1}(\nu_{n+d-1}(\mathbb{P}^1))$, there exist $P_1, \ldots, P_n, Q_1, \ldots, Q_n \in \nu_{n+d-1}(\mathbb{P}^1)$ such that $\mu_d(\pi_1) = \eta_{n+d-1}(< P_1, \ldots, P_n >)$ and $\mu_d(\pi_2) = \eta_{n+d-1}(< Q_1, \ldots, Q_n >)$. Therefore $\mu_d(< \pi_1, \pi_2 >) \subset \mu_d(\mathrm{Sec}_1(\nu_d(\mathbb{P}^n))) \subset \mathrm{Split}_d(\mathbb{P}^n)$. By the Lemma 4.4.13 $\mu_d(< \pi_1, \pi_2 >) \subset \mathbb{G}(n-1, n+d-1)$. The image of the span $\mu_d(< \pi_1, \pi_2 >)$ parameterizes a pencil of $(n-1)$-spaces contained in $\mathbb{P}^n \subset \mathbb{P}^{n+d-1}$ and containing a $\mathbb{P}^{n-2}$. Then $P_1, \ldots, P_n, Q_1, \ldots, Q_n$ lie on a $\mathbb{P}^n$ instead of being generic in $< \nu_{n+d-1}(\mathbb{P}^1) >= \mathbb{P}^{n+d-1}$, hence $\sharp\{P_1, \ldots, P_n, Q_1, \ldots, Q_n\} = n+1$. □

Chapter 5

Secant varieties of Segre Varieties

In this chapter we finally present the study of the last problem mentioned in the section 2.6.1. This section is of an expository nature; here we will describe two different methods of approaching the study of secant varieties to Segre varieties. The first one is finalized to the study of their dimensions, the second one presents an algorithm to compute their ideals, in particular it will allow to prove the Garcia, Stillman, Strumfeld conjecture (see [**GSS**]) on the generation of the ideal of the first secant variety to the Segre variety with three factors: $\mathrm{Sec}_1(Seg(\mathbb{P}(A_1) \otimes \mathbb{P}(A_2) \otimes \mathbb{P}(A_3)))$, where A_1, A_2, A_3 are three vector spaces.

5.1 Inverse System for Segre Varieties

In this first section we want to present how the Apolarity method was used in [**CGG1**] and [**CGG3**] in order to study the dimension of the secant varieties to Segre varieties.

We study the embedding of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ into $\mathbb{P}^N$, with $N = \Pi_{i=1}^k (1 + n_i) - 1$, given by the following embedding:

$$\begin{array}{rcl} \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k} & \to & \mathbb{P}^N \\ (x_{1,0}, \ldots, x_{1,n_1}; \ldots; x_{k,0}, \ldots x_{k,n_k}) & \mapsto & (x_{1,0} \cdots x_{k,0}, \ldots, x_{1,i_1} \cdots x_{k,i_k}, \ldots, x_{1,n_1} \cdots x_{k,n_k}) \end{array}$$

where $i_j \in \{0, \ldots, n_j\}$, $j = 1, \ldots, k$, and $\{x_{i,0}, \ldots, x_{i,n_i}\}$ are homogeneous coordinates in $\mathbb{P}^{n_i}$.

Another way of viewing the Segre variety is as the variety which parameterizes completely decomposable tensors.

Definition 5.1.1. *Let $A_1, \ldots, A_k$ be vector spaces; a tensor $T \in A_1 \otimes \cdots \otimes A_k$ is said to be decomposable if there exist vectors $v_i \in A_i$ for $i = 1, \ldots, k$ such that $T = v_1 \otimes \cdots \otimes v_k$.*

If $A_1, \ldots, A_k$ are of dimension $n_1+1, \ldots, n_k+1$ respectively, and we set $\{x_{i,0}, \ldots, x_{i,n_i}\}$ as basis of A_i, then any $T \in A_1 \otimes \cdots \otimes A_k$ can be written as

$$T = \sum_{0 \leq j_i \leq n_i;\ 1 \leq i \leq t} \alpha_{j_1,\ldots,j_t} x_{1,j_1} \otimes \cdots \otimes x_{k,j_k}.$$

Definition 5.1.2. *The Tensor Rank of $T \in A_1 \otimes \cdots \otimes A_k$ is the minimal s such that T is a sum of s decomposable tensors.*

Observe that the tensor rank of every vector in $A_1 \otimes \cdots \otimes A_k$ is at most $\Pi_{i=1}^{k-1}(n_i+1)$. Moreover for any $T \in A_1 \otimes \cdots \otimes A_k$ and any scalar $\lambda \neq 0$, both T and λT have the same tensor rank. Thus it makes sense to speak of the tensor rank of an element in $\mathbb{P}(A_1 \otimes \cdots \otimes A_k)$.

If $T \in A_1 \otimes \cdots \otimes A_k$, then T corresponds to a multi-linear form

$$A_1^* \times \cdots \times A_k^* \to K.$$

Then a tensor T is completely described by its values on k-uples of basis vector $\{x_{0,i}^*, \ldots, x_{i,n_i}^*\}$:

$$T(x_{1,j_1}^*, \ldots, x_{k,j_k}^*) = \alpha_{j_1,\ldots,j_k}.$$

Let $S^j := K[x_{j,0}, \ldots, x_{j,n_j}]$ for $j = 1, \ldots, k$, and $S := K[x_{1,0}, \ldots, x_{1,n_1}; \ldots; x_{k,0}, \ldots, x_{k,n_k}]$. Consider the usual identifications $A_i^* = S_1^i$ and $A_1^* \otimes \cdots \otimes A_k^* = S_{\underline{1}}$ where $\underline{1} = (1, \ldots, 1)$. With this point of view, we can describe the Segre variety as the image of the embedding

$$\begin{array}{rcl} (\mathbb{P}^{n_1})^* \times \cdots \times (\mathbb{P}^{n_k})^* \simeq \mathbb{P}(S_1^1) \times \cdots \times \mathbb{P}(S_1^k) & \to & \mathbb{P}(S_{\underline{1}}) \\ ([L_1], \ldots, [L_k]) & \mapsto & [L_1 \otimes \cdots \otimes L_k] \end{array}.$$

The image of this map is the classical Segre Variety $Seg(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k})$ and it is clear that it parameterizes decomposable tensors in $A_1 \otimes \cdots \otimes A_k$.

So the problem of finding the tensor rank s for a generic tensor $T \in A_1 \otimes \cdots \otimes A_k$ is equivalent to find the minimum integer s such that the $(s-1)$-secant variety to the Segre variety $Seg(\mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_k))$ fills up $\mathbb{P}^N$.

The idea of the method used in [**CGG1**] and [**CGG3**] is to use Terracini's Lemma (see Lemma 2.6.1) in order to translate the problem of determining the dimension of secant varieties into the one of determining the value of the Hilbert function of generic sets of 2-fat points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$.

Terracini's Lemma translates the problem from the study of $\mathrm{Sec}_{s-1}(Seg(\mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_k)))$ to the study of the vector space $T_{1,\ldots,s} :=< T_{P_1}(Seg(\mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_k))), \ldots, T_{P_s}(Seg(\mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_k))) >$ where $T_{P_i}(Seg(\mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_k)))$ is the tangent space to $Seg(\mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_k))$ at a generic point $P_i \in Seg(\mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_k))$ for $i = 1, \ldots, s$.

The Corollary 2.6.2 allows to translate again the problem into the study of a projective scheme of s generic 2 fat points in $Seg(\mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_k))$. If we apply that corollary to the Segre Variety we find that

$$\dim(\mathrm{Sec}_{s-1}(Seg(\mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_k)))) = N - \dim(H^0(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}, \mathcal{O}_{\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_k}}(1,\ldots,1))).$$

If Z is a sub-scheme of s generic 2 fat points in $X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, then the Hilbert function $H(Z, \underline{j})$, with $\underline{j} \in \mathbb{N}^k$, is

$$H(Z, \underline{j}) = \dim(S_{\underline{j}}) - \dim(H^0(X, \mathcal{I}_Z(\underline{j}))).$$

In particular the typical rank of the Segre variety $Seg(\mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_k))$ is the smallest s for which there are no $(1,\ldots,1)$-forms in the ideal of s generic 2-fat points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$.

It is classically known that the tangent space $T_P(Seg(\mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_k)))$ at a point $P \simeq L_1 \otimes \cdots \otimes L_k \in Seg(\mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_k))$ is isomorphic to:

$$\left\{ \sum_{j=1}^{k} L_1 \cdots L_{j-1} M_j L_{j+1} \cdots L_k \mid M_j \in R_1^j,\ j = 1,\ldots,k \right\} \subset S_{\underline{1}}.$$

Let $W_{\underline{1}}$ be the affine cone over the tangent space $T_P(Seg(\mathbb{P}(A_1)\times\cdots\times\mathbb{P}(A_k))) \subset S_{\underline{1}}$. It is not restrictive to suppose that $L_i = x_{i,0}$ for $i = 1,\ldots,k$. Hence, consider in $R := K[y_{1,0},\ldots,y_{1,n_1};\ldots;y_{k,0},\ldots,y_{k,n_k}]$ the ideal:

$$\wp^2 = (y_{1,1}, y_{1,2}, \ldots, y_{1,n_1}; \ldots; y_{k,1}, y_{k,2}, \ldots, y_{k,n_k})^2. \tag{5.1}$$

Its inverse system is such that $(\wp^2)^{-1}_{\underline{1}} = W_{\underline{1}}$.

Notice that $\wp^2$ is the ideal of a 2-fat point in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$.
The ideal $I \subset R_{\underline{1}}$ such that $(I^{-1})_{\underline{1}}$ is equal to the affine cone over $T_{1,\ldots,s}$ is

$$I = (\wp_1^2 \cap \cdots \cap \wp_s^2),$$

where each $\wp_i^2 \subset R_{\underline{1}}$ is defined as an ideal of the form (5.1) with support on a point P_i, and $P_1,\ldots,P_s$ are generic points.

Thus, if Z is the projective scheme defined by the ideal I, then

$$\dim(T_{1,\ldots,s}) = H(Z, (1,\ldots,1)). \tag{5.2}$$

The methods used in [**CGG1**] and in [**CGG3**] are slightly different.

In order to present the main result in [**CGG1**] we need some notation.

Notation:

- A coordinate point of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ is a point $P_{\underline{r}} = (P_{r_1}, \ldots, P_{r_k})$ where P_{r_j} is the r_j-th coordinate point of $\mathbb{P}^{n_j}$.
- Given $\underline{r}_1 = (r_{1,1}, \ldots, r_{1,k})$ and $\underline{r}_2 = (r_{2,1}, \ldots, r_{2,k})$ in $\underline{J} := \{\underline{r} = (r_1, \ldots, r_k) \mid 0 \leq r_i \leq n_i\}$, we say that the *Hamming distance* between $\underline{r}_1$ and $\underline{r}_2$ is l if $(r_{1,1} - r_{2,1}, \ldots, r_{1,k} - r_{2,k})$ has exactly l non-zero entries.

A result in [**CGG1**] gives a translation of this problem in terms of code theory:

Theorem 5.1.3. *Let $P_{\underline{r}_1}, \ldots, P_{\underline{r}_s}$ be a set of coordinate points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. Let $\wp_i$ be the ideal of $P_{\underline{r}_i}$ and let Z be the scheme defined by $\wp_1^2 \cap \cdots \cap \wp_s^2$. Then*

$$H(Z, \underline{1}) = |\{\underline{r} \in \underline{J} \mid \underline{r} \text{ has Hamming distance } \leq \underline{1} \text{ from at least one of } \underline{r}_1, \ldots, \underline{r}_s\}|.$$

This theorem allows the authors to get some result: especially in the monomial case.

In [**CGG3**] the authors present another technique to compute (5.2). That paper examines a more general problem: it studies the secant varieties of the so called Segre-Veronese varieties that are the embedding of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ into $\mathbb{P}^N$ given by $\mathcal{L} = O_{\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}}(a_1, \ldots, a_k)$ where $a_i \in \mathbb{N}$. It is clear that the Segre variety is the particular case of the Segre-Veronese where $(a_1, \ldots, a_k) = (1, \ldots, 1)$.

Let $n = n_1 + \cdots + n_k$ and consider the birational map:

$$\begin{array}{rcl} g : \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k} & \dashrightarrow & \mathbb{A}^n \\ ((x_{1,0}, \ldots, x_{1,n_1}), \ldots, (x_{k,1}, \ldots, x_{k,n_k})) & \mapsto & \left(\frac{x_{1,1}}{x_{1,0}}, \ldots, \frac{x_{1,n_1}}{x_{1,0}}; \ldots; \frac{x_{k,1}}{x_{k,0}}, \ldots, \frac{x_{k,n_k}}{x_{k,0}}\right) \end{array}$$

which is defined in the open subset of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ given by $\{x_{1,0}, \ldots, x_{k,0} \neq 0\}$.

Consider $K[z_0, z_{1,1}, \ldots, z_{1,n_1}, z_{2,1}, \ldots, z_{2,n_2}, \ldots, z_{k,1}, \ldots, z_{k,n_k}]$ as the coordinate ring of $\mathbb{P}^n$ and the embedding $\varphi : \mathbb{A}^n \to \mathbb{P}^n$ whose image is the chart $\mathbb{A}^n_0 = \{z_0 = 1\}$. By composing $\varphi \circ g$ we get:

$$\begin{array}{rcl} f : \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k} & \dashrightarrow & \mathbb{P}^n \\ ((x_{1,0}, \ldots, x_{1,n_1}), \ldots, (x_{k,1}, \ldots, x_{k,n_k})) & \mapsto & \left(1, \frac{x_{1,1}}{x_{1,0}}, \ldots, \frac{x_{1,n_1}}{x_{1,0}}; \ldots; \frac{x_{k,1}}{x_{k,0}}, \ldots, \frac{x_{k,n_k}}{x_{k,0}}\right) = \\ & & = (x_{1,0} \cdots x_{k,0}, x_{1,1}x_{2,0} \cdots x_{k,0}, \ldots, x_{1,0} \cdots x_{k-1,0}x_{k,n_k}). \end{array}$$

Let $Z \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ be a zero dimensional scheme which is contained in the affine chart $\{x_{1,0} \cdots x_{k,0} \neq 0\}$ and let $Z' = f(Z)$. We can construct now a scheme $W \subset \mathbb{P}^n$ such that $H(W, a) = H(Z, (a_1, \ldots a_k))$ where $a = a_1 + \cdots + a_k$.

Let $Q_0, Q_{1,1}, \ldots, Q_{1,n_1}, \ldots, Q_{k,1}, \ldots, Q_{k,n_k}$ be the coordinate points of $\mathbb{P}^n$, then consider the hyperplanes $\Pi_i \simeq \mathbb{P}^{n_i - 1}$, with $\Pi_i =< Q_{i,1}, \ldots, Q_{i,n_i} >$; let W_i be the scheme given by $(a - a_i)\Pi_i$, i.e. the scheme defined by the ideal $(I_{\Pi_i})^{a - a_i}$. Notice that $W_i \cap W_j = \emptyset$ for $i \neq j$.

Theorem of [**CGG3**] that allows to translate the problem of the study of the Hilbert function of a projective scheme in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ into the study of the Hilbert function of a projective scheme in $\mathbb{P}^n$ is the following:

Theorem 5.1.4. *Let Z and Z' be as above, let $W = Z' \cup W_1 \cup \cdots \cup W_k \subset \mathbb{P}^n$, then*

$$\dim(I_W)_a = \dim(I_Z)_{(a_1,\ldots,a_k)}$$

where $a = a_1 + \cdots + a_k$.

Corollary 5.1.5. *Let $Z \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ be a generic set of s 2-fat points, let $W \subset \mathbb{P}^n$ be as in Theorem 5.1.4, then*

$$\dim(\mathrm{Sec}_{s-1}(Seg(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}))) = H(Z, (a_1, \ldots, a_k)) - 1 = N - \dim(I_W)_a.$$

Then the authors use the "Lemme d'Horace différentiel" (see [**AH**]) to do computations in many cases.

What we have seen here are methods to compute the dimension of the $(s-1)$-secant variety to the Segre variety; in the next section we will see a method to determine the generators of the ideal of the secant variety to the Segre variety.

5.2 Representations of Finite Groups

In this section we present an introduction about the Representation Theory of Finite Groups in order to present the method used in [**LM1**] to compute the ideals of the secant varieties of Segre varieties. Moreover they have proved that the ideal of $\mathrm{Sec}_1(Seg(\mathbb{P}(A_1)\otimes\mathbb{P}(A_2)\otimes\mathbb{P}(A_3)))$ is generated in degree 3 (this is, for the case of three factors, the Garcia, Stillman, Strumfeld conjecture: see [**GSS**]).

This introductive section follows the exposition of [**FHar**].

5.2.1 Preliminary Definitions

Definition 5.2.1. *A Representation of a finite group G on a finite-dimensional complex vector space V is an homomorphism*

$$\rho : G \to GL(V)$$

on the group of automorphisms of V.

This map gives V a structure of a G-module on V. In the language of Representation Theory the G-module V, equipped with the homomorphism ρ, is called a representation of G.

Definition 5.2.2. *A map φ between two representations V and W of G is a vector space map $\varphi : V \to W$ such that the following diagram commutes for every $g \in G$:*

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & W \\ g\downarrow & & \downarrow g \\ V & \xrightarrow{\varphi} & W \end{array} .$$

Such a map is called a G-linear map between V and W.

Definition 5.2.3. *A Subrepresentation of a representation V is a vector subspace $W \subset V$ which is invariant under G.*

Definition 5.2.4. *A representation is called Irreducible if there is no proper nonzero invariant subspace W of V.*

Proposition 5.2.5. *If V and W are representations of G, then also $V \oplus W$, $V \otimes W$, $V^{\otimes m}$, $\mathrm{Sym}^m(V)$, $\bigwedge^m(V)$, $V^* = \mathrm{Hom}(V, \mathbb{C})$ and $\mathrm{Hom}(V, W)$ are representations of G.*

Proof. The representation $V \otimes W$ is induced by $g(v \otimes w) = gv \otimes gw$.
The only one that is not obtained in an obvious way is V^*. If $\rho : G \to GL(V)$ is a representation then $\rho^* : G \to GL(V^*)$ must satisfy the following relation for all $g \in G$, $v \in V$ and $w^* \in V^*$:

$$< \rho^*(g)(w^*), \rho(g)(v) >=< w^*, v >$$

where $< \, , \, >$ is the natural pairing between V^* and V. This forces us to define the dual representation by

$$\rho^*(g) = {}^t\rho(g^{-1}) : V^* \to V^*$$

for all $g \in G$; the meaning of ${}^t\rho(g^{-1})$ is given by the following:

$$< \rho^*(g)(w^*), v >=< w^*, \rho(g^{-1})(v) > .$$

Now, since $\mathrm{Hom}(V, W) \simeq V^* \otimes W$, we have

$$(g\varphi)(v) = g[\varphi(g^{-1}v)]$$

for all $\varphi \in \mathrm{Hom}(V, W)$ and $v \in V$. In other words the following diagram has to commute:

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & W \\ g\downarrow & & \downarrow g \\ V & \xrightarrow{g\varphi} & W \end{array}.$$

□

Definition 5.2.6. *If X is any finite set and G acts on the left on X, i.e. $G \to \mathrm{Aut}(X)$ is an homomorphism to the permutation group of X, there is an associated Permutation Representation: let V be a vector space with basis $\{e_x : x \in X\}$, then G acts on V by*

$$g \cdot \sum a_x e_x = \sum a_x e_{gx}.$$

The Regular Representation R_G corresponds to the left action of G on itself, i.e. it is the space of complex-valued functions on G, where an element $g \in G$ acts on a function α by $(g\alpha)(h) = \alpha(g^{-1}h)$.

5.2.2 Schur's Lemma

Definition 5.2.7. *A representation is said to be Indecomposable if it cannot be expressed as a direct sum of other representations.*

Proposition 5.2.8. *If W is a subrepresentation of a representation V of a finite group G, then there is a complementary invariant subspace W' of V, so that $V = W \oplus W'$.*

Proof. Let U be a complementary subspace of W in V and $\pi_0 : V \to W$ be the projection given by the direct sum decomposition $V = W \oplus U$. Let $\pi : V \to W$ be defined by $\pi(v) = \sum_{g \in G} g(\pi_0(g^{-1}v))$. This is a G-linear map from V onto W and it is the multiplication by $|G|$ on W; its kernel will be a subspace of V invariant under G and complementary to W. □

Corollary 5.2.9. *Any representation of a finite group is a direct sum of irreducible representations.*

This property is called "Complete Reducibility" or "Semisimplicity", but we will see it better in Definition 5.5.6.

We are now ready to state Schur's Lemma:

Lemma 5.2.10. (**Schur**) *If V and W are irreducible representations of a finite group G and $\varphi : V \to W$ is a G-module homomorphism, then*

1. *either φ is an isomorphism, or $\varphi = 0$;*
2. *if $V = W$, then $\varphi = \lambda \cdot I$ for some $\lambda \in \mathbb{C}$, where I is the identity map.*

Proof. The first claim is a consequence of the fact that $\mathrm{Ker}(\varphi)$ and $\mathrm{Im}(\varphi)$ are invariant subspaces. For the second one it is sufficient to observe that there exists a $\lambda \in \mathbb{C}$ such that $\varphi - \lambda I$ has a nonzero kernel and apply the first claim from which φ as to be equal to λI. □

One of the most important consequences of this theorem is the following proposition (when we will quote Schur's Lemma we will usually be referring to this proposition).

Proposition 5.2.11. *For any representation V of a finite group G, there is a decomposition*

$$V = V_1^{\oplus a_1} \oplus \cdots \oplus V_k^{\oplus a_k},$$

where the V_i are distinct irreducible representations. This decomposition of V is unique, as are the V_i that occur and their multiplicities a_i.

Proof. If W is another representation of G with decomposition $W = \oplus W_j^{b_j}$, and $\varphi : V \to W$ is a map of representations, then φ must map the summand $V_i^{\oplus a_i}$ into a summand $W_j^{\oplus b_j}$ for which $W_j \simeq V_i$; when applied to the identity map of V to V, the stated uniqueness follows. □

A consequence of character theory (that we have not used yet but that we will introduce in Section 5.6) is the following:

Proposition 5.2.12. *Any irreducible representation V of G appears in the Regular Representation* $\dim(V)$ *times.*

5.3 The Group Algebra

The group algebra $\mathbb{C}G$ associated to a finite group G that we are going to define can completely replace the group G itself when we consider representations of G, since any proposition we can formulate about representations of G has an exactly equivalent statement in terms of its group algebra.

The underlying vector space of the group algebra of G is a vector space with basis $\{e_g \mid g \in G\}$, i.e. the underlying vector space of the Regular Representation.

The algebra structure on that space is defined as follows:

$$e_g \cdot e_h = e_{gh}.$$

A representation of the algebra $\mathbb{C}G$ on a vector space V is just an algebra homomorphism:

$$\mathbb{C}G \to \mathrm{End}(V)$$

so that a representation V of $\mathbb{C}G$ is a left $\mathbb{C}G$-module.

Observe that a representation $\rho : G \to \mathrm{Aut}(V)$ will extend by linearity to a map

$$\widetilde{\rho} : \mathbb{C}G \to \mathrm{End}(V) \tag{5.3}$$

so that representations of $\mathbb{C}G$ correspond to representations of G; the left $\mathbb{C}G$-module given by $\mathbb{C}G$ itself corresponds to the regular representation.

By applying this linear extension to any W_i appearing in the irreducible decomposition of the regular representation of G:

$$R = \bigoplus (W_i)^{\oplus \dim(W_i)},$$

we get a canonical map

$$\varphi : \mathbb{C}G \to \bigoplus \mathrm{End}(W_i)$$

that is injective since the representation is faithful. Now $\dim(\mathbb{C}G) = \sum(\dim W_i)^2 = \dim(\bigoplus \mathrm{End}(W_i))$ then

$$\mathbb{C}G \simeq \bigoplus \mathrm{End}(W_i).$$

5.4 Symmetric group and its Representations

5.4.1 Definitions

Definition 5.4.1. *A Permutation Group is a finite group G whose elements are permutations of a given set and whose group operation is composition of permutations in G.*

Definition 5.4.2. *The Symmetric Group $\mathfrak{S}_m$ of degree m is the group of all permutations on m symbols.*

The group $\mathfrak{S}_m$ is therefore a permutation group of order $m!$ and it contains as subgroups every group of order m. The number of conjugacy classes of $\mathfrak{S}_m$ is given by the partition function $p(m)$ which gives the number of ways of writing the integer m as a sum of positive integers, where the order of addends is not considered significant and it is obtained from the following formula:

$$\sum_{m=0}^{\infty} p(m)t^m = \prod_{n=1}^{\infty} \left(\frac{1}{1+t^n} \right) = (1+t+t^2+\cdots)(1+t^2+t^4+\cdots)(1+t^3+\cdots). \tag{5.4}$$

5.4.2 Young tableaux

The conjugacy classes of $\mathfrak{S}_m$ correspond to the partitions of m. If $\lambda = (\lambda_1 \dots \lambda_k)$ is a partition of m (i.e. $m = \lambda_1 + \dots + \lambda_k$ and $\lambda_1 \geq \dots \geq \lambda_k$) then the corresponding conjugacy class is made by disjoint cycles of length $\lambda_1, \dots, \lambda_k$.

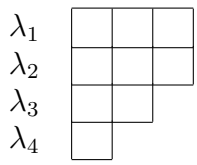
Example: $m = 3$

partitions of 3		conjugacy classes of $\mathfrak{S}_3$
111	$\leftrightarrow$	(1)
21	$\leftrightarrow$	(12)
3	$\leftrightarrow$	(123)

$m = 4$

partitions of 4		conjugacy classes of $\mathfrak{S}_4$
1111	$\leftrightarrow$	(1)
211	$\leftrightarrow$	(12)
31	$\leftrightarrow$	(123)
4	$\leftrightarrow$	(1234)
22	$\leftrightarrow$	(12)(34)

The number of irreducible representations of $\mathfrak{S}_m$ is the number $p(m)$, defined in (5.4), of conjugacy classes which is the number of partitions of m. Therefore we can give a one to one correspondence between partitions of m and representations of $\mathfrak{S}_m$.
In order to do that, we introduce *Young diagrams* and *Young tableau*. To a partition $\lambda = (\lambda_1 \dots \lambda_k)$ is associated a Young diagram such that the number of boxes in its j-th row is exactly λ_j; for example: if $m = 9$, the Young diagram associated to the partition (3321) is the following:



Now a Young tableau is obtained by numbering the boxes of the corresponding Young diagram from 1 to m starting from left top to the right bottom; for example the previous Young diagram becomes the following Young tableau $Y_{(3321)}$:

1	2	3
4	5	6
7	8	
9		

Therefore for every partition $\lambda = (\lambda_1 \dots \lambda_k)$ of m there is a well defined Young tableau Y_λ. Then for every λ we can define two subgroups of the symmetric group:

$$P_\lambda := \{g \in \mathfrak{S}_m \mid g \text{ preservs each row of } Y_\lambda\},$$
$$Q_\lambda := \{g \in \mathfrak{S}_m \mid g \text{ preservs each column of } Y_\lambda\}.$$

In correspondence of those subgroups we define the following two elements of the group algebra $\mathbb{C}\mathfrak{S}_m$:

$$a_\lambda := \sum_{g \in P_\lambda} e_g,$$
$$b_\lambda = \sum_{g \in Q_\lambda} \operatorname{sgn}(g) e_g,$$

then we can define the so called "Young symmetrizer":

$$c_\lambda = a_\lambda \cdot b_\lambda \in \mathbb{C}\mathfrak{S}_m.$$

Theorem 5.4.3. *For all partitions λ of $m \in \mathbb{N}$ there exists some scalar multiple of the Young symmetrizer c_λ which is idempotent, i.e., $c_\lambda^2 = n_\lambda c_\lambda$, and the image of c_λ (by right multiplication on $\mathbb{C}\mathfrak{S}_m$) is an irreducible representation V_λ of $\mathfrak{S}_m$. Every irreducible representation of $\mathfrak{S}_m$ can be obtained in this way for a unique partition.*

For the proof see [**FHar**].

This theorem allows us to write the one to one correspondence between partitions $\lambda = (\lambda_1 \dots \lambda_k)$ and irreducible representations of $\mathfrak{S}_m$ as follows:

$$\boxed{\lambda \longleftrightarrow V_\lambda := \mathbb{C}\mathfrak{S}_m \cdot c_\lambda} \tag{5.5}$$

where V_λ is the $\mathfrak{S}_m$-module corresponding to the partition λ.

Example: Consider the case $m = 3$. Then

$$\begin{array}{ll} a_{(111)} = e_{Id}, & b_{(111)} = \sum_{g \in \mathfrak{S}_3} \operatorname{sgn}(g) e_g, \\ a_{(21)} = e_{(12)} + e_{Id}, & b_{(21)} = -e_{(13)} + e_{Id}, \\ a_3 = \sum_{g \in \mathfrak{S}_3} e_g, & b_3 = e_{Id}; \end{array}$$

$$\begin{array}{l} c_{(111)} = b_{(111)}, \\ c_{(21)} = (e_{(12)} + e_{Id}) \cdot (e_{Id} - e_{(13)}) = 1 + e_{(12)} - e_{(13)} - e_{(12)}e_{(13)} = 1 + e_{(12)} - e_{(13)} - e_{132}, \\ c_3 = a_3. \end{array}$$

The previous theorem allows us to write all the possible irreducible representations of $\mathfrak{S}_3$:

1. $V_{(111)} = \mathbb{C}\mathfrak{S}_3 \cdot \sum_{g\in\mathfrak{S}_3} \operatorname{sgn}(g)e_g = \mathbb{C} \cdot \sum_{g\in\mathfrak{S}_3} \operatorname{sgn}(g)e_g$ that is the Alternating representation;

2. $V_3 = \mathbb{C}\mathfrak{S}_3 \cdot \sum_{g\in\mathfrak{S}_3} e_g = \mathbb{C} \cdot \sum_{g\in\mathfrak{S}_3} e_g$ that is the Trivial representation;

3. $V_{(21)} = \mathbb{C}\mathfrak{S}_3 \cdot (1+e_{(12)}-e_{(13)}-e_{(132)}) =< c_{(21)}, (13)\cdot c_{(21)} >$ that is the Standard representation.

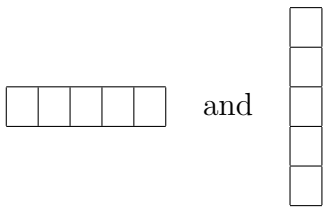
In general, for any vector space V, we can define an action of $\mathfrak{S}_m$ on $V^{\otimes m}$ (by permuting factors) such that the image of a_λ through the map $\widetilde{\rho} : \mathbb{C}\mathfrak{S}_m \to \operatorname{End}(V^{\otimes m})$, where $\widetilde{\rho}$ is defined as in (5.3), is the subspace

$$\widetilde{\rho}(a_\lambda) = \operatorname{Sym}^{\lambda_1} V \otimes \cdots \otimes \operatorname{Sym}^{\lambda_k} V$$

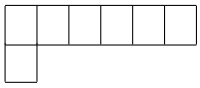
which we can view as a subspace of $V^{\otimes m}$ by grouping factors according to the rows of the Young tableau Y_λ. In the same way we can see that

$$\widetilde{\rho}(b_\lambda) = \wedge^{\mu_1} V \otimes \cdots \otimes \wedge^{\mu_k} V \subset V^{\otimes m}$$

where μ is the conjugate partition to λ.
From those simple observations we get that what we have seen in case 1. and 2. of the previous example is a general fact. If $\lambda = (m)$, then $c_{(m)} = a_{(m)} = \sum_{g\in\mathfrak{S}_m} e_g$ and the image of $c_{(m)}$ in $V^{\otimes m}$ is $\operatorname{Sym}^m V$. When $\lambda = (1\dots1)$, then $c_{(1\dots1)} = b_{(1\dots1)} = \sum_{g\in\mathfrak{S}_m} \operatorname{sgn}(g)e_g$, and the image of $c_{(1\dots1)}$ in $V^{\otimes m}$ is $\bigwedge^m V$. Therefore Young diagrams of the trivial and alternating representations of $\mathfrak{S}_m$ are always of the following forms respectively:



each one with m boxes.
One can also prove that the standard representation corresponds to the partition $m = (m-1)+1$, therefore its Young diagram is of the following form:



We are interested now in the image $\widetilde{\rho}(c_\lambda)$ when λ is a generic partition.

5.5 Decomposition of $\mathrm{Sym}^m(A_1 \otimes \cdots \otimes A_k)$ into irreducible modules

5.5.1 Schur power

Definition 5.5.1. *Let λ be a partition of the integer m and V a vector space. The λ-th* Schur power*, denoted by $\mathbb{S}_\lambda V$, of V is the image of c_λ in $V^{\otimes m}$ via the composition of the maps $\widetilde{\rho} : \mathbb{C}\mathfrak{S}_m \to \mathrm{End}(V)$ defined in (5.3), and $\mathrm{End}(V) \to V^{\otimes m}$ obtained by grouping factors:*

$$Im(c_\lambda|_{V^{\otimes m}}) := \mathbb{S}_\lambda V = V^{\otimes m} \otimes_{\mathbb{C}\mathfrak{S}_m} V_\lambda = V^{\otimes m} \cdot c_\lambda.$$

Three consequences of this definition are:

1. $\mathbb{S}_\lambda V = \mathrm{Hom}_{\mathfrak{S}_m}(V_\lambda, V^{\otimes m})$;
2. if $\lambda = (m)$ then the m-th Schur power of V is
$$\mathbb{S}_m V = \mathrm{Sym}^m(V)$$
because $\mathbb{S}_m V = V^{\otimes m} \cdot c_{(m)} = V^{\otimes m} \cdot a_m = Im(a_m) = \mathrm{Sym}^m(V)$;
3. if $\lambda = (1 \dots 1)$ then the $(1 \dots 1)$-th Schur power of V is $\mathbb{S}_{(1 \dots 1)} = \bigwedge^m V$.

The goal of this section is to prove the following result (see [**LM1**] Proposition 4.1):

Theorem 5.5.2. *Let $A_1, \dots, A_k$ be vector spaces. Then*

$$\mathrm{Sym}^m(A_1 \otimes \cdots \otimes A_k) = \bigoplus_{|\pi_1| = \cdots = |\pi_k| = m} (V_{\pi_1} \otimes \cdots \otimes V_{\pi_k})^{\mathfrak{S}_m} \mathbb{S}_{\pi_1} A_1 \otimes \cdots \otimes \mathbb{S}_{\pi_k} A_k \tag{5.6}$$

where $\pi_1, \dots, \pi_k$ are partitions of m and $(V_{\pi_1} \otimes \cdots \otimes V_{\pi_k})^{\mathfrak{S}_m}$ denotes the space of $\mathfrak{S}_m$-invariants in the tensor product.

5.5.2 Schur duality

Consider the action ρ_m of $GL(V)$, the General Linear Group, on $V^{\otimes m}$ via the m-th tensor power of its defining representation:

$$\rho_m(g)(v_1 \otimes \cdots \otimes v_m) = gv_1 \otimes \cdots \otimes gv_m \tag{5.7}$$

for $v_1, \ldots, v_m \in V$.

The symmetric group $\mathfrak{S}_m$ also acts on the tensor space $V^{\otimes m}$ (by permuting factors):

$$\sigma_m(s)(v_1 \otimes \cdots \otimes v_m) = v_{s^{-1}(1)} \otimes \cdots \otimes v_{s^{-1}(m)},$$

the notation $v_{s(i)}$ denotes just a permutation of factors: the vector v_i is sent to position $s(i)$.

These two actions clearly commute, in fact:

$$\sigma_m(s)\rho_m(g) = \rho_m(g)\sigma_m(s)$$

for all $s \in \mathfrak{S}_m$ and $g \in GL(V)$.

Definition 5.5.3. *Let V be a finite dimensional vector space. For any subset S of* $\mathrm{End}(V)$*, the Commutator of S is:*

$$\mathrm{Comm}(S) := \{x \in \mathrm{End}(V) \ : \ xs = sx \ \forall\, s \in S\}.$$

Remark: The commutator is an associative algebra.

Each one of the two actions of $GL(V)$ and of $\mathfrak{S}_m$ on $V^{\otimes m}$ generates the centralizer of the other, in fact any linear transformation on $V^{\otimes m}$ that commutes with $\sigma_m(\mathfrak{S}_m)$ is a linear combination of the transformation $\rho_m(g)$ with $g \in GL(V)$. This is the so called *Schur duality*:

Theorem 5.5.4. *If $\mathcal{A} = \rho_m(\mathbb{C}[GL(V)])$ and $\mathcal{B} = \sigma_m(\mathbb{C}[\mathfrak{S}_m])$. Then* $\mathrm{Comm}(\mathcal{B}) = \mathcal{A}$ *and* $\mathrm{Comm}(\mathcal{A}) = \mathcal{B}$.

Now we need two definitions before we can state a more general result (see [**Go1**]).

Definition 5.5.5. *An associative algebra $\mathcal{A}$ is called Simple if it contains no nontrivial ideals.*

Definition 5.5.6. *A finite dimensional associative algebra $\mathcal{A}$ with unit is said to be Semisimple if it is the direct sum of simple algebras.*

Proposition 5.5.7. *If $\mathcal{A} \subset \mathrm{End}(V)$ is a semisimple algebra with $I_V \in \mathcal{A}$ and irreducible decomposition $\mathcal{A} \simeq \bigoplus_{i=1}^r \mathrm{End}(U_i)$, then*

$$V \simeq \bigoplus_{i=1}^{r} U_i \otimes W_i$$

where $W_i = \mathrm{Hom}_{\mathcal{A}}\,(U_i, V)$.

Proof. The linear map wich turns out to be an $\mathcal{A}$-module isomorphism is the following:

$$\begin{array}{rcl} S \ : \bigoplus_i (W_i \otimes U_i) & \to & V \\ \sum_i w_i \otimes u_i & \to & \sum_i w_i(u_i) \end{array}.$$

□

Now we have all the ingredients to prove the following theorem:

Theorem 5.5.8. *If V is an n-dimensional vector space which is a representation of $\mathfrak{S}_m$, then*

$$\boxed{V^{\otimes m} \simeq \bigoplus_{|\lambda|=m} \mathbb{S}_\lambda(V) \otimes_{\mathbb{C}} V_\lambda} \tag{5.8}$$

where $\mathbb{S}_\lambda(V)$ is the λ-th Schur power of V defined in Definition 5.5.1.

Proof. First we have to observe that, if λ has more parts than m, then $\mathbb{S}_\lambda(V) = 0$.
By Schur's Lemma (5.2.10), if V_λ is defined as in (5.5), then $V \simeq \sum_\lambda V_\lambda$.
From the definition of $\mathbb{S}_\lambda(V)$ one immediately gets that $\mathbb{S}_\lambda(V) \simeq \mathrm{Hom}_{\mathfrak{S}_m}(V_\lambda, V^{\otimes m})$.
The actions of $GL(V)$ and of $\mathfrak{S}_m$ commute on $V^{\otimes m}$.
Now it is sufficient to apply Proposition 5.5.7 to $V^{\otimes m}$ in order to obtain what we wanted to prove. □

5.5.3 Decomposition of $\mathrm{Sym}^m(A_1 \otimes \cdots \otimes A_k)$

We are finally ready to prove Theorem 5.5.2.

Proof. Let $A_1, \ldots, A_k$ be vector spaces as in the statement. By definition of tensor power $(A_1 \otimes \cdots \otimes A_k)^{\otimes m} \simeq A_1^{\otimes m} \otimes \cdots \otimes A_k^{\otimes m}$. Let us apply (5.8) that says that each $A_i^{\otimes m}$ is isomorphic to $\bigoplus_{|\lambda|=m} V_\lambda \otimes \mathbb{S}_\lambda(A_i)$. Therefore

$$(A_1 \otimes \cdots \otimes A_k)^{\otimes m} \simeq \bigoplus_{|\pi_1|=\cdots=|\pi_k|=m} (V_{\pi_1} \otimes \cdots \otimes V_{\pi_k}) \otimes (\mathbb{S}_{\pi_1}(A_1) \otimes \cdots \otimes \mathbb{S}_{\pi_k}(A_k)) \tag{5.9}$$

where $\pi_1, \ldots, \pi_k$ are partitions of m.

Recall now that, if V is any finite dimensional vector space, the space $\mathrm{Sym}^m(V)$ is, by definition, the quotient of $V^{\otimes m}$ by the subspace

$$< \{v_1 \otimes \cdots \otimes v_m - v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(m)} \mid v_1, \ldots, v_m \in V,\ \sigma \text{ permutes two successive factors}\} >, \tag{5.10}$$

and also that, even if the usual immersion $i : \mathrm{Sym}^m(V) \hookrightarrow V^{\otimes m}$ is not canonical, $i(\mathrm{Sym}^m(V)) \subset V^{\otimes m}$ is the space of invariants for the right action of $\mathfrak{S}_m$ on $V^{\otimes m}$.
Therefore we have that $\mathrm{Sym}^m(A_1 \otimes \cdots \otimes A_k)$ is just the quotient of $(A_1 \otimes \cdots \otimes A_k)^{\otimes m}$ by the space (5.10) when $V = A_1 \otimes \cdots \otimes A_k$; and also that $\mathrm{Sym}^m(A_1 \otimes \cdots \otimes A_k)$ is the space of invariants for the right action of $\mathfrak{S}_m$ on $(A_1 \otimes \cdots \otimes A_k)$.
Now by using last observation and formula (5.9) we get that

$$\mathrm{Sym}^m(A_1 \otimes \cdots \otimes A_k) \simeq \bigoplus_{|\pi_1|=\cdots=|\pi_k|=m} ((V_{\pi_1} \otimes \cdots \otimes V_{\pi_k}) \otimes (\mathbb{S}_{\pi_1}(A_1) \otimes \cdots \otimes \mathbb{S}_{\pi_k}(A_k)))^{\mathfrak{S}_m}.$$

The last step is to recall that $\mathbb{S}_\lambda(V) = \mathrm{Hom}_{\mathfrak{S}_m}(V_\lambda, V^{\otimes m}) = \mathrm{Hom}^{\mathfrak{S}_m}(V_\lambda, V^{\otimes m})$, then

$$\mathrm{Sym}^m(A_1 \otimes \cdots \otimes A_k) \simeq \bigoplus_{|\pi_1|=\cdots=|\pi_k|=m} (V_{\pi_1} \otimes \cdots \otimes V_{\pi_k})^{\mathfrak{S}_m} \otimes (\mathbb{S}_{\pi_1}(A_1) \otimes \cdots \otimes \mathbb{S}_{\pi_k}(A_k))$$

that is exactly what we wanted to prove! □

5.6 Secant varieties of homogeneous varieties and their ideals

5.6.1 Some previous considerations

Let $S^m V^*$ be the set of homogeneous polynomials of degree m on V^*.

Definition 5.6.1. *If $A \subset S^m V^*$, the p-th prolongation of A is*

$$A^{(p)} := (A \otimes S^p V^*) \cap S^{p+m} V^*.$$

The meaning of the intersection above is not clear because a priori $S^{p+m}V^*$ is not contained in $S^pV^* \otimes S^mV^*$, so we need to explain what we actually mean.
If $q \in S^{p+m}V^*$ we can write:

$$q(u+v) = \sum_{d=0}^{p+m} \binom{p+m}{d} \sum_i R_{d,i}(u) Q_{p+m-d,i}(v) \tag{5.11}$$

where $R_{d,i} \in S^pV^*$ and $Q_{p+m-d,i} \in S^mV^*$ for all $u, v \in V^*$. Hence we will consider $S^{p+m}V^* \subset S^pV^* \otimes S^mV^*$ via the following immersion:

$$\begin{array}{rcl} S^{p+m}V^* & \hookrightarrow & S^pV^* \otimes S^mV^* \\ q & \mapsto & \sum \binom{p+m}{d} (R_{d,i} \otimes Q_{p+m-d,i}) \end{array}.$$

Example: Let V be a 2-dimensional vector space and $m = p = 1$. If $u = (u_1, u_2), v = (v_1, v_2) \in V$ and $q = xy$ then $q(u+v) = (u_1+v_1)(u_2+v_2) = u_1u_2 + u_1v_2 + u_2v_1 + v_1v_2$ then for $q = xy$ we can have the following three images:

$$xy \begin{array}{ccl} & \nearrow & xy \otimes 1 \in S^2V^* \otimes S^0V^* \\ \rightarrow & & \frac{1}{2}(x \otimes y + y \otimes x) \in S^1V^* \otimes S^1V^* \\ & \searrow & 1 \otimes xy \in S^0V^* \otimes S^2V^* \end{array} .$$

In the same way, if $q = x^2$, the evaluation $q(u+v) = u_1^2 + u_1v_1 + v_1^2$, therefore

$$x^2 \begin{array}{ccl} & \nearrow & x^2 \otimes 1 \in S^2V^* \otimes S^0V^* \\ \rightarrow & & x \otimes x \in S^1V^* \otimes S^1V^* \\ & \searrow & 1 \otimes x^2 \in S^0V^* \otimes S^2V^* \end{array} .$$

The same can be done for $q = y^2$. Now, since $\{x^2, xy, y^2\}$ is a base for S^2V^* and we are interested in its image into $S^1V^* \otimes S^1V^*$, we can write:

$$\begin{array}{rcl} S^2V^* & \rightarrow & S^1V^* \otimes S^1V^* \\ x^2 & \mapsto & x \otimes x \\ xy & \mapsto & \frac{1}{2}(x \otimes y + y \otimes x) \\ y^2 & \mapsto & y \otimes y \end{array} .$$

Now it should be more clear what $(A \otimes S^pV^*) \cap S^{p+m}V^*$ means.

Let now $A \subset S^2V^*$ and consider $A^{(p-1)} = (A \otimes S^{p-1}V^*) \cap S^{p+1}V^*$.

Example: If $A = (x^2) \subset (\mathbb{C}[x,y])_2$ then $A^{(p-1)}$ is given by:

$$\begin{array}{rcccl} S^{p+1}V^* & \hookrightarrow & S^2V^* \otimes S^{p-1}V^* & \rightarrow & (S^2V^* \otimes S^{p-1}V^*) \cap (A \otimes S^{p-1}V^*) \\ x^{p+1} & \mapsto & x^2 \otimes x^{p-1} & \mapsto & x^2 \otimes x^{p-1} \\ x^py & \mapsto & \frac{1}{2}(x^2 \otimes x^{p-2}y + xy \otimes x^{p-1}) & \mapsto & \frac{1}{2}x^2 \otimes x^{p-2}y \\ x^{p-1}y^2 & \mapsto & \frac{1}{3}(x^2 \otimes x^{p-3}y^2 + xy \otimes x^{p-2}y + y^2 \otimes x^{p-2}) & \mapsto & \frac{1}{3}x^2 \otimes x^{p-3}y^2 \\ \vdots & & \vdots & & \vdots \end{array}$$

In the same way the elements of $A^{(p)}$ are all of the form $x^2 \otimes x^{p-i}y^{i-1}$ with $0 \leq i \leq p$.

Remark: We can observe that $A^{(p-1)} = \left\{\frac{\partial q}{\partial v} \mid q \in A^{(p)}\right\}$.
Moreover, if we define:

$$Base(A) := \{[v] \in \mathbb{P}(V) \mid q(v) = 0 \; \forall q \in A\}$$

we can also observe that $A^{(p-1)} = I(Base(A^{(p)})_{Sing})$.

We can now state the following proposition (see [**LM1**], Lemma 3.1).

Proposition 5.6.2. *Let $A \subset S^2V^*$ be a system of quadrics with base locus $Base(A) \subset \mathbb{P}(V)$. Then*

$$Base(A^{k-1}) \supseteq \mathrm{Sec}_{k-1}(Base(A)).$$

Moreover if $Base(A)$ is linearly non-degenerate, then for $k \geq 2$, $I_k(\mathrm{Sec}_{k-1}(Base(A))) = 0$ and if $A = I_2(Base(A))$, then $I_{k+1}(\mathrm{Sec}_{k-1}(Base(A))) = A^{(k-1)}$.

The prove can be found in [**LM2**] (see Lemma 2.2). We give here a slightly different version.

Proof. First we give the proof for $k = 2$.

We denote with B the affine cone over $Base(A)$. Let $x, y \in B$, $s, t \in \mathbb{C}$ and $v = sx + ty \in \mathrm{Sec}_1(Base(A))$.

For the inclusion $A^{(1)} \subseteq I_3(\mathrm{Sec}_1(Base(A)))$ we need to prove that all $q \in A^{(1)}$ are zero on every element $v \in \mathrm{Sec}_1(Base(A))$.
If $q \in A^{(1)}$ then $\deg(q) = 3$.
The image of $q(v) = q(sx + ty) \in S^3V^*$ into $(S^2V^* \otimes S^1V^*) \cap (A \otimes S^1V^*)$ is

$$\sum_{d=0}^{3} \binom{3}{d} \sum_i R_{d,i}(sx)Q_{3-d,i}(ty) = \sum_i t^3Q_{3,i}(y) + 3st^2R_{1,i}(x)Q_{2,i}(y) + 3s^2tR_{2,i}(x)Q_{1,i}(y) + s^3R_{3,i}(x) = 0$$

because $Q_{j,i}(y)$ and $R_{j,i}(x)$ are zero if $j \geq 2$ since $q \in A^{(1)} = (A \otimes S^1V^*) \cap S^3V^*$.

Now we want to prove that $A^{(1)} \supset I_3(\mathrm{Sec}_1(Base(A)))$.
Consider $q \in I_3(\mathrm{Sec}_1(Base(A)))$, then

$$0 = q(sx + ty) = \sum_{j=0}^{3} \binom{3}{j} \sum_i R_{j,i}(sx)Q_{3-j,i}(ty)$$

for all $s, t \in \mathbb{C}$ and for all $x, y \in B$. In particular $q(x) = 0$ and $q(y) = 0$ therefore $R_{3,i}(x) = 0$ and $Q_{3,i}(y) = 0$, hence $q(sx + ty) = \sum_{j=1}^{2} \binom{3}{j} \sum_i R_{j,i}(sx)Q_{3-j,i}(ty)$ is equal to zero if and only if $R_{j,i}(sx)Q_{3-j,i}(ty) = 0$ for $j = 1, 2$ and for all $s, t \in \mathbb{C}$ and $x, y \in B$, in particular $R_{2,i}(x)Q_{1,i}(ty) = 0$ for all $x, y \in B$ hence for all $y \in V$.

Thus, for all $y \in V$, $R_{2,i}(\cdot)Q_{1,i}(y)$ is a quadric vanishing on $Base(A)$, hence it belongs to $A = I_2(Base(A))$, that is equivalent to say that $q \in A^{(1)}$.

If $k \geq 2$ the proof is not very different.

Consider $q \in A^{(k-1)} = (A \otimes S^{k-1}V^*) \cap S^{k+1}V^*$. We want to show that $q(s_1x_1 + \cdots + s_kx_k) = 0$ for $s_1, \ldots, s_k \in \mathbb{C}$ and for all $x_1, \ldots, x_k \in B$. The polynomial $q(s_1x_1 + \cdots + s_kx_k) = 0$ can be decomposed as

$$\sum_{j=0}^{k+1} \binom{k+1}{j} \sum_i R_{j,i}(s_1x_1 + \cdots + s_{k-1}x_{k-1})Q_{k+1-j,i}(s_kx_k) \tag{5.12}$$

that is equal to

$$\sum_{j=0}^{k+1} \binom{k+1}{j} \sum_i s_k^{k+1-j} R_{j,i}(s_1x_1 + \cdots + s_{k-1}x_{k-1})Q_{k+1-j,i}(x_k).$$

Since $q \in A^{(k-1)}$, the polynomial $Q_{k+1-j}(x_k) = 0$ if $k+1-j \geq 2$, i.e. for all $j \leq k-1$. Therefore $q(s_1x_1+\cdots+s_kx_k) = \sum_i((k+1)s_1R_{k,i}(s_1x_1+\cdots+s_{k-1}x_{k-1})Q_1(x_k)+R_{k+1,i}(s_1x_1+\cdots+s_{k-1}x_{k-1}))$.

Let us consider:

$$R_{k,i}(s_1x_1 + \cdots + s_{k-1}x_{k-1}).$$

The study of $R_{k+1,i}(s_1x_1 + \cdots + s_{k-1}x_{k-1})$ is similar to the study of q since they are polynomials of the same degree.

The decomposition of $R_{k,i}(s_1x_1 + \cdots + s_{k-1}x_{k-1})$ is

$$\sum_{j=0}^{k} \binom{k}{j} \sum_i R_{j,i}(s_1x_1 + \cdots + s_{k-2}x_{k-2})Q_{k-j,i}(s_{k-1}x_{k-1})$$

that is equal to

$$\sum_{j=0}^{k} \binom{k}{j} \sum_i s_{k-1}^{k-j} R_{j,i}(s_1x_1 + \cdots + s_{k-2}x_{k-2})Q_{k-j,i}(x_{k-1}).$$

Now $Q_{k-j,i}(x_{k-1}) = 0$ for all $j \leq k-2$, then

$$R_{k,i}(s_1x_1+\cdots+s_{k-1}x_{k-1}) = \sum_i (ks_{k-1}R_{k-1,i}(s_1x_1+\cdots+s_{k-2}x_{k-2})Q_{1,i}(x_{k-1})+R_{k,i}(s_1x_1+\cdots+s_{k-2}x_{k-2})).$$

We can continue in decomposing $R_{k-1,i}(s_1x_1+\cdots+s_{k-2}x_{k-2})$ and after it we will have to decompose $R_{k-2,i}(s_1x_1 + \cdots + x_{k-3}x_{k-3})$, and so on; we will arrive to

$$R_{3,i}(s_1x_1 + s_2x_2) = \sum_i (R_{2,i}(s_1x_1)Q_{1,i}(s_2x_2) + R_{3,i}(s_1x_1))$$

that is equal to $\sum_i (s_1^2 s_2 R_{2,i}(x_1) Q_{1,i}(x_2) + s_1^3 R_{3,i}(x_1)$ which is zero because $q \in A^{(k-1)}$, hence $R_{j,i}(x) = 0$ for all $j \geq 2$.
Therefore

$$A^{(k-1)} \subset I_{k-1}(\mathrm{Sec}_{k-1}(Base(A))).$$

For the other inclusion we consider $q \in I_{k+1}(\mathrm{Sec}_{k-1}(Base(A)))$, then $q(s_1x_1 + \cdots + s_kx_k) = 0$ for $s_1, \ldots, s_k \in \mathbb{C}$ and for all $x_1, \ldots, x_k \in B$ and it can be decomposed as in (5.12). Working as before, via $k-1$ decompositions we get at $R_{2,i}(x_1)Q_{1,i}(s_2x_2) = 0$ for all $x_2 \in B$ and $s_2 \in \mathbb{C}$, hence for all $x_2 \in V$, therefore $R_{2,i}(\cdot)Q_{1,i}(x)$ is a quadric vanishing on $Base(A)$. □

Corollary 5.6.3. *Let $X \subset \mathbb{P}(V)$ be a variety with $I(X)$ generated in degree d. Then for all $k \geq 0$, $I_{d+k-2}(\mathrm{Sec}_{k-1}(X)) = 0$.*

5.6.2 Homogeneous varieties and highest weight vectors

At this point we need to do a digression on what an homogeneous variety is and how it is related to the concept of highest weight vector.

Homogeneous spaces

The following description of Homogeneous spaces is from [**GW**] and we will refer to that book for the proofs that we omit here.

Definition 5.6.4. *A Quasiprojective Algebraic Set is a subset $M \subset \mathbb{P}^n$ defined by a finite set of equalities and inequalities of the form:*

$$f_i(x) = 0,\ i = 1, \ldots, k;$$

$$g_j(x) \neq 0,\ j = 1, \ldots, l,$$

where f_i and g_j are homogeneous polynomials in $\mathbb{C}[x_0, \ldots, x_n]$ and $x = (x_0, \ldots, x_n)$.

In topological terms, M is the intersection of the closed set

$$Y = \{[x] \in \mathbb{P}^n \mid f_i(x) = 0,\ i = 1, \ldots, k\}$$

and the open set

$$Z = \{[x] \in \mathbb{P}^n \mid g_j(x) \neq 0,\ j = 1, \ldots, l\}.$$

Definition 5.6.5. *An Algebraic Action of a linear algebraic group G on a quasiprojective algebraic set M is a regular map $\alpha : G \times M \to M$, written as $(g, m) \mapsto g \cdot m$, such that*

$$g \cdot (h \cdot m) = (gh) \cdot m, \quad 1 \cdot m = m$$

for all $g, h \in G$ and $m \in M$.

Theorem 5.6.6. *Let G be a group acting on a quasi projective algebraic set $M \subset \mathbb{P}^n$. For every $x \in M$, the stabilizer G_x of x is an algebraic subgroup of G and the orbit $G \cdot x$ is a smooth quasiprojective subset of M.*

Corollary 5.6.7. *There exists a point $x \in M$ so that $G \cdot x$ is closed in M.*

Let H be an algebraic subgroup of an algebraic group G. By the previous theorem there is a regular representation $\pi : G \to GL(V)$ of G and a point $x_0 \in \mathbb{P}(V)$ so that H is the stabilizer of x_0. The map $g \mapsto g \cdot x_0$ is a bijection from the coset space G/H to the orbit $G \cdot x_0$. So when we view G/H as a smooth quasiprojective algebraic set by identifying it with the orbit $G \cdot x_0$.

Theorem 5.6.8. *Let $H \subset G$ be an algebraic subgroup of an algebraic group G. Let $\pi : G \to GL(V)$ be a regular representation of G and $x_0 \in G$ be stable under the action of H. Then:*

1. *The quasiprojective algebraic set structure on G/H is independent on the choice of the representation π.*

2. *The quotient map from G to G/H is regular.*

3. *If M is any quasiprojective algebraic set on which G acts algebraically, and $x \in M$ is such that $H \subset G_x$, then the map $gH \mapsto g \cdot x$ from G/H to the orbit $G \cdot x$ is regular.*

Definition 5.6.9. *A quotient space G/H with the previous properties is called a Homogeneous Space.*

The vector x_0 will be called *highest weight vector* when $H \subset G$ is a parabolic subgroup.

Highest weight for $GL(n, \mathbb{C})$

We need now to introduce the *Theorem of the Highest Weight*. We will do it for the irreducible regular representations of $GL(n, \mathbb{C})$ as it is done in [**Go2**]; analogous results hold for any complex reductive algebraic group (see [**GW**], Chap. 5).

Let $H, N, \widetilde{N} \subset GL(n, \mathbb{C})$ be the subgroup of diagonal matrices, the subgroup of upper triangular unipotent matrices (all diagonal entries equal 1) and the subgroup of lower triangular unipotent matrices respectively. Then $\widetilde{N}HN$ is a Zariski dense open subset of $GL(n, \mathbb{C})$, and a generic element $g \in GL(n, \mathbb{C})$ has a unique factorization $g = \widetilde{n}hn$ for some $\widetilde{n} \in \widetilde{N}$, $h \in H$ and $n \in N$. Thus a regular representation of $GL(n, \mathbb{C})$ is completely determined by its restriction to the subgroups $\widetilde{N}, H$ and N.

Notation: Assume G is a reductive finite group, and let $\widehat{G}$ be the equivalence classes of irreducible finite-dimensional regular representations of G.

Definition 5.6.10. *An Algebraic Torus is an algebraic group T isomorphic to $\underbrace{\mathbb{C}^\times \times \cdots \times \mathbb{C}^\times}_{l}$. The integer l is called the rank of the torus.*

Definition 5.6.11. *If G is a linear algebraic group, then a torus $H \subset G$ is Maximal if it is not contained in any larger torus in G and it is diagonalizable.*

The subgroup H of diagonal matrices is a maximal algebraic torus in $GL(n, \mathbb{C})$. The irreducible representations of H are one dimensional and given by $h = diag[x_1, \ldots, x_n] \mapsto h^\mu = x_1^{m_1} \cdots x_n^{m_n}$ where $\mu = [m_1, \ldots, m_n] \in \mathbb{Z}^n$. Thus we may identify $\widehat{H}$ with $\mathbb{Z}^n$. If $\rho : G \to GL(V)$ is a regular representation of G, then the restriction of ρ to H decomposes into *weight spaces*:

$$V = \bigoplus_{\mu \in \Phi(V)} V(\mu)$$

where $V(\mu) \neq 0$ and $\rho(h)v = h^\mu v$ for $v \in V(\mu)$. We call $\Phi(V) \subset \widehat{H}$ the set of *weights* of V.

Let $Norm_G(H)$ be the *normalizer* of H in G (i.e. the set of all $g \in G$ such that $HgH = gH$) and $W = Norm_G(H)/H$ be the *Weyl group* of G. The elements of W permute the weight spaces and the weights of V. In this case, $W \simeq \mathfrak{S}_n$ may be identified with the group of *permutation matrices* in G, and the action of W on H and $\widehat{H}$ is by the usual permutation of coordinates. Every W orbit in $\widehat{H}$ contains a unique *dominant weight*

$$\mu = [m_1, \ldots, m_n]$$

with $m_1 \geq \cdots \geq m_n$. We denote by $\mathbb{Z}^n_{++}$ the set of all such $\mu \in \mathbb{Z}^n$, and the corresponding vector is called *highest weight vector.*

Example: Let $V = \mathbb{C}^n$ be the defining representation of G. Then $\Phi(V) = \{\varepsilon_1, \ldots, \varepsilon_n\}$ where $\varepsilon(h) = x_i$ for $h = diag[x_1, \ldots, x_n]$. Here $\Phi(V) = W \cdot \varepsilon_1$ is a single W orbit with dominant weight ε_1.

Example: Let $V = \bigotimes^k \mathbb{C}^n$. A basis $\{e_I\}$ diagonalizes $\rho_k(H)$ where ρ_k is defined as in (5.7). For an index $I = [i_1, \ldots, i_k]$, with $1 \leq i_j \leq n$, define

$$\mu_I = [\mu_1, \ldots, \mu_n]$$

where $\mu_p = \sharp\{j \mid i_j = p\}$. Then $\rho_k(h)e_I = h^{\mu_I} e_I$ for $h \in H$. Hence for $\lambda \in \widehat{H}$,

$$V(\lambda) = Span\{e_I \mid \mu_I = \lambda\}.$$

In particular $V(\lambda) \neq 0$ if and only if $\lambda_i \geq 0$ for $i = 1, \ldots, n$ and $|\lambda| = k$, where $|\lambda| = \lambda_1 + \cdots + \lambda_n$. Thus $\Phi(\bigotimes^k \mathbb{C}^n) = W \cdot Par(k, n)$ where $Par(k, n)$ is the set of all partitions of k with at most n parts. Each such partition defines a dominant weight μ of H such that $h \mapsto h^\mu$ is a polynomial function on H (no negative powers of the coordinates x_i).

Definition 5.6.12. *Let G be a C^∞ manifold such that the underlying set has a group structure. We write $m(x, y) = xy$ (the group multiplication) and $\eta(x) = x^{-1}$ (the group inverse). We say that G is a Lie group if $m : G \times G \to G$ and $\eta : G \to G$ are C^∞ maps.*

Example: The group $GL(n, \mathbb{R}) = \{M \in M_n(\mathbb{R}) \mid det(M) \neq 0\}$ is a Lie group.

Definition 5.6.13. *A vector space L over a field F, with an operation $L \times L \to L$, denoted $(x, y) \mapsto [xy]$, called Bracket or Commutator of x and y, is said to be a Lie algebra over F if the following properties are satisfied:*

1. *The bracket operation is bilinear.*
2. $[xx] = 0$ *for all* $x \in L$.
3. $[x[yz]] + [y[zx]] + [z[xy]] = 0$ *for all* $x, y, z \in L$.

The third axiom is called *Jacobi identity.* Notice that *1.* and *2.* applied to $[x + y, x + y]$ imply anticommutativity: $[xy] = -[yx]$.

There is a way to associate a Lie algebra to a Lie group. Let G be a Lie group. Let $L_g : G \to G$ be defined by $L_g x = gx$. Then L_g is of class C^∞ and $(L_g)^{-1} = L_{g^{-1}}$ by the associative rule.

Definition 5.6.14. *Let M be a C^∞ manifold. Then to give a Vector Field on M we ask to give, for all $p \in M$, an assignment $p \mapsto X_p \in T_p(M)$, in such a way that for all $f \in C^\infty(M)$ the function $p \mapsto X_p f$ is an element of $C^\infty(M)$. We write $(Xf)(x) = X_x(f)$. Thus a vector field defines an endomorphism of $C^\infty(M)$ as a vector space over $\mathbb{R}$.*

Example: Let $M = \mathbb{R}^n$ and $v \in \mathbb{R}^n$. If we define for any $f \in C^\infty(\mathbb{R}^n)$:

$$v_x \cdot f = \frac{d}{dt} f(x + tv)|_{t=0},$$

then $x \mapsto v_x$ is a vector field.

We can view a vector field on G as a derivation of $C^\infty(G)$. That is, if X is a vector field and $f \in C^\infty(G)$ then $Xf \in C^\infty(G)$, defined by $Xf(x) = X_x f$ can be considered as the derivative of f in X_x direction. We have:

$$X(fg) = (Xf)g + f(Xg).$$

Let G be a Lie group; set $L_g^* f = f \circ L_g$, then a vector field on G is said to be Left Invariant if, for each $g \in G$: $L_g^* \circ X = X \circ L_g^*$.

Definition 5.6.15. *We set $Lie(G)$ to be the space of all left invariant vector fields on G.*

Proposition 5.6.16. *The map $X \mapsto X_1$ defines a linear bijection between $Lie(G)$ and $T_1(G)$. If $X, Y \in Lie(G)$ then $[X, Y] \in Lie(G)$. Thus $Lie(G)$ is an n-dimensional Lie algebra over $\mathbb{R}$, where $n = \dim(G)$.*

Following the above proposition we call $Lie(G)$ the *Lie algebra* of G.

Example: Let $\mathfrak{g} = Lie(GL(n, \mathbb{C})) = M_n(\mathbb{C})$ be the Lie algebra of $G = GL(n, \mathbb{C})$, and let $Ad(g)x = gxg^{-1}$ be the adjoint representation. The weights are 0 and $\{\varepsilon_i - \varepsilon_j \mid 1 \leq i \neq j \leq n\}$. We call the non-zero weights the *roots* of $\mathfrak{h}$ on $\mathfrak{g}$ (the algebra $\mathfrak{h}$ is the Lie algebra of $H \subset G$ the subgroup of diagonal matrices). The corresponding *root spaces* are

$$\mathfrak{g}_0 = \mathfrak{h} = Lie(H)$$

and

$$\mathfrak{g}_{\varepsilon_i - \varepsilon_j} = \mathbb{C}E_{ij}$$

where E_{ij} is the elementary matrix with 1 in position (i, j) and zero elsewhere. If $\alpha = \varepsilon_i - \varepsilon_j$, then we say $\alpha > 0$ if $i < j$, and $\alpha < 0$ if $i > j$. We denote the set of positive roots by Φ^+ and the set of negative roots by Φ^-. Thus

$$\mathfrak{n} = Lie(N) = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha,$$

$$\overline{\mathfrak{n}} = Lie(\overline{N}) = \bigoplus_{\alpha\in\Phi^-} \mathfrak{g}_\alpha.$$

The Lie algebra additive version of the Gauss decomposition is the so called *triangular decomposition*:

$$\mathfrak{g} = \widetilde{\mathfrak{n}} \oplus \mathfrak{h} \oplus \mathfrak{n}.$$

If $\rho : G \to GL(V)$ is any regular representation of G, then there is an associated Lie algebra representation $d\rho$ of $\mathfrak{g}$ defined by:

$$d\rho(X)v = \frac{d}{dt}\rho(exp(tX))v|_{t=0}.$$

One can prove that

$$d\rho(\mathfrak{n})V(\mu) \subset \bigoplus_{\lambda\in\mu+\Phi^+} V(\lambda).$$

We call $\mu \in \Phi(V)$ an N-*extreme* weight if $\mu + \alpha \notin \Phi(V)$ for all $\alpha\Phi^+$.

Theorem 5.6.17. *Let $\rho : G \to GL(V)$ be an irreducible representation of $G = GL(n, \mathbb{C})$. There is an unique N-extreme weight $\mu_0 \in \Phi(V)$. This weight is dominant, the weight space $V(\mu_0) = V^N$ (the N-fixed vectors in V), and* $\dim(V^N) = 1$.

We call μ_0 the *highest weight* of the representation $\rho : G \to GL(V)$. It determines the representation uniquely up to isomorphism.

Extreme vectors and highest weight

The above construction that we have done in the particular case of $GL(n, \mathbb{C})$ is more general. We are going to present what happens in general without go into details. We will follow the presentation of [**Go1**].

Let G be a classical group whose Lie algebra is semisimple. We fix a set Φ^+ of positive roots. It is a general fact (see Theorem 8.9 in [**Go1**]) that there always exists a sort of triangular decomposition associated to $\mathfrak{g}$:

$$\mathfrak{g} = \overline{\mathfrak{n}} + \mathfrak{h} + \mathfrak{n}.$$

We set $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$ and call $\mathfrak{b}$ a *Borel subalgebra* of $\mathfrak{g}$. We have

$$[\mathfrak{b}, \mathfrak{b}] = \mathfrak{n}, \qquad [\mathfrak{h}, \mathfrak{n}] = \mathfrak{n}.$$

Let $P(\mathfrak{g})$ be the weight lattice and $P_{++}(\mathfrak{g})$ the dominant weights relative to the choice of Φ^+. If (π, V) is a finite-dimensional representation of $\mathfrak{g}$, then V has a weight-space decomposition

$$V = \bigoplus_{\mu\in P(\mathfrak{g})} V(\mu),$$

where $V(\mu) = \{v \in V \mid \pi(Y)v = \mu(Y)v \text{ for all } Y \in \mathfrak{h}\}$. We denote by

$$\chi(V) = \{\mu \in P(\mathfrak{g}) \mid V(\mu) \neq 0\}$$

the set of weights of the $\mathfrak{g}$-module V.
Let $\{\alpha_1, \ldots, \alpha_l\}$ be the simple roots in Φ^+ and let $Q_+(\mathfrak{g}) = \mathbb{N}\alpha_1 + \cdots + \mathbb{N}\alpha_l$ be the semigroup generated by the positive roots. We define a *partial order* on $P(\mathfrak{g})$ by

$$\lambda \prec \mu \text{ if } \lambda = \mu - \beta \text{ for some } \beta \in Q_+(\mathfrak{g}) \smallsetminus \{0\}.$$

Let (π, V) be a representation of $\mathfrak{g}$ (not necessarily finite-dimensional). A non-zero vector $v_0 \in V$ is called $\mathfrak{b}$-extreme if $\pi(\mathfrak{b})v_0 \subset \mathbb{C}v_0$. A vector $v_0 \in V$ is $\mathfrak{g}$-cyclic if V is spanned by v_0 together with the vectors $\pi(x_1) \cdots \pi(x_p)v_0$, where $x_i \in \mathfrak{g}$ and $p = 1, 2, \ldots$.

Proposition 5.6.18. *Let (π, V) be a finite-dimensional representation of $\mathfrak{g}$.*

1. *A vector v_0 is $\mathfrak{b}$-extreme if and only if $\pi(\mathfrak{n})v_0 = 0$ and there exists $\mu \in P_{++}(\mathfrak{g})$ such that $\pi(H)v_0 =< \mu, H > v_0$ for all $H \in \mathfrak{h}$.*

2. *The $\mathfrak{b}$-extreme vectors in V span the subspace*

$$V^{\mathfrak{n}} = \{v \in V \mid \pi(\mathfrak{n})v = 0\}.$$

3. *Suppose μ is a maximal element of $\chi(V)$ relative to the partial order $\prec$. Then μ is dominant and $V(\mu) \subset V^{\mathfrak{n}}$. In particular $V^{\mathfrak{n}} \neq 0$.*

4. *Suppose $v_0 \in V$ is $\mathfrak{b}$-extreme of weight μ and is cyclic under $\mathfrak{g}$. Then π is irreducible, $V(\mu) = \mathbb{C}v_0$, and $\chi(V) \subset \mu - Q_+(\mathfrak{g})$.*

Theorem 5.6.19. (**Highest Weight**) *Suppose (π, V) is an irreducible finite-dimensional representation of $\mathfrak{g}$. Then V has a unique highest weight μ such that $\lambda \prec \mu$ for all other weights λ of V. One has $\mu \in P_{++}(\mathfrak{g})$ and $\dim(V(\mu)) = 1$. A non-zero vector $v_0 \in V(\mu)$ is called a highest weight vector of V . If U is another irreducible finite-dimensional $\mathfrak{g}$-module with highest weight μ, then $U \simeq V$.*

The definition of *highest weight* depends on the choice of the set of positive roots. However, the elements of $P_{++}(\mathfrak{g})$ are in one to one correspondence with the Weyl group orbits in $P(\mathfrak{g})$. Thus every irreducible finite-dimensional representation of $\mathfrak{g}$ corresponds to a unique W_G-orbit in $P^{\mathfrak{g}}$, namely the orbit of the highest weight.

We want to give some geometric interpretations of these and similar facts. Instead of looking at the action of a Lie algebra on a representation, we look at the action of a group on the associated projective vector space. In this context, it is natural to look at various geometric objects associated to the action: for example, we look at closures of orbits of the action, which all turns out to be algebraic variety, i.e. definable by polynomial equations.

The important fact we want to point out in this section is that an homogeneous variety defined as the orbit of a point x_0 stable under the action of the subgroup $H \subset G$ in nothing else than the closure of the highest weight vector's orbit.

Theorem 5.6.20. *Let G be a connected classical group. There is a projective algebraic set X_G on which G acts algebraically and transitively and there is a point $x_0 \in X_G$ so that the stabilizer $B = G_{x_0}$ has Lie algebra $\mathfrak{b}$.*

As example we want to study what happens if we look at the action of the group $SL_2(\mathbb{C})$ or $PGL_2(\mathbb{C})$ on the associated projective spaces $\mathbb{P}(W)$.

Example: How can we embed an homogeneous variety into $\mathbb{P}(V)$? Let us do the example of $G = SL(2,\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 1 \right\}$. The Borel subgroup $B \subset G$ is $B = \left\{ \begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix} \right\}$. Consider the $(n+1)$-dimensional standard representation o $SL(2,\mathbb{C})$:

$$\rho : SL(2) \to GL(V) = GL(n)$$

it is $S^n(\mathbb{C}^2) =< x^n, x^{n-1}y, x^{n-2}y^2, \ldots, y^n >$.
We need to find the highest weight vector: it is a $P \in S^n(V^*)$ such that $bP = \lambda P$ for all $b \in B$. The dimension of V is two, let $\{e, f\}$ be a base of V and $V^* =< x, y >$. An element $P \in S^nV^*$ is $P = x^k y^{n-k}$. By definition $(b \cdot P)(\alpha e + \beta f) = P(b^{-1}(\alpha e + \beta f))$ that is equal to $P\left(\begin{pmatrix} 1/a & -b \\ 0 & a \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}\right)$ since $e = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $f = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then we have that $(bP)(\alpha e + \beta f) = P((\frac{\alpha}{a} - b\beta)e + a\beta f) = (\frac{\alpha}{a} - b\beta)^k (a\beta)^{n-k}$ that is $\left(\sum_{i=0}^k (-1)^i \binom{k}{i} \alpha^{k-i} a^{i-k} \beta^i b^i\right) a^{n-k}\beta^{n-k}$ that we can write as $\sum_{i=0}^k (-1)^i \binom{k}{i} a^{n+i-2k} b^i \alpha^{k-i} \beta^{n-k+i}$. Therefore

$$bP = \begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix} x^k y^{n-k} = \sum_{i=0}^{k} (-1)^i \binom{k}{i} a^{n+i-2k} b^i x^{k-i} y^{n-k+i}.$$

Then the only $P \in S^nV^*$ for which $bP = \lambda P$ can be obtained for $k = 0$, this means that the highest weight vector is y^n.

Now we need to find the orbit of y^n. If $g \in G$ then $gP = \begin{pmatrix} a & b \\ c & d \end{pmatrix} y^n$. By definition $\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} y^n\right)\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = y^n\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}\right) = y^n\begin{pmatrix} d\alpha - b\beta \\ -c\alpha + a\beta \end{pmatrix} = (-c\alpha + a\beta)^n$ that is $\sum_{i=0}^n (-1)^i \binom{n}{i} a^{n-i}\beta^{n-i}c^i\alpha^i$ then $\begin{pmatrix} a & b \\ c & d \end{pmatrix} y^n = \sum_{i=0}^n (-1)^i \binom{n}{i} a^{n-i}c^i x^i y^{n-i}$. Therefore the immersion we are looking for is:

$$\begin{array}{ccccc} \begin{pmatrix} a & 0 \\ c & 1/a \end{pmatrix} \in & SL(2)/B & \hookrightarrow & \mathbb{P}(V) \\ \downarrow & \downarrow\simeq & & \downarrow\simeq \\ & \mathbb{P}^1 & \hookrightarrow & \mathbb{P}^n \\ (a:c) \mapsto & (a:c) & \mapsto & (a^n : -na^{n-1}c : \binom{n}{2}a^{n-2}c^2 : -\binom{n}{3}a^{n-3}c^3 : \cdots : (-1)^n c^n) \end{array}.$$

Example: For any vector space V and any positive integer n, we have a natural map, called the *Veronese embedding*;

$$\mathbb{P}(V)^* \hookrightarrow \mathbb{P}(\mathrm{Sym}^n(V^*))$$

that maps the line spanned by $v \in V^*$ to the line spanned by $v^n \in Sym^n(V^*)$. If $\dim(\mathbb{P}(V^*)) = 1$ the image of the previous map is called the *rational normal curve* $C = C_n$ of degree n. Choosing a base $\{\alpha, \beta\}$ for V^* and $\{\ldots, [n!/k!(n-k)!]\alpha^k\beta^{n-k}, \ldots\}$ for $\mathrm{Sym}^n(V^*)$ and expanding out $(x\alpha + y\beta)^n$ we see that in coordinates this map may be given as

$$[x, y] \mapsto [x^n, x^{n-1}y, \ldots, y^n].$$

From the definition, the action of $PGL_2(\mathbb{C})$ on $\mathbb{P}^n$ preserves C_n; conversely, since any automorphism of $\mathbb{P}^n$ fixing C_n pointwise is the identity and since the group of automorphisms of $\mathbb{P}^n$ is $PGL_{n+1}(\mathbb{C})$, the group G of automorphisms of $\mathbb{P}^n$ that preserve C_n is $PGL_2(\mathbb{C})$. Conversely if W is any $(n+1)$-dimensional representation of $SL_2(\mathbb{C})$ and $\mathbb{P}(W) \simeq \mathbb{P}^n$ contains a rational normal curve of degree n preserved by the action of $PGL_2(\mathbb{C})$, then we must have $W \simeq \mathrm{Sym}^n(V)$.

5.6.3 Ideals of secant varieties of homogeneous varieties

Consider now the case $X = G/H$ is an homogeneous variety, embedded as the orbit of the highest weight vector $v_l \in \mathbb{P}^n$: if $V_l =< v_l >$ then $G/H \subset \mathbb{P}(V_l)$.

We will need to use the following unpublished Theorem of Konstant and a generalization of it.

Theorem 5.6.21. (Konstant) *If $X = G/H \subset \mathbb{P}V_l$ is a homogeneous variety which is the orbit of the highest weight vector $v_l \in \mathbb{P}^n$ then*

$$I_2(X) = (V_{2l})^\perp \subset S^2V^*$$

where $V_{2l} \subset S^2V^*$ is generated by $v_l \circ v_l$.

Theorem 5.6.22. *(Generalization) If $X = G/H \subset \mathbb{P}(V_l)$ is a homogeneous variety which is the orbit of the highest weight vector $v_l \in \mathbb{P}^n$ then*

$$I_k(X) = (V_{kl})^\perp \subset S^kV^*$$

where V_{kl} is generated by $v_l \circ \cdots \circ v_l$, k times.

Proof. Let $p \in S^kV^*$ and consider p as a multi-linear form on K. Then $p(v_l \circ \cdots \circ v_l) = 0$ means that p annihilates the vectors of weight kl in S^kV. An irreducible module $W \subset S^kV^*$ having this property for all $p \in W$ satisfies $W \subset V_{kl}^\perp$. (This proof is from notes by J. M. Landsberg on Secant varieties, Lie algebra and Rational Homogeneous varieties, see *http://www.math.tamu.edu/∼jml/*.) □

We are now interested in studying the degree d part of the ideal $I(\mathrm{Sec}_{k-1}(X))$ where X is an embedded homogeneous variety.

Suppose that $p \in S^dV^*$, then, for every $m = 0, \ldots, d$, there exist $R_i \in S^mV^*$ and $Q_{d-i} \in S_{d-m}V^*$ such that:

$$\begin{array}{rcl} S^dV^* & \to & S^mV^* \otimes S^{d-m}V^* \\ p & \mapsto & \sum_i R_i \otimes Q_{d-i} \end{array}.$$

For example, if $\dim(V) = 2$, $d = 3$, $m = 2$ and $p = x^2y$, then

$$\begin{array}{rcl} S^3V^* & \to & S^2V^* \otimes S^1V^* \\ x^2y & \mapsto & \frac{1}{2}(x^2 \otimes y + xy \otimes x) \end{array}.$$

More generally (see also Section 5.6.1), one can construct the following map:

$$\begin{array}{rcl} S^dV^* & \to & \bigoplus_{m=0}^d S^mV^* \otimes S^{d-m}V^* \\ p(u+v) & \mapsto & \sum_{i=1}^d \binom{d}{i} \sum_j R_{i,j}(u) \otimes Q_{d-i,j}(v) \end{array}; \tag{5.13}$$

For example, with the same conditions of the last example, if we write $u = (u_1, u_2)$ and $v = (v_1, v_2)$ then $p(u+v) = p((u_1+v_1, u_2+v_2)) = (u_1+v_1)^2(u_2+v_2) = [u_1^2u_2 + u_1^2v_2 + 2u_1u_2v_1 + 2u_1v_1v_2 + v_1^2u_2 + v_1^2v_2] \mapsto [(1 \otimes x^2y) + (2x \otimes xy + y \otimes x^2) + (x^2 \otimes y + 2xy \otimes x) + (x^2y \otimes 1)] \in (S^0V^* \otimes S^3V^*) \oplus (S^1V^* \otimes S^2V^*) \oplus (S^2V^* \otimes S^1V^*) \oplus (S^3V^* \otimes S^0V^*)$.

Now, for every $m = 0, \dots, d$, we can decompose again:

$$S^m V^* \to \bigotimes_{i=1}^{k} S^{m_i} V^* \tag{5.14}$$

for all $m_i \in \mathbb{Z}$ such that $\sum_{i=1}^k m_i = m$.

For every decomposition $S^d V^* \to S^{m_1} V^* \otimes \cdots \otimes S^{m_k} V^*$ we can consider the evaluation map at the last argument:

$$S^{m_1} V^* \otimes \cdots \otimes S^{m_k} V^* \to S^{m_1} V^* \otimes \cdots \otimes S^{m_{k-1}} V^*.$$

We can also consider successive evaluations maps:

$$S^{m_1} V^* \otimes \cdots \otimes S^{m_k} V^* \to S^{m_1} V^* \otimes \cdots \otimes S^{m_{k-1}} V^* \to \cdots \to S^{m_1} V^* \otimes S^{m_2} V^* \to S^{m_1} V^* \to \mathbb{C} \tag{5.15}$$

at the end of those we will have the evaluation of a polynomial $p \in S^d V^*$.

Now $p \in I_d(\mathrm{Sec}_{k-1}(X))$ if and only if $p \in S^d V^*$ and for all $v_1, \dots, v_k \in \hat{X}$ (where $\hat{X} \subset \mathbb{A}^{n+1}$ is the affine cone over $X \subset \mathbb{P}^n$) and for all $\lambda_1, \dots, \lambda_k \in \mathbb{C}$

$$p(\lambda_1 v_1 + \cdots + \lambda_k v_k) = 0. \tag{5.16}$$

By (5.13) and (5.14), $S^d V^*$ can be "naturally" embedded in the sum of all possible decomposition of type (5.14), while $(m_1, \dots, m_k)$ varies in the set of partitions of d. Therefore the condition (5.16) is equivalent to ask that the image of $p(\lambda_1 v_1 + \cdots + \lambda_k v_k) \in S^d V^*$ in each one of the decompositions of type (5.14) composed with the successive evaluation maps of type (5.15) has to be zero: for all $m_1, \dots, m_k \in \mathbb{Z}$ such that $\sum_{i=1}^k m_i = d$

$$\begin{array}{ccccccc}
S^d V^* & \to & S^{m_1} V^* \otimes \cdots \otimes S^{m_k} V^* & \to & S^{m_1} V^* \otimes \cdots \otimes S^{m_{k-1}} V^* & \to & \cdots \\
p(\sum_{i=1}^k \lambda_i v_i) & \mapsto & p(\sum_{i=i}^k \lambda_i v_i) & \mapsto & \text{eval in } \lambda_k v_k & \mapsto & \cdots
\end{array}$$

$$\begin{array}{ccccc}
\cdots & \to & S^{m_1} & \to & \mathbb{C} \\
\cdots & \mapsto & \text{eval in } \sum_{i=2}^k \lambda_i v_i & \mapsto & 0
\end{array}$$

i.e. $p \in I_d(\mathrm{Sec}_{k-1}(X))$ if and only if $p \in \mathrm{Ker}(S^{m_1} V^* \otimes \cdots \otimes S^{m_k} V^* \to \mathbb{C})$ for all $(m_1, \dots, m_k)$ partitions of d.

For the generalized Theorem of Konstant we know that if $X \subset \mathbb{P}(V_l)$ is an homogeneous variety then

$$I_d(X) = (V_{dl})^\perp \subset S^d V^*. \tag{5.17}$$

This fact has some important implications.

Example: Suppose we want to study $I_d(\mathrm{Sec}_1(X))$, we do not have to control that all the contractions, for all m, $S^mV^* \otimes S^{d-m}V^* \to S^mV^* \to \mathbb{C}$ are the zero map, because $p \in I_d(\mathrm{Sec}_1(X))$ if and only if $S^dV^* \ni p(u+v) = \sum_{i=1}^d \binom{d}{i} \sum_j R_{i,j}(u)Q_{d-i,j}(v) \mapsto 0$ iff both $R_{i,j}(u) \mapsto 0$ and $Q_{d-i,j}(v) \mapsto 0$, i.e. $R_{i,j} \in I_i(X)$ and $Q_{d-i,j} \in I_{d-i}(X)$, that is the same to ask that for all $i = 1, \dots, d$ the $R_{i,j}$ annihilate on $(V_{il})^\perp$ and $Q_{d-i,j}$ annihilate on $(V_{(d-i)l})^\perp$, i.e. all contractions, for $i = 1, \dots, d$, $(V_{il})^\perp \otimes (V_{(d-i)l})^\perp \to \mathbb{C}$ are the zero map; but now this is equivalent to ask that all contractions

$$S^{a_1}(V_l) \otimes S^{a_2}(V_{2l}) \otimes \cdots \otimes S^{a_p}(V_{pl}) \to \mathbb{C}$$

such that $a_1 + \cdots + a_p = 2$, $a_1 + 2a_2 + \cdots + pa_p = d$, are the zero map.

What we pointed out in this example holds in general (the following proposition is the same of Prop 3.3 on [**LM1**])

Proposition 5.6.23. *If $X \subset \mathbb{P}(V_l)$ is a rational homogeneous variety then a module $W \subset S^dV^*$ is contained in $I_d(\mathrm{Sec}_{k-1}(X))$ if and only if for all $(a_1, \dots, a_p)$ partitions of $k \in \mathbb{Z}$ such that $a_1 + 2a_2 + \cdots + pa_p = d$, the contraction*

$$W \otimes S^{a_1}(V_l) \otimes S^{a_2}(V_{2l}) \otimes \cdots \otimes S^{a_p}(V_{pl}) \to \mathbb{C} \tag{5.18}$$

is the zero map.

Example: Suppose that $k = 3$ and $d = 5$, the only two partitions $(a_1, \dots, a_p)$ of 3 such that $\sum_{i=1}^p ia_i = 5$ are $(a_1, a_2, a_3) = (2, 0, 1), (1, 2, 0)$. Therefore if $X \subset \mathbb{P}(V_l)$ is an homogeneous variety then a module $W \subset S^5(V)$ is contained in $I_5(\mathrm{Sec}_2(X))$ if and only if the following two contractions are the zero map (and only those).

$$S^2(V_l) \otimes S^0(V_{2l}) \otimes S^1(V_{3l}) \to \mathbb{C},$$

$$S^1(V_l) \otimes S^2(V_{2l}) \to \mathbb{C}.$$

Corollary 5.6.24. *Let $X = G/P \subset \mathbb{P}(V)$ be a rational homogeneous variety. Then for all $d > 0$*

1. *$I_d(\mathrm{Sec}_{d-1}(X)) = 0$;*
2. *if $f \in I_{d+1}(\mathrm{Sec}_{d-1}(X))$ then, for all $v \in V^2$, the element $v \otimes f$ belongs to the kernel of the map $V^2 \otimes S^{d+1}V^* \to S^{d-1}V^*$;*
3. *let W be an irreducible component of S^dV^* and suppose that for all $(a_1 \dots a_p)$ partitions of k such that $\sum_{i=1}^p ia_i = d$, W^* is not an irreducible component of $S^{a_1}(V) \otimes S^{a_2}(V^2) \otimes \cdots \otimes S^{a_p}(V^p)$. Then $W \subset I_d(\mathrm{Sec}_{k-1}(X))$.*

Proof. 1. It is a consequence of Corollary 5.6.3.

2. It follows from Proposition 5.6.2 and Proposition 5.6.23.

3. It follows from (5.18) and Schur's Lemma because if an irreducible submodule $W \subset S^dV^*$ does not belong to $I_d(\mathrm{Sec}_{k-1}(X))$, one of the contraction maps (5.18) must be non-zero.

□

The following two "inheritance" will allow us to apply an algorithm for computing the ideal of the secant varieties to Segre varieties.

Proposition 5.6.25. (**First inheritance**) *Let $A_1, \dots, A_k$ be vector spaces and $\pi_1, \dots, \pi_k$ be partitions of d. Suppose that an $\mathfrak{S}_d$-invariant I of $V_{\pi_1} \otimes \cdots \otimes V_{\pi_k}$ defines a non-zero embedding of I into $\mathbb{S}_{\pi_1}A_1^* \otimes \cdots \otimes \mathbb{S}_{\pi_k}A_k^* \subset S^d(A_1 \otimes \cdots \otimes A_k)^*$. Then for any vector spaces $A'_1, \dots, A'_k$ such that $\dim(A'_i) \geq \dim(A_i)$ for all i, the image of the embedding of $(\mathbb{S}_{\pi_1}A'_1)^* \otimes \cdots \otimes (\mathbb{S}_{\pi_k}A'_k)^*$ in $S^d(A'_1 \otimes \cdots \otimes A'_k)^*$ defined by I, is in $I_d(\mathrm{Sec}_{s-1}(Seg(\mathbb{P}(A'_1) \times \cdots \times \mathbb{P}(A'_k))))$ if and only if the image of the embedding of $\mathbb{S}_{\pi_1}A_1^* \otimes \cdots \otimes \mathbb{S}_{\pi_k}A_k^*$ in $S^d(A_1 \otimes \cdots \otimes A_k)^*$ defined by I is in $I_d(\mathrm{Sec}_{s-1}(Seg(\mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_k))))$.*

Proof. The proof is a consequence of the fact that the action of $\mathfrak{S}_d$ commutes both with the action of $GL(A_i)$ and the action of $GL(A'_i)$, hence the invariant that defines the embedding of $\mathbb{S}_{\pi_1}A_1^* \otimes \cdots \otimes \mathbb{S}_{\pi_k}A_k^*$ into $S^d(A_1 \otimes \cdots \otimes A_k)^*$ is the same invariant that defines the embedding of $(\mathbb{S}_{\pi_1}A'_1)^* \otimes \cdots \otimes (\mathbb{S}_{\pi_k}A'_k)^*$ into $S^d(A'_1 \otimes \cdots \otimes A'_k)^*$. Therefore we can choose vector spaces A_i of dimensions as small as possible. □

Proposition 5.6.26. (**Second inheritance**) *Let X be the Segre variety $Seg(\mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_k))$ and $\widetilde{X}$ be the Segre variety $Seg(\mathbb{P}(A_2) \times \cdots \times \mathbb{P}(A_k))$. Then $I_d(\mathrm{Sec}_{d-2}(X)) \cap (S^dA_1^* \otimes S^d(A_2^* \otimes \cdots \otimes A_k^*)) = S^dA_1^* \otimes I_d(\mathrm{Sec}_{d-2}(\widetilde{X}))$.*

Proof. One of the inclusions is quite obvious:

$$I_d(\mathrm{Sec}_{d-2}(X)) \cap (S^dA_1^* \otimes S^d(A_2^* \otimes \cdots \otimes A_k^*)) \subseteq S^dA_1^* \otimes I_d(\mathrm{Sec}_{d-2}(\widetilde{X})).$$

If $A := A_1$ and $B := A_2 \otimes \cdots \otimes A_k$, the inclusion above follows immediately from the standard way to embed $S^dA^* \otimes S^dB^*$ into $S^d(A \otimes B)^*$. Since $\mathbb{S}_{(d)}V^* = S^dV^*$ for any vector space V, Theorem 5.5.2 shows that $S^dA^* \otimes S^dB^* \hookrightarrow S^d(A \otimes B)^*$. The embedding $S^dA^* \otimes S^dB^* \hookrightarrow S^d(A \otimes B)^*$ is given by Theorem 5.5.7 applied to the particular case of the decomposition into irreducible modules of $S^d(A \otimes B)^*$ obtained by the formula (5.6).

The less obvious inclusion is

$$S^d A_1^* \otimes I_d(\mathrm{Sec}_{d-2}(\widetilde{X})) \subseteq I_d(\mathrm{Sec}_{d-2}(X)) \cap (S^d A_1^* \otimes S^d(A_2^* \otimes \cdots \otimes A_k^*)).$$

Let $\mathrm{Sec}^0_{d-2}(Seg(\mathbb{P}(A_1)\otimes\cdots\otimes\mathbb{P}(A_k)))$ be the dense open subset of $\mathrm{Sec}_{d-2}(Seg(\mathbb{P}(A_1)\otimes\cdots\otimes\mathbb{P}(A_k)))$ such that if $P \in \mathrm{Sec}^0_{d-2}(Seg(\mathbb{P}(A_1)\otimes\cdots\otimes\mathbb{P}(A_k)))$ then $P = \lambda_1 P_1 + \cdots + \lambda_{d-1} P_{d-1}$ for distinct points $P_1, \ldots, P_{d-1} \in Seg(\mathbb{P}(A_1) \otimes \cdots \otimes \mathbb{P}(A_k))$. We can write $\lambda_i P_i = e_i \otimes f_i$ with each $e_i \in A_1$ and $f_i \in Seg(\mathbb{P}(A_2)\otimes\cdots\otimes\mathbb{P}(A_k))$, i.e. $P = e_1\otimes f_1 + \cdots + e_{d-1}\otimes f_{d-1}$. So any polynomial on A_1 multiplied by a polynomial in $I_d(\mathrm{Sec}_{d-2}(Seg(\mathbb{P}(A_1)\otimes\cdots\otimes\mathbb{P}(A_k))))$ will vanish on $Sec_{d-2}(Seg(\mathbb{P}(A_1)\otimes\cdots\otimes\mathbb{P}(A_k)))$. In fact if $F \in S^d(A_1^*)$ and $G \in I_d(\mathrm{Sec}_{d-2}(Seg(PA_2 \otimes \cdots \otimes PA_k)))$ we can write $(FG)(P) = F(e_1)G(f_1) + \cdots + F(e_{d-1})G(f_{d-1})$. Since $G(f) = 0$ for all $x \in \mathrm{Sec}_{d-2}(Seg(\mathbb{P}(A_2) \otimes \cdots \otimes \mathbb{P}(A_k)))$, we get $FG(x) = 0$ for all $x \in \mathrm{Sec}^0_{d-2}(Seg(\mathbb{P}(A_1) \otimes \cdots \otimes \mathbb{P}(A_k)))$, and now clearly FG is zero also on the closure of $\mathrm{Sec}^0_{d-2}(Seg(\mathbb{P}(A_1) \otimes \cdots \otimes \mathbb{P}(A_k)))$ that is $\mathrm{Sec}_{d-2}(Seg(\mathbb{P}(A_1) \otimes \cdots \otimes \mathbb{P}(A_k)))$.

□

ALGORITHM

- Fix $\pi_1, \ldots, \pi_k$ partitions of d.
- Compute the dimension m of $(V_{\pi_1} \otimes \cdots \otimes V_{\pi_k})^{\mathfrak{S}_d}$.
- Explicitly realize the representations V_{π_j} of $\mathfrak{S}_d$.
- Take independent elements $e_j \in V_{\pi_j}$ and average $e_1 \otimes \cdots \otimes e_k$ over $\mathfrak{S}_d$. The result is either a nontrivial invariant I or zero.
- Continue finding such elements I until one has m independent such.
- Choose embeddings $\mathbb{S}_{\pi_j}(A_j) \to A_j^{\otimes d}$, the images of the invariants I_r, $1 \le r \le m$ give the modules.

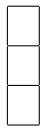
Example: Let $k = 4$ and $d = 3$. We want to study the secant of lines $\mathrm{Sec}_{2-1}(Seg(\mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_4)))$.

The partitions of $d = 3$ are (3), (111) and (21). They correspond to the trivial representation, the alternating representation and the standard representation of $\mathfrak{S}_3$ respectively:

The trivial representation $U = V_{Id}$ is $\{v \in \mathbb{C}^3 \mid gv = v,\ g \in \mathfrak{S}_3\}$; its Young diagram is:

 .

The alternating representation $U' = V_{(111)}$ is $\{v \in \mathbb{C}^3 \mid gv = sgn(g)v,\ g \in \mathfrak{S}_3\}$; its Young diagram is:



.

The standard representation $V' = V_{(12)}$ is $\{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid z_1+z_2+z_3 = 0\}$; its Young diagram is:

.

There are many ways to compute the dimension of a representation V_π of $\mathfrak{S}_d$. The most intuitive is a one using the notion of "hook length".

Definition 5.6.27. *The hook length of a box in a Young diagram is the number of squares directly below and directly to the right of the box, including the box itself.*

The hook length formula is

$$\dim(V_\pi) = \frac{d!}{\Pi(\text{hook length})}.$$

If $d = 3$ we have:
$\dim(U) = \frac{6}{3\cdot 2\cdot 1} = 1$,
$\dim(U') = \frac{6}{3\cdot 2\cdot 1} = 1$,
$\dim(V') = \frac{6}{3\cdot 1\cdot 1} = 2$.

Definition 5.6.28. *If V is a representation of a group G, its character χ_V is the complex-valued function on the group defined by:*

$$\chi_V(g) := Tr(g|_V),$$

the trace of g on V.

In particular we have $\chi_V(hgh^{-1}) = \chi_V(g)$, so that χ_V is constant on the conjugacy classes of a group G. Note that $\chi_V(1) = \dim(V)$.

Let us compute $\chi_U, \chi_{U'}, \chi_{V'}$:
$\chi_U(g) = 1$ for all $g \in \mathfrak{S}_3$,
$\chi_{U'}(1) = 1$, $\chi_{U'}((12)) = -1$, $\chi_{U'}((123)) = 1$ since $gv = sgn(v)$ and $(1)v = v$, $(12)v = -v$ and $(123)v = v$.
The standard representation can be obtained as:

$$\mathbb{C}^3 = U \oplus V'$$

where $\mathbb{C}^3$ is the permutation representation V, i.e. $V = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid g \cdot (z_1, z_2, z_3) = (z_{g^{-1}(1)}, z_{g^{-1}(2)}, z_{g^{-1}(3)}),\ g \in \mathfrak{S}_3\}$. The characters $\chi_V(g)$ of the permutation representation are always the number of independent elements of X that are fixed by g. In our case $X = \mathbb{C}^3 =< (1,0,0), (0,1,0), (0,0,1) >$, then $Id(v) = v$ for all $v \in \mathbb{C}^3$ hence $\chi_V(1) = 3$; then $\chi_V((12)) = 1$ since $(12)(1,0,0) = (0,1,0) \neq (1,0,0)$, $(12)(0,1,0) = (1,0,0) \neq (0,1,0)$ and $(12)(0,0,1) = (0,0,1)$; eventually $\chi_V((123)) = 0$.
Now we apply the property that if V and W are two representations of a finite group G then

$$\chi_{(V \oplus W)} = \chi_V + \chi_W,$$

therefore $\chi_V = \chi_U + \chi_{V'}$ then, $\chi_{V'} = (3,1,0) - (1,1,1) = (2,0,-1)$.
We can draw the table of characters for $\mathfrak{S}_3$ (in the first line we write the number of elements in each conjugacy class, and in the second line the conjugacy classes):

	1	3	2
	1	(12)	(123)
U	1	1	1
U'	1	-1	1
V'	2	0	-1

Which is the dimension of $(V' \otimes V' \otimes V' \otimes V')^{\mathfrak{S}_3}$?

It is a general fact that the dimension of the space of invariants by the action of a group G is:

$$\dim(V^G) = \frac{1}{|G|} \sum_{g \in G} \chi_V(g).$$

The order of $\mathfrak{S}_3$ is 6.
If V and W are two representations of a finite group G then

$$\chi_{(V \otimes W)} = \chi_V \cdot \chi_W.$$

Hence $\chi_{(V' \otimes V' \otimes V' \otimes V')} = (\chi_{V'})^4 = (2^4, 0^4, 1^4) = (16, 0, 1)$.
Therefore

$$m := \dim(V'^{\otimes 4}) = \frac{1}{6}(16 \cdot 1 + 0 \cdot 3 + 1 \cdot 2) = 3.$$

Hence we have to find three independent invariants.

The space V' can be realized as $\{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid z_1 + z_2 + z_3 = 0\}$, we consider the following two independent generators: $e = (1, -1, 0)$ and $f = (0, 1, -1)$, then a base for $V'^{\otimes 4}$ is $\{e \otimes e \otimes e \otimes e, e \otimes e \otimes e \otimes f, \ldots, f \otimes f \otimes f \otimes f\}$.
The invariants of $V'^{\otimes 4}$ can be obtained by applying the averaging operator to the elements of the

base of $V'^{\otimes 4}$; we will find only three of them that are independent.
The averaging operator is defined as:

$$\varphi = \frac{1}{|G|}\sum_{g\in G} g.$$

In the case of $\mathfrak{S}_3$ one has $\varphi = \frac{1}{6}(Id + (12) + (13) + (23) + (123) + (132))$. In the following table we list how an element of $\mathfrak{S}_3$ acts on $e, f \in V'$:

	e	f
Id	e	f
(12)	$-e$	$e+f$
(13)	$-f$	$-e$
(23)	$e+f$	$-f$
(123)	$-e-f$	e
(132)	f	$-e-f$

By applying the averaging operator at $e\otimes e\otimes e\otimes e$, $e\otimes e\otimes f\otimes f$ and $e\otimes e\otimes e\otimes f$ respectively, we obtain three independent generators for the space of the invariants $(V'\otimes V'\otimes V'\otimes V')^{\mathfrak{S}_3}$:

$$\varphi(e\otimes e\otimes e\otimes e) = \tfrac{1}{3}(e\otimes e\otimes e\otimes e + (e+f)\otimes(e+f)\otimes(e+f)\otimes(e+f) + f\otimes f\otimes f\otimes f) := I_1,$$

$$\varphi(e\otimes e\otimes f\otimes f) = \tfrac{1}{6}(2e\otimes e\otimes e\otimes e + e\otimes e\otimes e\otimes f + e\otimes e\otimes f\otimes e + e\otimes f\otimes e\otimes e + f\otimes e\otimes e\otimes e + 3e\otimes e\otimes e\otimes f\otimes f + 3f\otimes f\otimes e\otimes e + e\otimes f\otimes f\otimes f + f\otimes e\otimes f\otimes f + f\otimes f\otimes e\otimes f + f\otimes f\otimes f\otimes e + 2f\otimes f\otimes f\otimes f) := I_2,$$

$$\varphi(e\otimes e\otimes e\otimes f) = -\tfrac{1}{6}(2e\otimes e\otimes e\otimes e\otimes e + e\otimes e\otimes e\otimes f + e\otimes e\otimes f\otimes e + e\otimes f\otimes e\otimes e + e\otimes e\otimes f\otimes f + e\otimes f\otimes e\otimes f + 2f\otimes e\otimes e\otimes f + e\otimes f\otimes f\otimes e + 2f\otimes f\otimes e\otimes e + 2f\otimes e\otimes f\otimes e + e\otimes f\otimes f\otimes f + 2f\otimes e\otimes f\otimes f + 2f\otimes f\otimes e\otimes f + 2f\otimes e\otimes e\otimes e + f\otimes f\otimes f\otimes e + 2f\otimes f\otimes f\otimes f) := I_3.$$

Now we have all the informations we need on $(V'^{\otimes 4})^{\mathfrak{S}_3}$. Let us study, for any vector space V, the Schur power $\mathbb{S}_{(21)}V = \mathrm{Hom}_{\mathfrak{S}_3}(V_{(21)}, V^{\otimes 3})$. An element $u \in \mathbb{S}_{(21)}V$ is an homomorphism $u : V_{(21)} \to V^{\otimes 3}$ invariant by the action of $\mathfrak{S}_3$. We define $u(e) = E \in V^{\otimes 3}$, $s_1 := (12)$ and $s_2 := (23)$, then $s_1 e = -e$, hence $s_1 E = u(s_1 e) = -E$. Moreover $f = s_2 e - e$, hence $u(f) = s_2 E - E$ and $s_1 f = e + f$. Finally $s_1 s_2 E + E = s_1 u(f) = s_2 E$. Therefore

$$\mathbb{S}_{(21)}V^* \simeq \{E \in V^{\otimes 3} \mid s_1 E = -E,\ s_1 s_2 E + E - s_2 E = 0\}.$$

The proof of Proposition 5.5.7, applied to our case, gives the following isomorphism:

$$\begin{array}{rcl} \bigoplus (V_{\pi_1}\otimes V_{\pi_2}\otimes V_{\pi_3}\otimes V_{\pi_4})^{\mathfrak{S}_3}\otimes(\mathbb{S}_{\pi_1}A_1\otimes\mathbb{S}_{\pi_2}A_2\otimes\mathbb{S}_{\pi_3}A_3\otimes\mathbb{S}_{\pi_4}A_4) & \to & S^3(A_1\otimes A_2\otimes A_3\otimes A_4) \\ \sum J\otimes(u_1\otimes u_2\otimes u_3\otimes u_4) & \mapsto & \sum J(u_1\otimes u_2\otimes u_3\otimes u_4) \end{array}$$

where $J \in (V_{\pi_1} \otimes V_{\pi_2} \otimes V_{\pi_3} \otimes V_{\pi_4})^{\mathfrak{S}_3}$ and $u_i \in \mathbb{S}_{\pi_i} A_i$. Anytime we fix an invariant J there is an associate immersion:

$$\begin{array}{rcl} P^J : \mathbb{S}_{\pi_1} A_1 \otimes \mathbb{S}_{\pi_2} A_2 \otimes \mathbb{S}_{\pi_3} A_3 \otimes \mathbb{S}_{\pi_4} A_4 & \hookrightarrow & S^3(A_1 \otimes A_2 \otimes A_3 \otimes A_4) \\ (u_1 \otimes u_2 \otimes u_3 \otimes u_4) & \mapsto & \sum J(u_1 \otimes u_2 \otimes u_3 \otimes u_4) \end{array}.$$

Fix now the $u_i \in S_{\pi_i} A_i$ and consider the corresponding polynomial $P^J_{(u_1,u_2,u_3,u_4)}$.
We are studying the case when $\pi_i = (21)$ for $i = 1, \dots, 4$, and $J \in < I_1, I_2, I_3 >$; we will write $J = \alpha_1 e \otimes e \otimes e \otimes e + \cdots + \alpha_{16} f \otimes f \otimes f \otimes f$ for some coefficients $\alpha_1, \dots, \alpha_{16}$.

An element $v = \lambda_1 v_1 + \lambda_2 v_2$ belongs to $\mathrm{Sec}_{2-1}(Seg(\mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_4)))$ if and only if $P^J_{u_1,u_2,u_3,u_4}(\lambda_1 v_1 + \lambda_2 v_2) = 0$ for all $\lambda_1, \lambda_2 \in K$ and $v_1, v_2 \in Seg(\mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_4))$. By the decomposition (5.11), there exist polynomials $Q_{j,i}, R_{j,i}$ of degree j for $j = 0, \dots, 3$ such that $P^J_{u_1,u_2,u_3,u_4}(\lambda_1 v_1 + \lambda_2 v_2) = \sum_{d=0}^{3} \binom{3}{d} \sum_i R_{d,i}(\lambda_1 v_1) Q_{3-d,i}(\lambda_2 v_2)$. Hence $P^J_{u_1,u_2,u_3,u_4} \in I_3(\mathrm{Sec}_1(Seg(\mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_4))))$ if and only if

$$\begin{cases} R_{3,i}(v_1) = 0, \\ R_{2,i}(v_1) Q_{1,i}(v_2) = 0, \\ R_{1,i}(v_1) Q_{2,i}(v_2) = 0, \\ Q_{3,i}(v_2) = 0 \end{cases}$$

for all $v_1, v_2 \in Seg(\mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_4))$. Therefore it is sufficient to ask that $R_{2,i}(v_1) Q_{1,i}(v_2) = 0$ for all $v_1, v_2 \in Seg(\mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_4))$.

Now, if J is any invariant of $(V_{\pi_1} \otimes V_{\pi_2} \otimes V_{\pi_3} \otimes V_{\pi_4})^{\mathfrak{S}_3}$ then

$$P^J_{u_1,u_2,u_3,u_4}(\lambda_1 v_1 + \lambda_2 v_2) = (\alpha_1 e \otimes e \otimes e \otimes e + \cdots + \alpha_{16} f \otimes f \otimes f \otimes f)(u_1 \otimes u_2 \otimes u_3 \otimes u_4)(\lambda_1 v_1 + \lambda_2 v_2) =$$

$$= (\alpha_1 e u_1 \otimes e u_2 \otimes e u_3 \otimes e u_4 + \cdots + \alpha_{16} f u_1 \otimes f u_2 \otimes f u_3 \otimes f u_4)(\lambda_1 v_1 + \lambda_2 v_2) =$$

$$= \alpha_1 (u_1(e) \otimes u_2(e) \otimes u_3(e) \otimes u_4(e))(\lambda_1 v_1 + \lambda_2 v_2) + \cdots + \alpha_{16} (u_1(f) \otimes u_2(f) \otimes u_3(f) \otimes u_4(f))(\lambda_1 v_1 + \lambda_2 v_2).$$

So $P^J_{u_1,u_2,u_3,u_4}(\lambda_1 v_1 + \lambda_2 v_2) = 0$ if and only if

$$\begin{cases} (u_1(e) \otimes u_2(e) \otimes u_3(e) \otimes u_4(e))(\lambda_1 v_1 + \lambda_2 v_2) = 0 \\ \vdots \\ (u_1(f) \otimes u_2(f) \otimes u_3(f) \otimes u_4(f))(\lambda_1 v_1 + \lambda_2 v_2) = 0 \end{cases}$$

Now we apply the decomposition (5.11) to all those sixteen polynomials. Let the decomposition of the j-th of those polynomials be $\sum_{d=0}^{3} \binom{3}{d} \sum_i R^{(j)}_{d,i}(\lambda_1 v_1) Q^{(j)}_{3-d,i}(\lambda_2 v_2)$. By the consideration above it is sufficient to look at the vanishing of $R^{(j)}_{2,i}(v_1) Q^{(j)}_{1,i}(v_2)$.

Remind that $u_i(e)$ is skew symmetric in the first two arguments, hence the contributions of $\alpha_j (u_i(e) \otimes u_2(e) \otimes u_2(e) \otimes u_4(e))(\lambda_1 v_1 + \lambda_2 v_2)$ are zero for all $j = 1, \dots, 15$.

Therefore $P^J_{u_1,u_2,u_3,u_4} \in I_3(\mathrm{Sec}_1(Seg(\mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_4))))$ if and only if $\alpha_{16}(u_1(f) \otimes u_2(f) \otimes u_3(f) \otimes u_4(f))(\lambda_1 v_1 + \lambda_2 v_2) = 0$ for all $\lambda_1, \lambda_2 \in K$ and $v_1, v_2 \in Seg(\mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_4))$. This is equivalent to asking that $\alpha_{16} = 0$.

This condition allows us to write down explicitly the invariant $J \in < I_1, I_2, I_3 >$:

$$J = \alpha I_1 + \beta I_2 + (\alpha + \beta) I_3$$

for $\alpha, \beta \in K$.

Note that we have never used the fact that $k = 4$, hence we can state the following proposition:

Proposition 5.6.29. *The space of modules in $I_3(\mathrm{Sec}_1(Seg(\mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_k))))$ induced from the representation $(V_{(21)})^{\otimes k}$ is a codimension one subspace of the space of modules in S^3V^* induced from $(V_{(21)})^{\otimes k}$.*

Now we have all the ingredients to determine which of the irreducible modules that appear in the decomposition of $S^3(A_1 \otimes \cdots \otimes A_k)^*$ are in the decomposition into irreducible modules of $I_3(\mathrm{Sec}_1(Seg(\mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_4))))$.

- Every component of $S^3(A_1 \otimes \cdots \otimes A_k)^*$ involving a wedge power is in the space of cubics vanishing on $\mathrm{Sec}_1(Seg(\mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_k)))$ because the assumption that $P^J_{u_1,u_2,u_3,u_4} = 0$ on $\mathrm{Sec}_1(Seg(\mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_k)))$ is equivalent to $R_{2,i}(v_1)Q_{1,i}(v_2) = 0$ for all $v_1, v_2 \in Seg(\mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_k))$: a cubic in $\Lambda^3 V$ is always zero on an element of the form (v_1, v_1, v_2).

- Every component involving a symmetric power is determined inductively by Proposition 5.6.26.

- The only remaining term is $\mathbb{S}_{(21)}A_1 \otimes \cdots \otimes \mathbb{S}_{(21)}A_k$ that appears in the decomposition of $S^3(A_1 \otimes \cdots \otimes A_k)^*$ with a certain multiplicity l, hence, by Proposition 5.6.29, the subspace that vanishes on $\mathrm{Sec}_1(Seg(\mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_k)))$ has multiplicity $l - 1$.

Let us study the decomposition of $S^3(A_1 \otimes \cdots \otimes A_k)$ in detail.

We have already observed that the irreducible representation of $\mathfrak{S}_3$ are $V_{(3)} = S^3V$, $V_{(21)}$ and $V_{(1,1,1)} = \wedge^3(V)$. Let us compute the characters of $(V_{(3)})^{\otimes\alpha}$, $(V_{(21)})^{\otimes\beta}$ and $(V_{(111)})^{\otimes\gamma}$ for some $\alpha, \beta, \gamma \in \mathbb{N}$:
$\chi_{((V_{(3)})^{\otimes\alpha})} = (1^\alpha, 1^\alpha, 1^\alpha) = (1, 1, 1)$,
$\chi_{((V_{(21)})^{\otimes\beta})} = (2^\beta, 0, (-1)^\beta)$,
$\chi_{((V_{(111)})^{\otimes\gamma})} = (1, (-1)^\gamma, 1)$.
Then if $\beta \geq 1$ the character of $(V_{(3)})^{\otimes\alpha} \otimes (V_{(21)})^{\otimes\beta} \otimes (V_{111})^{\otimes\gamma}$ for $\alpha+\beta+\gamma = k$ is $\chi_{((V_{(3)})^{\otimes\alpha} \otimes (V_{(21)})^{\otimes\beta} \otimes (V_{111})^{\otimes\gamma})}$ $(1, 1, 1) \cdot (2^\beta, 0, (-1)^\beta) \cdot (1, (-1)^\gamma, 1) = (2^\beta, 0, (-1)^\beta)$. Then $\dim(((V_{(3)})^{\otimes\alpha} \otimes (V_{(21)})^{\otimes\beta} \otimes (V_{111})^{\otimes\gamma})^{\mathfrak{S}_3}) =$

$\frac{1}{6}(2^\beta \cdot 1 + 0 \cdot 3 + (-1)^\beta \cdot 2) = \frac{2^{\beta-1}+(-1)^\beta}{3}$.
Moreover if $\beta = 0$ we have that $\chi_{((V_{(3)})^{\otimes\alpha}\otimes(V_{111})^{\otimes\gamma})} = (1,1,1)\cdot(1,(-1)^\gamma,1) = (1,(-1)^\gamma,1)$; then $\dim(((V_{(3)})^{\otimes\alpha}\otimes(V_{111})^{\otimes\gamma})^{\mathfrak{S}_3}) = \frac{1}{6}(1\cdot 1 + (-1)^\gamma \cdot 3 + 1\cdot 2) = \frac{1+(-1)^\gamma}{2}$ that is zero if γ is odd, and it is 1 if γ is even.

We can therefore write down the decomposition of $S^3(A_1 \otimes \cdots \otimes A_k)$ into irreducible modules. Let Δ, Θ and Γ be multi-indices such that $\Delta \cup \Theta \cup \Gamma = \{1,\ldots,k\}$ and $\alpha \in \Delta$, $\beta \in \Theta$ and $\gamma \in \Gamma$; with the notation $\mathbb{S}_\pi A_\Delta$ we indicate $\otimes_{\alpha\in\Delta}\mathbb{S}_\pi A_\alpha$.

$$S^3(A_1\otimes\cdots\otimes A_k) \simeq \bigoplus_{\substack{|\Delta|+|\Theta|+|\Gamma|=k\\ |\Delta|,|\Gamma|\geq 0;\ |\Theta|\geq 1}} \frac{2^{\beta-1}+(-1)^\beta}{3}\mathbb{S}_{(3)}A_\Delta\otimes\mathbb{S}_{(21)}A_\Theta\otimes\mathbb{S}_{(111)}A_\Gamma \bigoplus_{\substack{|\Delta|+|\Gamma|=k\\ |\Gamma|\text{ even}}} \mathbb{S}_{(3)}A_\Delta\otimes\mathbb{S}_{(111)}A_\Gamma$$

Now we have the decomposition of $S^3(A_1 \otimes \cdots \otimes A_k)$ into irreducible modules, then, by considerations above, we can write down the decomposition of

$$I_3(\mathrm{Sec}_1(Seg(\mathbb{P}(A_1)\times\cdots\times\mathbb{P}(A_k))))$$

in the following way:

$$\left(\bigoplus_{\substack{|\Delta|+|\Theta|+|\Gamma|=k\\ |\Theta|\geq 1,\ |\Gamma|\geq 1}} \frac{2^{\beta-1}+(-1)^\beta}{3}\mathbb{S}_{(3)}A_\Delta\otimes\mathbb{S}_{(21)}A_\Theta\otimes\mathbb{S}_{(111)}A_\Gamma\right)\oplus$$

$$\oplus\left(\bigoplus_{\substack{|\Delta|+|\Theta|=k\\ |\Theta|\geq 1}} \left(\frac{2^{\beta-1}+(-1)^\beta}{3}-1\right)\mathbb{S}_{(3)}A_\Delta\otimes\mathbb{S}_{(21)}A_\Theta\right)\oplus$$

$$\oplus\left(\bigoplus_{\substack{|\Delta|+|\Gamma|=k\\ |\Gamma|\text{ even}}} \mathbb{S}_{(3)}A_\Delta\otimes\mathbb{S}_{(111)}A_\Gamma\right).$$

This algorithm is efficient in a low degree and for small k. In order to show how the computational problem increases by increasing k we treat the examples $k = 3$ and $k = 4$.

$$I_3(\mathrm{Sec}_1(Seg(\mathbb{P}(A)\times\mathbb{P}(B)\times\mathbb{P}(C)))) =$$

$$= (\mathbb{S}_{(21)}A\otimes\mathbb{S}_{(21)}B\otimes\Lambda^3C)^*\oplus(\mathbb{S}_{(21)}A\otimes\Lambda^3B\otimes\mathbb{S}_{(21)}C)^*\oplus(\Lambda^3A\otimes\mathbb{S}_{(21)}B\otimes\mathbb{S}_{(21)}C)^*\oplus \tag{5.19}$$

$$\oplus(S^3A\otimes\Lambda^3B\otimes\Lambda^3C)^*\oplus(\Lambda^3A\otimes S^3B\otimes\Lambda^3C)^*\oplus(\Lambda^3A\otimes\Lambda^3B\otimes S^3C).$$

$$I_3(\mathrm{Sec}_1(Seg(\mathbb{P}(A_1)\times\cdots\times\mathbb{P}(A_4)))) =$$

$$= (\mathbb{S}_{(21)}A_1\otimes\mathbb{S}_{(21)}A_2\otimes\mathbb{S}_{(21)}A_3\otimes\Lambda^3A_4)^*\oplus(\mathbb{S}_{(21)}A_1\otimes\mathbb{S}_{(21)}A_2\otimes\Lambda^3A_3\otimes\mathbb{S}_{(21)}A_4)^*\oplus$$
$$\oplus(\mathbb{S}_{(21)}A_1\otimes\Lambda^3A_2\otimes\mathbb{S}_{(21)}A_3\otimes\mathbb{S}_{(21)}A_4)^*\oplus(\Lambda^3A_1\otimes\mathbb{S}_{(21)}A_2\otimes\mathbb{S}_{(21)}A_3\otimes\mathbb{S}_{(21)}A_4)^*\oplus$$
$$\oplus(\mathbb{S}_{(21)}A_1\otimes\mathbb{S}_{(21)}A_2\otimes\Lambda^3A_3\otimes\Lambda^3A_4)^*\oplus(\mathbb{S}_{(21)}A_1\otimes\Lambda^3A_2\otimes\mathbb{S}_{(21)}A_3\otimes\Lambda^3A_4)^*\oplus$$
$$\oplus(\mathbb{S}_{(21)}A_1\otimes\Lambda^3A_2\otimes\Lambda^3A_3\otimes\mathbb{S}_{(21)}A_4)^*\oplus(\Lambda^3A_1\otimes\mathbb{S}_{(21)}A_2\otimes\mathbb{S}_{(21)}A_3\otimes\Lambda^3A_4)^*\oplus$$
$$\oplus(\Lambda^3A_1\otimes\mathbb{S}_{(21)}A_2\otimes\Lambda^3A_3\otimes\mathbb{S}_{(21)}A_4)^*\oplus(\Lambda^3A_1\otimes\Lambda^3A_2\otimes\mathbb{S}_{(21)}A_3\otimes\mathbb{S}_{(21)}A_4)^*\oplus$$
$$\oplus(S^3A_1\otimes\mathbb{S}_{(21)}A_2\otimes\mathbb{S}_{(21)}A_3\otimes\Lambda^3A_4)^*\oplus(\Lambda^3A_1\otimes\mathbb{S}_{(21)}A_2\otimes\mathbb{S}_{(21)}A_3\otimes S^3A_4)^*\oplus$$
$$\oplus(\mathbb{S}_{(21)}A_1\otimes\mathbb{S}_{(21)}A_2\otimes S^3A_3\otimes\Lambda^3A_4)^*\oplus(\mathbb{S}_{(21)}A_1\otimes\mathbb{S}_{(21)}A_2\otimes\Lambda^3A_3\otimes S^3A_4)^*\oplus$$
$$\oplus(\mathbb{S}_{(21)}A_1\otimes S^3A_2\otimes\mathbb{S}_{(21)}A_3\otimes\Lambda^3A_4)^*\oplus(\mathbb{S}_{(21)}A_1\otimes\Lambda^3A_2\otimes\mathbb{S}_{(21)}A_3\otimes S^3A_4)^*\oplus$$
$$\oplus(\mathbb{S}_{(21)}A_1\otimes S^3A_2\otimes\Lambda^3A_3\otimes\mathbb{S}_{(21)}A_4)^*\oplus(\mathbb{S}_{(21)}A_1\otimes\Lambda^3A_2\otimes S^3A_3\otimes\mathbb{S}_{(21)}A_4)^*\oplus$$
$$\oplus(S^3A_1\otimes\mathbb{S}_{(21)}A_2\otimes\Lambda^3A_3\otimes\mathbb{S}_{(21)}A_4)^*\oplus(\Lambda^3A_1\otimes\mathbb{S}_{(21)}A_2\otimes S^3A_3\otimes\mathbb{S}_{(21)}A_4)^*\oplus$$
$$\oplus(S^3A_1\otimes\Lambda^3A_2\otimes\mathbb{S}_{(21)}A_3\otimes\mathbb{S}_{(21)}A_4)^*\oplus(\Lambda^3A_1\otimes S^3A_2\otimes\mathbb{S}_{(21)}A_3\otimes\mathbb{S}_{(21)}A_4)^*\oplus$$
$$\oplus 2(\mathbb{S}_{(21)}A_1\otimes\mathbb{S}_{(21)}A_2\otimes\mathbb{S}_{(21)}A_3\otimes\mathbb{S}_{(21)}A_4)^*\oplus(\Lambda^3A_1\otimes\Lambda^3A_2\otimes\Lambda^3A_3\otimes\Lambda^3A_4)^*\oplus$$
$$\oplus(S^3A_1\otimes S^3A_2\otimes\Lambda^3A_3\otimes\Lambda^3A_4)^*\oplus(S^3A_1\otimes\Lambda^3A_2\otimes S^3A_3\otimes\Lambda^3A_4)^*\oplus$$
$$\oplus(S^3A_1\otimes\Lambda^3A_2\otimes\Lambda^3A_3\otimes S^3A_4)^*\oplus(\Lambda^3A_1\otimes S^3A_2\otimes S^3A_3\otimes\Lambda^3A_4)^*\oplus$$
$$\oplus(\Lambda^3A_1\otimes S^3A_2\otimes\Lambda^3A_3\otimes S^3A_4)^*\oplus(\Lambda^3A_1\otimes\Lambda^3A_2\otimes S^3A_3\otimes S^3A_4)^*.$$

Now, if one wants to compute the dimensions of $I_3(\mathrm{Sec}_1(Seg(\mathbb{P}(A)\times\mathbb{P}(B)\times\mathbb{P}(C))))$ and of $I_3(\mathrm{Sec}_1(Seg(\mathbb{P}(A_1)\times\cdots\times\mathbb{P}(A_4))))$, one need the fact that if V is a k-dimensional representation of $\mathfrak{S}_d$, then

$$\dim(\mathbb{S}_\pi V) = \Pi\frac{(k-i+j)}{h_{i,j}},$$

where the products are over the d pairs (i,j) that number the row and the column of boxes in π, and $h_{i,j}$ is the hook number of the corresponding box.
If V is vector space of dimension n, then

$$\dim(S^3V) = \binom{n+2}{3},$$

$$\dim(\Lambda^3V) = \binom{n}{3},$$

$$\dim(\mathbb{S}_{(21)}V) = \frac{n(n-1)(n+1)}{3}.$$

If A, B, C are vector spaces of dimensions a, b, c respectively, then:

$$\dim(I_3(\mathrm{Sec}_1(Seg(\mathbb{P}(A)\times\mathbb{P}(B)\times\mathbb{P}(C)))) =$$

$$= \frac{1}{72}acb(2a^2c + 2a^2c^2 + 16 - 8a - 6ab - 6ac + 27acb - 5a^2bc^2 - 3a^2bc + 2a^2b - 8c + 2bc^2 - 6bc+$$
$$-8b + 5a^2b^2c^2 + 2a^2b^2 + 2b^2c^2 - 5ac^2b^2 + 2ab^2 + 2ac^2 + 2b^2c - 5a^2b^2c - 3ac^2b - 3acb^2)$$

If $A_1, \dots A_4$ are vector spaces of dimension a, b, c, d respectively, then:

$$\dim(I_3(\mathrm{Sec}_1(Seg(\mathbb{P}(A_1)\times\cdots\times\mathbb{P}(A_4))))) =$$

$$= \frac{1}{1296}abcd(368 + 10a^2c^2d^2 + 10a^2b^2d^2 + 10b^2c^2d^2 + 10a^2b^2c^2 + 18bc^2d^2 + 18ac^2d^2+$$
$$+18a^2cd^2 + 18a^2bd^2 + 18b^2cd^2 + 18a^2c^2d + 18a^2b^2d + 18ab^2c^2 + 18a^2bc^2 + 18a^2b^2c+$$
$$+18bcd^2 + 18acd^2 + 18abd^2 + 18bc^2d + 18ac^2d + 18b^2cd + 18a^2cd + 18a^2bd+$$
$$+18ab^2d + 18abc^2 - 54bcd + 18ab^2c - 54acd - 54abd - 54abc - 72a+$$
$$-72c - 72d - 72b + 18ab^2d^2 + 18b^2c^2d + 8a^2c^2 + 8a^2d^2 + 8b^2c^2+$$
$$+8b^2d^2 - 72cd + 8a^2b^2 + 567abcd + 143a^2b^2c^2d^2 - 63a^2b^2c^2d + 18a^2bc - 72ab+$$
$$-72bd - 72bc - 72ad - 72ac - 8a^2 - 8b^2 - 8c^2 - 8d^2+$$
$$-27ab^2cd - 27a^2bcd - 27abcd^2 - 45ab^2cd^2 - 45a^2bcd^2 - 63ab^2c^2d^2 - 63a^2bc^2d^2 - 63a^2b^2cd^2+$$
$$+8c^2d^2 - 45ab^2c^2d - 45abc^2d^2 - 45a^2bc^2d - 45a^2b^2cd - 27abc^2d).$$

Now, the authors of [**LM1**], by using the decomposition (5.19), can give the decomposition of the degree 3 part of the ideal of $\mathrm{Sec}_1(Seg(\mathbb{P}(A)\times\mathbb{P}(B)\times\mathbb{P}(C)))$.

Definition 5.6.30. *Given $V = A_1 \otimes \cdots \otimes A_k$, a Flattening of V is a decomposition*

$$V = (A_{i_1} \otimes \cdots \otimes A_{i_q}) \otimes (A_{j_1} \otimes \cdots \otimes A_{j_{k-q}}) = A_I \otimes A_J$$

where $I + J = \{1, \ldots, k\}$ is a partition of $\{1, \ldots, k\}$ into two subsets.

Corollary 5.6.31. *Let $X = Seg(\mathbb{P}(A) \times \mathbb{P}(B) \times \mathbb{P}(C))$. Then (5.19) holds and $I_3(\mathrm{Sec}_1(X))$ is the space of 3×3 minors of the three possible flattenings of $A \otimes B \otimes C$.*

This Corollary gives the decomposition of $I_3(\mathrm{Sec}_1(Seg(\mathbb{P}(A) \times \mathbb{P}(B) \times \mathbb{P}(C))))$, but the main result of the paper [**LM1**] is to solve the G.S.S. conjecture (see [**GSS**]) in the case of the Segre variety of three factors.

Conjecture 5.6.32. (**L.D. Garcia, M. Stillman, B. Strumfeld**) *The ideal of $\mathrm{Sec}_1(Seg(\mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_k)))$ is generated by the 3×3 minors of flattenings, i.e. $\mathrm{Sec}_1(Seg(\mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_k)))$ is intersection as a scheme of the varieties $\mathrm{Sec}_1(\mathbb{P}(A_I) \times \mathbb{P}(A_J))$.*

For the proof of the case $k = 3$ of this conjecture, the authors of [**LM1**] introduced another algorithm that is longer than the one we have shown in this section, but it is more efficient in higher degrees. In fact the algorithm we presented here is very useful to compute the decomposition of the degree d part of the ideal of the secant variety to the Segre variety when d is not "too big" but it does not give any information on what happens for big values of d.

The main result of [**LM1**] is given by the following two theorems (for their proofs we refer to the paper mentioned above).

Theorem 5.6.33. *Let $X = Seg(\mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_k)) \subset \mathbb{P}(A_1 \otimes \cdots \otimes A_k)$ be a Segre product of projective spaces. Then the first secant variety $\mathrm{Sec}_1(X)$ is defined set theoretically by the 3×3 minors of flattenings. Moreover $I_3(\mathrm{Sec}_1(X))$ is spanned by the 3×3 minors of flattenings.*

Theorem 5.6.34. *Let $X = Seg(\mathbb{P}(A_1) \times \mathbb{P}(A_2) \times \mathbb{P}(A_3)) \subset \mathbb{P}(A_1 \otimes A_2 \otimes A_3)$ be a triple Segre product. Then the ideal of the secant variety $\mathrm{Sec}_1(X)$ is generated by cubics.*

Bibliography

[**Ad**] B. Ådlandsvik. *Varieties with an extremal number of degenerate higher secant varieties.* J. Reine Angew. Math. **392** (1988), 16-26.

[**AH**] J. Alexander, A. Hirschowitz; *Polynomial interpolation in several variables.*, J. Alg. Geom. **4** (1995), 201-222.

[**AP**] E. Arrondo, R. Paoletti, *Characterization of Veronese varieties via projections in Grassmannians*, to appear in the proceedings of the conference Projective varieties with unexpected properties (Siena, June 8-13 2004), edited by De Gruyter.

[**Ba**] E. Ballico; *On the secant varieties to the tangent developable of a Veronese variety*, Preprint.

[**BC**] A. Bernardi, M. V. Catalisano, *Some defective secant varieties to osculating varieties of Veronese surfaces*, Scientific Technical Notes, Sez M, DIPEM. Univ. di Genova.

[**BCGI**] A. Bernardi, M. V. Catalisano, A. Gimigliano, M. Idà, *Osculating varieties and their higher secant varieties.* Canadian J. Math. To appear.

[**BCS**] P. Bürgisser, M. Clausen, M. A. Shokrollahi, M. Idà, *Algebraic Complexity Theory.* Vol. 315, Grund. der Math. Wiss., Springer, 1997.

[**BF1**] E. Ballico, C. Fontanari; *On the secant varieties to the osculating variety of a Veronese surfaces*, Central Europ. J. of Math. **1** (2003), 315-326.

[**BF2**] E. Ballico, C. Fontanari; *A Terracini Lemma for osculating spaces with applications to Veronese surfaces*, aeXiv:math.AG/0406321, 16 June 2004.

[**CCMO**] C. Ciliberto, F. Cioffi, R. Miranda, F. Orecchia; *Bivariate Hermite interpolation and linear systems of plane curves with base fat points*, Proc. ASCM 2003, Lecture note series on Computing 10, World Scientific Publ. (2003), 87-102.

[**CGG1**] M.V. Catalisano, A.V. Geramita, A. Gimigliano; *Rank of tensors, secant variety of Segre varieties and fat points*, Lin. Alg. and Applic. **355** (2002), 263-285.

[**CGG2**] M. V. Catalisano, A. V. Geramita, A. Gimigliano, *On the secant varieties to the tangential varieties of a Veronesean*, Proc. Am. Math. Soc, **130**, (2002), no. 4, 975-985.

[**CGG3**] M. V. Catalisano, A. V. Geramita, A. Gimigliano, *Higher Secant Varieties of Segre and Segre-Veronese varieties*, preprint arXiv:math.AG/0309399 (2003), To appear on J. of Pure and Appl. Algebra.

[**CGG4**] M. V. Catalisano, A. V. Geramita, A. Gimigliano, *Secant varieties of Grassmann Varieties*, Proc. Am. Math. Soc, **133**, (2005), no. 3, 633-642.

[**Ch**] L. Chiantini; *Lectures on the structure of Projective Embeddings.*, to appear on Rendiconti del Seminario Matematico Univ. Politecnico Torino.

[**CLO**] D. Cox, J. Little, D. O' Shea; *Ideal, varieties and algorithms.*, Springer.

[**CoCoA**] A. Capani - G. Niesi - L. Robbiano, *CoCoA, A system for doing Computations in Commutative Algebra*, Available via anonymous ftp from: cocoa.dima.unige.it..

[**Du**] H. J. A. Duparc, *On Canonical Forms*, Indagationes Math. **14**, (1952), 474-482.

[**Eh**] R. Ehrenborg, *On Apolarity and Generic Canonical Forms*, Journal of Algebra **213**, 167-194 (1999).

[**EH**] D. Eisenbud, J. Harris; *The Geometry of Schemes*, Springer-Verlag.

[**ER**] R. Ehrenborg, G. C. Rota; *Apolarity and Canonical Forms for Homogeneus Polynomials*, European Jo. of Comb., 1993.

[**Ev**] L. Evain; *Computing limit linear series with infinitesimal methods*, arXiv:math.AG:0407143, 8 Jul 2004.

[**FHan**] W. Fulton, J. Hansen, *A connectedness theorem for projective varieties,with applications to intersections and singularities of mappings*, Ann. of Math. **110** n. 1, 159- 166, (1979).

[**FHar**] W. Fulton, J. Harris, *Representation Theory. A first Course*, 1991 Springer-Verlag New Yprk Inc.

[**GSS**] L.D. Garcia, M. Stillman, B. Sturmfels *Algebraic Geometry of Bayesian Networks*, preprint arXiv:math.AG/0301255 v1, 22 Jan 2003.

[**Ge**] A.V. Geramita; *Inverse system of fat points: Waring's problem, Secant varieties of Veronese Varieties and parameter spaces for Gorenstain ideals*, 1999 Queen's Paper in Pure and Appl. Math. No.102, 1-131.

[**Go1**] R. Goodman, *Classical groups, Representations and Invariants*, Hong Kong University Programme on Lie groups, April-June 1999.

[**Go2**] R. Goodman, *Multipicity-free spaces and Schur-Weyl-Howe duality, Representation af real and p-adic groups*, 305-415, Lect. Notes Ser. Inst. Math. Sci. Nat. Univ. Singap., **2**, Singapore University press, Singapore, 2004.

[**Gu**] S. Guazzone, *Canonical Forms for general polynomials*, J. London Math. Soc. **(2)**, 10 (1975).

[**GW**] R. Goodman N. R. Wallach, *Representations and Invariants of the Classical Groups*, Cambridge U. Press, 1998. Hong Kong University Programme on Lie groups, April-June 1999.

[**Harb**] B. Harbourne, *Problems and progress: A survey on fat points in* $\mathbb{P}^2$, Queen's Pap. Pure Appl. Math. (Queen's University, Kingston, CA), **123** (2002), 87-132.

[**Harr**] J. Harris, *Algebraic Geometry, A First Course*, Springer.

[**Hart**] Robin Hartshorne; *Algebraic Geometry*, 1977 Springer-Verlag.

[**Hi**] A.Hirschowitz. *La méthode de Horace pour linterpolation à plusieurs variables*, Manuscripta Math. 50 (1985), 337-388.

[**Il**] G. Ilardi *Sulle ipersuperfici algebriche di* $\mathbb{P}^4(\mathbb{C})$ *contenute multiplamente nella propria hessiana*, Le Matematiche, Vol. XLVIII (1993) - Fasc. II, 297-309.

[**KR**] J. P. S. Kung, G. C. Rota *The invariant theory of binary forms*, Aequationes Math. **57**, (1999) no. 2-3.

[**LM1**] J.M. Landsberg, L. Manivel, *On the ideals of secant varieties of Segre varieties*, Found Comput. Math. 397-422 (2004).

[**LM2**] J.M. Landsberg, L. Manivel, *On the projective geometry of homogeneous varieties*, Comm. Math. Helv. **78**, (2003), 65-100.

[**Macaulay**] D. Bayer, M. Stillman, *Macaulay* Version 3.0, created 12 September 1994, available at http://www.math.columbia.edu/ bayer/Macaulay.

[**Mi**] R. Miranda, *Linear system of plane curves*, Notices of the AMS, Vol 46, No 2.

[**Se**] B. Segre, *Bertini forms and hessian matrices*, J. London Math. Soc. **26**, (1951), 164-176.

[**Te**] A.Terracini. *Sulle V_k per cui la varietà degli S_h $(h+1)$-seganti ha dimensione minore dell'ordinario.* Rend. Circ. Mat. Palermo **31** (1911), 392-396.

[**Wak**] K. Wakeford, *On Canonical Forms*, Proc. London Math. Soc. 18 (1918-19), 403-410.

[**War**] E. Waring, *Meditationes Algebricae*, 1770. Translated from the Latin by D. Weeks, American Mathematical Sociey, Providence, RI, 1991. I× +459 pp, ISBN: 0-8218-0169-4.