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Abstract

It is known that the identifiability of the structural parameters of the class of Linear(ized) Rational Expectations (LRE) models currently used in monetary policy and business cycle analysis may change dramatically across different regions of the theoretically admissible parameter space. This paper derives novel necessary and sufficient conditions for local identifiability which hold irrespective of whether the LRE model as a determinate (unique stable) reduced form solution or indeterminate (multiple stable) reduced form solutions. These conditions can be interpreted as prerequisite for the likelihood-based (classical or Bayesian) empirical investigation of determinacy/indeterminacy in stationary LRE models and are particular useful for the joint estimation of the Euler equations comprising the LRE model by 'limited-information' methods because checking their validity does not require the knowledge of the full set of reduced form solutions.

1 Introduction

As is known, Linear Rational Expectations (LRE) models may have a unique stable solution (determinacy), multiple stable solutions (indeterminacy), or multiple explosive solutions (commonly referred to as 'rational' bubbles). While there exists a large literature on testing rational bubbles,[1] the investigation of determinacy/indeterminacy in LRE models with stable (asymptotically stationary) reduced form solutions has only recently received attention in the econometric literature because of its connection with the analysis of the 'U.S. Great Moderation', see e.g. Lubik and Schorfheide (2004) and Benati and Surico (2009) and references therein.

[1] See, among many others, Flood and Garber (1980), Hamilton and Whiteman (1985), West (1987), Evans (1991) and Imrohoroğlu (1993).

A related crucial issue is whether the structural parameters of a LRE model can be estimated consistently from the data, i.e. their identifiability. Since Pesaran (1981, 1987), the tight connection between solution properties and identification issues in LRE models is a well documented fact. In particular, it is known that the identifiability of the structural parameters may change quite dramatically across different regions of the parameter space. Notably, this phenomenon is independent on the occurrence of the arbitrary auxiliary parameters that appear under indeterminacy and index solution multiplicity (Whiteman, 1983; Pesaran, 1987; Broze and Szafarz, 1991).

It is usually argued that one of the advantages of estimating LRE models by 'limited-information' techniques (instrumental variables, generalized method of moments), is that these methods are robust to determinacy/indeterminacy (Wickens, 1982). Actually, such a strategy is prone to incorrect inference if one disregards solution properties and their identifiability (Pesaran, 1987). Unfortunately, in the recent literature on local identification in (linearized) dynamic stochastic general equilibrium models, the common practice is to focus on the identifiability of the system under determinacy and the fact that in many cases the space of theoretically admissible values of the structural parameters contains also points that lead to indeterminacy is ignored in practice, see e.g. Canova and Sala (2009) and Iskerev (2010a, 2010b).[2] Thus one runs the risk of dealing with 'partially identified' (Phillips, 1989) LRE models.

This paper derives necessary and sufficient conditions for the local identifiability (Rothemberg, 1971) of the structural parameters of a class of multivariate LRE models typically used in monetary policy and business cycle analysis, which hold irrespective of whether the system has a determinate or indeterminate reduced form representation. These 'robust' identifying conditions, which are new in the literature, can be interpreted as prerequisite for testing determinacy *vs* indeterminacy in stable LRE models and, more generally, for the joint estimation of the structural parameters by 'full-' (maximum likelihood) as well as 'limited-information' (generalized method of moments) methods. However, they are particular useful when one estimates the Euler equations of the system jointly by 'limited-information' methods because all that is needed to apply these conditions is solving a quadratic matrix equation.

It is worth stressing that the robust conditions for local identification derived in this paper reflect the understanding of identifiability of an econometric model as a population, not a finite sample issue, see e.g. Rothemberg (1971) and Hsiao (1983). This means that the necessary and sufficient conditions can be checked prior to confront the LRE model with the data. This concept of identification is also referred to as 'mathematical' identification, Johansen (2010, p. 262).

To obtain our necessary and sufficient conditions for identification, we derive the full set of stable finite order reduced form solutions associated with the class of

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[2] Cochrane (2007) is a remarkable exception. Cochrane's (2007) analysis, however, is mainly focused on showing that in New Keynesian models the parameters of the Taylor rule relating interest rates to inflation and other variables are not identified at all without unrealistic assumptions.

multivariate LRE models of interest. Special cases such as 'purely forward-looking' LRE models and Minimum State Variable (MSV) (McCallum, 1983) reduced form solutions are discussed. Albeit there are many solution methods available in the literature and some of them explicitly account for the time series representation of equilibria under indeterminacy (e.g. Binder and Pesaran, 1995, Broze *et al.* 1995, and Lubik and Schorfheide, 2003), none of them provides a reduced form characterization of multiplicity similar to ours. Several examples drawn from the recent literature on macroeconomic dynamic modelling are discussed.

The paper is organized as follows. Section 2 introduces the problem by presenting some examples of stable LRE model which are identifiable under one regime (determinacy) but not in the other (indeterminacy). Section 3 summarizes the assumptions upon which the results of the paper are based and derives the reduced form solutions associated with the class of LRE models under study. Section 4 reports the main result on identification and Section 5 shows how the robust necessary and sufficient conditions for identification can be applied in practice. All profs are contained in the Appendix.

Throughout the paper we use the following notation and conventions. $\mathcal{M}_{p,p} \subseteq \mathbb{R}^{p\times p}$ is the space of all $p \times p$ real matrices. Given $M \in \mathcal{M}_{p,p}$, $sr[M] := \{\lambda,\, \lambda \in \mathbb{C} \text{ is an eigenvalue of } M\}$ denotes the spectrum of M, $r[M] := \max\{|\lambda|,\, \lambda \in sr(M)\}$ its spectral radius and $r_{\min}[M] := \min\{|\lambda|,\, \lambda \in sr(M)\}$ its lowest eigenvalue in absolute value. A stable matrix is defined as a matrix $M \in \mathcal{M}_{p,p}$ such that $r[M] < 1$. The rank of M is denoted by $rank(M)$. If the elements of the matrix $M \in \mathcal{M}_{p,p}$ depend nonlinearly on the elements of the vector $v \in \mathbb{R}^r$, $r \leq p^2$, we write $M := M(v)$. Given the set $\mathcal{C}$ and the vector v, card$(\mathcal{C})$ denotes the cardinality of $\mathcal{C}$ and dim(v) the dimension of v. 'vec' is the usual column-stacking operator, '$\otimes$' is the Kronecker product, diag(M) collects the diagonal elements of M in the a $p \times 1$ vector and the symbol '$\prec$' denotes vector inequality. L is the lag/lead operator, $L^h X_t := X_{t-h}$.

2 Problem

Let X_t be a n-dimensional vector of observable time series and consider the multivariate Linear Rational Expectations (LRE) model

$$\Gamma_0 X_t = \Gamma_f E_t X_{t+1} + \Gamma_b X_{t-1} + \omega_t \tag{1}$$

$$\omega_t = R\omega_{t-1} + u_t \quad , \quad u_t \sim \text{WN}(0, \Sigma_u) \tag{2}$$

where Γ_0, Γ_f and Γ_b are $n \times n$ matrices containing the structural parameters, $E_t \cdot := E(\cdot \mid \mathcal{F}_t)$ is the conditional expectations operator, $\mathcal{F}_t$ is a nondecreasing information set containing the sigma-field $\sigma(X_t, X_{t-1}, ..., X_1)$, ω_t is the $n \times 1$ vector of structural disturbances which is assumed to obey a vector autoregressive (VAR) process of order one, R is a $n \times n$ stable matrix and u_t is a n-dimensional white noise process with positive definite covariance matrix Σ_u. The unrestricted parameters in the matrices Γ_0, Γ_f, Γ_b and R are collected in the $m \times 1$ vector θ whose 'true' value is θ_0 and lies in the 'theoretically admissible' parameter space $\mathcal{P} \subset \mathbb{R}^m$. Further lags and/or leads of X_t can be included in the system by considering a 'canonical' (state-space) representation of the model, see Eq. (12) below.

The structural form (1)-(2), or system of Euler equations, covers a class of (linearized) models currently used in monetary policy and business cycle analysis, known as small-scale dynamic stochastic general equilibrium models.

Before proceeding we distinguish, inspired by Broze and Szafarz (1991), the concept of solution of a LRE model from that of reduced form solution and then define the identifiability of θ in 'broad' sense.

Definition 1 [Solutions] A solution to the LRE model (1)-(2) is any stochastic process $\{X_t^*\}_{t=1}^{\infty}$ such that $E(X_{t+1}^* \mid \mathcal{F}_t)$ exists and, for conventionally fixed initial conditions, if $X_t := X_t^*$ is substituted into the structural equations, the model is verified for each t.

Definition 2 [Reduced form solution] Given the set of all solutions of the LRE model (1)-(2) consistent with Definition 1, a linear reduced form solution is an exhaustive time series representation of the solution set obtained by expressing, provided it is possible, any $X_t := X_t^*$ in terms of lags of X_t, lags (and leads) of u_t and, possibly, other arbitrary 'external' (to the model) stochastic disturbances independent on u_t.

Definition 3 [Identifiability in 'broad' sense] The vector of structural parameters θ is identifiable in 'broad' sense if all of its elements can be estimated consistently from the observations $X_1, ..., X_T$ generated from a reduced form solution of the LRE model.

The definition of identifiability of θ will be specialized technically in the Definition 3' of Section 4.

The two examples that follow present LRE models in which θ is identifiable in the sense of Definition 3 under indeterminacy but not under determinacy.

Example 1 [Scalar LRE model] Consider the LRE model

$$X_t = \gamma_f E_t X_{t+1} + \gamma_b X_{t-1} + \omega_t \tag{3}$$

where X_t is a scalar ($n := 1$), ω_t is a scalar white noise process with variance 1, and where the parameters γ_f and γ_b belong to the space $\mathcal{P} := \left\{\gamma_f, \gamma_b,\ 0<\gamma_f<1, 0<\gamma_b<1\right\} \subset \mathbb{R}^2$. Using the notation of system (1)-(2), $\Gamma_0 := 1$, $\Gamma_f =: \gamma_f$, $\Gamma_b := \gamma_b$, $R := 0$ and $\theta := (\gamma_f, \gamma_b)'$. If $\gamma_f + \gamma_b < 1$, the reduced form solution of the LRE model (3) takes the form

$$[1 - \phi_1(\theta)L]X_t = \frac{\omega_t}{1 - \gamma_f \phi_1(\theta)} \tag{4}$$

where $\phi_1(\theta)$ is a real stable (-1<$\phi_1(\theta)$<1) root of the quadratic polynomial

$$\gamma_f \phi^2 - \phi + \gamma_b = 0. \tag{5}$$

If $\gamma_f+\gamma_b>1$ and $1/2<\gamma_f<1$,[3] the reduced form solutions can be represented as

$$[1-\phi_2(\theta)L][1-\phi_1(\theta)L]X_t=\frac{1}{1-\phi_1(\theta)\gamma_f}[\kappa-\phi_2(\theta)L]\omega_t+s_t \quad (6)$$

where $\phi_1(\theta)$ and $\phi_2(\theta)$ are two stable real roots of the polynomial (5) that satisfy

$$\phi_1(\theta)+\phi_2(\theta)=\gamma_f^{-1} \quad , \quad \phi_1(\theta)\phi_2(\theta)=\gamma_f^{-1}\gamma_b,$$

$\kappa\in\mathbb{R}$ is and arbitrary auxiliary parameter unrelated to θ, and s_t is a martingale difference sequence (MDS) with respect to $\mathcal{I}_t$ ($E(s_{t+1}\mid\mathcal{I}_t)=0$) independent on ω_t, usually referred to as 'sunspot shock', see e.g. Evans and Honkapohja (1986) and Pesaran (1987). It can be noticed that while θ, other than the auxiliary parameter κ, is identifiable in the sense of Definition 3 (it can potentially be recovered from the estimation of Eq. (6)), the parameter γ_b (other than κ) is not identifiable from Eq. (4).

Example 2 [Unidentifiability under determinacy] Consider the following two-equation ($n:=2$) LRE model taken from Lubik and Surico (2010, Section 2.2) and here reported with a slight change of notation:

$$z_t = \alpha E_t z_{t+1}+\beta E_t y_{t+1}+\omega_{z,t} \quad (7)$$

$$y_t = \psi z_t+\omega_{y,t} \quad (8)$$

where $\omega_t:=(\omega_{z,t},\omega_{y,t})'$ is a white noise process with covariance matrix $\Sigma_\omega:=dg(\sigma_z^2,\sigma_y^2)$, $\theta:=(\alpha,\beta,\psi)'$, $X_t:=(z_t,y_t)'$ and

$$\Gamma_0:=\begin{bmatrix}1 & 0\\ -\psi & 1\end{bmatrix} \quad , \quad \Gamma_f:=\begin{bmatrix}\alpha & \beta\\ 0 & 0\end{bmatrix} \quad , \quad \Gamma_b:=0_{2\times 2}=:R.$$

It is further assumed that the admissible parameter space is given by $\mathcal{P}:=\left\{\theta,\ 0<\alpha<1, 0<\beta<1, \psi\in\mathbb{R}^{+}\backslash\left\{\frac{1-\alpha}{\beta}\right\}\right\}\subset\mathbb{R}^3$.[4] If $0<\psi<\frac{1-\alpha}{\beta}$, the reduced form solution is determinate and can be represented as

$$X_t:=\begin{bmatrix}1 & 0\\ \psi & 1\end{bmatrix}\omega_t; \quad (9)$$

if $\psi>\frac{1-\alpha}{\beta}$, the reduced form solution is indeterminate and can be represented as the VARMA(1,1)-type process

$$X_t:=\begin{bmatrix}\frac{1}{\alpha+\beta\psi} & 0\\ \frac{1}{\alpha+\beta\psi} & 0\end{bmatrix}X_{t-1}+\begin{bmatrix}\kappa & 0\\ \psi\kappa & 1\end{bmatrix}\omega_t-\begin{bmatrix}\frac{1}{\alpha+\beta\psi} & 0\\ \frac{\psi}{\alpha+\beta\psi} & 0\end{bmatrix}\omega_{t-1}+\begin{bmatrix}0 & 1\\ \psi & \psi\end{bmatrix}\begin{pmatrix}0\\ \frac{1}{\psi}s_t\end{pmatrix} \quad (10)$$

[3] The case $\gamma_f+\gamma_b=1$ implies a unit root in Eq. (5) and is potentially consistent with both determinacy and indeterminacy, depending on whether $\gamma_f<1/2$ or $\gamma_f\nless 1/2$ .We rule out these situations from the analysis, see Assumption 1 below.

[4] We deliberately exclude the point $\psi:=\frac{1-\alpha}{\beta}$ from $\mathcal{P}$ to rule out the occurrence of unit roots in the reduced form representations of $X_t:=(z_t,y_t)'$ that would contrast with the assumptions in Section 3.

where κ is an arbitrary auxiliary parameter unrelated to θ and s_t is an arbitrary MDS with respect to $\mathcal{I}_t$ independent on ω_t. In this example, θ is not identifiable under determinacy in the sense of Definition 3 because α and β are not recoverable from Eq. (9). Conversely, θ is identifiable (along κ) under indeterminacy in the sense of Defintion 3.

The two examples above clarify that the identification issues we deal with in this paper are not related to the presence/absence of the auxiliary parameters κ that are unidentifiable under determinacy and govern solution multiplicity under indeterminacy; in general, the parameters in κ are identifiable under indeterminacy if θ is identifiable (Broze and Szafarz, 1991). Rather, our objective is to find conditions that ensure that θ is identifiable irrespective of whether its true value lies in the determinacy or indeterminacy region of the parameter space.

3 Assumptions and reduced form solutions

Given the multivariate LRE model (1)-(2), let $\mathcal{P} \subset \mathbb{R}^m$, $m := \dim(\theta)$, the open space of all theoretically admissible values of θ. We consider the following assumptions:

Assumption 1 *θ_0 is an interior point of $\mathcal{P}$ and for each $\theta \in \mathcal{P}$ and X_0, X_{-1} and X_{-2} given, any reduced form of interest of system (1)-(2) belongs to the class of covariance stationary processes with $E(X_t) = 0$ and whose distribution depends on all or part of the elements of θ.*

Assumption 2 *The elements of the matrix $\Gamma := [\Gamma_0 : \Gamma_f : \Gamma_b : R] \in \mathcal{M}_{n,4n}$ depend on θ through the function $vec(\Gamma) = q(\theta)$, where $q(\cdot)$ is such that the $4n^2 \times m$ Jacobian $Q(\theta) := \frac{\partial q(\theta)}{\partial \theta'}$ matrix has full column rank m.*

Assumption 3 *For $\theta \in \mathcal{P}$, the matrices Γ_0, $\Gamma_0^R := (\Gamma_0 + R\Gamma_f)$, $\Theta := \Theta(\theta) := (\Gamma_0^R - \Gamma_f \Phi_1)$ and $\Upsilon := \Upsilon(\theta) := (\Theta - R\Gamma_f) := (\Gamma_0 - \Gamma_f \Phi_1)$ are non-singular, where $\Phi_1 := \Phi_1(\theta) \in \mathcal{M}_{n,n}$ is defined below.*

Assumptions 1 provides a regularity condition and rules out reduced form solutions that embody unit or explosive roots as well as deterministic components; moreover, the space $\mathcal{P}$ contains only points that generate stationary reduced form solutions. Assumption 2 is an auxiliary necessary identification condition. The non-singularity of Γ_0 and Γ_0^R in Assumption 3 does not imply any loss of generality, while, as shown in Example 3 below, the class of small-scale LRE models used in the monetary and business cycle literature are usually based on a non-singular Θ matrix.

A further assumption will be used from Section 4 onwards to rule out the occurrence of ‘purely forward-looking’ models from the class of LRE models with respect to which we derive robust identification conditions.

Assumption 4 *Given the LRE model (1)-(2), either* $\Gamma_b \neq 0_{n\times n}$ *or* $R \neq 0_{n\times n}$*, or* $\Gamma_b \neq 0_{n\times n}$ *and* $R \neq 0_{n\times n}$.

To discuss the identifiability of θ, we preliminarily derive the reduced form solutions associated with the LRE model (1)-(2) and then discuss the implied likelihood functions.

We write system (1)-(2) in the form

$$\Gamma_0^R X_t = \Gamma_f E_t X_{t+1} + \Gamma_{b,1}^R X_{t-1} + \Gamma_{b,2}^R X_{t-2} + u_t^R \tag{11}$$

$$\Gamma_{b,1}^R := (\Gamma_b + R\Gamma_0) \quad , \quad \Gamma_{b,2}^R := -R\Gamma_b$$

where Γ_0^R is defined in Assumption 3, $\eta_t := (X_t - E_{t-1}X_t)$ and $u_t^R := u_t + R\Gamma_f \eta_t$ are MDS with respect to $\mathcal{I}_t$. Next we express system (11) in canonical form

$$\mathring{\Gamma}_0 \mathring{X}_t = \mathring{\Gamma}_f E_t \mathring{X}_{t+1} + \mathring{\Gamma}_b \mathring{X}_{t-1} + \mathring{u}_t \tag{12}$$

where $\mathring{X}_t := (X_t', X_{t-1}')'$, $\mathring{u}_t := (u_t', 0_{n\times 1}')'$ and

$$\mathring{\Gamma}_0 := \begin{bmatrix} \Gamma_0^R & 0_{n\times n} \\ 0_{n\times n} & I_n \end{bmatrix} \quad , \quad \mathring{\Gamma}_f := \begin{bmatrix} \Gamma_f & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} \end{bmatrix} \; , \; \mathring{\Gamma}_b := \begin{bmatrix} \Gamma_{b,1}^R & \Gamma_{b,2}^R \\ I_n & 0_{n\times n} \end{bmatrix}.$$

Observe that if in Eq. (2) $R := 0_{n\times n}$, the canonical form based on $\mathring{X}_t := X_t$, $\mathring{u}_t := u_t$, $\mathring{\Gamma}_0 := \Gamma_0$, $\mathring{\Gamma}_f := \Gamma_f$ and $\mathring{\Gamma}_b := \Gamma_b$ collapses to system (1).

Using Assumption 3,

$$\mathring{X}_t = \mathring{\Gamma}_0^{-1}\mathring{\Gamma}_f E_t \mathring{X}_{t+1} + \mathring{\Gamma}_0^{-1}\mathring{\Gamma}_b \mathring{X}_{t-1} + \mathring{\Gamma}_0^{-1}\mathring{u}_t \tag{13}$$

and for given initial conditions $\mathring{X}_0 := (X_0', X_{-1}')'$, any solution $\left\{\mathring{X}_t\right\}_{t=1}^{\infty}$ to Eq. (13) is linked to the solutions $\{X_t\}_{t=1}^{\infty}$ of system (12) (or (1)-(2)) by the relationship $X_t := W\mathring{X}_t$, where $W := [I_n : 0_{n\times n}]$ is a selection matrix. Finding a reduced form solution to system (13) is equivalent to finding a reduced form solution to system (12) (or (1)-(2)).

Following Binder and Pesaran (1995, 1997), we impose that any $\mathring{X}_t$ that solves Eq. (13) be uncoupled as

$$\mathring{X}_t := \mathring{X}_{B,t} + \mathring{X}_{F,t} \quad , \quad \mathring{X}_{B,t} := \mathring{\Phi}\mathring{X}_{t-1} \tag{14}$$

where the process $\left\{\mathring{X}_{B,t}\right\}_{t=1}^{\infty}$ can be interpreted as the 'backward' part of the solution, the process $\left\{\mathring{X}_{F,t}\right\}_{t=1}^{\infty}$ as its 'forward' part and $\mathring{\Phi} \in \mathcal{M}_{2n\times 2n}$ is defined implicitly as the solution to the quadratic matrix equation

$$\mathring{\Gamma}_f \mathring{\Phi}^2 - \mathring{\Gamma}_0 \mathring{\Phi} + \mathring{\Gamma}_b = 0_{2n\times 2n}. \tag{15}$$

Define the matrix $\mathring{\Theta} := (\mathring{\Gamma}_0 - \mathring{\Gamma}_f \mathring{\Phi})$ which is non-singular under Assumption 3 and the set

$$\mathcal{F} := \left\{ \mathring{\Phi},\ \mathring{\Phi} \text{ solves Eq. (15)},\ r[\mathring{\Phi}] < 1,\ \det(\mathring{\Theta}) \neq 0 \right\}$$

$$\equiv \left\{ \mathring{\Phi},\ \mathring{\Phi} = (\mathring{\Gamma}_0 - \mathring{\Gamma}_f \mathring{\Phi})^{-1} \mathring{\Gamma}_b,\ r[\mathring{\Phi}] < 1, \right\} \subset \mathcal{M}_{2n \times 2n};$$

as is known, the elements of $\mathcal{F}$ are intimately related to the solution to a generalized eigenvalue/eigenvector problem, see Binder and Pesaran (1995), Binder and Pesaran (1997, Proposition 1) and Uhlig (1999). However, differently from what happens when $n := 1$, when $n \geq 2$, necessary and sufficient conditions for $\text{card}(\mathcal{F}) = 1$ or, alternatively, sufficient conditions for $\text{card}(\mathcal{F}) = 0$ or $\text{card}(\mathcal{F}) = \infty$ are not available unless the matrices $\mathring{\Gamma}_f$, $\mathring{\Gamma}_0$ and $\mathring{\Gamma}_b$ fulfill constraints that hardly can be justified on theoretical grounds, see e.g. Higham and King (2000). If $\text{card}(\mathcal{F}) \neq 0$, the non-zero (and different from 1) elements of $\mathring{\Phi}$ depend nonlinearly on θ, $\mathring{\Phi} := \mathring{\Phi}(\theta)$; moreover, by construction the block structure of $\mathring{\Phi}$ is given by

$$\mathring{\Phi} := \begin{bmatrix} \Phi_1 & \Phi_2 \\ I_n & 0_{n \times n} \end{bmatrix} \tag{16}$$

where $\Phi_1 := \Phi_1(\theta)$ is the $n \times n$ matrix which enters the definition of Θ in Assumption 3, and $\Phi_2 := \Phi_2(\theta)$. If $R := 0_{n \times n}$ in Eq. (1) the matrix $\mathring{\Phi}$ collapses to $\mathring{\Phi} := \Phi_1$.

From Eq. (14) it turns out that

$$E_t \mathring{X}_{F,t+1} = E_t \mathring{X}_{t+1} - \mathring{\Phi} \mathring{X}_t \tag{17}$$

hence by using Eq.s (14) and (17) in Eq. (13) the solution becomes

$$\mathring{X}_{B,t} + \mathring{X}_{F,t} = \mathring{\Gamma}_0^{-1} \mathring{\Gamma}_f [E_t \mathring{X}_{F,t+1} + \mathring{\Phi} \mathring{X}_t] + \mathring{\Gamma}_0^{-1} \mathring{\Gamma}_b \mathring{X}_{t-1} + \mathring{\Gamma}_0^{-1} \mathring{u}_t$$

and by imposing the restrictions in Eq. (15) and using Assumption 3, this system can be re-arranged in the form

$$\mathring{X}_{F,t} = \mathring{G} E_t \mathring{X}_{F,t+1} + \mathring{\Theta}^{-1} \mathring{u}_t \tag{18}$$

where $\mathring{G} := (\mathring{\Gamma}_0 - \mathring{\Gamma}_f \mathring{\Phi})^{-1} \mathring{\Gamma}_f := \mathring{\Theta}^{-1} \mathring{\Gamma}_f$. The LRE model (18) is known as the 'Cagan multivariate model' (Broze and Szafarz, 1991); inspection of the $\mathring{G}$ matrix in Eq. (18) reveals that

$$\mathring{G} := \mathring{G}(\theta) := \begin{bmatrix} S & 0_{n \times n} \\ 0_{n \times n} & 0_{n \times n} \end{bmatrix} \tag{19}$$

where $S := S(\theta) := (\Gamma_0^R - \Gamma_f \Phi_1)^{-1} \Gamma_f := \Theta^{-1} \Gamma_f$, hence S and $\mathring{G}$ have identical non-zero eigenvalues. Given the block structure of $\mathring{G}$, $\mathring{\Theta}$ and $\mathring{u}_t$, system (18) has the block structure

$$\mathring{X}_{F,t} = \begin{bmatrix} S & 0_{n \times n} \\ 0_{n \times n} & 0_{n \times n} \end{bmatrix} E_t \mathring{X}_{F,t+1} + \begin{bmatrix} \Theta^{-1} & \Theta^{-1} \Gamma_f \Phi_2 \\ 0_{n \times n} & I_n \end{bmatrix} \begin{pmatrix} u_t^R \\ 0_{n \times 1} \end{pmatrix}$$

hence the solution properties of the Cagan multivariate model depend on the solution properties of its upper sub-system

$$X^S_{F,t} = SE_t X^S_{F,t+1} + \Theta^{-1} u^R_t \quad (20)$$

where $\mathring{X}_{F,t} := (X^{S\prime}_{F,t}, 0_{1\times n})'$.

In light of Eq. (14), if card$(\mathcal{F}) \neq 0$, a solution to the LRE model is obtained as the sum, for each t, of the solution to Eq. (20) and $\mathring{X}_{B,t} := \mathring{\Phi}\mathring{X}_{t-1}$; it turns out that the elements of the $n \times 2n$ sub-matrix $\Phi := [\Phi_1 : \Phi_2]$ of $\mathring{\Phi}$ play the role of reduced form coefficients subject to the implicit set of nonlinear (cross-equation) restrictions stemming from Eq. (15). The stability/instability of the matrix $S(\theta)$ plays a crucial role in determining solution properties. Consider the Jordan normal form of $S(\theta)$:

$$S(\theta) := P(\theta) \begin{bmatrix} \Lambda_1 & 0_{n_1\times n_2} \\ 0_{n_2\times n_1} & \Lambda_2 \end{bmatrix} P^{-1}(\theta) \quad (21)$$

where $P(\theta) \in \mathcal{M}_{n,n}$ is non-singular, $\Lambda_1 \in \mathcal{M}_{n_1,n_1}$ is the Jordan block containing the n_1 eigenvalues of $S(\theta)$ that lie inside the unit disk and $\Lambda_2 \in \mathcal{M}_{n_2,n_2}$ is the Jordan block containing the $n_2 := n - n_1$ eigenvalues that lie outside the unit disk; by construction $rank(\Lambda_2) = n_2$. If $r[S(\theta)] < 1$, the decomposition in Eq. (21) collapses to $S(\theta) := P(\theta)\Lambda_1 P^{-1}(\theta)$, $n = n_1$ and $r[S(\theta)]$:=diag(Λ_1); conversely, if $r[S(\theta)] > 1$, diag$(\Lambda_2) \in sr[S(\theta)]$ are the eigenvalues of $S(\theta)$ that lie outside the unit disk.

The next proposition establishes that the dynamic structure of the solutions to the LRE model (1)-(2) depends on the location of the eigenvalues of the matrix $S(\theta)$ in the unit disk.

Proposition 1 [Reduced form solutions] Consider the LRE model (1)-(2) with initial conditions X_0, X_{-1} and X_{-2} fixed, and suppose that, under Assumptions 1-3, card$(\mathcal{F}) \neq 0$.

(a) If $r[S(\theta)] < 1$, the reduced form solution can be represented, if it exists, in the form

$$[I_n - \Phi_1(\theta)L - \Phi_2(\theta)L^2]X_t = \Upsilon(\theta)^{-1}u_t \quad (22)$$

where $\Phi_1(\theta)$ and $\Phi_2(\theta)$ are the sub-matrices of $\mathring{\Phi}$ defined in Eq. (16).

(b) If $r[S(\theta)] > 1$ and dim(diag(Λ_2)):=n_2, $1 \leq n_2 \leq n$, the reduced form solutions can be represented, if they exist, in the form

$$[I_n - \Pi(\theta)L][I_n - \Phi_1(\theta)L - \Phi_2(\theta)L^2]X_t = [M(\theta,\kappa) - \Pi(\theta)L]\Psi(\theta,\kappa)u_t$$
$$+ [M(\theta,\kappa) - \Pi(\theta)L]\Theta^{-1}R\Gamma_f V(\theta,\kappa)\tau_t + \tau_t \quad (23)$$

where

$$\Pi(\theta) := P(\theta) \begin{bmatrix} \Lambda_1 & 0_{n_1\times n_2} \\ 0_{n_2\times n_1} & \Lambda_2^{-1} \end{bmatrix} P^{-1}(\theta),$$

$$M(\theta,\kappa) := P(\theta) \begin{bmatrix} I_{n_1} & 0_{n_1\times n_2} \\ 0_{n_2\times n_1} & \kappa \end{bmatrix} P^{-1}(\theta),$$

$$\Psi(\theta,\kappa) := [I_n + \Theta^{-1} R\Gamma_f V(\theta,\kappa) M(\theta,\kappa)]\Theta^{-1} \quad , \quad V(\theta,\kappa) := [I_n - M(\theta,\kappa)\Theta^{-1} R\Gamma_f]^{-1}$$

$$\tau_t := P(\theta)\zeta_t$$

and $n_1 := n - n_2$, κ is a $n_2 \times n_2$ matrix containing arbitrary elements not related to θ and such that $vec(\kappa) \in \mathcal{K} \subseteq \mathbb{R}^{n^2}$, and ζ_t is a $n \times 1$ MDS with respect to $\mathcal{I}_t$, called 'sunspot shock', which is independent on u_t and whose components are given by $\zeta_t := (0'_{n_1 \times 1}, s'_t)' := (0, ...0, s_{1,t}, .., s_{n_2,t})'$.

Proof: Appendix.

Some remarks are in order.

Remark 1 Proposition 1 covers the full set of reduced form equilibria consistent with Assumptions 1-3. It is shown that these solutions have a finite order vector autoregressive moving average (VARMA)-type representation: while the time series representation in Eq. (22) belongs to the class of parametrically constrained VAR models, the time series representation in Eq. (23) is obtained as the sum of a parametrically constrained VARMA model and a MVA term involving n_2 arbitrary MDS processes (when $R \neq 0_{n\times n}$), where n_2 is the number of eigenvalues of the matrix $S(\theta)$ that lie outside the unit disk. The model in Eq. (22) involves only the state variables and parameters of the LRE model and reads as the determinate reduced form solution of the LRE model. System (23) shows that there exist two sources of indeterminacy: the presence of the $(n_2)^2$ arbitrary auxiliary parameters κ (parametric indeterminacy) and the presence of the n_2 arbitrary sunspot shocks embodies in the vector τ_t (stochastic indeterminacy) that may alter the dynamics and volatility of the system generated by the 'fundamental' disturbance u_t. According to Proposition 1, when $r[S(\theta)] > 1$ the system is indeterminate even if $s_t := 0$ a.s. $\forall$ t, implying $\tau_t := 0$ a.s. $\forall$ t, a situation referred to as 'indeterminacy without sunspots' (Lubik and Schorfheide, 2004).

Remark 2 It is easily seen that if the eigenvalues of the matrix $\kappa\Lambda_2$ lie outside the unit disk the VMA polynomial of system (23) is invertible and the indeterminate reduced form can be represented as a linear process and potentially used for estimation.[5]

Remark 3 Let $\mathcal{P}^D$ be the determinacy region of the theoretically admissible parameter space, i.e. the subspace of $\mathcal{P}$ containing points that lead to a unique

[5]Indeed one has

$$\det[M(\theta,\kappa) - \Pi(\theta)z] = \det\left\{\begin{bmatrix} I_{n_1} & 0_{n_1\times n_2} \\ 0_{n_2\times n_1} & \kappa \end{bmatrix} - \begin{bmatrix} \Lambda_1 & 0_{n_1\times n_2} \\ 0_{n_2\times n_1} & \Lambda_2^{-1} \end{bmatrix} z\right\}$$

$$= \det\left\{\begin{bmatrix} I_{n_1} - \Lambda_1 z & 0_{n_1\times n_2} \\ 0_{n_2\times n_1} & \kappa - \Lambda_2^{-1} z \end{bmatrix}\right\} = \det(I_{n_1} - \Lambda_1 z)\det(\kappa. - \Lambda_2^{-1} z).$$

stable reduced form solution. In principle, a possible characterization of $\mathcal{P}^D$ is $\mathcal{P}^D := \{\theta \in \mathcal{P},\ h(\theta) \prec 0_\ell\}$, where $h(\cdot)$ is a ℓ-dimensional differentiable function. Manageable closed-form expressions for $h(\cdot)$ are generally not available unless the elements of θ are highly restricted. Proposition 1 suggests that a 'natural' choice of $h(\cdot)$ is given by the scalar function $h(\theta) := r[S(\theta)] - 1$ and the condition $h(\theta) := r[S(\theta)] - 1 < 0$ is sufficient for determinacy. Under Assumptions 1-3, the set $\mathcal{P}\backslash\mathcal{P}^D$ corresponds to the indeterminacy region of the parameter space.

Before deriving our main results, it is instructive to focus on two interesting classes of reduced form solutions nested in the set derived in Proposition 1. The former is obtained from Eq. (23) when $\kappa := I_{n_2}$; in this case, which is treated in detail in Corollary 1, there are common ('cancelling') roots in the autoregressive and moving average polynomials characterizing the reduced form solutions. The latter refers to the reduced form solutions obtained when $\mathring{\Gamma}_b := 0$ (i.e. when both $R := 0_{n\times n}$ and $\Gamma_b := 0_{n\times n}$ in the structural equations (1)-(2)); Corollary 2 derives the reduced form solutions of the 'purely forward-looking' counterpart of system (1)-(2) and paves the way to a first simple result on identification.

Corollary 1 [MSV reduced form solutions] Consider the LRE model (1)-(2) with initial conditions X_0, X_{-1} and X_{-2} fixed, and suppose that, under Assumptions 1-3, card$(\mathcal{F}) \neq 0$. If $r[S(\theta)] > 1$ and $\kappa := I_{n_2}$, the reduced form solutions take the form

$$[I_n - \Phi_1(\theta)L - \Phi_2(\theta)L^2]X_t = \Upsilon(\theta)^{-1}u_t + [\tilde{V}(\theta,\kappa) - I_n]\tau_t + [I_n - \Pi(\theta)L]^{-1}\tau_t \quad (24)$$

where $\Phi_1(\theta)$, $\Phi_2(\theta)$, $\Pi(\theta)$ and τ_t are defined as in Proposition 1 and $\tilde{V}(\theta,\kappa) := [I_n - \Theta^{-1}R\Gamma_f]^{-1}$.

Proof: Appendix.

Corollary 2 [Purely forward-looking model] Consider the LRE model (1)-(2) with initial conditions X_0, X_{-1} and X_{-2} fixed, and suppose that, in addition to Assumptions 1-3, it holds $\mathring{\Gamma}_b := 0$ (meaning that both $R := 0_{n\times n}$ and $\Gamma_b := 0_{n\times n}$) and one of the following conditions: (c1) Γ_f is singular; (c2) $r[\Gamma_f\Gamma_0^{-1}] > 1$. (a) If $r[S(\theta)] < 1$, the determinate reduced form solution collapses to

$$X_t = \Gamma_0^{-1}u_t.$$

(b) If $r[S(\theta)] > 1$ the indeterminate reduced form solutions collapse to

$$\begin{aligned}[I_n - \Pi(\theta)L]X_t &= [M(\theta,\kappa) - \Pi(\theta)L]\Psi^*(\theta,\kappa)u_t \\ &\quad + [M(\theta,\kappa) - \Pi(\theta)L]\Theta^{*-1}R\Gamma_f V^*(\theta,\kappa)\tau_t + \tau_t\end{aligned}$$

where $\Pi(\theta)$, $M(\theta,\kappa)$ and τ_t are defined as in Proposition 1, $\Psi^*(\theta,\kappa) := [I_n + \Theta^{*-1}R\Gamma_f V^*(\theta,\kappa)M(\theta,\kappa)]\Theta^{*-1}$, $V^*(\theta,\kappa) := [I_n - M(\theta,\kappa)\Theta^{*-1}R\Gamma_f]^{-1}$ and $\Theta^* := (\Gamma_0 + R\Gamma_f)$.

Proof: Appendix.

Remark 4 Corollary 1 shows that is necessary to impose the absence of sunspot shocks in Eq. (24) ($\tau_t := 0$ a.s. $\forall\ t$) to obtain a an indeterminate reduced form solution that has the same dynamic representation as the determinate reduced form in Eq. (22). The reduced form solution in Eq. (24) is obtained, however, in correspondence of a point of zero-Lebesque measure in the space $\mathcal{K}$.

Remark 5 A direct consequence of Corollary 2 is that in 'purely forward-looking' models, which are largely used to discuss monetary policy stabilization (Woodford, 2003, Ch. 4), θ is generally unidentifiable under determinacy.[6] More precisely, all elements of θ that enter the matrix Γ_f but not the matrix Γ_0 are not identifiable in the sense of Definition 3 (hene it is now clear why θ is not identifiable in the LRE model of Example 2 under determinacy). It may happen that once a subset of the elements of θ has been fixed, then the other elements are identifiable; whether subset of θ are identifiable under determinacy in 'purely forward-looking' LRE models is an issue that must be addressed on a case by case basis. Conversely, θ and κ are generally identifiable under indeterminacy.

Remark 6 Corollary 2 suggests that in order to derive 'robust' identification conditions for the LRE model (1)-(2), i.e. conditions that hold irrespective of whether the reduced form solution belongs to the class of models defined in Eq. (22) or in Eq. (23), it is necessary to avoid that $0_{2n\times 2n} \in \mathcal{F}$. This objective will be achieved by considering the class of LRE models (1)-(2) under Assumption 4.

4 Main results

Consider the multivariate LRE model (1)-(2) under Assumptions 1-4 and the point of view of an econometrician whose ultimate objective is the estimation of θ.

Define the $a^2 \times 1$ vector $\phi := vec[\mathring{\Phi}]$, where hereafter $a := n$ if $R := 0_{n\times n}$ in Eq. (2) (which implies $\mathring{\Phi} := \Phi_1$), and $a := 2n$ otherwise. Partition the $\mathring{\Phi}$ matrix as $\mathring{\Phi}' := \left[\Phi' : \Pi_{0,1}'\right]$, where $\Phi := [\Phi_1 : \Phi_2]$ and $\Pi_{0,1}$ is the block that contains zeros and ones (obviously when $R := 0_{n\times n}$, $\Phi := \Phi_1$ and there exists no $\Pi_{0,1}$ sub-matrix); then denote by $\tilde{\phi} := vec[\Phi]$ the sub-vector of ϕ that does not contain zeros and ones and $d := \dim(\tilde{\phi})$; one has $d < a^2$ when $R \neq 0_{n\times n}$, and $\phi := \tilde{\phi}$ and $d := a^2$ when $R := 0_{n\times n}$. The relationship between $\tilde{\phi}$ and ϕ can be formalized as

$$\phi := H \begin{pmatrix} \tilde{\phi} \\ \pi_{0,1} \end{pmatrix} := H_1\tilde{\phi} + H_2\pi_{0,1} \tag{25}$$

[6] Observe that a LRE model based on a non-singular Γ_f matrix would be highly restrictive from the economic viewpoint.

where H is a $a^2\times a^2$ permutation matrix (that will correspond to the identity matrix when $\phi := \tilde{\phi}$), H_1 and H_2 are $a^2 \times d$ and $a^2 \times (a^2 - d)$ sub-matrices of H where H_1 has full column rank d, and $\pi_{0,1} := vec(\Pi_{0,1})$.

If one of the reduced form solutions derived in Proposition 1 exists, then $\tilde{\phi}$ depends on θ, i.e. $\tilde{\phi} = g(\theta)$, where $g(\cdot)$ is a nonlinear function. Intuitively, the identifiability of θ requires that the mapping $\tilde{\phi} = g(\theta)$ be unique in a neighborhood $\mathcal{O}(\theta_0) \subset \mathcal{P}$ of θ_0. More precisely, Proposition 1 suggests that given the observations X_1,..., X_T and a distribution for u_t and s_t, the (concentrated) log-likelihood function of any reduced form solution to the LRE model (1)-(2) can be written as

$$\begin{aligned}\ell_T(\theta, vec(\kappa)) &:= \ell_{D,T}(g(\theta)) \times \mathbb{I}(r[S(\theta)] < 1) \\ &\quad + \ell_{I,T}(g(\theta), vec(\kappa)) \times \mathbb{I}(r[S(\theta)] > 1)\end{aligned} \tag{26}$$

where $\mathbb{I}(\cdot)$ is the indicator function and $\ell_{D,T}(\tilde{\phi})$ and $\ell_{I,T}(\tilde{\phi}, vec(\kappa))$ are the log-likelihoods associated with the unrestricted counterparts of systems (22) and (23), respectively. The parameters of the LRE model are unidentifiable if for fixed $\breve{\theta}_1, \breve{\theta}_2 \in \mathcal{P}$, $\breve{\theta}_1 \neq \breve{\theta}_2$, $\tilde{\phi} = g(\breve{\theta}_1) = g(\breve{\theta}_2)$ leading, for any fixed κ, to $\ell_T(g(\breve{\theta}_1), vec(\kappa)) = \ell_T(g(\breve{\theta}_2), vec(\kappa))$.

We are now in the position to refine the Definition 3 of identifiability of θ introduced in Section 2 with the concept of local identifiability taken from Rothemberg (1971, Theorem 1, Theorem 2).

Definition 3' [Local identifiability] The vector of structural parameters θ is locally identifiable if the information matrix associated with the log-likelihood $\ell_T(\theta, vec(\kappa))$ in Eq. (26) is non-singular in a neighborhood $\mathcal{O}(\theta_0) \subset \mathcal{P}$ of θ_0.

Under standard regularity conditions,[7] the information matrix associated with the unrestricted counterpart of the reduced form solution in Eq. (22) is given by

$$\mathcal{I}_{D,T}(\tilde{\phi}_u) := E\left[\left\{\frac{\partial \ell_{D,T}(\tilde{\phi}_u)}{\partial \tilde{\phi}_u'}\right\}\left\{\frac{\partial \ell_{D,T}(\tilde{\phi}_u)}{\partial \tilde{\phi}_u'}\right\}'\right] \tag{27}$$

where the symbol $\tilde{\phi}_u$ denotes the unrestricted VAR coefficients. The $d \times d$ matrix $\mathcal{I}_{D,T}(\tilde{\phi}_u)$ has full rank. Defined $\tilde{\phi}_u^* := (\tilde{\phi}_u', vec(\kappa)')'$, under standard regularity conditions the information matrix associated with the unrestricted counterpart of the reduced form solutions in Eq. (23) is given by

$$\mathcal{I}_{I,T}(\tilde{\phi}_u^*) := \begin{bmatrix} \mathcal{I}_{11} & \mathcal{I}_{12} \\ \mathcal{I}_{12}' & \mathcal{I}_{22} \end{bmatrix} \qquad \begin{matrix} d \times d & d \times (n_2)^2 \\ (n_2)^2 \times d & (n_2)^2 \times (n_2)^2 \end{matrix} \tag{28}$$

where dimensions of blocks have been reported alongside matrices and

$$\mathcal{I}_{11} := \mathcal{I}_{11}(\tilde{\phi}_u^*) := E\left[\left\{\frac{\partial \ell_{I,T}(\tilde{\phi}_u^*)}{\partial \tilde{\phi}_u'}\right\}\left\{\frac{\partial \ell_{I,T}(\tilde{\phi}_u^*)}{\partial \tilde{\phi}_u'}\right\}'\right] \quad ,$$

[7] See e.g. Assumptions 1-5 in Rothemberg (1971).

$$\mathcal{I}_{12} := \mathcal{I}_{12}(\tilde{\phi}_u^*) := E\left[\left\{\frac{\partial \ell_{I,T}(\tilde{\phi}_u^*)}{\partial \tilde{\phi}_u'}\right\}\left\{\frac{\partial \ell_{I,T}(\tilde{\phi}_u^*)}{\partial vec(\kappa)'}\right\}'\right],$$

$$\mathcal{I}_{22} := \mathcal{I}_{22}(\tilde{\phi}_u^*) := E\left[\left\{\frac{\partial \ell_{I,T}(\tilde{\phi}_u^*)}{\partial vec(\kappa)'}\right\}\left\{\frac{\partial \ell_{I,T}(\tilde{\phi}_u^*)}{\partial vec(\kappa)'}\right\}'\right].$$

Also the information matrix $\mathcal{I}_{I,T}(\tilde{\phi}_u^*)$ has full rank if the unconstrained counterpart of the VARMA-type system in Eq. (23) is identified. It follows that the information matrix associated with the reduced form solutions of the LRE model can be written as

$$\mathcal{I}_T(\theta, vec(\kappa)) := B(\theta)'\mathcal{I}_{D,T}^* B(\theta) \times \mathbb{I}(r[S(\theta)] < 1) + \begin{bmatrix} B(\theta)'\mathcal{I}_{11}^* B(\theta) & B(\theta)'\mathcal{I}_{12}^* \\ \mathcal{I}_{12}^{*\prime} B(\theta) & \mathcal{I}_{22}^* \end{bmatrix} \times \mathbb{I}(r[S(\theta)] > 1) \quad (29)$$

where $B(\theta) := \frac{\partial g(\theta)}{\partial \theta'}$ is a $d \times m$ Jacobian matrix, $\mathcal{I}_{D,T}^* := \mathcal{I}_{D,T}(g(\theta))$ and $\mathcal{I}_{ij}^* := \mathcal{I}_{ij}(g(\theta), vec(\kappa)')$, $i, j = 1, 2$.

From Eq. (29), necessary and sufficient conditions for θ to be identifiable is that the Jacobian matrix $B(\theta)$ has full-column rank m in a neighborhood of θ_0. Proposition 2 shows how this condition specializes in practice.

Proposition 2 [Necessary and sufficient conditions] Given the LRE model (1)-(2), Assumptions 1-4, and the set of stable reduced form solutions derived in Proposition 1, let $\mathcal{O}(\theta_0) \subset \mathcal{P}$ be a neighborhood of θ_0 and $\mathcal{L}(\phi_0) \subset R^a$ a neighborhood of ϕ_0, where ϕ_0 is the 'true' ϕ.

(a) θ is locally identifiable iff

$$rank\left[I_{a^2} - \mathring{N}(\theta)\right] = a^2 \quad , \quad \theta \in \mathcal{O}(\theta_0) \quad (30)$$

where $\mathring{N}(\theta) := [\mathring{\Phi}(\theta)' \otimes \mathring{G}(\theta)]$, $\mathring{\Phi}(\theta) \in \mathcal{F}$ is unique (i.e. card($\mathcal{F}$)=1) for $\theta \in \mathcal{O}(\theta_0)$, the matrix $\mathring{G}(\theta)$ is defined in Eq. (19) and $a := \dim(\phi)$.

(b) Necessary order condition is $m \leq a^2$.

Proof: Appendix.

Proposition 2 clarifies that the vector θ is not identifiable in Example 1 of Section 2 under determinacy because of the failure of the necessary order condition (b).

An immediate consequence of Proposition 2 is stated in the next Corollary.

Corollary 3 [Identifiability under determinacy] Let $\mathcal{P}^D := \{\theta,\ r[S(\theta)] < 1\} \subset \mathcal{P}$ be the determinacy region of the theoretically admissible parameter space. If for $\breve{\theta} \in \mathcal{P}^D$ the reduced form solution in Eq. (22) exists $\breve{\theta}$ is locally identifiable.

Proof: Appendix.

Some remarks are in order.

Remark 7 The necessary and sufficient conditions for identification derived in Proposition 2 are 'robust' to determinacy/indeterminacy in the sense that they hold for the entire class of reduced form solution derived in Proposition 1 provided Assumption 4 is added to rule out the occurrence of 'purely forward-looking' models.

Remark 8 Given the definition of $\mathring{G}(\theta)$ in Eq. (19), the non zero elements of $sr[\mathring{N}(\theta)]$ are the same as the non-zero elements of $sr[\mathring{\Phi}(\theta)' \otimes S(\theta)]$. Thus the crucial ingredients of the rank condition in Eq. (30) are the two matrices $\mathring{\Phi}(\theta)$ and $S(\theta)$. The former is the stable matrix that solves the cross-equation restrictions implied by the quadratic matrix equation (15) while the eigenvalues of the latter matrix, whose elements are function of the elements of the sub-matrix $\Phi_1(\theta)$ of $\mathring{\Phi}(\theta)$, govern the determinacy/indeterminacy of the LRE model (Proposition 1). There are situations in which the quadratic matrix equation (15) can be solved analytically (e.g. Example 1 above and Example 3' below) and others in which it is either necessary to apply generalized eigenvalues techniques or to resort to numerical (iterative) solutions (e.g. Example 4 below) along the lines discussed by Higham and King (2000).

Remark 9 The main advantage of the result derived in Proposition 2 is that in order to check the validity of the necessary and sufficient identification condition in Eq. (30) all that is needed is solving the quadratic matrix equation (15). This result is particularly useful if one wishes to estimate jointly the Euler equations comprising the LRE model by 'limited-information' methods because it is not necessary to know in detail the form of the reduced form solutions associated with the LRE model to check the identifiability of the system, see Section **??** and Section 5.

5 Identification check

If the conditions (a) and (b) of Proposition 2 are fulfilled, one can recover consistent estimates of the parameters of the multivariate LRE model. One possibility is to apply 'limited-information' techniques directly to the Euler equations of system (11) by using $r \geq m$ instruments selected from X_{t-1}, $X_{t-2}, ...$ For instance, under Assumptions 1-4 and other standard regularity conditions (including the maintained of correct specification of the LRE model), the generalized method of moments estimator of θ is root-T consistent for θ_0 if instruments are properly selected and is robust to determinacy/indeterminacy, see Fanelli (2010).

Alternatively, one can maximize either $\ell_{D,T}(g(\theta))$ or $\ell_{I,T}(g(\theta), vec(\kappa))$ in Eq. (26) and obtain a maximum likelihood estimator of θ. The maximum likelihood estimator will be consistent and fully efficient under Assumptions 1-4 (and other regularity conditions) if the likelihood is maximized under the 'correct' regime and will be inconsistent if determinacy is erroneously assumed.

The next example shows how the necessary and sufficient conditions in Eq. (30) can be applied to study the identifiability of a multivariate LRE model typically used in monetary policy analysis.

Example 3 [New-Keynesian monetary policy model] We consider a three-equation New Keynesian business cycle monetary model which takes the form (1)-(2) and is obtained by slightly modifying the notation of system (1)-(3) in Benati and Surico (2009). The LRE model is based on the following matrices

$$\Gamma_0 := \begin{bmatrix} 1 & 0 & \delta \\ -\varrho & 1 & 0 \\ -(1-\rho)\varphi_{\tilde{y}} & -(1-\rho)\varphi_{\pi} & 1 \end{bmatrix}, \Gamma_f := \begin{bmatrix} \gamma & \delta & 0 \\ 0 & \lambda_f & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Gamma_b := \begin{bmatrix} 1-\gamma & 0 & 0 \\ 0 & \lambda_b & 0 \\ 0 & 0 & \rho \end{bmatrix}, \quad R := \begin{bmatrix} \xi_{\tilde{y}} & 0 & 0 \\ 0 & \xi_{\pi} & 0 \\ 0 & 0 & \xi_R \end{bmatrix}, \quad \Sigma_u := \begin{bmatrix} \sigma^2_{\tilde{y}} & 0 & 0 \\ 0 & \sigma^2_{\pi} & 0 \\ 0 & 0 & \sigma^2_R \end{bmatrix}$$

where $X_t = (\tilde{y}_t, \pi_t, R_t)'$ and $\tilde{y}_t$, π_t, and R_t are the output gap, inflation, and the nominal interest rate. The interpretation of the structural parameters $\theta := (\varrho, \delta, \gamma, \lambda_f, \lambda_b, \rho, \varphi_{\tilde{y}}, \varphi_{\pi}, \xi_{\tilde{y}}, \xi_{\pi}, \xi_R)'$ may be found in Benati and Surico (2009). In this case, $n := 3$ and $R \neq 0_{3\times 3}$, hence $a := 2n = 6$ and the necessary identification order condition (b) of Proposition 2 is met as $m := \dim(\theta) := 11$. Assume first that the data are generated in correspondence of the structural parameters $\bar{\theta}_0$:=(0.044, $(8.062)^{-1}$, 0.744, 0.57, 0.048, 0.595, 0.527, 0.821, 0.796, 0.418, 0.404) (plus $\sigma^2_{\tilde{y}}$:=0.055, σ^2_{π}:=0.391, σ^2_R:=0.492). This point corresponds to the median of the 90 percent coverage percentiles of the Bayesian estimates reported in Table 1 of Benati and Surico (2009) for the period before October 1979. Assumptions 1-4 are in this case satisfied, in particular $\det[\Theta(\bar{\theta}_0)] = 0.799$ and the solution $\mathring{\Phi}(\bar{\theta}_0)$ to the quadratic matrix equation (15) is such that $r[\mathring{\Phi}(\bar{\theta}_0)]$:=0.796.[8] Moreover, $r[S(\bar{\theta}_0)]$:=1.013 and we know from Proposition 1 that the reduced form solution is indeterminate with a single arbitrary parameter governing indeterminacy and, possibly, a single MDS independent on u_t. It is seen that $rank\left[I_{36} - \mathring{N}(\bar{\theta}_0)\right]$ =36=a^2 (in particular det $\left[I_{36} - \mathring{N}(\bar{\theta}_0)\right]$:=0.00094 and $r_{\min}\left[I_{36} - \mathring{N}(\bar{\theta}_0)\right]$:=0.19) hence by condition (a) of Proposition 2 the point $\bar{\theta}_0$ is locally identifiable. Next consider the point $\breve{\theta}_0$:=(0.044, $(8.062)^{-1}$, 0.744, 0.57, 0.048, 0.834, 1.146, 1.749, 0.796, 0.418, 0.404) ($\sigma^2_{\tilde{y}}$, σ^2_{π} and σ^2_R are as before) that corresponds to the median of the 90 percent coverage percentiles of the Bayesian estimates reported in Table 1 of Benati and Surico (2009) for the period after the Volcker stabilization. Also in this case Assumptions 1-4 are satisfied, $\det[\Theta(\breve{\theta}_0)] = 0.85$, $r[\mathring{\Phi}(\breve{\theta}_0)]$=0.796 and $r[S(\breve{\theta}_0)]$:=0.793 hence we know from Proposition 1 that the reduced form solution is determinate and takes the form of a constrained VAR system with two lags; moreover, $rank\left[I_{36} - \mathring{N}(\breve{\theta}_0)\right]$ =36 (in

[8] In this case the stable solution of the quadratic matrix equation (15) has been obtained numerically along the lines of Higham and King (2000).

particular $\det\left[I_{36}-\mathring{N}(\breve{\theta}_0)\right]$:=0.00068 and $r_{\min}\left[I_{36}-\mathring{N}(\breve{\theta}_0)\right]$:=0.225) hence also $\breve{\theta}_0$ is locally identifiable. In principle, it is possible to consider many theoretically admissible values of θ and check the validity of the condition in Eq. (30) prior to estimation.

Another use of the result in Proposition 2 is when a reduced form representation of the LRE model does not exist in correspondence of a point θ that is erroneously assumed to belong to the theoretically admissible parameter space. In these cases θ is (trivially) unidentifiable hence the rank condition in Eq. (30) does not hold. A situation of this type is investigated in detail in the next example.

Example 4 [No reduced form solutions] Consider the following two-equation ($n := 2$) LRE model

$$z_t = \alpha E_t z_{t+1} + E_t y_{t+1} + \omega_{z,t} \qquad (31)$$
$$y_t = \psi y_{t-1} + \omega_{y,t} \qquad (32)$$

where $\omega_t := (\omega_{z,t}, \omega_{y,t})'$ is a MDS with covariance matrix $\Sigma_\omega := dg(\sigma_z^2, \sigma_y^2)$, $\theta := (\alpha, \psi)'$, α>0, 0<ψ<1, $X_t := (z_t, y_t)'$ and

$$\Gamma_0 := I_2, \quad \Gamma_f := \begin{bmatrix} \alpha & \beta \\ 0 & 0 \end{bmatrix}, \quad \Gamma_b := \begin{bmatrix} 0 & 0 \\ 0 & \psi \end{bmatrix}, R := 0_{2\times 2}.$$

Suppose that the space of theoretically admissible values of θ is given by $\mathcal{P} := \{(\alpha, \psi)', \alpha>0, 0<\psi<1\} \subset \mathbb{R}^2$. In this case, a:=n :=2 and dim(θ):=2 hence the necessary identification order condition (b) of Proposition 2 is met. The unique stable solution to the quadratic matrix equation (15) can be determined analytically and is given by

$$\mathring{\Phi}(\theta) \equiv \Phi(\theta) := \begin{bmatrix} 0 & -\frac{\psi^2}{\alpha\psi-1} \\ 0 & \psi \end{bmatrix} \qquad (33)$$

and it easily seen that this matrix is stable. Moreover,

$$\mathring{G}(\theta) \equiv S(\theta) := \Theta^{-1}\Gamma_f := \begin{bmatrix} \alpha & 1 \\ 0 & 0 \end{bmatrix}$$

so that by Proposition 1 a unique stable reduced form solution is obtained if 0<α<1 and multiple stable reduced form solutions if α>1.[9] Moreover,

$$\det\left[I_4 - \mathring{N}(\theta)\right] = \det \begin{bmatrix} 1 & 0 & \alpha\frac{\psi^2}{\alpha\psi-1} & \frac{\psi^2}{\alpha\psi-1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\alpha\psi+1 & -\psi \\ 0 & 0 & 0 & 1 \end{bmatrix} = 1 - \alpha\psi$$

[9] We deliberately rule out the point $\alpha := 1$ from the set of theoretically admissible values $\mathcal{P}$ for the same reason as in footnote 4.

hence one is reasonably tempted to claim that all θs that lie in the determinacy region $\mathcal{P}^D := \{(\alpha,\psi)', 0<\alpha<1, 0<\psi<1\} \subset \mathcal{P}$ are locally identifiable, while all zero-Lebesque measure points θ that lie in the region $\mathcal{P}^* := \{(\alpha,\psi)', \alpha>1, 0<\psi<1\}$ and fulfill the restriction $\alpha\psi := 1$ are unidentifiable. Actually, while this is true, the structure of the matrix $\Phi(\theta)$ in Eq. (33) and the form of the indeterminate solutions derived in Eq. (23) (Proposition 1) reveal that in correspondence of the points $\theta \in \mathcal{P}^*$ such that $\alpha\psi := 1$ a reduced form solution does not exist for the LRE model hence θ is trivially unidentifiable. It turns out that $\mathcal{P}^I := \{(\alpha,\psi)', \alpha>1, 0<\psi<1, \alpha\psi \neq 1\} \subset \mathcal{P}$ is the indeterminacy region of the parameter space and all $\theta \in \mathcal{P}^I$ are locally identifiable.

Example 3 and Example 4 suggest that one may potentially use the same method proposed in Amisano and Giannini (1997) for checking the identifiability of structural VARs. Indeed, it is in principle possible to assess the validity of the rank condition in Eq. (30) at many randomly drawn points from the theoretically admissible space $\mathcal{P}$, or from a suitably chosen subset $\mathcal{Z}$ of $\mathcal{P}$, prior to estimation. Such an approach generalizes the identification analysis procedure advocated by Iskerev (2010) who discusses the identifiability of dynamic stochastic geneal equilibrium models only for the case of determinacy. It may involve considerable computation costs.

On the inferential side, it would be natural to think of using a root-T consistent asymptotically Gaussian estimator $\hat{\theta}_T$ and existing techniques for testing the rank of the matrix $I_{a^2} - \mathring{N}(\theta)$, see e.g. Cragg and Donald (1997), Robin and Smith (2000), Kleiberger and Paap (2006) and references therein. The drawback in this case is that all methods for inferring the rank of a matrix are based on the null hypothesis that $rank\left[I_{a^2} - \mathring{N}(\theta)\right]:=h$ against the alternative $rank\left[I_{a^2} - \mathring{N}(\theta)\right] \geq h$, where or $h < a^2$, and $T^{1/2}(\hat{\theta}_T - \theta_0)$ is no longer asymptotically Gaussian under this type of null.

Appendix: Proofs

Proof of Proposition 1. (a) Since $r[S(\theta)] < 1$, $S(\theta)$ is absolutely summable. Using this property and Assumptions 3, system (20) can be solved as

$$X_{F,t}^S = \sum_{j=0}^{\infty}[S(\theta)]^j E_t \Theta^{-1} u_{t+j}^R = \Theta^{-1} u_t^R. \tag{34}$$

By combining Eq. (34) with $\mathring{X}_{B,t} := \mathring{\Phi}\mathring{X}_{t-1}$ yields

$$\begin{pmatrix} X_t \\ X_{t-1} \end{pmatrix} = \begin{bmatrix} \Phi_1 & \Phi_2 \\ I_n & 0_{n\times n} \end{bmatrix} \begin{pmatrix} X_{t-1} \\ X_{t-2} \end{pmatrix} + \begin{pmatrix} X_{F,t}^S \\ 0_{n\times 1} \end{pmatrix}$$

$$= \begin{bmatrix} \Phi_1 & \Phi_2 \\ I_n & 0_{n\times n} \end{bmatrix} \begin{pmatrix} X_{t-1} \\ X_{t-2} \end{pmatrix} + \begin{pmatrix} \Theta^{-1}u_t^R \\ 0_{n\times 1} \end{pmatrix}$$

hence the first block of n equations reads as

$$X_t = \Phi_1 X_{t-1} + \Phi_2 X_{t-1} + \Theta^{-1}u_t^R.$$

Since $\eta_t := X_t - E_{t-1}X_t = (X_t - \Phi_1 X_{t-1} - \Phi_2 X_{t-2})$, from the expression $\eta_t = \Theta^{-1}u_t^R$ and the definition $u_t^R := u_t + R\Gamma_f \eta_t$ it follows that $(I - \Theta^{-1}R\Gamma_f)\eta_t = \Theta^{-1}u_t$ which becomes, under Assumption 3, $\eta_t := (\Theta - R\Gamma_f)^{-1}u_t = \Upsilon^{-1}u_t$. Thus the solution is given by Eq. (22).

(b) Using the Jordan decomposition in Eq. (21), system (20) can be written as

$$X_{F,t}^S = P\Lambda P^{-1}E_t X_{F,t+1}^S + \Theta^{-1}u_t^R,$$

transformed into

$$P^{-1}X_{F,t}^S = \Lambda P^{-1}E_t X_{F,t+1}^S + P^{-1}\Theta^{-1}u_t^R$$

and finally partitioned as

$$\begin{matrix} n_1 \times 1 \\ n_2 \times 1 \end{matrix} \begin{pmatrix} X_{F,t}^{S_1} \\ X_{F,t}^{S_2} \end{pmatrix} = \begin{bmatrix} \Lambda_1 & 0_{n_1\times n_2} \\ 0_{n_2\times n_1} & \Lambda_2 \end{bmatrix} E_t \begin{pmatrix} X_{F,t+1}^{S_1} \\ X_{F,t+1}^{S_2} \end{pmatrix} + \begin{pmatrix} \vartheta_t^{S_1} \\ \vartheta_t^{S_2} \end{pmatrix} \quad (35)$$

where

$$\begin{pmatrix} X_{F,t}^{S_1} \\ X_{F,t}^{S_2} \end{pmatrix} := P^{-1}X_{F,t}^S \quad , \quad \begin{pmatrix} \vartheta_t^{S_1} \\ \vartheta_t^{S_2} \end{pmatrix} := P^{-1}\Theta^{-1}u_t^R.$$

The first sub-system in Eq. (35)

$$X_{F,t}^{S_1} = \Lambda_1 E_t X_{F,t+1}^{S_1} + \vartheta_t^{S_1} \quad (36)$$

can be regarded as a multivariate LRE model based on $\Gamma_0^* := I_{n_1}, \Gamma_f^* := \Lambda_1$ and $\Gamma_b^* := 0_{n_1\times n1}$. By part (a) of Proposition 1 its reduced form solution is given by

$$X_{F,t}^{S_1} = \vartheta_t^{S_1}. \quad (37)$$

The second sub-system in Eq. (35)

$$X_{F,t}^{S_2} = \Lambda_2 E_t X_{F,t+1}^{S_2} + \vartheta_t^{S_2} \quad (38)$$

can be written as

$$E_t X_{F,t+1}^{S_2} = (\Lambda_2)^{-1} X_{F,t}^{S_2} - (\Lambda_2)^{-1} \vartheta_t^{S_2}$$

or, equivalently, in the form

$$X_{F,t+1}^{S_2} = (\Lambda_2)^{-1} X_{F,t}^{S_2} - (\Lambda_2)^{-1} \vartheta_t^{S_2} + \eta_{t+1}^{S_2} \quad (39)$$

where $\eta_{t+1}^{S_2} := X_{F,t+1}^{S_2} - E_t X_{F,t+1}^{S_2}$ is an arbitrary $n_2 \times 1$ MDS with respect to $\mathcal{I}_t$. Consider the following representation of $\eta_{t+1}^{S_2}$:

$$\eta_{t+1}^{S_2} \ := \ \kappa\vartheta_{t+1}^{S_2} + \ s_{t+1} \tag{40}$$

where κ is a $n_2 \times n_2$ matrix whose elements are arbitrary and not related to θ, and $s_{t+1} := (s_{1,t+1,}s_{2,t+1}, ..., s_{n_2,t+1,}) \in \mathcal{I}_{t+1}$ is a $n_2 \times 1$ 'extraneous' MDS with respect to $\mathcal{I}_t$ independent on $\vartheta_{t+1}^{S_2}$. If Eq. (39) is a linear solution to Eq. (38), also the sub-system

$$X_{F,t+1}^{S_2} = (\Lambda_2)^{-1} X_{F,t}^{S_2} + \kappa \ \vartheta_{t+1}^{S_2} - (\Lambda_2)^{-1} \vartheta_t^{S_2} + \ s_{t+1} \tag{41}$$

obtained by using Eq. (40) in Eq. (39) will be a linear solution of Eq. (38).By coupling Eq. (37) with Eq. (41) expressed at time t, we obtain

$$\begin{pmatrix} X_{F,t}^{S_1} \\ X_{F,t}^{S_2} \end{pmatrix} = \begin{bmatrix} 0_{n_1\times n_1} & 0_{n_1\times n_2} \\ 0_{n_2\times n_1} & \Lambda_2^{-1} \end{bmatrix} \begin{pmatrix} X_{F,t-1}^{S_1} \\ X_{F,t-1}^{S_2} \end{pmatrix} + \begin{bmatrix} I_{n_1} & 0_{n_1\times n_2} \\ 0_{n_2\times n_1} & \kappa \end{bmatrix} \begin{pmatrix} \vartheta_t^{S_1} \\ \vartheta_t^{S_2} \end{pmatrix}$$
$$- \begin{bmatrix} 0_{n_1\times n_1} & 0_{n_1\times n_2} \\ 0_{n_2\times n_1} & \Lambda_2^{-1} \end{bmatrix} \begin{pmatrix} \vartheta_{t-1}^{S_1} \\ \vartheta_{t-1}^{S_2} \end{pmatrix} + \begin{pmatrix} 0_{n_1\times 1} \\ s_t \end{pmatrix}$$

which is equivalent to

$$\begin{aligned} X_{F,t}^{S} \ &= \ P \begin{bmatrix} 0_{n_1\times n_1} & 0_{n_1\times n_2} \\ 0_{n_2\times n_1} & \Lambda_2^{-1} \end{bmatrix} P^{-1} X_{F,t-1}^{S} + P \begin{bmatrix} I_{n_1} & 0_{n_1\times n_2} \\ 0_{n_2\times n_1} & \kappa \end{bmatrix} P^{-1}\Theta^{-1}u_t^R \\ &\quad - P \begin{bmatrix} 0_{n_1\times n_1} & 0_{n_1\times n_2} \\ 0_{n_2\times n_1} & \Lambda_2^{-1} \end{bmatrix} P^{-1}\Theta^{-1}u_{t-1}^R + P\zeta_t \end{aligned} \tag{42}$$

where $\zeta_t := (0'_{n_1\times 1}, s'_t)'$. Eq. (42) can be simplified in the expression

$$X_{F,t}^{S} = \Pi(\theta) \ X_{F,t-1}^{S} + M(\theta,\kappa)\Theta^{-1}u_t^R - \Pi(\theta)\Theta^{-1}u_{t-1}^R + P(\theta)\zeta_t \tag{43}$$

where the matrices $\Pi(\theta)$ and $M(\theta,\kappa)$ are defined as

$$\Pi(\theta) := P \begin{bmatrix} 0_{n_1\times n_1} & 0_{n_1\times n_2} \\ 0_{n_2\times n_1} & \Lambda_2^{-1} \end{bmatrix} P^{-1} \tag{44}$$

$$M(\theta,\kappa) := P \begin{bmatrix} I_{n_1} & 0_{n_1\times n_2} \\ 0_{n_2\times n_1} & \kappa \end{bmatrix} P^{-1}. \tag{45}$$

In terms of $\mathring{X}_{F,t} = (X_{F,t}^{S'}, 0'_{n\times 1})'$ and $\mathring{u}_t := (u_t^{R\prime}, 0'_{n\times 1})'$, the solutions in Eq. (43) become

$$\begin{pmatrix} X_{F,t}^{S} \\ 0 \end{pmatrix} = \begin{bmatrix} \Pi(\theta) & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} \end{bmatrix} \begin{pmatrix} X_{F,t-1}^{S} \\ 0 \end{pmatrix}$$
$$+ \begin{bmatrix} M(\theta,\kappa)\Theta^{-1} & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} \end{bmatrix} \begin{pmatrix} u_t^R \\ 0_{n\times 1} \end{pmatrix} - \begin{bmatrix} \Pi(\theta)\Theta^{-1} & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} \end{bmatrix} \begin{pmatrix} u_{t-1}^R \\ 0_{n\times 1} \end{pmatrix} + \begin{pmatrix} \tau_t \\ 0_{n\times 1} \end{pmatrix}$$

and for $\tau_t := P(\theta)\zeta_t$ can be compacted in the expression

$$\mathring{X}_{F,t} = \mathring{\Pi}\mathring{X}_{F,t-1} + \mathring{M}\mathring{\Xi}\mathring{u}_t - \mathring{\Pi}\mathring{\Xi}\mathring{u}_{t-1} + \mathring{\xi}_t \tag{46}$$

where $\mathring{\xi}_t := (\tau_t', 0_{n\times 1}')'$ and

$$\mathring{\Pi} := \begin{bmatrix} \Pi(\theta) & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} \end{bmatrix} \ , \ \mathring{M} := \begin{bmatrix} M(\theta,\kappa) & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} \end{bmatrix} \ , \mathring{\Xi} := \begin{bmatrix} \Theta^{-1}(\theta) & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} \end{bmatrix}.$$

By combining Eq. (46) with $\mathring{X}_{B,t} := \mathring{\Phi}\mathring{X}_{t-1}$ one obtains

$$\mathring{X}_t \ = (\mathring{\Pi} + \mathring{\Phi})\mathring{X}_{t-1} - \mathring{\Pi}\ \mathring{\Phi}\mathring{X}_{t-2} + \mathring{M}\mathring{\Xi}\ \mathring{u}_t - \mathring{\Pi}\mathring{\Xi}\ \mathring{u}_{t-1} + \mathring{\xi}_t \tag{47}$$

and this system reads, using the lag operator, as

$$(I_{2n} - \mathring{\Pi}L)(I_{2n} - \mathring{\Phi}L)\mathring{X}_t = (\mathring{M} - \mathring{\Pi}L)\mathring{\Xi}\ \mathring{u}_t + \mathring{\xi}_t. \tag{48}$$

The first block of n equations of system (48) is given by

$$[I_n - \Pi(\theta)L][I_n - \Phi_1(\theta)L - \Phi_2(\theta)L^2]X_t = [M(\theta,\kappa) - \Pi(\theta)L]\Theta^{-1}u_t^R + \tau_t. \tag{49}$$

Since $\eta_t := X_t - E_{t-1}X_t := M(\theta,\kappa)\Theta^{-1}u_t^R + \tau_t$; from the definition $u_t^R := u_t + R\Gamma_f\eta_t$ it follow that $[I_n - M(\theta,\kappa)\Theta^{-1}R\Gamma_f]\eta_t = M(\theta,\kappa)\Theta^{-1}u_t + \tau_t$; therefore, using Assumption 3 it holds the relationship

$$\eta_t := V(\theta,\kappa)[M(\theta,\kappa)\Theta^{-1}u_t + \tau_t] \equiv M(\theta,\kappa)\Theta^{-1}u_t^R + \tau_t \tag{50}$$

where

$$V(\theta,\kappa) := [I_n - M(\theta,\kappa)\Theta^{-1}R\Gamma_f]^{-1}. \tag{51}$$

Write the right-hand side of system (49) as

$$MA_t := [M(\theta,\kappa) - \Pi(\theta)L]\Theta^{-1}[u_t + R\Gamma_f\eta_t] + \tau_t$$

and use Eq. (50) obtaining

$$MA_t := [M(\theta,\kappa) - \Pi(\theta)L]\Theta^{-1}[u_t + R\Gamma_f\left\{V(\theta,\kappa)[M(\theta,\kappa)\Theta^{-1}u_t + \tau_t]\right\}] + \tau_t;$$

now, rearranging terms and using some algebra

$$MA_t := [M(\theta,\kappa) - \Pi(\theta)L][\Theta^{-1}u_t + \Theta^{-1}R\Gamma_f V(\theta,\kappa)M(\theta,\kappa)\Theta^{-1}u_t + \Theta^{-1}R\Gamma_f V(\theta,\kappa)\tau_t] + \tau_t$$

$$= [M(\theta,\kappa) - \Pi(\theta)L][I_n + \Theta^{-1}R\Gamma_f V(\theta,\kappa)M(\theta,\kappa)]\Theta^{-1}u_t + [M(\theta,\kappa) - \Pi(\theta)L][\Theta^{-1}R\Gamma_f V(\theta,\kappa)\tau_t] + \tau_t$$

$$= [M(\theta,\kappa) - \Pi(\theta)L]\Psi(\theta,\kappa)u_t + [M(\theta,\kappa) - \Pi(\theta)L][\Theta^{-1}R\Gamma_f V(\theta,\kappa)\tau_t] + \tau_t.$$

By substituting the right-hand-side of system (49) with this expression for MA_t, Eq. (23) in the text is obtained. This completes the proof ■.

Proof of Corollary 1. If $\kappa := I_n$, from the definitions in Proposition 1, $M(\theta,\kappa) := I_n$ and $\tilde{V}(\theta,\kappa) := [I_n - \Theta^{-1}R\Gamma_f]^{-1}$. Given system (49), the relationship in Eq. (50) collapses to

$$\tilde{V}(\theta,\kappa)[\Theta^{-1}u_t + \tau_t] \equiv \Theta^{-1}u_t^R + \tau_t$$

hence $\Theta^{-1}u_t^R = \tilde{V}(\theta,\kappa)\Theta^{-1}u_t + [\tilde{V}(\theta,\kappa) - I_n]\tau_t$. By using the last expression, system (49) can be written

$$\begin{aligned}[I_n - \Pi(\theta)L][I_n - \Phi_1(\theta)L - \Phi_2(\theta)L^2]X_t &= [I_n - \Pi(\theta)L]\tilde{V}(\theta,\kappa)\Theta^{-1}u_t \\ &\quad +[I_n - \Pi(\theta)L][\tilde{V}(\theta,\kappa) - I_n]\tau_t + \tau_t\end{aligned}$$

and since the polynomial $[I_n - \Pi(\theta)L]$ is invertible by construction, the model can be represented as

$$[I_n - \Phi_1(\theta)L - \Phi_2(\theta)L^2]X_t = \tilde{V}(\theta,\kappa)\Theta^{-1}u_t + [\tilde{V}(\theta,\kappa) - I_n]\tau_t + [I_n - \Pi(\theta)L]^{-1}\tau_t.$$

Finally, $\tilde{V}(\theta,\kappa)\Theta^{-1} := [I_n - \Theta^{-1}R\Gamma_f]^{-1}\Theta^{-1} := [\Theta - R\Gamma_f]^{-1} := \Upsilon^{-1}(\theta)$. This completes the proof. ■.

Proof of Corollary 2. (a) Since $\Gamma_b := 0_{n\times n}$, $\mathring{\Phi} := 0_{2n\times 2n} \in \mathcal{F}$. Moreover, card$(\mathcal{F}) = 1$ if c1 or c2 or both hold. $\mathring{\Phi} := 0_{2n\times 2n}$ implies $\Phi_1 := 0_{n\times n} =: \Phi_2$, hence the autoregressive polynomial of the reduced form collapses to I_n, and $\Theta(\theta) := \Gamma_0^R := (\Gamma_0 + R\Gamma_f) := \Theta^*$ and $\Upsilon(\theta) := (\Theta^* - R\Gamma_f) := (\Gamma_0 + R\Gamma_f - R\Gamma_f) = \Gamma_0$, respectively. By substituting these expressions into Eq. (22) the result is obtained. (b) The result is obtained by replacing Θ with Θ^* in the definition of $\Psi(\theta,\kappa)$ and $V(\theta,\kappa)$ of Proposition 1. This completes the proof ■.

Proof of Proposition 2. (a) By applying the vec operator to both sides of Eq. (15) one gets

$$f(\tilde{\phi},\theta) := vec(\mathring{\Gamma}_f\mathring{\Phi}^2 - \mathring{\Gamma}_0\mathring{\Phi} + \mathring{\Gamma}_b) = 0_{a^2\times 1} \tag{52}$$

where the vector function $f(\cdot,\cdot) : \mathcal{A} \to \mathbb{R}^{a^2}$ is defined in an open set $\mathcal{A}$ in $\mathbb{R}^{a^2+m}$. We next define the function

$$\tilde{f}(\tilde{\phi},\theta) := Kf(\tilde{\phi},\theta)$$

where K is a $d \times a^2$ full row rank matrix which selects the sub-set of relationships that do not involve trivial identities from $f(\tilde{\phi},\theta)$. Observe that: (i) $\tilde{f}(\tilde{\phi}_0,\theta_0) = 0_{d\times 1}$; (ii) $\tilde{f}(\cdot,\cdot)$ is differentiable at $(\tilde{\phi}_0,\theta_0)$; (iii) Assumption 4 ensures that $\tilde{\phi} := 0_{d\times 1}$ is not consistent with Eq. (52); (iv) using Eq. (25),

$$J(\tilde{\phi},\theta) := \frac{\partial\tilde{f}(\tilde{\phi},\theta)}{\partial\tilde{\phi}'} := K\frac{\partial f(\tilde{\phi},\theta)}{\partial\phi'}H_1$$

$$= K\left[(I_a \otimes \mathring{\Gamma}_f)[(\mathring{\Phi}' \otimes I_a) + (I_a \otimes \mathring{\Phi})] - (I_a \otimes \mathring{\Gamma}_0)\right]H_1$$

$$= K\left[(I_a \otimes \mathring{\Gamma}_f \mathring{\Phi}) - (I_a \otimes \mathring{\Gamma}_0) + (\mathring{\Phi}' \otimes \mathring{\Gamma}_f) = -[I_a \otimes (\mathring{\Gamma}_0 - \mathring{\Gamma}_f \mathring{\Phi})] + (\mathring{\Phi}' \otimes \mathring{\Gamma}_f)\right] H_1$$
$$= -K[I_a \otimes (\mathring{\Gamma}_0 - \mathring{\Gamma}_f \mathring{\Phi})]\left\{I_{a^2} - [I_a \otimes (\mathring{\Gamma}_0 - \mathring{\Gamma}_f \mathring{\Phi})]^{-1}(\mathring{\Phi}' \otimes \mathring{\Gamma}_f)\right\} H_1$$
$$= -K[I_a \otimes \mathring{\Theta}]\left[I_{a^2} - \mathring{N}\right] H_1. \quad (53)$$

We first prove that the condition in Eq. (30) is sufficient for local identifiability. By the implicit function theorem, if the $a^2 \times a^2$ matrix $[I_a \otimes \mathring{\Theta}]\left[I_{a^2} - \mathring{N}\right]$ is non-singular at $(\tilde{\phi}_0, \theta_0)$, it is possible to derive from $\tilde{f}(\tilde{\phi}_0, \theta_0) = 0_{d\times 1}$ the unique mapping

$$\tilde{\phi} = g(\theta) \quad (54)$$

which holds for $\theta \in \mathcal{O}(\theta_0)$ and $\tilde{\phi} \in \mathcal{L}(\tilde{\phi}_0)$, and such that $rank[B(\theta)] = m$ for $\theta \in \mathcal{O}(\theta_0)$, where $B(\theta) := \frac{\partial g(\theta)}{\partial \theta'}$. Since the matrix $[I_a \otimes \mathring{\Theta}]$ is non-singular by Assumption 3, the validity of the rank condition in Eq. (30) is sufficient for $\det J(\tilde{\phi}_0, \theta_0) \neq 0$. To prove that the rank condition in Eq. (30) is also necessary for identification, observe that if it exists a unique mapping between reduced form and structural parameters in Eq. (54), then by the implicit function theorem

$$\underset{d\times m}{B(\theta)} := \underset{d\times d}{\left[J(\tilde{\phi}, \theta)\right]^{-1}} \times \underset{d\times m}{\frac{\partial \tilde{f}(\tilde{\phi}, \theta)}{\partial \theta'}}$$

must have full column rank m. Therefore, if $\frac{\partial \tilde{f}(\tilde{\phi}, \theta)}{\partial \theta'}$ is proved to have full column rank m, the result is obtained. Using the chain rule we have that

$$\frac{\partial \tilde{f}(\tilde{\phi}, \theta)}{\partial \theta'} := K\frac{\partial f(\tilde{\phi}, \theta)}{\partial \theta'} := \underset{d\times a^2}{K} \times \underset{a^2\times 12n^2}{C_1(\tilde{\phi})} \times \underset{3a^2\times a^2}{C_2} \times \underset{a^2\times a^2}{C_3(\theta)} \times \underset{a^2\times m}{Q(\theta)}$$

where

$$C_1(\tilde{\phi}) := \frac{\partial f(\phi, \theta)}{\partial vec(\mathring{\Gamma}_0 : \mathring{\Gamma}_f : \mathring{\Gamma}_b)'}, \quad C_2 := \frac{\partial vec(\mathring{\Gamma}_0 : \mathring{\Gamma}_f : \mathring{\Gamma}_b)}{\partial vec(\Gamma_0^R : \Gamma_f : \Gamma_{b,1}^R : \Gamma_{b,2}^R)'}$$
$$C_3(\theta) := \frac{\partial vec(\Gamma_0^R : \Gamma_f : \Gamma_{b,1}^R : \Gamma_{b,2}^R)}{\partial vec(\Gamma)'},$$

and by using standard derivative rules

$$C_1(\tilde{\phi}) := \left[(\mathring{\Phi}')^2 \otimes I_{2n} : -\mathring{\Phi}' \otimes I_{2n} : I_{2n}\right], \quad C_2 := \begin{bmatrix} I_n & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & I_n & 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} & I_n & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & I_n \end{bmatrix}$$

$$C_3(\theta) := \begin{bmatrix} I_{n^2} & (I_n \otimes R) & 0_{n^2\times n^2} & 0_{n^2\times n^2} \\ 0_{n^2\times n^2} & I_{n^2} & 0_{n^2\times n^2} & 0_{n^2\times n^2} \\ 0_{n^2\times n^2} & (I_n \otimes R) & I_{n^2} & (\Gamma_f' \otimes I_n) \\ 0_{n^2\times n^2} & 0_{n^2\times n^2} & -(I_n \otimes R) & -(\Gamma_b' \otimes I_n) \end{bmatrix}.$$

It is evident that $C_1(\tilde{\phi}_0)$ and $C_3(\theta_0)$ have full row rank while C_2 has full column rank. It follows that the matrix $\frac{\partial \tilde{f}(\tilde{\phi},\theta)}{\partial \theta'}$ has full-column rank m at $(\tilde{\phi}_0, \theta_0)$. (b) In the absence of the order condition $m \leq a^2$ the reduced form coefficients could not be expressed as function of θ and the implicit function theorem could not be applied. This completes the proof ■.

Proof of Corollary 3. If $\breve{\theta} \in \mathcal{P}^D$, $r[S(\breve{\theta})]$<1 and, accordingly, $r[\mathring{G}(\breve{\theta})]$<1 so that $r[\mathring{\Phi}(\breve{\theta}) \otimes \mathring{G}(\breve{\theta})]$<1 and then the rank condition in Eq. (30) is satisfied. This completes the proof ■.

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