TENSOR RANKS ON TANGENT DEVELOPABLE OF SEGRE VARIETIES

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Abstract. We describe the stratification by tensor rank of the points belonging to the tangent developable of any Segre variety. Moreover we prove the Comon's conjecture on the rank of symmetric tensors for those tensors belonging to tangential varieties to Veronese varieties.

Introduction

In this paper we want to address the problem of tensor decomposition over an algebraically closed field K of characteristic 0 for tensors belonging to a tangent space of the projective variety that parameterizes completely decomposable tensors.

Let $V_1, \ldots, V_d$ be K-vector spaces of dimensions $n_1+1, \ldots, n_d+1$ respectively; the projective variety $X_{n_1,\ldots,n_d} \subset \mathbb{P}(V_1 \otimes \cdots \otimes V_d)$ that parameterizes projective classes of completely decomposable tensors $v_1 \otimes \cdots \otimes v_d \in V_1 \otimes \cdots \otimes V_d$ is classically known as a Segre variety (see Definition 1). Given a tensor $T \in V_1 \otimes \cdots \otimes V_d$, finding the minimum number of completely decomposable tensors such that T can be written as a linear combination of them (see Definition 2 for the notion of "tensor rank") is related to the tensor decomposition problem that nowadays seems to be crucial in many applications like Signal Processing (see eg. [1], [15], [11]), Algebraic Statistics ([14], [20]), Neuroscience (eg. [3]). The specific case of tensors belonging to tangential varieties to Segre varieties (Notation 1) is studied in [8] and it turns out to be of certain interest in the context of Computational Biology. In fact in [12] a particular class of statistical models (namely certain context-specific independence model – CSI) is shown to be crucial in machine learning and computational biology. L. Oeding has recently shown in [17] how to interpret the CSI model performed by [12] in terms of tangential variety to Segre variety. In this setting B. Sturmfels and P. Zwiernik in a very recent paper ([18]) show how to derive parametrizations and implicit equations in cumulants for the tangential variety of the Segre variety $X_{1,\ldots,1}$ and for certain CSI models (see [6] for a combinatorial point of view on cumulants).

In this paper, after a preliminary section, we give a complete classification of the tensor rank of an element belonging to the tangent developable of any Segre variety. In particular in Theorem 1 we will prove that if $P \in T_O(X_{n_1,\ldots,n_d})$ for certain point $O = (O_1, \ldots, O_d) \in X_{n_1,\ldots,n_d}$, then the minimum number r of completely decomposable tensors $v_{1,i} \otimes \cdots \otimes v_{d,i} \in V_1 \otimes \cdots \otimes V_d$ such that $P = \sum_{i=1}^{r} [v_{1,i} \otimes \cdots \otimes v_{d,i}]$ is equal to the minimum number $\eta_{X_{n_1,\ldots,n_d}}(P)$ for which there exist $E \subseteq \{1, \ldots, d\}$ such that $\sharp(E) = \eta_{X_{n_1,\ldots,n_d}}(P)$ and $T_O(X_{n_1,\ldots,n_d}) \subseteq$

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$\langle \cup_{i\in E} Y_{O,i}\rangle$ where $Y_{O,i}$ the n_i-dimensional linear subspace obtained by fixing all coordinates $j \in \{1,\dots,d\}\setminus\{i\}$ equal to $O_j \in \mathbb{P}^n_i$ (see Notation 3).

In the last section of this paper we show how to use this theorem in order to prove the so called "Comon's conjecture" in the particular case in which the points $P \in \tau(X_{n_1,\dots,n_d})$ parameterize symmetric tensors. Let us give more details on that.

Let $V_1 = \cdots = V_d = V$ be a vector space of dimension $n+1$ and consider the subspace $S^dV \subset V^{\otimes d}$ of symmetric tensors. The intersection between the Segre variety $X_{n,\dots,n}$ and $\mathbb{P}(S^dV)$ is a way to interpret the classical Veronese embedding of $\mathbb{P}^n$ via the sections of the sheaf $\mathcal{O}(d)$. Therefore an element of the Veronese variety $\nu_d(\mathbb{P}^n) = X_{n,\dots,n} \cap \mathbb{P}(S^dV)$ is the projective class of a completely decomposable symmetric tensor. Now, given a point $P \in \mathbb{P}(S^dV)$ that parameterizes a projective class of a symmetric tensor, we can look at two different decompositions of it. Let $v_{1,i}\otimes\cdots\otimes v_{d,i} \in V^{\otimes d}$ and let $w_j^{\otimes d} \in S^dV$, and ask for the minimum r and the minimum r' such that $P = \sum_{i=1}^r [v_{1,i}\otimes\cdots\otimes v_{d,i}] = \sum_{j=1}^{r'} [w_j^{\otimes d}]$. In 2008, at the AIM workshop in Palo Alto, USA (see the report [16]), Pierre Comon stated the following:

Conjecture 1. **[Comon's Conjecture]** The minimum integer r such that a symmetric tensor $T \in S^dV$ can be written as

$$T = \sum_{i=1}^{r} v_{1,i}\otimes\cdots\otimes v_{d,i}$$

for $v_{1,i}\otimes\cdots\otimes v_{d,i} \in V^{\otimes d}$, $i = 1,\dots,r$, is equal to the minimum integer r' for which there exist $w_j^{\otimes d} \in S^dV$, $j = 1,\dots,r'$ such that

$$T = \sum_{j=1}^{r'} w_j^{\otimes d}.$$

As far as we know this conjecture is proved if $r \le \dim(V)$ (for a general d-tensor, d even and large) and if $r = 1,2$ (see [10]).

In Section 3 we show that our Theorem 1 implies that this conjecture is true also for $[T] \in \tau(X_{n,\dots,n})$ (Corollary 2).

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1. Preliminaries

Let us start with the classical definition of the Segre varieties.

Definition 1. For all positive integers d and n_i, $1 \le i \le d$, let

$$j_{n_1,\dots,n_d} : \mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_d} \to \mathbb{P}^{N(n_1,\dots,n_d)},$$

with $N(n_1,\dots,n_d) := (\prod_{i=1}^d (n_i+1)) - 1$, denote the Segre embedding of $\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_d}$ obtained by the section of the sheaf $\mathcal{O}(1,\dots,1)$. Set $X_{n_1,\dots,n_d} := j_{n_1,\dots,n_d}(\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_d})$.

Observe that if we identify each $\mathbb{P}^{n_i}$ with $\mathbb{P}(V_i)$ for certain (n_i+1)-dimensional vector space V_i for $i = 1,\dots,d$, then an element $[T] \in X_{n_1,\dots,n_d}$ can be interpreted as the projective class of a completely decomposable tensor $T \in V_1\otimes\ldots\otimes V_d$, i.e. there exist $v_i \in V_i$ for

$i = 1, \dots, d$ such that $T = v_1 \otimes \cdots \otimes v_d$.
We can give now the definition of the rank of an element $P \in \mathbb{P}^{N(n_1,\dots,n_d)} = \mathbb{P}(V_1 \otimes \cdots \otimes V_d)$.

Definition 2. For each $P \in \mathbb{P}^{N(n_1,\dots,n_d)}$ the rank (or tensor rank) $r_{X_{n_1,\dots,n_d}}(P)$ of P is the minimal cardinality of a finite set $S \subset X_{n_1,\dots,n_d}$ such that $P \in \langle S \rangle$, where $\langle \ \ \rangle$ denote the linear span.

Notation 1. Let $\tau(X_{n_1,\dots,n_d})$ denote the tangent developable of $X_{n_1,\dots,n_d}$, i.e. the union of all tangent spaces $T_P X_{n_1,\dots,n_d}$ of $X_{n_1,\dots,n_d}$. Since $\tau(X_{n_1,\dots,n_d})$ is closed in the Zariski topology, this is equivalent to the usual definition of the tangent developable of a submanifold of a projective space as the closure of the union of all tangent spaces.

Remark 1. Observe that $X_{n_1,\dots,n_d}$ is the singular locus of $\tau(X_{n_1,\dots,n_d})$.
Now fix any $P \in \tau(X_{n_1,\dots,n_d}) \setminus X_{n_1,\dots,n_d}$. There is a unique $O \in X_{n_1,\dots,n_d}$ and a unique zero-dimensional scheme $Z \subset X_{n_1,\dots,n_d}$ such that $Z_{red} = \{O\}$, $\deg(Z) = 2$ and P is contained in the line $\langle Z \rangle$:

$$P \in T_O X_{n_1,\dots,n_d} = \langle Z \rangle \tag{1}$$

Notation 2. Let $\tilde{O} = (O_1, \dots, O_d) \in \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_d}$. With an abuse of notation we will write the point $O = j_{n_1,\dots,n_d}(\tilde{O}) \in X_{n_1,\dots n_d}$ as $O = (O_1, \dots, O_d)$.

Notation 3. Fix $O = (O_1, \dots, O_d) \in X_{n_1,\dots n_d}$ as above, we indicate with $Y_{O,i} \subset \mathbb{P}^{N(n_1,\dots,n_d)}$ the n_i-dimensional linear subspace obtained by fixing all coordinates $j \in \{1, \dots, d\} \setminus \{i\}$ equal to $O_j \in \mathbb{P}^n_i$. To be precise:

$$Y_{O,i} = j_{n_1,\dots,n_d}(O_1, \cdots, O_{i-1}, \mathbb{P}^{n_i}, O_{i+1}, \cdots, O_d).$$

Remark 2. Let $Y_{O,i} \subset \mathbb{P}^{N(n_1,\dots,n_d)}$ the n_i-dimensional linear subspace just defined. Observe that, as scheme-theoretic intersection, we have that:

$$T_O X_{n_1,\dots,n_d} \cap X_{n_1,\dots,n_d} = \cup_{i=1}^d Y_{O,i}. \tag{2}$$

Moreover, if $Z \subset X_{n_1,\dots,n_d}$ is the unique degree 2 zero-dimensional scheme such that $\langle Z \rangle = T_O X_{n_1,\dots,n_d}$ as in Remark 1, then there is a unique minimal subset $E \subseteq \{1, \dots, d\}$ such that $\langle Z \rangle \subseteq \langle \cup_{i \in E} Y_{O,i} \rangle$.
The integer $\sharp(E)$ will be called the type $\eta_{X_{n_1,\dots,n_d}}(P)$ of P:

$$\eta_{X_{n_1,\dots,n_d}}(P) := \min\{\sharp(E) \mid E \subseteq \{1, \dots, d\}, \langle Z \rangle \subseteq \langle \cup_{i \in E} Y_{O,i} \rangle\}. \tag{3}$$

Notice that $2 \le \eta_{X_{n_1,\dots,n_d}}(P) \le d$. Moreover for a general $Q \in T_O X_{n_1,\dots,n_d}$ we have that $\eta_{X_{n_1,\dots,n_d}}(Q) = d$. Not only but every integer $k \in \{2, \dots, d\}$ is the type of some point of $\tau(X_{n_1,\dots,n_d}) \setminus X_{n_1,\dots,n_d}$. Finally for all $Q \in X_{n_1,\dots,n_d}$ we write $\eta_{X_{n_1,\dots,n_d}}(Q) = 1$ and say that Q has type 1.

In Theorem 1 we will actually prove that if $P \in \tau(X_{n_1,\dots,n_d})$, then the integer $\eta_{X_{n_1,\dots,n_d}}(P)$ just introduced in (3) is actually the rank of P.
Before proving that theorem we need to introduce the notion of secant varieties and other related objects.

Definition 3. For each integer $t \ge 2$ let $\sigma_t(X_{n_1,\dots,n_d})$ denote the Zariski closure in $\mathbb{P}^{N(n_1,\dots,n_d)}$ of the union of all $(t-1)$-dimensional linear subspaces of $\mathbb{P}^{N(n_1,\dots,n_d)}$ spanned by t points of $X_{n_1,\dots,n_d}$. This object is classically known as the t-secant variety of $X_{n_1,\dots,n_d}$.

Notation 4. For each $t \ge 2$ there is a non-empty open subset of $\sigma_t(X_{n_1,\dots,n_d})$, that we indicate with $\sigma_t^0(X_{n_1,\dots,n_d})$, whose elements are points of rank exactly equal to t.

We want to focus our attention on the case $t = 2$ that is very particular. In fact it is classically known that each element of $\sigma_2(X_{n_1,\dots,n_d}) \backslash X_{n_1,\dots,n_d}$ is in the linear span of a unique zero-dimensional scheme $Z \subset X_{n_1,\dots,n_d}$ (this classical assertion uses only that $X_{n_1,\dots,n_d}$ is a smooth submanifold of a projective space; see [4], Proposition 11, and [7], Lemma 2.1.5, for much more). Hence Theorem 1 will give the complete stratification by ranks of $\sigma_2(X_{n_1,\dots,n_d})$ (see also 1). Indeed, fix $P \in \sigma_2(X_{n_1,\dots,n_d})$. Obviously $\tau(X_{n_1,\dots,n_d}) \subset \sigma_2(X_{n_1,\dots,n_d})$. If $P \notin \tau(X_{n_1,\dots,n_d})$, then $r_{X_{n_1,\dots,n_d}}(P) = 2$. If $P \in \tau(X_{n_1,\dots,n_d})$, then in Theorem 1 we will show that $r_{X_{n_1,\dots,n_d}}(P) \in \{1, \dots, d\}$. In particular for each $k \in \{1, \dots, d\}$, Theorem 1 will also imply the existence of $P \in \tau(X_{n_1,\dots,n_d})$ such that $r_{X_{n_1,\dots,n_d}}(P) = k$.

Definition 4. For any $P \in \mathbb{P}^{N(n_1,\dots,n_d)}$ the border rank, or border tensor rank, $b_{X_{n_1,\dots,n_d}}(P)$ is the minimal integer t such that $P \in \sigma_t(X_{n_1,\dots,n_d})$.

Notice that
$$b_{X_{n_1,\dots,n_d}}(P) = 1 \iff r_{X_{n_1,\dots,n_d}}(P) = 1 \iff P \in X_{n_1,\dots,n_d}.$$
Thus Theorem 1 may be considered as the description of the ranks of all points with border rank 2 (Corollary 1).

For the case of Veronese varieties, i.e. the case of symmetric tensors, and symmetric border rank 2 or 3, see [4] and references therein.

2. Proof of Theorem 1.

We use the following elementary lemma (see e.g. [2], Lemma 1).

Lemma 1. *Fix any $P \in \mathbb{P}^{N(n_1,\dots,n_d)}$ and two zero-dimensional subschemes A, B of $X_{n_1,\dots,n_d}$ such that $A \neq B$, $P \in \langle A \rangle$, $P \in \langle B \rangle$, $P \notin \langle A' \rangle$ for any $A' \subsetneq A$ and $P \notin \langle B' \rangle$ for any $B' \subsetneq B$. Then $h^1(\mathbb{P}^{N(n_1,\dots,n_d)}, \mathcal{I}_{A\cup B}(1)) > 0$.*

Lemma 2. *Fix a zero-dimensonal scheme $\tilde{W} \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_d}$. Then $h^1(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_d}, \mathcal{I}_{\tilde{W}}(1, \dots, 1)) = h^1(\mathbb{P}^{N(n_1,\dots,n_d)}, \mathcal{I}_{j_{n_1,\dots,n_d}(\tilde{W})}(1))$.*

Proof. It is sufficient to observe that $j_{n_1,\dots,n_d}$ is the linearly normal embedding induced by the complete linear system $|\mathcal{O}_{\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_d}}(1, \dots, 1)|$ and that $h^1(\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_d}, \mathcal{O}_{\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_d}}(1, \dots, 1)) = 0$. □

We are now ready to prove the main theorem of this paper.

Theorem 1. *Let $\tau(X_{n_1,\dots,n_d})$ be the tangential variety of the Segre variety $X_{n_1,\dots,n_d}$. For each $P \in \tau(X_{n_1,\dots,n_d})$ we have that the tensor rank of P is:*
$$r_{X_{n_1,\dots,n_d}}(P) = \eta_{X_{n_1,\dots,n_d}}(P)$$
where the integer $\eta_{X_{n_1,\dots,n_d}}(P)$ is the type of P defined in (3).

This result was independently proved by J. Buczyński and J. M. Landsberg (their proof was before ours).

Proof. Fix $P \in \tau(X_{n_1,\dots,n_d})$ and look for $r_{X_{n_1,\dots,n_d}}(P)$.
Since $\eta_{X_{n_1,\dots,n_d}}(P) = 1 \iff P \in X_{n_1,\dots,n_d} \iff r_{X_{n_1,\dots,n_d}}(P) = 1$, the case $P \in X_{n_1,\dots,n_d}$ is obvious. Hence we may assume $P \notin X_{n_1,\dots,n_d}$. Take $O \in X_{n_1,\dots,n_d}$ and $Z \subset X_{n_1,\dots,n_d}$ with $Z_{red} = \{O\}$, $\deg(Z) = 2$ and $P \in \langle Z \rangle$, hence, as in (1), we have that
$$P \in T_O X_{n_1,\dots,n_d} = \langle Z \rangle.$$

As we have seen above, both the point $O \in X_{n_1,\dots,n_d}$ and the degree 2 zero-dimensional scheme $Z \subset X_{n_1,\dots,n_d}$ that satisfy (1) exist, and moreover they are uniquely determined by P. Moreover we can think at $Z \subset X_{n_1,\dots,n_d}$ as

$$Z = j_{n_1,\dots,n_d}(\widetilde{Z})$$

with $\widetilde{Z} \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_d}$ and $\widetilde{Z} \cong Z$.
Now, as in Remark 2, fix $E \subseteq \{1, \dots, d\}$ such that

$$\sharp(E) = \eta_{X_{n_1,\dots,n_d}}(P)$$

and

$$P \in \langle \cup_{i\in E} Y_{O,i} \rangle$$

(where $Y_{O,i}$ are defined as in Notation 3).
Since each $Y_{O,i} \subset \mathbb{P}^{n_1,\dots,n_d}$ is a linear subspace, then for each $i \in E$ there is $Q_i \in Y_{O,i}$ such that $P \in \langle \cup_{i\in E} Q_i \rangle$. Thus

$$r_{X_{n_1,\dots,n_d}}(P) \le \eta_{X_{n_1,\dots,n_d}}(P).$$

Therefore we need simply to prove the opposite inequality.

For each $j \in \{1, \dots, d\}$ and each $Q_j \in \mathbb{P}^{n_j}$ (or, with the same abuse of notation as in Notation 2, we can think at a point Q in the Segre variety obtained as $j_{n_1\dots,n_d}(\tilde{Q})$ with $\tilde{Q} = (Q_1, \dots, Q_d) \in \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_d}$ and then write $Q = (Q_1, \dots, Q_d) \in X_{n_1,\dots,n_d}$), set:

$$X_{n_1,\dots,n_d}(Q_j, j) := \{(A_1, \dots, A_d) \in X_{n_1,\dots,n_d} : A_j = Q_j\}. \tag{4}$$

Hence $X(Q_j, j)$ is an $(n_1 + \cdots + n_d - n_j)$-dimensional product of $d-1$ projective spaces embedded as a Segre variety in a linear subspace of $\mathbb{P}^{N(n_1,\dots,n_d)}$.

Now our proof splits in two parts: in the first one ((a) together with (b)) we study the case of the Segre product of d copies of $\mathbb{P}^1$'s (i.e. we proove the theorem for $\tau(X_{1,\dots,1})$); in part (c) we generalize the result obtained for $X_{1,\dots,1}$ to the general case $X_{n_1,\dots,n_d}$ with $n_i \ge 1$, $i = 1, \dots, d$.

(a) Here we assume $n_i = 1$ for all i and $\eta_{X_{n_1,\dots,n_d}}(P) = d$.

Assume $r := r_{X_{1,\dots,1}}(P) < d$ and fix a 0-dimensional scheme $S \subset X_{n_1,\dots,n_d}$ that computes the rank r of P, i.e. fix

$$\tilde{S} \subset \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$$

such that

$$j_{n_1,\dots,n_d}(\tilde{S}) = S,\ P \in \langle S \rangle \text{ and } \sharp(j_{n_1,\dots,n_d}(S)) = r.$$

Write

$$S = \{Q_1, \dots, Q_r\}$$

and let $(Q_{i,1}, \dots, Q_{i,d})$ be the components of each $Q_i \in X_{1,\dots,1}$ with $i = 1 \dots, r$, i.e. let $\tilde{Q}_i = (Q_{i,1}, \dots, Q_{i,d}) \in \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ s.t. $j_{n_1,\dots,n_d}(\tilde{Q}_i) = Q_i$ and then, according with Notation 2, write $Q_i = (Q_{i,1}, \dots, Q_{i,d})$.
Now write

$$\tilde{O} = (O_1, \dots, O_d) \in \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$$

and

$$O = j_{n_1,\dots,n_d}(\tilde{O}).$$

Choose homogeneous coordinates on $\mathbb{P}^1$. Since $X_{1,\dots,1}$ is homogeneous, it is sufficient to prove the case $O_i = [1, 0]$ for all $i = 1, \dots, d$.
Notice that $\deg(Z \cup S) = r + 2$ if $O \notin S$ and $\deg(Z \cup S) = r + 1$ if $O \in S$.
Since S computes $r_{X_{1,\dots,1}}(P)$, we have $P \notin \langle j_{n_1,\dots,n_d}(\tilde{S}') \rangle$ for any $\tilde{S}' \subsetneq \tilde{S}$. Since $P \ne O$ and

$\{O\}$ is the only proper subscheme of $Z = T_O(X_{n_1,\dots,n_d})$, we have $P \notin \langle Z' \rangle$ for all proper subschemes Z' of Z. Since $P \in \langle Z \rangle \cap \langle S \rangle$, then, by Lemma 1, we have $h^1(\mathcal{I}_{S\cup Z}(1)) > 0$. Thus to get a contradiction and prove Theorem 1 in the case $n_i = 1$ for all $i = 1,\dots,d$ and $\eta_{X_{1,\dots,1}}(P) = d$, it is sufficient to prove $h^1(\mathcal{I}_{S\cup Z}(1)) = 0$, i.e. $h^1(\mathbb{P}^1 \times \cdots \times \mathbb{P}^1, \mathcal{I}_{\widetilde{Z}\cup\tilde{S}}(1,\dots,1)) = 0$ where, as above, $\widetilde{Z} \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_d}$ s.t. $Z = j_{n_1,\dots,n_d}(\widetilde{Z})$.
First assume the existence of an integer $j \in \{1,\dots,d\}$ such that $Q_{i,j} = [1,0]$ for all $i \in \{1,\dots,r\}$.
We get $S \subset X_{1,\dots,1}([1,0],j)$, where $X_{n_1,\dots,n_d}(Q_j,j)$ is defined in (4). Hence $P \in \langle X_{1,\dots,1}([1,0],j)\rangle$. However $T_O X_{1,\dots,1} \cap X_j = \langle \cup_{i\neq j} Y_{O,i}\rangle$. Hence $\eta(P) \le d-1$, but this is a contradiction.

Thus for each $j \in \{1,\dots,d\}$ there is $Q_{i_j} \in S$ such that $Q_{i_j,j} \neq [1,0]$.

(a1) Here we assume $O \notin \langle S \rangle$.
Since S computes $r_{X_{1,\dots,1}}(P)$, it is linearly independent, i.e. (by Lemma 2) $h^1(\mathbb{P}^1 \times \cdots \times \mathbb{P}^1, \mathcal{I}_{\tilde{S}}(1,\dots,1)) = 0$.
Since $O \notin \langle S \rangle$, we get that $\tilde{S}\cup\{\tilde{O}\}$ is linearly independent, i.e. $h^1(\mathbb{P}^1\times\cdots\times\mathbb{P}^1, \mathcal{I}_{\tilde{S}\cup\{\tilde{O}\}}(1,\dots,1)) = 0$.
We fix $i \in \{1,\dots,r\}$ such that $Q_{i,1} \neq [1,0]$ (we just saw the existence of such an integer i).
Write $S_1 := S \cap X_{1,\dots,1}(Q_{i,1},1)$, where $X_{n_1,\dots,n_d}(Q_j,j)$ is defined in (4). By construction $i \in S_1$ and hence $\sharp(S_1) \ge 1$.
Assume for now that $S_1 \neq S$ and that there exist $j \in S \setminus S_1$ such that $Q_{j,2} \neq [1,0]$. Set $S_2 := S \cap X_{1,\dots,1}(Q_{j,2})$. And so on constructing subsets $S_1,\dots,S_j$ of S such that:

- $S_j \nsubseteq \cup_{1\le i<j} S_i$,
- $Q_{k,i} \neq [1,0]$ for all $k \in S_i$,
- $S_i = S \cap X_{1,\dots,1}(Q_{h,i},i)$ for all $h \in S_i$,

until we arrive at one of the following cases:

(i) $S_1 \cup \cdots \cup S_j = S$;
(ii) $S_1 \cup \cdots \cup S_j \neq S$ and $Q_{k,j+1} = [1,0]$ for all $k \in S \setminus (S_1 \cup \cdots \cup S_j)$.

Now fix an index $m_{i+1} \in S_{i+1} \setminus S_i$, $1 \le i \le j-1$, and set

$$D_i := X_{1,\dots,1}(Q_{m_i,i}, i),\ 1 \le i \le j,$$

i.e. according with (4), $D_i := \{(A_1,\dots,A_d) \in X_{n_1,\dots,n_d} : A_i = Q_{m_i} \text{ with } m_i \in S_i \setminus S_{i-1}\}$ for $1 \le i \le j$.

First assume that (i) occurs (with j minimal).
Fix $B_i \in \mathbb{P}^1 \setminus \{[1,0]\}$, $j+1 \le i \le d-1$ and set:

- $D_i := X_{1,\dots,1}(B_i,i)$, if $j+1 \le i \le d-1$;
- $D_d := X_{1,\dots,1}(O_d,d)$;
- $D := \cup_{i=1}^d D_i$.

Notice that obviously $D \in |\mathcal{O}_{X_{1,\dots,1}}(1)|$ and also that $S \cup \{O\} \subset D$.
Moreover observe that $O \in D_i$ if and only if $i = d$.
Finally, D_d is smooth at O and $T_O D$ is spanned by $\cup_{i=1}^{d-1} X_{1,\dots,1}(O_1,i)$.
Therefore $Z \nsubseteq D$ and $Z\cup S$ imposes one more condition to $|\mathcal{O}_{\mathbb{P}^1\times\cdots\times\mathbb{P}^1}(1,\dots,1)|$ than $S\cup\{O\}$.
Since $j_{1,\dots,1}(\tilde{S} \cup \{\tilde{O}\})$ is linearly independent, we get $h^1(\mathbb{P}^1 \times \cdots \times \mathbb{P}^1, \mathcal{I}_{\widetilde{Z}\cup\tilde{S}}(1,\dots,1)) = 0$ that is a contradiction.

Now assume that (ii) occurs and set:

- $M_{j+1} = X_{1,\dots,1}([1,0],j+1)$;
- $M_h := X_{1,\dots,1}([1,0],h)$, for all $h \in \{j+2,\dots d\}$;
- $D' := \bigcup_{i=1}^j D_i \cup \bigcup_{h=j+1}^d M_h$.

Notice that $D' \in |\mathcal{O}_{X_{1,\dots,1}}(1)|$ and that $S \cup \{O\} \subset D$.
The hypersurface M_{j+1} is the unique irreducible component of D' containing O.
Since M_{j+1} is smooth at O and $T_O M_{j+1}$ is spanned by $\cup_{i\neq j+1}^{d-1} X_{1,\dots,1}(O_1,i)$, we get as above that $h^1(\mathbb{P}^1\times\cdots\times\mathbb{P}^1, \mathcal{I}_{\widetilde{Z}\cup\tilde{S}}(1,\dots,1)) = 0$, and than another contradiction.

(a2) Here we assume $O \in S$.
Hence $S\cup\{O\} = S$ and $j_{1,\dots,1}(\tilde{S}\cup\{\tilde{O}\})$ is linearly independent. Set $S' := S\setminus\{O\}$. We make the construction of step (a1) with S' instead of S, defining the subsets S_i of S' until we get an integer j such that either $S' = S_1\cup\cdots\cup S_j$ or $S_1\cup\cdots\cup S_j \neq S'$ and $Q_{j+1,i} = [1,0]$ for all $i \in S'\setminus(S_1\cup\cdots\cup S_j)$. In both cases we add the other $d-j$ hypersurfaces, exactly one of them containing O. Since $\deg(Z\cup S) = \deg(S\cup\{O\})+1$, we get $h^1(\mathbb{P}^1\times\cdots\times\mathbb{P}^1, \mathcal{I}_{\widetilde{Z}\cup\tilde{S}}(1,\dots,1)) = 0$ as in step (a1) and hence we get a contradiction.

(a3) Here assume $O \notin S$ and $O \in \langle S\rangle$.
Hence $\langle Z\rangle \subset \langle S\rangle$. Thus there is $S' \subset S$ such that $\sharp(S') = \sharp(S)-1$ and $\langle S'\cup\{O\}\rangle = \langle S\rangle$. Hence the set $S_1 := S'\cup\{O\}$ computes $r_{X_{1,\dots,1}}(P)$. Apply step (a2) to the set S_1.

(b) Here we assume $n_i = 1$ for all i and $r := \eta_{X_{1,\dots,1}}(P) < d$.

Let $E \subset \{1,\dots,d\}$ be the minimal subset such that $P \in \langle \cup_{i\in E} Y_{O,i}\rangle$. By the definition of the type $\eta_{X_{1,\dots,1}}(P)$ of P we have $\sharp(E) = \eta_{X_{1,\dots,1}}(P)$. Set $X' := \{(U_1,\dots,U_d) \in X_{1,\dots,1} : U_i = [1,0] \text{ for all } i \notin E\}$. We identify X' with a Segre product of r copies of $\mathbb{P}^1$. Obviously $\eta_{X'}(P) = \eta_{X_{1,\dots,1}}(P)$. By step (a) we have $r_{X'}(P) = \eta_{X'}(P)$. We have $r_{X_{1,\dots,1}}(P) = r_{X'}(P)$ by the concision property of tensors ([8], Corollary 2.2, or [13], Proposition 3.1.4.1).

(c) Here we assume $n_i \geq 2$ for some i.

Since $P \in \langle \cup_{i=1}^d Y_{O,i}\rangle$, there is $U_i \in Y_{O,i}$ such that $P \in \langle\{U_1,\dots,U_d\}\rangle$. Let $U_i^i \in \mathbb{P}^{n_i}$ be the i-th component of U_i. The line $L_i \subseteq \mathbb{P}^{n_i}$ is the line spanned by O_i and U_i^i. We have $P \in \langle \prod_{i=1}^d L_i\rangle$ and $\eta_{X_{n_1,\dots,n_d}}(P) = \eta_{X_{1,\dots,1}}(P)$, where we identify $j_{n_1,\dots,n_d}(\prod_{i=1}^d L_i)$ with the Segre variety $X_{1,\dots,1}$. By parts (a) and (b) we have $r_{X_{1,\dots,1}}(P) = \eta_{X_{1,\dots,1}}(P)$. We have $r_{X_{n_1,\dots,n_d}}(P) = r_{X_{1,\dots,1}}(P)$ by the concision property of tensors ([8], Corollary 2.2, or [13], Proposition 3.1.4.1). □

Corollary 1. *Let $P \in \sigma_2(X_{n_1,\dots,n_d})$, then:*

- $r_{X_{n_1,\dots,n_d}} = 1$ *iff* $P \in X_{n_1,\dots,n_d}$*;*
- $r_{X_{n_1,\dots,n_d}} = 2$ *iff either* $P \in \sigma_2(X_{n_1,\dots,n_d})\setminus\tau(X_{n_1,\dots,n_d})$ *or there exist* $O \in X_{n_1,\dots,n_d}$*,* $O \neq P$*, and* $Y_{O,i}, Y_{O,j} \subset \mathbb{P}^{N(n_1,\dots,n_d)f}$ *as in Notation 3, such that* $P \in T_O(X_{n_1,\dots,n_k}) \subset Y_{O,i}\cup Y_{O,j}$ *for certain* $i \neq j \in \{1,\dots,d\}$*;*
- $r_{X_{n_1,\dots,n_d}} = k$ *with* $3 \leq k \leq d$ *iff* k *is the minimum integer s.t. there exist* $Y_{O,i_1},\dots,Y_{O,i_k} \subset \mathbb{P}^{N(n_1,\dots,n_d)}$ *as in Notation 3, such that* $P \in T_O(X_{n_1,\dots,n_k}) \subset \cup_{j=1,\dots k} Y_{O,i_j}$ *for certain* $i_j \in \{1,\dots,d\}$*,* $j = 1,\dots,k$*.*

Proof. This corollary follows straightforward from Theorem 1 and the fact that $\sigma_2(X_{n_1,\dots,n_d})\setminus \tau(X_{n_1,\dots,n_d}) = \sigma_2^0(X_{n_1,\dots,n_d})$ when it is not empty. □

3. On Comon's conjecture

In this section we want to relate the result obtained in Theorem 1 to the Comon's conjecture stated in the Introduction.

Let $\nu_d(\mathbb{P}^n)$ be the classical Veronese embedding of $\mathbb{P}^n$ into $\mathbb{P}^{\binom{n+d}{d}-1}$ via the sections of the sheaf $\mathcal{O}(d)$. As pointed out in the introduction if $\mathbb{P}^n \simeq \mathbb{P}(V)$ with V an $(n+1)$-dimensional vector space, then $\nu_d(\mathbb{P}^n) \subset \mathbb{P}(S^dV)$ can be interpreted as the variety that parameterizes projective classes of completely decomposable symmetric tensors $T \in S^dV$. Moreover

$$\nu_d(\mathbb{P}^n) = X_{n,\dots,n} \cap \mathbb{P}(S^dV) \subset \mathbb{P}(V^{\otimes d}).$$

Definition 5. Let $P \in \mathbb{P}(S^dV)$ be a projective class of a symmetric tensor. We define the symmetric rank $r_{\nu_d(\mathbb{P}^n)}(P)$ of P as the minimum number of r of points $P_i \in \nu_d(\mathbb{P}^n)$ whose linear span contains P.

With this definition, **Comon's conjecture** (Conjecture 1) can be rephrased as follows:

$$\text{if } P \in \mathbb{P}(S^dV) \text{ then } r_{\nu_d(\mathbb{P}^n)}(P) = r_{X_{n,\dots,n}}(P).$$

Obviously $r_{X_{n,\dots,n}}(P) \le r_{\nu_d(\mathbb{P}^n)}(P)$. In [10] the authors prove the reverse inequality for a general d-tensor (d even and large) with rank at most n (Proposition 5.3) and for $r_{X_{n,\dots,n}}(P) = 1, 2$.

With Theorem 1 we can prove that conjecture for all symmetric tensors of border rank 2.

Corollary 2. *Let $P \in \sigma_2(\nu_d(\mathbb{P}^n))$. Then $r_{\nu_d(\mathbb{P}^n)}(P) = r_{X_{n,\dots,n}}(P)$.*

Proof. For any projective variety X we can observe that $\sigma_2(X) = X \cup \tau(X) \cup \sigma_2^0(X)$.
If $P \in \nu_d(\mathbb{P}^n) \subset X_{n,\dots,n}$ then there exist $v \in V$ such that $P = [v^{\otimes d}] \in \nu_d(\mathbb{P}^n) \subset X_{n,\dots,n}$, therefore obviously $r_{X_{n,\dots,n}}(P) = r_{\nu_d(\mathbb{P}^n)}(P) = 1$.

If $P \in \sigma_2^0(\nu_d(\mathbb{P}^n))$ then $r_{\nu_d(\mathbb{P}^n)}(P) = 2$, that implies that $r_{X_{n,\dots,n}}(P) \le 2$, and therefore by [10], that we have that $r_{X_{n,\dots,n}}(P) = r_{\nu_d(\mathbb{P}^n)}(P) = 2$.

Now assume that $P \in \tau(\nu_d(\mathbb{P}^n)) \setminus \nu_d(\mathbb{P}^n)$ and that $\sigma_2(\nu_d(\mathbb{P}^n)) \ne \tau(\nu_d(\mathbb{P}^n))$. For such a P we know that $r_{\nu_d(\mathbb{P}^n)}(P) = d$ (see [19], [9], [5], [4]). Any point $P \in \tau(\nu_d(\mathbb{P}^n)) \setminus \nu_d(\mathbb{P}^n)$ can be thought as the projective class of a homogeneous degree d polynomial in $n+1$ variables for which there exist two linear forms L, M in $n+1$ variables such that $P = [L^{d-1}M]$; hence d is the minimum integer k such that $P \in \langle \nu_k(\mathbb{P}^n) \rangle$. Therefore $\eta_{X_{n,\dots,n}}(P) = d$. Since obviously $\tau(\nu_d(\mathbb{P}^n)) \subset \tau(X_{n,\dots,n})$ we have that, by Theorem 1, $r_{X_{n,\dots,n}}(P) = \eta_{X_{n,\dots,n}}(P)$. $\square$

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