The hyperelliptic integrals and π

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Abstract

A class of hyperelliptic integrals are expressed through hypergeometric functions, like those of Gauss, Lauricella and Appell, namely multiple power series. Whenever they can on their own be reduced to elliptic integrals through an algebraic transformation, we obtain a two-fold representation of the same mathematical object, and then several completely new π determinations through the above special functions and/or Euler integrals. All our π formulae have been succesfully tested by means of convenient Mathematica®'s packages and enter in a wide historical/sound context of π- formulae quite far from being exhausted. Due to their structure, the formulae's practical value does not lie in computing π, but in allowing, through π, a benchmark for computing the involved special functions, particularly those less elementary.

KEYWORD: Complete Elliptic Integral of first kind, Hypergeometric Function, π, Appell Function, Lauricella-Saran Function

NOTE: this is the post print version of the paper appeaered on Journal of Number Theory **129** (2009) 3094–3108 doi:10.1016/j.jnt.2009.06.002

1 π and the integrals: the literary background and our aim

This paper originates from the need, common to pure and applied mathematics, to compute hyperelliptic integrals in a simple fashion, avoiding Riemann two-variable Theta functions. In both fields the Exton book, [9] pages 264-266, and some papers of the cosmologist Kraniotis, [16], served as references. The latter explicitly computed several hyperelliptic integrals by applying Appell and Lauricella functions; following these tracks, we succeeded in computing further integrals. They will be tackled in our next specific papers covering the single applications. The outcome was that some identities have been established providing π through elliptic integrals and hypergeometric functions as ${}_2F_1$, F_1, $F_D^{(3)}$. Some of them are not new, like the Legendre relationship, see [24] page 524, deduced from formula (13). But in other cases, relations have been obtained as in (19), (20), (21), (22), (27), (28), (35), (38), which were not known up to now.

One of the beauties of mathematics is that it promises us eternal truths that do not depend on opinions or trends, and in doing so, the most beautiful solution is often the simplest one, but not necessarily the most obvious, as various methods are capable of leading to the same truth, as by Sir Michael Atiyah declared in an interview [21]:

> *Any good theorem should have several proofs, the more the better. For two reasons: usually, different proofs have different strengths and weaknesses, and they generalize in different directions: they are not just repetitions of each other.*

Moreover, mathematical works can take the form of a surprising connection between two different areas of mathematics previously appearently unrelated. A remarkable example is the link (Euler, 1735) between two of the most organic concepts in mathematics: the series of integers $1, 2, 3, 4, \ldots$ and the irrational number π:

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots=\frac{\pi^2}{6}.$$

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As a matter of fact, π, whose irrationality was proved by Lambert, 1761, is a constant subject quite far from being exhausted. We consider both the previous and the so called Ivory (1671)-Leibniz (1674) odd series equally interesting:

$$\sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4};$$

but the reader will judge. A piece of recent research, see [4], was carried out by Bailey-Borwein-Plouffe, hereinafter referred as BBP:

$$\pi = \sum_{n=0}^{\infty} \frac{1}{16^n} \left(\frac{4}{8n+1} - \frac{2}{8n+4} - \frac{1}{8n+5} - \frac{1}{8n+6} \right). \tag{1}$$

Such a formula (1) was discovered using the so called *Ferguson's PSLQ integer relation finding algorithm* [5], [14]. As shown in [4] formula (1) allows the calculation of the n-th hexadecimal or binary digit of π, without computing any of the first $n-1$ digits, by means of a simple algorithm. A very good analysis of how the BBP formulae appear directly from Calculus is given in the textbook [15]. The relevant comments held in [1], [2] also prove to be interesting. Formula (1) together with many other, always arises by computing a definite integral through two different paths: for instance (1) comes from:

$$\pi = \int_0^{1/\sqrt{2}} \frac{4\sqrt{2} - 8x^3 - 4\sqrt{2}\,x^4 - 8x^5}{1-x^8}\,\mathrm{d}x$$

and from the identity, [4] page 905,

$$\int_0^{1/\sqrt{2}} \frac{x^{n-1}}{1-x^8}\,\mathrm{d}x = \frac{1}{2^{n/2}} \sum_{k=0}^{\infty} \frac{1}{16^k(8k+n)},$$

obtained by developing a geometrical series. Another recent contribution, see [13], presents π as an infinite series of powers of the reciprocal of the Golden Ratio $\phi = \frac{1}{2}(1+\sqrt{5})$, following techniques of integral calculus similar to [4]. Almkvist, Krattenthaler, and Petersson, [2] presented new sums for π involving binomial coefficients and stemming from the Gosper formula (1974)

$$\pi = \sum_{n=0}^{\infty} \frac{50n-6}{2^n \binom{3n}{n}}, \tag{2}$$

so that they succeed in providing formulae like this:

$$\pi = \sum_{n=0}^{\infty} \frac{S(n)}{a^n \binom{m\,n}{p\,n}} \tag{3}$$

where $S(n)$ is a polynomial in n computed by means of the LLL algorithm, [18]. They used the so-called Beta method which is based on Beta function $B(p,q)$ for positive integer values p and q. The recent contributions of Gourétovich and Guillera, [11], presents a variant of the Beta method, which builds other series, similar to (3), e.g.:

$$\pi = \frac{1}{16807} \sum_{n=0}^{\infty} \frac{x_n}{2^n \binom{7n}{2n}}, \tag{4}$$

where:

$$x_n = \left(\frac{59296}{7n+1} - \frac{10326}{7n+2} - \frac{3200}{7n+3} - \frac{1352}{7n+4} - \frac{792}{7n+5} + \frac{552}{7n+6} \right).$$

However equations, (2), (3) (4) also result from an integral representation of a series.

Throughout this paper new formulae will be provided connecting π to elliptic, hyperelliptic and hypergeometric integrals. Classic formulae have defined π as half area of the unit disk or half circumference of the unit circle:

$$\int_{-1}^{1} \sqrt{1-x^2}\,\mathrm{d}x = \frac{\pi}{2}, \quad \int_{-1}^{1} \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \pi.$$

Other suggestive and well-known $\pi/2$ expressions can be:

$$\frac{\pi}{2}\,{}_2F_1\left(\frac{1}{2};\frac{1}{2};1,k^2\right)=\boldsymbol{K}(k) \tag{5}$$

$$\frac{\pi}{2}\,{}_2F_1\left(-\frac{1}{2};\frac{1}{2};1,k^2\right)=\boldsymbol{E}(k) \tag{6}$$

where $\boldsymbol{K}(k)$ and $\boldsymbol{E}(k)$ are respectively the first and second kind complete elliptic integral of module $0<k<1$:

$$\boldsymbol{K}(k)=\int_0^1\frac{\mathrm{d}x}{\sqrt{(1-x^2)(1-k^2x^2)}},\quad \boldsymbol{E}(k)=\int_0^1\sqrt{\frac{1-k^2x^2}{1-x^2}}\,\mathrm{d}x,$$

and ${}_2F_1$ is the Gauss hypergeometric series where $|x|<1$:

$${}_2F_1(a,b;c;x)=\sum_{n=0}^{\infty}\frac{(a)_n\,(b)_n}{(c)_n}\frac{x^n}{n!},$$

where $(a)_k$ is a Pochhammer symbol: $(a)_k=a(a+1)\cdots(a+k-1)$. For ${}_2F_1$ a powerful integral representation theorem, hereinafter referred as IRT, is available:

$${}_2F_1\left(a,b;c;x\right)=\frac{\Gamma(c)}{\Gamma(c-a)\Gamma(a)}\int_0^1\frac{t^{a-1}(1-t)^{c-a-1}}{(1-xt)^b}\,\mathrm{d}t,$$

whose validity ranges are: $\operatorname{Re}c>\operatorname{Re}a>0$, $|x|<1$. The IRT provides the way for extending the region where the (complex) hypergeometric function is defined, namely for its analytical continuation to the (almost) whole complex plane excluding the half line $]1,\infty[$.

First of all we prove (5) and (6) as a useful introduction to our incoming theorems. We know that, see [6], n. 233.00, page 72:

$$\int_c^b\frac{\mathrm{d}t}{\sqrt{(a-t)(b-t)(t-c)}}=\frac{2}{\sqrt{a-c}}\,\boldsymbol{K}\left(\sqrt{\frac{b-c}{a-c}}\right) \tag{7}$$

where $c<b<a$. By carrying out the transformation $t=c+(b-c)u$ in (7), we obtain:

$$a-t=(a-c)\left[1-\frac{b-c}{a-c}\,u\right],\quad b-t=(b-c)(1-u),\quad t-c=(b-c)u,$$

and consequently, via the IRT:

$$\int_c^b\frac{\mathrm{d}t}{\sqrt{(a-t)(b-t)(t-c)}}=\frac{\Gamma^2(1/2)}{\Gamma(1)}\,\frac{1}{\sqrt{a-c}}\,{}_2F_1\left(\frac{1}{2};\frac{1}{2};1,\frac{b-c}{a-c}\right). \tag{8}$$

But $\Gamma^2(1/2)=\pi$, there-after, by posing $k^2=(b-c)/(a-c)$ we obtain (5). Equation (6) can be analogously proved starting from the integral [6] 233.01 page 72:

$$\int_c^b\sqrt{\frac{a-t}{(b-t)(t-c)}}\,\mathrm{d}t=2\sqrt{a-c}\,\boldsymbol{E}\left(\sqrt{\frac{b-c}{a-c}}\right) \tag{9}$$

where $c<b<a$, using the same variables transformation as for obtaining the formula for the first kind elliptic integral.

Our method is then founded on a double evaluation of the same elliptic, or which can be turned to elliptic, integral: the first consists of the usual reduction to its standard form, see Byrd Friedman, [6]. The second manages hypergeometric functions and therefore is deeply different from strategies like BBP, which applies integration by series. Before introducing our original formulae, it is important to demonstrate how our approach can operate in finding a well-known identity introduced by Legendre. Starting from the complete Beta function definition:

$$B(p,q)=\int_0^1 t^{p-1}(1-t)^{q-1}\,\mathrm{d}t=\frac{\Gamma\left(p\right)\Gamma\left(q\right)}{\Gamma\left(p+q\right)},$$

where $p,\, q > 0$, we have, see [10] 3.249.7 page 321, some remarkable formulae, like:

$$\int_0^1 (1-x^a)^{b-1}\,\mathrm{d}x = \frac{1}{a}B\left(\frac{1}{a}, b\right), \tag{10}$$

allowing the integration of a class of binomial hyperelliptic differentials for any $n \in \mathbb{N}$:

$$\int_0^1 \frac{\mathrm{d}t}{\sqrt{1-t^n}} = \frac{\Gamma\left(1+\frac{1}{n}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{n}\right)}\sqrt{\pi}. \tag{11}$$

The hyperelliptic integral on the left hand side of (11), through a suitable change of variable, is reduced to elliptic ($n = 3,\, 4,\, 6$), so that we obtain an expression which can be solved to π. In fact the following identities hold:

Theorem 1.1.

$$\pi = \frac{1}{\sqrt{3}}\left[\frac{\Gamma(5/6)}{\Gamma(4/3)} F\left(\arccos\left(2-\sqrt{3}\right), \sqrt{\frac{2+\sqrt{3}}{4}}\right)\right]^2 \tag{12}$$

$$\pi = \frac{1}{2}\left[\frac{\Gamma(3/4)}{\Gamma(5/4)}\boldsymbol{K}\left(\frac{1}{\sqrt{2}}\right)\right]^2 \tag{13}$$

$$\pi = \frac{1}{\sqrt{3}}\left[\frac{\Gamma(2/3)}{\Gamma(7/6)}\boldsymbol{K}\left(\sqrt{\frac{2-\sqrt{3}}{4}}\right)\right]^2 \tag{14}$$

where

$$F(\varphi, k) = \int_0^{\varphi} \frac{\mathrm{d}\theta}{\sqrt{1-k^2\sin^2\theta}} = \int_0^{\sin\varphi} \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-k^2t^2)}}.$$

Proof. Using formulae n. 244.00 page 92; n. 213.00 page 48; n. 576.00 page 256 of [6] three integrals are evaluated:

$$\int_0^1 \frac{\mathrm{d}t}{\sqrt{1-t^3}} = \frac{1}{\sqrt[4]{3}} F\left(\arccos\left(2-\sqrt{3}\right), \sqrt{\frac{2+\sqrt{3}}{4}}\right)$$

$$\int_0^1 \frac{\mathrm{d}t}{\sqrt{1-t^4}} = \frac{1}{\sqrt{2}}\boldsymbol{K}\left(\frac{1}{\sqrt{2}}\right)$$

$$\int_0^1 \frac{\mathrm{d}t}{\sqrt{1-t^6}} = \frac{1}{\sqrt[4]{3}}\boldsymbol{K}\left(\sqrt{\frac{2-\sqrt{3}}{4}}\right)$$

afterwards, evaluating the same integrals through (11) with $n = 3,\, 4,\, 6$ respectively then formulae (12), (13), (14) follow immediately by equating. □

It is worth noting that by formula (12) it is possible to obtain the Legendre relationship, see [24] page 524.

Corollary 1.1.1.

$$\boldsymbol{K}\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{4\sqrt{\pi}}\left(\Gamma\left(\frac{1}{4}\right)\right)^2.$$

Proof. Recalling the Gamma "one-half formula":

$$\Gamma\left(a+\tfrac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2a-1}}\frac{\Gamma(2a)}{\Gamma(a)} \tag{15}$$

and minding that $\Gamma\left(\frac{3}{4}\right) = \Gamma\left(\frac{1}{4}+\frac{1}{2}\right), \quad \Gamma\left(\frac{5}{4}\right) = \Gamma\left(1+\frac{1}{4}\right)$, applying (15), we get:

$$\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{5}{4}\right)} = \frac{4\pi\sqrt{2}}{\left(\Gamma\left(\frac{1}{4}\right)\right)^2}. \tag{16}$$

Inserting in (12) the (16), the Legendre relationship is soon obtained:

$$\pi = \frac{1}{2}\left[\frac{4\pi\sqrt{2}}{\left(\Gamma\left(\frac{1}{4}\right)\right)^2}\boldsymbol{K}\left(\frac{1}{\sqrt{2}}\right)\right]^2.$$

□

Let us remark that the proof given in [24], is founded on the hyperelliptic integral

$$\int_0^1 \frac{\mathrm{d}t}{\sqrt{1-t^4}}$$

evaluated through the Beta function. On the other side [24] makes use of the lemniscatic integrals. We follow a different approach: changing variables, see [6], page 48: $t = \mathrm{cn}(u|\frac{1}{\sqrt{2}})$, we get:

$$\int_0^1 \frac{\mathrm{d}t}{\sqrt{1-t^4}} = \frac{1}{\sqrt{2}}\int_0^{\boldsymbol{K}\left(\frac{1}{\sqrt{2}}\right)} \frac{\mathrm{dn}(u|\frac{1}{\sqrt{2}})}{\sqrt{1+\mathrm{cn}^2(u|\frac{1}{\sqrt{2}})}}\mathrm{d}u$$

and the thesis follows thanks to identity:

$$1+\mathrm{cn}^2(u|\tfrac{1}{\sqrt{2}}) = 2\mathrm{dn}^2(u|\tfrac{1}{\sqrt{2}}).$$

where $\mathrm{cn}\,(\cdot|k)$ and $\mathrm{dn}\,(\cdot|k)$ are Jacobi elliptic functions of module k. The method followed in theorem 1.1 can be used even when the hyperelliptic differentials are much more difficult: in such a case we will apply the less known Lauricella hypergeometric functions.

By transforming some elliptic integrals in hypergeometric integrals, as in [7], we will obtain newly-coined relationships on π involving elliptic integrals and the gaussian ${}_2F_1$, and some linking π to $\boldsymbol{K}$ and to the Appell hypergeometric function [3]. The hypergeometric method to π is also demostrated in [8]. In section 4, different and more intricate hyperelliptic integrals will be reduced to the Lauricella-Saran hypergeometric functions, [17, 22], following [9], section 8.4 pages 264-266, and [16], pages 1791-1792. Whenever the above hyperelliptic integrals can be reduced to the first kind complete elliptic integral $\boldsymbol{K}(k)$, via a suitable transformation, e.g. Jacobi's transformation considered further in the paper, we obtain a π-formula involving both the Lauricella function $F_D^{(3)}$ and $\boldsymbol{K}(k)$ itself.

For reader's convenience we recall the definition and the main properties of the special function used in the following sections.

1.1 Appell function

The two-variable Appell hypergeometric function, see [3], for $|x| < 1$, $|y| < 1$ is defined as a double x, y-power series:

$$F_1\left(a, b_1, b_2; c; x, y\right) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b_1)_m(b_2)_n}{(c)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!},$$

whose IRT is:

$$F_1\left(a, b_1, b_2; c; x, y\right) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 \frac{u^{a-1}\left(1-u\right)^{c-a-1}}{\left(1-ux\right)^{b_1}\left(1-uy\right)^{b_2}}\,\mathrm{d}u, \tag{17}$$

with $\mathrm{Re}\,a > 0$, $\mathrm{Re}(c-a) > 0$. It should be observed that F_1 and its IRT can be fruitfully used for treating very intricate integrals of algebraic functions, e.g. the square root of a fourth degree polynomial divided to another quartic, see e.g. [19] page 10, where the following integration is established:

$$\int_\alpha^\beta \frac{\sqrt{1+x^4}}{px^4-q}\,\mathrm{d}x = \frac{1}{q}\left\{\alpha\, F_1\left(\frac{1}{4}, 1, -\frac{1}{2}; \frac{5}{4}; -\alpha^4, \frac{p}{q}\alpha^4\right) - \beta\, F_1\left(\frac{1}{4}, 1, -\frac{1}{2}; \frac{5}{4}; -\beta^4, \frac{p}{q}\beta^4\right)\right\},$$

with p, q real numbers such that $pq > 0$ and the real root of the equation $px^4 - q = 0$ lies outside the interval $[\alpha, \beta]$.

1.2 Lauricella function

The hypergeometric Lauricella function $F_D^{(n)}$ of $n \in \mathbb{N}^+$ variables, [17, 22], is defined as:

$$F_D^{(n)}\left(a, b_1, \ldots, b_n; c; x_1, \ldots, x_n\right) := \sum_{m_1,\ldots,m_n \in \mathbb{N}} \frac{(a)_{m_1+\cdots+m_n}(b_1)_{m_1}\cdots(b_n)_{m_n}}{(c)_{m_1+\cdots+m_n} m_1! \cdots m_n!} x_1^{m_1}\cdots x_n^{m_m}$$

where $(x)_k$ is again the Pochhammer symbol, and with the hypergeometric series usual convergence requirements $|x_1| < 1, \ldots, |x_n| < 1$. If $\operatorname{Re} c > \operatorname{Re} a > 0$, the relevant IRT provides:

$$F_D^{(n)}\left(a, b_1, \ldots, b_n; c; x_1, \ldots, x_n\right) = \frac{\Gamma(c)}{\Gamma(a)\,\Gamma(c-a)} \int_0^1 \frac{u^{a-1}(1-u)^{c-a-1}\,\mathrm{d}u}{(1-x_1 u)^{b_1}\cdots(1-x_n u)^{b_n}} \tag{18}$$

allowing the analytic continuation to $\mathbb{C}^n$ deprived of the cartesian n-dimensional product of the interval $]1,\infty[$ with itself.

2 π and the Gauss function ${}_2F_1$

In this section we provide some new formulae linking π, like done by (5), to the first and the second kind complete elliptic integrals of module k, and to the Gauss hypergeometric function ${}_2F_1\left(1/2, 3/2; 2; \xi\right)$ or ${}_2F_1\left(1/2, 1/2; 2; \xi\right)$, being k and ξ algebraic functions of a triple of given numbers. Such formulae will be found by computing through the hypergeometric way the elliptic integrals one can find, for instance, in [6] at entries 237.05 page 82, 233.04 page 72, 236.03 page 79, 233.03 page 82. All this is ruled by:

Theorem 2.1. *For each triple of real numbers $c < b < a$, one can compute π through whichever of the following formulae involving $\boldsymbol{K}$, $\boldsymbol{E}$, ${}_2F_1$:*

$$\pi = \frac{4}{b-c}\,\frac{(a-c)\boldsymbol{E}\left(\sqrt{\dfrac{b-c}{a-c}}\right) + (b-a)\boldsymbol{K}\left(\sqrt{\dfrac{b-c}{a-c}}\right)}{{}_2F_1\left(\dfrac{1}{2}, \dfrac{1}{2}; 2; \dfrac{b-c}{a-c}\right)} \tag{19}$$

$$\pi = \frac{4(a-c)}{b-c}\,\frac{\boldsymbol{K}\left(\sqrt{\dfrac{b-c}{a-c}}\right) - \boldsymbol{E}\left(\sqrt{\dfrac{b-c}{a-c}}\right)}{{}_2F_1\left(\dfrac{1}{2}, \dfrac{3}{2}; 2; \dfrac{b-c}{a-c}\right)} \tag{20}$$

$$\pi = \frac{4\sqrt{(a-c)(b-c)}}{a-b}\,\frac{\boldsymbol{K}\left(\sqrt{\dfrac{a-b}{a-c}}\right) - \boldsymbol{E}\left(\sqrt{\dfrac{a-b}{a-c}}\right)}{{}_2F_1\left(\dfrac{1}{2}, \dfrac{1}{2}; 2; \dfrac{a-b}{c-b}\right)} \tag{21}$$

$$\pi = \frac{4}{a-b}\sqrt{\frac{b-c}{a-c}}\,\frac{(a-c)\boldsymbol{E}\left(\sqrt{\dfrac{a-b}{a-c}}\right) + (c-b)\boldsymbol{K}\left(\sqrt{\dfrac{a-b}{a-c}}\right)}{{}_2F_1\left(\dfrac{1}{2}, \dfrac{3}{2}; 2; \dfrac{b-a}{b-c}\right)} \tag{22}$$

Proof. Identities (19) and (20) come down as follows. Using the IRT for ${}_2F_1$, and passing from x to u through $x = c + (b-c)u$, we find:

$$\int_c^b \sqrt{\frac{b-x}{(a-x)(x-c)}}\,\mathrm{d}x = \frac{\pi(b-c)}{2\sqrt{a-c}}\,{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 2; \frac{b-c}{a-c}\right). \tag{23}$$

On the other hand, see [6], n. 233.04 page 72:

$$\int_c^b \sqrt{\frac{b-x}{(a-x)(x-c)}}\,\mathrm{d}x = 2\sqrt{a-c}\,\boldsymbol{E}\left(\sqrt{\frac{b-c}{a-c}}\right) - \frac{2(a-b)}{\sqrt{a-c}}\boldsymbol{K}\left(\sqrt{\frac{b-c}{a-c}}\right). \tag{24}$$

Thesis (19) follows by equating (23) and (24) and then obtaining π. The other thesis (20) can be obtained starting analogously from integral 233.03 page 72 of [6]:

$$\int_c^b \sqrt{\frac{x-c}{(a-x)(b-x)}}\,\mathrm{d}x = 2\sqrt{a-c}\left\{\boldsymbol{K}\left(\sqrt{\frac{b-c}{a-c}}\right) - \boldsymbol{E}\left(\sqrt{\frac{b-c}{a-c}}\right)\right\}. \tag{25}$$

The same transformation $x \leftrightarrow u$ and the IRT for ${}_2F_1$ will provide:

$$\int_c^b \sqrt{\frac{x-c}{(a-x)(b-x)}}\,\mathrm{d}x = \frac{\pi(b-c)}{2\sqrt{a-c}}\,{}_2F_1\left(\frac{1}{2},\frac{3}{2};2;\frac{b-c}{a-c}\right), \tag{26}$$

so that, equating (26) and (25), formula (20) is immediate. The identity (21) is founded on the elliptic integral evaluation 237.05, [6] page 82, when compared to its hypergeometric counterpart:

$$\int_b^a \sqrt{\frac{a-x}{(x-b)(x-c)}}\,\mathrm{d}x = 2\sqrt{a-c}\left\{\boldsymbol{K}\left(\sqrt{\frac{a-b}{a-c}}\right) - \boldsymbol{E}\left(\sqrt{\frac{a-b}{a-c}}\right)\right\}$$

$$\int_b^a \sqrt{\frac{a-x}{(x-b)(x-c)}}\,\mathrm{d}x = \frac{\pi(a-b)}{2\sqrt{b-c}}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};2;\frac{a-b}{c-b}\right).$$

The same for (22), with reference to integral 236.03 page 79 of [6]:

$$\int_b^a \sqrt{\frac{x-b}{(a-x)(x-c)}}\,\mathrm{d}x = 2\sqrt{a-c}\,\boldsymbol{E}\left(\sqrt{\frac{a-b}{a-c}}\right) - \frac{2(b-c)}{\sqrt{a-c}}\,\boldsymbol{K}\left(\sqrt{\frac{a-b}{a-c}}\right)$$

$$\int_b^a \sqrt{\frac{x-b}{(a-x)(x-c)}}\,\mathrm{d}x = \frac{\pi(a-b)}{2\sqrt{b-c}}\,{}_2F_1\left(\frac{1}{2},\frac{3}{2};2;\frac{a-b}{c-b}\right).$$

□

3 π and the Appell function F_1

Our next formulae are founded on entries 252.00 page 103 and 256.00 page 120 of [6]. The link of F_1 to π comes from the:

Theorem 3.1. *For each set of four real numbers, say $d < c < b < a$ the following identities hold:*

$$\pi = 2\sqrt{\frac{a-d}{a-c}}\,\frac{\boldsymbol{K}\left(\sqrt{\frac{(a-b)(c-d)}{(a-c)(b-d)}}\right)}{F_1\left(\frac{1}{2};\frac{1}{2},\frac{1}{2};1;\frac{c-d}{a-d},\frac{c-d}{b-d}\right)}, \tag{27}$$

$$\pi = 2\sqrt{\frac{b-c}{a-c}}\,\frac{\boldsymbol{K}\left(\sqrt{\frac{(a-b)(c-d)}{(a-c)(b-d)}}\right)}{F_1\left(\frac{1}{2};\frac{1}{2},\frac{1}{2};1;\frac{a-b}{c-b},\frac{a-b}{d-b}\right)}. \tag{28}$$

Proof. Changing the variable from x to u, $x = d + (c-d)u$, we are led, using (17), to the hypergeometric identity:

$$\int_d^c \frac{\mathrm{d}x}{\sqrt{(a-x)(b-x)(c-x)(x-d)}} = \frac{\pi}{\sqrt{(a-d)(b-d)}}\,F_1\left(\frac{1}{2};\frac{1}{2},\frac{1}{2};1;\frac{c-d}{a-d},\frac{c-d}{b-d}\right) \tag{29}$$

but the elliptic counterpart gives:

$$\int_d^c \frac{\mathrm{d}x}{\sqrt{(a-x)(b-x)(c-x)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}}\,\boldsymbol{K}\left(\sqrt{\frac{(a-b)(c-d)}{(a-c)(b-d)}}\right). \tag{30}$$

Thesis (27) follows getting π by (29) and (30). All the same, thesis (28) follows after considering the chain:

$$\begin{aligned}&\int_b^a \frac{\mathrm{d}x}{\sqrt{(a-x)(b-x)(c-x)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}}\,\boldsymbol{K}\left(\sqrt{\frac{(a-b)(c-d)}{(a-c)(b-d)}}\right)\\ &= \frac{\pi}{\sqrt{(b-c)(b-d)}}\,F_1\left(\frac{1}{2};\frac{1}{2},\frac{1}{2};1;\frac{a-b}{c-b},\frac{a-b}{d-b}\right).\end{aligned}$$

□

4 Hyperelliptic integrals, Lauricella function $F_D^{(3)}$ and π

Even if hyperelliptic integrals appear in many branches of applied mathematics, they are seldom computed via explicit formulae due to severe obstacles encountered along the way. Nevertheless some particular hyperelliptic integrals can be reduced to hypergeometric functions. For instance in [20], a series integration leads to:

$$\int_1^u \frac{\mathrm{d}\tau}{\sqrt{\tau(\tau^4-1)}} = \frac{\sqrt{\pi}}{4}\frac{\Gamma(3/8)}{\Gamma(7/8)} - \frac{2}{3u\sqrt{u}}\,{}_2F_1\left(\frac{3}{8},\frac{1}{2};\frac{11}{8};\frac{1}{u^4}\right),$$

true if $u > 1$. The above cannot be considered a reduction formula in the strict sense of the term: in fact *reduction* passes through a convenient new variable, lowering the degree within a integration problem, but keeping the algebraic nature of the integrand. This is achieved through a rational change of variable $\xi = \varphi_\mu(\zeta)/\psi_\nu(\zeta)$, and $\varphi_\mu(\zeta)$ and $\psi_\nu(\zeta)$ are polynomials of degrees μ, ν. Among the γ. th degree rational transformations, $\gamma = \max(\mu,\nu)$, we can find several hyperelliptic reduction formulae developed in XIX century by Legendre (1828), Jacobi (1832), Hermite (1876), Königsberger, Goursat (1885), Bolza, Richelot, Burnside (1892) and others. We will briefly discuss the Jacobi reduction: it operates on integrals like:

$$\int \frac{\mathrm{d}\zeta}{\sqrt{P_5(\zeta)}}, \tag{31}$$

where $P_5(\zeta)$ is a fifth degree polynomial whose roots are all real and given by: 0, 1, a, b, ab, and provides (31) in terms of elliptic integrals. For any couple of real numbers, $a > b > 1$, then, through a second degree rational transformation:

$$\xi = \frac{(1-a)(1-b)\zeta}{(\zeta-a)(\zeta-b)}$$

one can obtain, after some work, the Jacobi reduction formula:

$$\int_0^1 \frac{\left(\sqrt{ab}+\zeta\right)\,\mathrm{d}\zeta}{\sqrt{\zeta(\zeta-1)(\zeta-ab)(\zeta-a)(\zeta-b)}} = \frac{1}{\sqrt{(1-a)(1-b)}}\int_0^1 \frac{\mathrm{d}\xi}{\sqrt{\xi(1-\xi)(1-c\xi)}}. \tag{32}$$

where

$$c = -\frac{\left(\sqrt{a}-\sqrt{b}\right)^2}{(1-a)(1-b)}. \tag{33}$$

A Jacobi formula variant is also available, where the linear term (instead of $\sqrt{ab}+\zeta$) is $\varepsilon+\zeta$ with ε independent on a and b. The latter, taking $\varepsilon = 0$, $a = -1$, $1/b = \beta$, restores the hyperelliptic integral of Legendre:

$$\int \frac{\mathrm{d}y}{\sqrt{y(1-y^2)(1-\beta^2y^2)}},$$

Jacobi was the first to reduce it to a sum of elliptic ones.

Using the IRT for Lauricella function, formula (18), we will succeed in computing two hyperelliptic integrals in order to find two new relationships linking π, $\boldsymbol{K}(k)$ and the three-variable Lauricella function $F_D^{(3)}$. This will be covered by the:

Theorem 4.1. *Let* $0<\alpha<\beta<\gamma<\delta<\varepsilon$ *real numbers. Let*

$$\mathcal{I}_1^{(1/2)}(\alpha,\beta,\gamma,\delta,\varepsilon)=\int_\alpha^\beta \frac{\mathrm{d}x}{\sqrt{(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)(x-\varepsilon)}}$$

then we have:

$$\mathcal{I}_1^{(1/2)}(\alpha,\beta,\gamma,\delta,\varepsilon)=\frac{\pi}{\sqrt{(\gamma-\alpha)(\delta-\alpha)(\varepsilon-\alpha)}}\,F_D^{(3)}\left(\frac{1}{2};\frac{1}{2},\frac{1}{2},\frac{1}{2};1;\frac{\beta-\alpha}{\gamma-\alpha},\frac{\beta-\alpha}{\delta-\alpha},\frac{\beta-\alpha}{\varepsilon-\alpha}\right). \tag{34}$$

Proof. We use the change of variables $x=\alpha+(\beta-\alpha)u$, which implies:

$$x-\alpha=(\beta-\alpha)\,u, \qquad x-\beta=(\beta-\alpha)(1-u),$$
$$x-\gamma=(\gamma-\alpha)\left(\frac{\beta-\alpha}{\gamma-\alpha}\,u-1\right), \qquad x-\delta=(\delta-\alpha)\left(\frac{\beta-\alpha}{\delta-\alpha}\,u-1\right),$$
$$x-\varepsilon=(\varepsilon-\alpha)\left(\frac{\beta-\alpha}{\varepsilon-\alpha}\,u-1\right),$$

one can see that:

$$\mathcal{I}_1^{(1/2)}(\alpha,\beta,\gamma,\delta,\varepsilon)=\frac{1}{\sqrt{(\gamma-\alpha)(\delta-\alpha)(\varepsilon-\alpha)}}\int_0^1 \frac{u^{-1/2}(1-u)^{-1/2}\,\mathrm{d}u}{\left(1-\frac{\beta-\alpha}{\gamma-\alpha}\,u\right)^{1/2}\left(1-\frac{\beta-\alpha}{\delta-\alpha}\,u\right)^{1/2}\left(1-\frac{\beta-\alpha}{\varepsilon-\alpha}\,u\right)^{1/2}}$$

and then (34) follows in a similar way as for (8). □

Our proof does not depend on the root index, and at this point, assuming, for $y<1$

$$\mathcal{I}_1^{(y)}(\alpha,\beta,\gamma,\delta,\varepsilon)=\int_\alpha^\beta \frac{\mathrm{d}x}{\left[(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)(x-\varepsilon)\right]^y},$$

one easy can see that:

$$\mathcal{I}_1^{(y)}(\alpha,\beta,\gamma,\delta,\varepsilon)=\frac{1}{(\gamma-\alpha)^y(\delta-\alpha)^y(\varepsilon-\alpha)^y}\,\frac{\Gamma^2(1-y)}{\Gamma(2-2y)}\times$$
$$\times F_D^{(3)}\left(1-y;y,y,y;2-2y;\frac{\beta-\alpha}{\gamma-\alpha},\frac{\beta-\alpha}{\delta-\alpha},\frac{\beta-\alpha}{\varepsilon-\alpha}\right).$$

Furthermore, the approach of theorem 4.1 allows the analogous treatment when the integration interval is $[\gamma,\delta]$ or $[\varepsilon,\infty[$. Let us provide, for example, the

Corollary 4.1.1. *If* $0<\alpha<\beta<\gamma<\delta<\varepsilon$, *defining*

$$\mathcal{I}_2^{(1/2)}(\alpha,\beta,\gamma,\delta,\varepsilon)=\int_\gamma^\delta \frac{\mathrm{d}x}{\sqrt{(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)(x-\varepsilon)}}$$

then:

$$\mathcal{I}_2^{(1/2)}(\alpha,\beta,\gamma,\delta,\varepsilon)=\frac{\pi}{\sqrt{(\gamma-\alpha)(\gamma-\beta)(\varepsilon-\gamma)}}F_D^{(3)}\left(\frac{1}{2};\frac{1}{2},\frac{1}{2},\frac{1}{2};1;\frac{\delta-\gamma}{\alpha-\gamma},\frac{\delta-\gamma}{\beta-\gamma},\frac{\delta-\gamma}{\varepsilon-\gamma}\right).$$

To our π-formulae is strictly linked the following:

Theorem 4.2. *Let* $\alpha<\beta<\gamma<\delta<\varepsilon$ *and define:*

$$\mathcal{J}_1^{(1/2)}(\alpha,\beta,\gamma,\delta,\varepsilon)=\int_\alpha^\beta \frac{x\,\mathrm{d}x}{\sqrt{(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)(x-\varepsilon)}}.$$

Then:

$$\mathcal{J}_1^{(1/2)}(\alpha,\beta,\gamma,\delta,\varepsilon)=\frac{\pi}{\sqrt{(\gamma-\alpha)(\delta-\alpha)(\varepsilon-\alpha)}}\left(\mathcal{J}_{1,\alpha}^{(1/2)}+\mathcal{J}_{1,\alpha\beta}^{(1/2)}\right)$$

where:

$$\begin{aligned}\mathcal{J}_{1,\alpha}^{(1/2)}&=\alpha\,F_D^{(3)}\left(\frac12;\frac12,\frac12,\frac12;1;\frac{\beta-\alpha}{\gamma-\alpha},\frac{\beta-\alpha}{\delta-\alpha},\frac{\beta-\alpha}{\varepsilon-\alpha}\right)\\ \mathcal{J}_{1,\alpha\beta}^{(1/2)}&=(\beta-\alpha)\,F_D^{(3)}\left(\frac32,\frac12,\frac12,\frac12,2,\frac{\beta-\alpha}{\gamma-\alpha},\frac{\beta-\alpha}{\delta-\alpha},\frac{\beta-\alpha}{\varepsilon-\alpha}\right).\end{aligned}$$

Proof. It is enough to change variable $x=\alpha+(\beta-\alpha)u$ and then invoke the IRT for $F_D^{(3)}$. □

Recalling the Jacobi formula (32) for the hyperelliptic to elliptic reduction, we get:

Theorem 4.3. *If $a>b>1$, then:*

$$\pi=\frac{4ab}{\left(\sqrt{ab}-1\right)(A+B)}\boldsymbol{K}\left(\frac{\sqrt{a}-\sqrt{b}}{\sqrt{ab}-1}\right)\tag{35}$$

where

$$A=2\sqrt{ab}\,F_D^{(3)}\left(\frac12;\frac12,\frac12,\frac12;1;\frac1b,\frac1a,\frac1{ab}\right),\quad B=F_D^{(3)}\left(\frac32,\frac12,\frac12,\frac12,2;\frac1b,\frac1a,\frac1{ab}\right).$$

Proof. Looking at the right hand side of (32) and using entry 235.00 page 77 of [6], we get:

$$\mathcal{H}=\int_0^1\frac{\left(\zeta+\sqrt{ab}\right)\,\mathrm{d}\zeta}{\sqrt{(\zeta-1)\zeta(\zeta-a)(\zeta-b)(\zeta-ab)}}=\frac{2}{\sqrt{ab}-1}\boldsymbol{K}\left(\frac{\sqrt{a}-\sqrt{b}}{\sqrt{ab}-1}\right).\tag{36}$$

By the theorems 4.1 and 4.2 the integral at left hand side in (36) can be converted in hypergeometric terms, and noting that $\alpha=0$, $\beta=1$, $\gamma=b$, $\delta=a$, $\varepsilon=ab$, we have:

$$\mathcal{H}=\frac{\pi}{\sqrt{ab}}F_D^{(3)}\left(\frac12;\frac12,\frac12,\frac12;1;\frac1b,\frac1a,\frac1{ab}\right)+\frac{\pi}{2ab}F_D^{(3)}\left(\frac32,\frac12,\frac12,\frac12,2;\frac1b,\frac1a,\frac1{ab}\right)\tag{37}$$

Thesis (35) follows solving to π after eliminating $\mathcal{H}$ between (36) and (37). □

All the same, if $0<b<a<1$ and c is the constant introduced in (33) we obtain:

Theorem 4.4.

$$\pi=\frac{4}{\left(1-\sqrt{ab}\right)(A+B)}\boldsymbol{K}\left(\frac{\sqrt{a}-\sqrt{b}}{1-\sqrt{ab}}\right)\tag{38}$$

where:

$$A=2\,F_D^{(3)}\left(\frac12;\frac12,\frac12,\frac12;1;ab,a,b\right),\quad B=\sqrt{ab}\,F_D^{(3)}\left(\frac32,\frac12,\frac12,\frac12,2;ab,a,b\right).$$

Proof. It is enough to observe that:

$$\int_0^1\frac{\mathrm{d}\xi}{\sqrt{\xi(1-\xi)(1-c\xi)}}=\sqrt{(1-a)(1-b)}\int_0^1\frac{(\sqrt{ab}+\zeta)\,\mathrm{d}\zeta}{\sqrt{\zeta(\zeta-1)(\zeta-ab)(\zeta-a)(\zeta-b)}}$$

and go on as before. □

5 Conclusions

A large class of hyperelliptic integrals are represented, via some suitable IRT, through hypergeometric functions, like those of Gauss, Lauricella and Appell, namely multiple power series. Whenever a hyperelliptic integral can in addition be reduced to an elliptic integral, which can be done through an algebraic transformation, e.g. quadratic for the Jacobi reduction, we obtain a two-fold representation of the same mathematical object. By equating them, one can gain some completely new π determinations through the mentioned special functions or the eulerian complete Gamma and Beta integrals. Each of our π-formulae has been tested through Mathematica®'s packages for special functions. A convenient routine has only been implemented for the Lauricella's triple series: agreements were found successful in all cases. The practical value of our π-formulae is not in computing π, but in allowing a benchmark evaluation of the practical literary formulae of ${}_2F_1$, F_1, $F_D^{(3)}$, F, E, $\boldsymbol{K}$, being π known beyond one billion of digits.

One of the most remarkable hypergeometric formulae, not surprisingly due to Ramanujan, see formula (45) page 1359 of [23], gives 14 digits of π per term. The same equation in another form was given by the Chudnovsky brothers quoted by [8]:

$$\pi = \frac{426880\sqrt{10005}}{A\left[{}_3F_2\left(\frac{1}{6},\frac{1}{2},\frac{5}{6};1,1;B\right) - C\,{}_3F_2\left(\frac{7}{6},\frac{3}{2},\frac{11}{6};2,2;B\right)\right]},$$

with:

$$A = 13591409;\ B = \frac{-1}{151931373056000};\ C = \frac{30285563}{1651969144908540723200}.$$

Such a formula, already used by Mathematica® itself to calculate π, is founded on the six-argument univariate hypergeometric function:

$${}_3F_2(a_1,a_2,a_3;b_1,b_2;x) = \sum_{n=0}^{\infty}\frac{(a_1)_n\,(a_2)_n\,(a_3)_n}{(b_1)_n\,(b_2)_n}\frac{x^n}{n!},$$

whose order is 2 steps higher than ${}_2F_1$. By all means there are many formulae involving π, and one can consult [23] or [12] for a review of such identities. However our main π-formulae: (19), (20), (21), (27), (28), (35), (38), are not included in these repertories.

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