π and the hypergeometric functions of complex argument

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Abstract

In this article we derive some new identities concerning π, algebraic radicals and some special occurrences of the Gauss hypergeometric function ${}_2\mathrm{F}_1$ in the analytic continuation. All of them have been derived by tackling some elliptic or hyperelliptic known integral, and looking for another representation of it by means of hypergeometric functions like those of Gauss, Appell or Lauricella. In any case we have focused on integrand functions having at least one couple of complex-conjugate roots. Founding upon a special hyperelliptic reduction formula due to Hermite, [6], π is obtained as a ratio of a complete elliptic integral and the four-variable Lauricella function. Furthermore, starting with a certain binomial integral, we succeed in providing $\sqrt{2}/3$ as a ratio of a linear combination of complete elliptic integrals of the first and second kinds to the Appell hypergeometric function of two complex-conjugate arguments. Each of the formulae we found theoretically has been satisfactorily tested by means of Mathematica®

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1 Introduction

In this article some new identities concerning π and other relevant numbers are obtained following a methodology like that which appeared in our previous paper [9], which the reader is referred to, also for a review on the recent literature on π formulae. Hereinafter we will focus on some integrals not considered in [9], namely those with complex-conjugate roots, i.e. like

$$\int \frac{p(x)}{\sqrt{q(x)}}\,\mathrm{d}x$$

being $q(x)$ a third/fourth degree real coefficients polynomial with almost one couple of complex-conjugate roots and $p(x)$ with degree 0 or 1.

In the second section, by means of elliptic integrals, whose radicands always have complex-conjugate roots, evaluated by [3], we obtain further identities of elliptic-hypergeometric nature, not only to π but also to algebraic radicals like $\sqrt{2}$, $\sqrt[4]{3}$. Our most prominent outcomes appear to be where the hypergeometric functions ${}_2\mathrm{F}_1$ (Gauss), F_1 (Appell) and $\mathrm{F}_D^{(n)}$ (Lauricella) are theoretically found inside the unit disk, or in their analytic continuation. In the third section, starting from the integral where $2\alpha-\beta>1$:

$$\int_0^\infty \frac{t^\beta}{(1+t^2)^\alpha}\,\mathrm{d}t$$

we will establish some determinations, completely new as we deem, on the Gauss hypergeometric function ${}_2\mathrm{F}_1$ with argument 2, without making use of any formulae on analytic continuation.

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In the fourth section, a hyperelliptic subject takes place. A special hyperelliptic integral can in fact be reduced to an elliptic one, by the variable transformation:

$$y = \frac{2(z^3 - b^3)}{3(z^2 - a^2)}$$

leading to the reduction (Hermite 1876, [6]):

$$\int_{z_1}^{\infty} \frac{z}{\sqrt{(z^2 - a^2)(4z^3 - 3a^2 z - b^3)}}\, \mathrm{d}z = \frac{1}{\sqrt{6}} \int_{-2z_1}^{\infty} \frac{\mathrm{d}y}{\sqrt{y^3 - 3a^2 y + 2b^3}}. \tag{1}$$

Founding upon (1), if $b > a > 0$, defining $q_1(z) = 4z^3 - 3a^2 z - b^3$, $q_2(y) = y^3 - 3a^2 y + 2b^3$ we have two polynomials that have only one real root: moreover, there exists $z_1 > 0$ such that $q_1(z_1) = 0$, $q_2(-2z_1) = 0$. Notice that each root (real or complex) of $q_2(y)$ can be obtained by multiplying the $q_1(z)$ roots by -2. Next we will use the double approach (hypergeometric and elliptic) to compute some integrals which we have not dealt with before, [9], where all the roots of $q(x)$ are real and the interval of integration is bounded. In [9], integrals with real roots had been carried out over bounded intervals as per the famous hypergeometric-elliptic Jacobi (1832) reduction shown in [7]:

$$\int_0^1 \frac{\left(\sqrt{ab} + z\right)\, \mathrm{d}z}{\sqrt{z(z-1)(z-ab)(z-a)(z-b)}} = \frac{1}{\sqrt{(1-a)(1-b)}} \int_0^1 \frac{\mathrm{d}y}{\sqrt{y(1-y)(1-c_{ab}\, y)}}$$

where

$$c_{ab} = -\frac{\left(\sqrt{a} - \sqrt{b}\right)^2}{(1-a)(1-b)}.$$

We will see how the integral on the left-hand side of (1) can be evaluated via the same methods described in [9] through the Lauricella hypergeometric function $\mathrm{F}_D^{(4)}$, some argument of which shall be necessarily complex. As a consequence, a new π identity will follow once the (1) integral has been expressed as a complete first kind elliptic integral, formula 241.00 page 88 of [3]. In this case the transformation of the hyperelliptic integral in (1) to a hypergeometric one, is more difficult since the roots of the cubic equation require Cardano's solution. Furthermore we will need further assumptions on the real parameter a in (1) in order to ensure the convergence of the hypergeometric series to which the hyperelliptic integral has been switched into. Since we lack these assumptions, it will not be possible to employ the classic hypergeometric expansion in multiple power series; therefore, the identity we are going to prove will be true based on the analytic continuation of $\mathrm{F}_D^{(4)}$.

Notations

For the reader's best convenience we recall the main notations of the special functions involved throughout the paper: the reader is referred to [14, 1, 11, 4, 8, 10] for further information.

Euler-Legendre integral (Gamma function), defined for $x > 0$:

$$\Gamma(x) = \int_0^{\infty} e^{-u} u^{x-1} \mathrm{d}u.$$

Pochhammer symbol:

$$(a)_m = \frac{\Gamma(a+m)}{\Gamma(a)} = a(a+1)\cdots(a+m-1).$$

Complete elliptic integral of first kind, with modulus $|k| < 1$:

$$\boldsymbol{K}(k) = \int_0^1 \frac{\mathrm{d}u}{\sqrt{(1-u^2)(1-k^2u^2)}}.$$

Complete elliptic integral of second kind, with modulus $|k| < 1$:

$$\boldsymbol{E}(k) = \int_0^1 \sqrt{\frac{1-k^2u^2}{1-u^2}}\, \mathrm{d}u.$$

Gauss hypergeometric series defined for $|x_1| < 1$

$$ {}_2\mathrm{F}_1\left(\begin{array}{c} a;b \\ c \end{array}\middle| x_1\right) = \sum_{m=0}^{\infty} \frac{(a)_m\,(b)_m}{(c)_m}\frac{x_1^m}{m!}, $$

Recall the Integral Representation Theorem, acronym IRT, which holds for $\operatorname{Re} a > 0$, $\operatorname{Re}(c-a) > 0$, $|x_1| < 1$:

$$ {}_2\mathrm{F}_1\left(\begin{array}{c} a;b \\ c \end{array}\middle| x_1\right) = \frac{\Gamma(c)}{\Gamma(c-a)\Gamma(a)}\int_0^1 \frac{u^{a-1}(1-u)^{c-a-1}}{(1-x_1\,u)^b}\,\mathrm{d}u $$

Appell two-variables hypergeometric function defined for $|x_1| < 1$, $|x_2| < 1$:

$$ \mathrm{F}_1\left(\begin{array}{c} a;b_1,b_2 \\ c \end{array}\middle| x_1,x_2\right) = \sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty} \frac{(a)_{m_1+m_2}(b_1)_{m_1}(b_2)_{m_2}}{(c)_{m_1+m_2}}\frac{x_1^{m_1}}{m_1!}\frac{x_2^{m_2}}{m_2!}, $$

whose IRT reads, if $\operatorname{Re} a > 0$, $\operatorname{Re}(c-a) > 0$, as:

$$ \mathrm{F}_1\left(\begin{array}{c} a;b_1,b_2 \\ c \end{array}\middle| x_1,x_2\right) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 \frac{u^{a-1}\,(1-u)^{c-a-1}}{(1-x_1\,u)^{b_1}\,(1-x_2\,u)^{b_2}}\,\mathrm{d}u. \tag{2} $$

We will employ several times the general reduction formula expressing F_1 in terms of ${}_2\mathrm{F}_1$, see [11] formula (8.3.4) page 218, namely:

$$ \mathrm{F}_1\left(\begin{array}{c} a,b_1,b_2 \\ b_1+b_2 \end{array}\middle| x,y\right) = \frac{1}{(1-y)^a}\,{}_2\mathrm{F}_1\left(\begin{array}{c} a,b_1 \\ b_1+b_2 \end{array}\middle| \frac{x-y}{1-y}\right) \tag{3} $$

Lauricella hypergeometric functions of n variables

$$ \mathrm{F}_D^{(n)}\left(\begin{array}{c} a;b_1,\ldots,b_n \\ c \end{array}\middle| x_1,\ldots,x_n\right) = \sum_{m_1=0}^{\infty}\cdots\sum_{m_n=0}^{\infty} \frac{(a)_{m_1+\cdots+m_n}(b_1)_{m_1}\cdots(b_n)_{m_n}}{(c)_{m_1+\cdots+m_n}m_1!\cdots m_n!}\,x_1^{m_1}\cdots x_n^{m_m} $$

whose IRT for $\operatorname{Re} a > 0$, $\operatorname{Re}(c-a) > 0$ is:

$$ \mathrm{F}_D^{(n)}\left(\begin{array}{c} a;b_1,\ldots,b_n \\ c \end{array}\middle| x_1,\ldots,x_n\right) = \frac{\Gamma(c)}{\Gamma(a)\,\Gamma(c-a)}\int_0^1 \frac{u^{a-1}(1-u)^{c-a-1}}{(1-x_1u)^{b_1}\cdots(1-x_nu)^{b_n}}\,\mathrm{d}u. \tag{4} $$

A reduction formula of the same nature as that in (3) keeps its validity even while reducing the number of hypergeometric variables when the coefficient c equates the sum of all the b_j. For example the three-variable Lauricella $\mathrm{F}_D^{(3)}$ can be reduced to the Appell F_1 according to the

Lemma 1.1. *The reduction formula holds:*

$$ \mathrm{F}_D^{(3)}\left(\begin{array}{c} a;b_1,b_2,b_3 \\ b_1+b_2+b_3 \end{array}\middle| x_1,x_2,x_3\right) = \frac{1}{(1-x_3)^a}\,\mathrm{F}_1\left(\begin{array}{c} a,b_2,b_3 \\ b_1+b_2+b_3 \end{array}\middle| \frac{x_1-x_3}{1-x_3},\frac{x_2-x_3}{1-x_3}\right) \tag{5} $$

More generally:

$$ \mathrm{F}_D^{(n)}\left(\begin{array}{c} a;b_1,\ldots,b_n \\ b_1+\cdots+b_n \end{array}\middle| x_1,\ldots,x_n\right) = \frac{1}{(1-x_n)^a}\,\mathrm{F}_D^{(n-1)}\left(\begin{array}{c} a,b_2,\ldots,b_n \\ b_1+\cdots+b_n \end{array}\middle| \frac{x_1-x_n}{1-x_n},\ldots,\frac{x_{n-1}-x_n}{1-x_n}\right) \tag{6} $$

Proof. In the integral giving $\mathrm{F}_D^{(3)}$ the same change $u \rightsquigarrow v$ shown by Slater in [11] p. 217-218:

$$ u = \frac{v}{(1-x_3)\left(1-\dfrac{x_3}{x_3-1}\,v\right)} \implies \mathrm{d}u = \frac{\mathrm{d}v}{(1-x_3)\left(1-\dfrac{x_3}{x_3-1}\,v\right)^2}. $$

shall be used. The outcome follows by the direct computation keeping in mind that:

$$1-x_3\,u=\frac{1}{1-\dfrac{x_3}{x_3-1}\,v},\ 1-x_2\,u=\frac{1-\dfrac{x_3-x_2}{x_3-1}\,v}{1-\dfrac{x_3}{x_3-1}\,v},\ 1-x_1\,u=\frac{1-\dfrac{x_3-x_1}{x_3-1}\,v}{1-\dfrac{x_3}{x_3-1}\,v},\ 1-u=\frac{1-v}{1-\dfrac{x_3}{x_3-1}\,v}$$

□

All these hypergeometric functions can be analytically continued by the IRT to the complex field as a whole excluding the real positive axis for each of the variables x_j.

2 Elliptic-hypergeometric identities

Starting from the cubic case, let us consider the elliptic integral 241.00 of [3]:

$$\int_a^\infty \frac{\mathrm{d}t}{\sqrt{(t-a)\,(t^2+b^2)}}=\frac{2}{\sqrt[4]{a^2+b^2}}\,\boldsymbol{K}\left(\sqrt{\frac{\sqrt{a^2+b^2}-a}{2\sqrt{a^2+b^2}}}\right) \tag{7}$$

with a third degree polynomial under the square root and a couple of complex-conjugate roots. Also by computing (7) hypergeometrically, we will find our first new π formulae:

Theorem 2.1. *Assume $a>0$ and b any real number. The following identity holds:*

$$\pi=\frac{2\sqrt[4]{a^2+b^2}}{\sqrt{a}}\,\frac{\boldsymbol{K}\left(\sqrt{\dfrac{\sqrt{a^2+b^2}-a}{2\sqrt{a^2+b^2}}}\right)}{\mathrm{F}_1\left(\begin{array}{c}\frac12,\frac12,\frac12\\ 1\end{array}\middle|\,\dfrac{b^2}{a^2+b^2}+\dfrac{ab}{a^2+b^2}\,i,\dfrac{b^2}{a^2+b^2}-\dfrac{ab}{a^2+b^2}\,i\right)} \tag{8}$$

Moreover we also have:

$$\pi=\sqrt{2}\left(\frac{\sqrt{\sqrt{a^2+b^2}+a}}{\sqrt[4]{a^2+b^2}}+i\,\frac{\sqrt{\sqrt{a^2+b^2}-a}}{\sqrt[4]{a^2+b^2}}\right)\frac{\boldsymbol{K}\left(\sqrt{\dfrac{\sqrt{a^2+b^2}-a}{2\sqrt{a^2+b^2}}}\right)}{{}_2\mathrm{F}_1\left(\begin{array}{c}\frac12,\frac12\\ 1\end{array}\middle|\,\dfrac{2b^2}{a^2+b^2}+\dfrac{2ab}{a^2+b^2}\,i\right)} \tag{9}$$

Proof. Starting from formula (7), evaluating the left hand side integral and passing to the variable u:

$$u=1-\frac{a}{t}\implies \mathrm{d}t=\frac{a}{(1-u)^2}\,\mathrm{d}u,$$

we see that:

$$\int_a^\infty \frac{\mathrm{d}t}{\sqrt{(t-a)\,(t^2+b^2)}}=\sqrt{\frac{a}{a^2+b^2}}\int_0^1\frac{u^{-1/2}\,(1-u)^{-1/2}}{\left[1-\left(\frac{b^2}{a^2+b^2}+\frac{ab}{a^2+b^2}\,i\right)\,u\right]^{1/2}\left[1-\left(\frac{b^2}{a^2+b^2}-\frac{ab}{a^2+b^2}\,i\right)\,u\right]^{1/2}}\,\mathrm{d}u$$

so using the relevant IRT (2) for the Appell function, we find out:

$$\pi\,\mathrm{F}_1\left(\begin{array}{c}\frac12,\frac12,\frac12\\ 1\end{array}\middle|\,\frac{b^2+ab\,i}{a^2+b^2},\frac{b^2-ab\,i}{a^2+b^2}\right)=\int_0^1\frac{u^{-1/2}\,(1-u)^{-1/2}}{\left[1-\left(\frac{b^2+ab\,i}{a^2+b^2}\right)\,u\right]^{1/2}\left[1-\left(\frac{b^2-ab\,i}{a^2+b^2}\right)\,u\right]^{1/2}}\,\mathrm{d}u. \tag{10}$$

Thesis (8) follows by comparing (10) and (7). The second statement follows from the general reduction formula expressing F_1 in terms of ${}_2\mathrm{F}_1$, see formula (3); taking $a=b_1=b_2=1/2$ we find:

$$\mathrm{F}_1\left(\begin{array}{c}\frac12,\frac12,\frac12\\ 1\end{array}\middle|\,\frac{b^2+ab\,i}{a^2+b^2},\frac{b^2-ab\,i}{a^2+b^2}\right)=\left(\sqrt{\frac{\sqrt{a^2+b^2}+a}{2a}}-i\sqrt{\frac{\sqrt{a^2+b^2}-a}{2a}}\right){}_2\mathrm{F}_1\left(\begin{array}{c}\frac12;\frac12\\ 1\end{array}\middle|\,\frac{2b^2+2ab\,i}{a^2+b^2}\right).$$

In such a way, substituting in (8), after some algebraic work, formula (9) follows.

□

An analogous outcome is true for fourth degree polynomial. We know that, see [3] 213.00 page 48 with $b > 0$:

$$\int_{-b}^{b} \frac{\mathrm{d}t}{\sqrt{(a^2+t^2)(b^2-t^2)}} = \frac{2}{\sqrt{a^2+b^2}} \boldsymbol{K}\left(\frac{b}{\sqrt{a^2+b^2}}\right). \tag{11}$$

We have the:

Theorem 2.2. *The identity holds:*

$$\pi = \frac{2\boldsymbol{K}\left(\dfrac{b}{\sqrt{a^2+b^2}}\right)}{\mathrm{F}_1\left(\begin{array}{c} \frac{1}{2}; \frac{1}{2}, \frac{1}{2} \\ 1 \end{array} \middle| \dfrac{2b^2-2ab\,i}{a^2+b^2}, \dfrac{2b^2+2ab\,i}{a^2+b^2}\right)} \tag{12}$$

If $0 < b\sqrt{3} < a$, the hypergeometric series defining the Appell F_1 converges; if, on the contrary the condition $b\sqrt{3} \geq a > 0$ holds, the relation (12) *is meaningful whenever Appell F_1 function is analytically continued. Furthermore, by* (3) *it follows that:*

$$\pi = \frac{2(a-i\,b)}{\sqrt{a^2+b^2}} \frac{\boldsymbol{K}\left(\dfrac{b}{\sqrt{a^2+b^2}}\right)}{{}_2\mathrm{F}_1\left(\begin{array}{c} \frac{1}{2}, \frac{1}{2} \\ 1 \end{array} \middle| -\dfrac{4ab\,i}{(a-i\,b)^2}\right)} \tag{13}$$

Proof. Let us perform on the left-hand side of (11) the change of variable $t = b(2u-1)$ so that:

$$\frac{1}{\sqrt{a^2+b^2}} \int_0^1 \frac{u^{-1/2}(1-u)^{-1/2}}{\sqrt{\left(1-\frac{2b^2-2ab\,i}{a^2+b^2}\,u\right)\left(1-\frac{2b^2+2ab\,i}{a^2+b^2}\,u\right)}}\,\mathrm{d}u = \frac{2}{\sqrt{a^2+b^2}} \boldsymbol{K}\left(\frac{b}{\sqrt{a^2+b^2}}\right)$$

Thesis (12) follows by IRT (2) after noting that the module of the two complex conjugate arguments of F_1 is given by:

$$\left|\frac{2b^2+2ab\,i}{a^2+b^2}\right| = \left|\frac{2b^2-2ab\,i}{a^2+b^2}\right| = \frac{2b}{\sqrt{a^2+b^2}}.$$

To get the second identity, one shall apply only the transformation (3) and simplify.

□

Even if we start with an integral similar to (11), the next hypergeometric transformation does not lead to any π identity, but allows us to evaluate the Lauricella's hypergeometric function $\mathrm{F}_D^{(4)}$ analytic continuation for some special set of the variables. The start-up is formula 221.00 page 61 of [3]: if $a > b > 0$, then:

$$\int_0^\infty \frac{\mathrm{d}t}{\sqrt{(t^2+a^2)\,(t^2+b^2)}} = \frac{1}{a} \boldsymbol{K}\left(\sqrt{\frac{a^2-b^2}{a^2}}\right) \tag{14}$$

Theorem 2.3. *If $a > b > 0$ then:*

$$\mathrm{F}_D^{(4)}\left(\begin{array}{c} 1; \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 2 \end{array} \middle| 1+i\,a, 1-i\,a, 1+i\,b, 1-i\,b\right) = \frac{1}{a} \boldsymbol{K}\left(\sqrt{\frac{a^2-b^2}{a^2}}\right) \tag{15}$$

Owing to four of Lauricella's arguments that remain outside the unit disk, the left-hand side of (15) *has to be defined in the $\mathrm{F}_D^{(4)}$ analytic continuation.*

Proof. Changing the variable in (14) using $t = (1-u)/u$ we get:

$$\int_0^\infty \frac{\mathrm{d}t}{\sqrt{(t^2+a^2)\,(t^2+b^2)}} = \int_0^1 \frac{\mathrm{d}u}{\sqrt{[1-(1-ia)u]\,[1-(1+ia)u]\,[1-(1-ib)u]\,[1-(1+ib)u]}}$$

and hereinafter the thesis (15) follows.

□

In (15) π does not appear because parameters a and c of $\mathrm{F}_D^{(4)}$ have the values 1 and 2 respectively, so that:

$$\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} = 1$$

On the contrary, in all the previous formulae (10), (9), (12), (13) we always have $a = 1/2$ and $c = 1$. The above mentioned case also occurs in the next identity where the square root of 2 will be given as a ratio of an elliptic integral to a Lauricella function.

Theorem 2.4. *Let a, $b \in \mathbb{R}$ be two positive numbers. Then:*

$$\sqrt{2} = \frac{\boldsymbol{K}\left(\dfrac{a}{\sqrt{a^2+b^2}}\right)}{\mathrm{F}_D^{(3)}\left(\begin{array}{c}\frac{1}{2};\frac{1}{2},\frac{1}{2},\frac{1}{2}\\ \frac{3}{2}\end{array}\middle|\,\dfrac{a^2-i\,ab}{a^2+b^2},\dfrac{a^2+i\,ab}{a^2+b^2},\dfrac{1}{2}\right)} \tag{16}$$

By the reduction formula (5) *we get:*

$$2 = \frac{\boldsymbol{K}\left(\dfrac{a}{\sqrt{a^2+b^2}}\right)}{\mathrm{F}_1\left(\begin{array}{c}\frac{1}{2};\frac{1}{2},\frac{1}{2}\\ \frac{3}{2}\end{array}\middle|\,\dfrac{a-i\,b}{a+i\,b},\dfrac{a+i\,b}{a-i\,b}\right)} \tag{17}$$

Proof. Starting from the complete elliptic integral 212.00 page 47 of [3]:

$$\int_b^\infty \frac{\mathrm{d}t}{\sqrt{(a^2+t^2)(t^2-b^2)}} = \frac{1}{\sqrt{a^2+b^2}}\,\boldsymbol{K}\left(\frac{a}{\sqrt{a^2+b^2}}\right)$$

changing variables again:

$$t = \frac{b}{1-u} \implies \mathrm{d}t = \frac{b}{(1-u)^2}$$

so that:

$$\begin{aligned}\frac{1}{\sqrt{a^2+b^2}}\,\boldsymbol{K}\left(\frac{a}{\sqrt{a^2+b^2}}\right) &= \int_0^1 \frac{\mathrm{d}u}{\sqrt{(2u-u^2)\,(a^2u^2-2a^2u+a^2+b^2)}}\\ &= \frac{1}{\sqrt{2(a^2+b^2)}}\int_0^1 \frac{u^{-1/2}}{\sqrt{\left(1-\frac{1}{2}\,u\right)\left[1-\left(\frac{a^2-ab\,i}{a^2+b^2}\right)u\right]\left[1-\left(\frac{a^2+ab\,i}{a^2+b^2}\right)u\right]}}\,\mathrm{d}u\end{aligned}$$

Thesis (16) follows after the IRT. Again, thesis (17) after (16) and (5).

□

For the lack of computer algebra packages for the Lauricella functions, the above formula 16 will be the best benchmark in view of its future, hopefully near, implementation.

3 Some new results on the analytic continuation of ${}_2F_1$

Our next outcome will consist of an analogous formula providing once again an arithmetic radical through the complete elliptic integral **K** and of the Appell F_1 function computed in its analytic continuation. In this case, thanks to the special structure of parameters of F_1, we will obtain a further relationship involving the ${}_2F_1$analytic continuation. Notice that the analytic continuation of ${}_2F_1$ outside the unit disk, i.e. $|z| > 1$, namely for values not allowing the IRT, is ensured by the classic reflection formulae like those in [12] on page 120:

$$
\begin{aligned}
{}_2F_1\left(\begin{matrix} a;b \\ c \end{matrix} \middle| z\right) &= \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a}\,{}_2F_1\left(\begin{matrix} a;a-c+1 \\ a-b+1 \end{matrix} \middle| \frac{1}{z}\right) \\
&+ \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}\,{}_2F_1\left(\begin{matrix} b-c-1;b \\ b-a+1 \end{matrix} \middle| \frac{1}{z}\right)
\end{aligned}
\tag{18}
$$

The analytic continuation ${}_2F_1$ is a subject of recent research like [2], or [13], where a complicated formula is found involving three determinations of ${}_2F_1$. Therefore we deem our finding on the value of ${}_2F_1$ for $z = 2$ is of some interest not only to the analytic number theory, but also to fixing a benchmark for any computational efforts. We start with the integral, $2\alpha - \beta > 1$:

$$
\int_0^\infty \frac{t^\beta}{(1+t^2)^\alpha}\,\mathrm{d}t = \frac{1}{2}\,\frac{\Gamma\left(\frac{\beta+1}{2}\right)\Gamma\left(\frac{2\alpha-\beta-1}{2}\right)}{\Gamma(\alpha)}
\tag{19}
$$

notice that (19) can be found for instance by passing to $u = \arctan t$ in the integral, keeping in mind one of the expressions of the Euler Beta function:

$$
\mathrm{B}(x,y) = 2\int_0^{\frac{\pi}{2}} (\sin u)^{2x-1}(\cos u)^{2y-1}\mathrm{d}u
$$

and the Beta-Gamma theorem: $\mathrm{B}(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$. Computing (hypergeometrically) again the integral on the left hand side of (19), one gets:

Theorem 3.1. *If* $2\alpha - \beta > 1$ *then:*

$$
{}_2F_1\left(\begin{matrix} 2\alpha-\beta-1;\alpha \\ 2\alpha \end{matrix} \middle| 2\right) = (-i)^{2\alpha-\beta-1}\sqrt{\pi}\,\frac{\Gamma\left(\alpha+\frac{1}{2}\right)}{\Gamma\left(1+\frac{\beta}{2}\right)\Gamma\left(\alpha-\frac{\beta}{2}\right)}
\tag{20}
$$

where the complex powers are referred to the principal argument.

Proof. Changing the variables on left-hand side of (19) by means of the usual transformation $t = (1-u)/u$ we get:

$$
\int_0^\infty \frac{t^\beta}{(1+t^2)^\alpha}\,\mathrm{d}t = \int_0^1 \frac{(1-u)^\beta u^{2\alpha-\beta-2}}{[1-2u+2u^2]^\alpha}\,\mathrm{d}u
$$

Splitting $1-2u+2u^2 = [1-(1-i)u]\,[1-(1+i)u]$ we can use the IRT to obtain the Appell function:

$$
\int_0^\infty \frac{t^\beta}{(1+t^2)^\alpha}\,\mathrm{d}t = \frac{\Gamma(2\alpha-\beta-1)\Gamma(\beta+1)}{\Gamma(2\alpha)}\,F_1\left(\begin{matrix} 2\alpha-\beta-1;\alpha,\alpha \\ 2\alpha \end{matrix} \middle| 1-i, 1+i\right)
\tag{21}
$$

with conjugate complex arguments. By the reduction (3):

$$
\int_0^\infty \frac{t^\beta}{(1+t^2)^\alpha}\,\mathrm{d}t = \frac{\Gamma(2\alpha-\beta-1)\Gamma(\beta+1)}{\Gamma(2\alpha)}\,\frac{1}{(-i)^{2\alpha-\beta-1}}\,{}_2F_1\left(\begin{matrix} 2\alpha-\beta-1;\alpha \\ 2\alpha \end{matrix} \middle| 2\right)
\tag{22}
$$

equating to (19), thesis (20) follows, using the duplication formula, which holds for $2z \neq 0, -1, -2, \ldots$

$$
\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}}\,\Gamma(z)\Gamma\left(z+\frac{1}{2}\right).
$$

to simplify some coefficients. □

For special values of α parameter, integral (19) can be given by means of elliptic integrals, then producing further remarkable equalities where ${}_2\mathrm{F}_1$ of argument 2 can be computed with the knowledge of some complete elliptic integrals. Next let us provide some integral formulae taken by [3] and [5]. The first is [3] 273.00 page 152:

$$\int_0^\infty \frac{\mathrm{d}t}{\sqrt[4]{(t^2+1)^3}} = \sqrt{2}\,\boldsymbol{K}\left(\frac{1}{\sqrt{2}}\right) \tag{23}$$

the second is [3] 274.00 page 154:

$$\int_0^\infty \frac{\mathrm{d}t}{\sqrt[3]{(t^2+1)^2}} = \frac{3}{\sqrt[4]{3}}\boldsymbol{K}\left(\sqrt{\frac{2-\sqrt{3}}{4}}\right) \tag{24}$$

the third one is [3] 273.54 page 154 or, better, [5] 3.185.5 page 311:

$$\int_0^\infty \frac{\mathrm{d}t}{\sqrt[4]{(t^2+1)^5}} = 2\sqrt{2}\,\boldsymbol{E}\left(\frac{1}{\sqrt{2}}\right) - \sqrt{2}\,\boldsymbol{K}\left(\frac{1}{\sqrt{2}}\right) \tag{25}$$

and the last one is [3] integrals 273.53 and 361.02 (we take here the opportunity to point out a misprint in the integral 3.185.7 page 312 of [5]):

$$\int_0^\infty \frac{t^2}{\sqrt[4]{(1+t^2)^7}}\,\mathrm{d}t = \frac{2\sqrt{2}}{3}\,\boldsymbol{K}\left(\frac{1}{\sqrt{2}}\right) \tag{26}$$

In such a way, by comparison of the hypergeometrical solution in (22) with the special cases $\alpha = 3/4,\ 2/3,\ 5/4$ and $\beta = 0$ and $\alpha = 7/4$ and $\beta = 2$ computed via complete elliptic integrals in (23), (24), (25) and (26), one gets immediately that:

Theorem 3.2. *The following four formulae for ${}_2\mathrm{F}_1$ hold, the first from* (23)*:*

$${}_2\mathrm{F}_1\left(\begin{array}{c}\frac{1}{2};\frac{3}{4}\\ \frac{3}{2}\end{array}\middle|\,2\right) = \frac{1}{2}(1-i)\,\boldsymbol{K}\left(\frac{1}{\sqrt{2}}\right) \tag{27}$$

Same way, a start from (24) *will lead to:*

$${}_2\mathrm{F}_1\left(\begin{array}{c}\frac{1}{3};\frac{2}{3}\\ \frac{4}{3}\end{array}\middle|\,2\right) = \frac{\frac{\sqrt{3}}{2}-\frac{1}{2}i}{\sqrt[4]{3}}\,\boldsymbol{K}\left(\sqrt{\frac{2-\sqrt{3}}{4}}\right) \tag{28}$$

From (25) *we infer:*

$${}_2\mathrm{F}_1\left(\begin{array}{c}\frac{3}{2};\frac{5}{4}\\ \frac{5}{2}\end{array}\middle|\,2\right) = \frac{3}{2}(1+i)\left[\boldsymbol{K}\left(\frac{1}{\sqrt{2}}\right) - 2\boldsymbol{E}\left(\frac{1}{\sqrt{2}}\right)\right] \tag{29}$$

Eventually from (26) *we infer:*

$${}_2\mathrm{F}_1\left(\begin{array}{c}\frac{1}{2};\frac{7}{4}\\ \frac{7}{2}\end{array}\middle|\,2\right) = \frac{5}{8}(1-i)\,\boldsymbol{K}\left(\frac{1}{\sqrt{2}}\right) \tag{30}$$

Stopping just a step before applying the reduction formula (3) and evaluating the integral at the left-hand side of (19) via the formula (21) instead of (22), one finds:

$$\sqrt{2} = \frac{\boldsymbol{K}\left(\frac{1}{\sqrt{2}}\right)}{\mathrm{F}_1\left(\begin{array}{c}\frac{1}{2};\frac{3}{4},\frac{3}{4}\\ \frac{3}{2}\end{array}\middle|\,1-i,1+i\right)} \tag{31}$$

$$\sqrt[4]{3}=\frac{\boldsymbol{K}\left(\sqrt{\dfrac{2-\sqrt{3}}{4}}\right)}{\mathrm{F}_1\left(\begin{array}{c}\frac{1}{3};\frac{2}{3},\frac{2}{3}\\ \frac{4}{3}\end{array}\middle|\,1-i,1+i\right)} \tag{32}$$

$$\frac{\sqrt{2}}{3}=\frac{2\boldsymbol{E}\left(\dfrac{1}{\sqrt{2}}\right)-\boldsymbol{K}\left(\dfrac{1}{\sqrt{2}}\right)}{\mathrm{F}_1\left(\begin{array}{c}\frac{3}{2};\frac{5}{4},\frac{5}{4}\\ \frac{5}{2}\end{array}\middle|\,1-i,1+i\right)} \tag{33}$$

$$\frac{4\sqrt{2}}{5}=\frac{\boldsymbol{K}\left(\dfrac{1}{\sqrt{2}}\right)}{\mathrm{F}_1\left(\begin{array}{c}\frac{1}{2};\frac{7}{4},\frac{7}{4}\\ \frac{7}{2}\end{array}\middle|\,1-i,1+i\right)} \tag{34}$$

Lastly, by comparing (27) with (30) and (31) with (34) we obtain:

$$ {}_2\mathrm{F}_1\left(\begin{array}{c}\frac{1}{2};\frac{3}{4}\\ \frac{3}{2}\end{array}\middle|\,2\right)=\frac{4}{5}\,{}_2\mathrm{F}_1\left(\begin{array}{c}\frac{1}{2};\frac{7}{4}\\ \frac{7}{2}\end{array}\middle|\,2\right) \tag{35}$$

$$\mathrm{F}_1\left(\begin{array}{c}\frac{1}{2};\frac{3}{4},\frac{3}{4}\\ \frac{3}{2}\end{array}\middle|\,1-i,1+i\right)=\frac{4}{5}\,\mathrm{F}_1\left(\begin{array}{c}\frac{1}{2};\frac{7}{4},\frac{7}{4}\\ \frac{7}{2}\end{array}\middle|\,1-i,1+i\right) \tag{36}$$

4 Hyperelliptic-hypergeometric identities

Up to now we have managed only elliptic integrals: in this section the hyperelliptic ones will appear. We are going to construct a new identity linking π to a complete elliptic integral and to the hypergeometric Lauricella's function of four variables.

Theorem 4.1. *Let $0<a<b$, and*

$$S:=S(a,b)=\frac{1}{2}\sqrt[3]{b^3+\sqrt{b^6-a^6}},\quad T:=T(a,b)=\frac{1}{2}\sqrt[3]{b^3-\sqrt{b^6-a^6}}$$

so that the real root z_1 and the couple of complex-conjugate roots $z_2=\bar{z}_3$ of $q_1(z)=4z^3-3a^2z-b^3$ are given by:

$$z_1=S+T,\ z_2=-\frac{1}{2}(S+T)+\frac{\sqrt{3}}{2}(S-T)\,i,\ z_3=-\frac{1}{2}(S+T)-\frac{\sqrt{3}}{2}(S-T)\,i$$

Then, putting

$$\boldsymbol{K}_a^b=\boldsymbol{K}\left(\sqrt{\frac{1}{2}+\frac{\sqrt{3}(S+T)}{4\sqrt{S^2+ST+T^2}}}\right),\quad \boldsymbol{F}_a^b=\mathrm{F}_D^{(4)}\left(\begin{array}{c}\frac{1}{2};\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\\ 1\end{array}\middle|\,x_1,x_2,x_3,x_4\right)$$

where the arguments x_1, x_2, x_3, x_4 are dependent on T, S and then on a, b according to:

$$x_1=\frac{a}{a-(S+T)},\ x_2=\frac{a}{a+S+T},\ x_3=\frac{S^2+T^2}{2(S^2+ST+T^2)}-\frac{(S^2-T^2)\,i}{2\sqrt{3}(S^2+ST+T^2)},\ x_4=\bar{x}_3$$

In such a way we find that:

$$\pi=\frac{2\sqrt{(S^2+ST+T^2)\,[(S+T)^2-a^2]}}{\sqrt[4]{3}\sqrt{(S+T)^3}\sqrt[4]{S^2+ST+T^2}}\,\frac{\boldsymbol{K}_a^b}{\boldsymbol{F}_a^b} \tag{37}$$

Proof. Starting from (1) of Hermite, in order to compute the hypergeometric integral on the left-hand side, we need the roots of the cubic $q_1(z) = 4z^3 - 3a^2z - b^3$. We assume $b > a > 0$ and we know that $q_1(z)$ has only one real zero (let it be z_1), whilst $z_2 = \bar{z}_3$ are complex conjugate roots. We have $0 < a < z_1$. The customary change from z to u, $z = z_1/(1-u)$ normalizing to $[0,1]$ the left integral in (1), provides:

$$\int_{z_1}^{\infty} \frac{z}{\sqrt{(z^2-a^2)(4z^3-3a^2z-b^3)}}\,\mathrm{d}z =$$

$$\frac{z_1^{3/2}}{2\sqrt{(z_1^2-a^2)(z_1-z_2)(z_1-z_3)}}\int_0^1 \frac{u^{-1/2}(1-u)^{-1/2}}{\sqrt{\left(1-\frac{a}{a-z_1}u\right)\left(1-\frac{a}{z_1+a}u\right)\left(1-\frac{z_2}{z_2-z_1}u\right)\left(1-\frac{z_3}{z_3-z_1}u\right)}}\,\mathrm{d}u \tag{38}$$

$$\frac{\pi\, z_1^{3/2}}{2\sqrt{(z_1^2-a^2)(z_1-z_2)(z_1-z_3)}}\mathrm{F}_D^{(4)}\left(\begin{matrix}\frac12;\frac12,\frac12,\frac12,\frac12\\ 1\end{matrix}\,\middle|\,\frac{a}{a-z_1},\frac{a}{a+z_1},\frac{z_2}{z_2-z_1},\frac{z_3}{z_3-z_1}\right)$$

where the hyperelliptic integral has by IRT been expressed through a suitable Lauricella function. We see that the arguments of $\mathrm{F}_D^{(4)}$ in (38) are exactly the same conjugate complex numbers x_3 and x_4 that were introduced in the theorem statement; such numbers have modulus < 1 because:

$$\left|\frac{z_2}{z_2-z_1}\right| = \left|\frac{z_3}{z_3-z_1}\right| = \frac{1}{\sqrt{3}}\sqrt{\frac{S^2-ST+T^2}{S^2+ST+T^2}}$$

The second argument of $\mathrm{F}_D^{(4)}$ in (38) is < 1, whilst the first is < 1 if $2a < z_1 = S+T$, which ensures that the hypergeometric series converges. If such a condition is not met, (38) has to be meant in the analytic continuation of the Lauricella function.

The integral on the right hand side of (1) can be evaluated through the cardanic formula for the real roots y_1, and the complex-conjugate ones $y_2 = \bar{y}_3$ of the cubic $q_2(y) = y^3 - 3a^2y + 2b^3$ and via the formula [3] 241.00 page 88:

$$\int_{\alpha}^{\infty} \frac{\mathrm{d}y}{\sqrt{(y-\alpha)\left[(y-p)^2+q^2\right]}} = \frac{2}{\sqrt{A}}\boldsymbol{K}\left(\sqrt{\frac{A+p-\alpha}{2A}}\right)$$

where $A = \sqrt{(p-\alpha)^2+q^2}$. Observing that $q_1(r) = 0 \iff q_2(-2r) = 0$, we have:

$$q_2(y) = [y + 2(S+T)]\left[(y-S-T)^2 + 3(S-T)^2\right]$$

Therefore the right-hand side integral in (1) is given by:

$$\int_{-2(S+T)}^{\infty} \frac{\mathrm{d}y}{\sqrt{y^3-3a^2y+2b^3}} = \frac{\sqrt{2}}{\sqrt[4]{3}\,(S^2+ST+T^2)}\boldsymbol{K}\left(\sqrt{\frac12 + \frac{\sqrt{3}(S+T)}{4\sqrt{S^2+ST+T^2}}}\right) \tag{39}$$

Thesis (37) follows solving to π after having equated (38) to (39) via (1).

□

5 Conclusions

Several new identities have been found following our previous article [9] approach, by hypergeometrically treating some known integrals in terms of special functions, complete elliptic integrals or eulerian functions. In this article the treatment has been focused on the hypergeometrical complex arguments. Even though our aim is not to provide new numerical methods for computing remarkable mathematical constants, all the displayed identities have beeen checked numerically by means of Mathematica® where almost all the invoked special functions are implemented. The exception is the $\mathrm{F}_D^{(n)}$ Lauricella family, computed with a *naive* implementation through a partial sum of the hypergeometric series by which it is defined.

It would be an invaluable asset to the applications for both Mathematical Physics and pure Mathematics, if the wealth of functional identities on Lauricella functions, readable in [4], were embedded in a computer algebra system.

Our third research step will be the detection of analogue identities using the arithmetic of function fields over a finite field.

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