Probability of digits by dividing random numbers: a ψ and ζ functions approach

Alessandro Gambini[*] Giovanni Mingari Scarpello[†] and Daniele Ritelli [‡]

Abstract

This paper begins with the statistics of the decimal digits of n/d with $(n,d) \in \mathbb{N}^2$ randomly chosen. Starting with a statement by E. Cesàro on probabilistic number theory, see [3] and [4], we evaluate, through the Euler ψ function, an integral appearing there. Furthermore the probabilistic statement itself is proved, using a different approach: in any case the probability of a given digit r to be the first decimal digit after dividing a couple of random integers is

$$p_r = \frac{1}{20} + \frac{1}{2}\left\{\psi\left(\frac{r}{10}+\frac{11}{10}\right) - \psi\left(\frac{r}{10}+1\right)\right\}.$$

The theorem is then generalized to real numbers (Theorem 1, holding a proof of both $\frac{n}{d}$ results) and to the αth power of the ratio of integers (Theorem 2), via an elementary approach involving the ψ function and the Hurwitz ζ function. The article provides historic remarks, numerical examples, and original theoretical contributions: also it complements the recent renewed interest in Benford's law among number theorists.

KEYWORD: Elementary probability, Euler ψ function, Hurwitz ζ function.

AMS SUBJECT CLASSIFICATION 33B15, 11M35, M11K99

NOTE: this is the post print version of the paper appeaered on Applied Mathematical Modelling volume 30 (2012), Page 223-238 doi: 10.1016/j.exmath.2012.03.001

1 Introduction: the $\frac{n}{d}$ theorems

At the end of the 18$^{\text{th}}$ century, Gauss was one of the first to use probabilistic arguments for investigating the number of products consisting of exactly k distinct prime factors below a given bound. The case $k = 1$ led to the prime number theorem (proved in 1896), a milestone of analytic number theory. Instead of probabilistic number theory one should better speak about studying arithmetic functions with probabilistic methods. Anyway, the first approach in this direction dates back to Gauss, and to Ernesto Cesàro[1] who observed, 1881, that the probability that two randomly chosen integers are coprime is $6/\pi^2$. Substantial developments in probabilistic number theory started with the prime divisor counting function $\Omega(n)$ of a positive integer n, but only in 1917 did Hardy and Ramanujan discover the first deep results on such a function. The first of Cesàro's papers on the theory of numbers was followed in 1885 by a 117 pages book issued in Paris, [3], entitled: *Excursions arithmetiques a l'infini*, which reproduces nine articles from *Annali di matematica pura ed applicata* about some relevant problems in arithmetic. They looked at problems concerning the number of common divisors

[*] Università di Bologna alessandro.gambini4@unibo.it

[†] Via Negroli, 6 20136 Milano giovannimingari@yahoo.it

[‡] Dipartimento di Statistica, Università di Bologna daniele.ritelli@unibo.it

[1] E. Cesàro (1859-1906) entered the École des Mines where he studied mathematics with Catalan. In Paris he was particularly struck by lectures of Darboux on geometry which led him to his own studies of intrinsic geometry. Supported by Cremona, Battaglini, and Dini, he undertook research at the University of Rome from 1884 where the doctorate was awarded to him in 1887. He also developed the study of divergent series; furthermore his interest in mathematical physics is evident from two successful calculus texts he wrote. Cesàro was offered the chair of mathematics at Palermo where he remained until 1891, moving then to Naples where he held the chair of mathematical analysis until his death in 1906.

of two integers, determination of the values of the sum of their squares, the probability of the incommensurability of three arbitrary numbers, and so on; to these he attempted to apply results obtained in the theory of Fourier series. In this paper we focus on that collection's third article (pp. 35–56), entitled *Eventualités de la division arithmétique*, originally appearing in [4]. In addition to the topics mentioned above, Cesàro contributed to founding probabilistic number theory studying the distribution of primes and trying to improve results obtained there by Chebyshev.

1.1 Cesàro's statements and the aim of the paper

In [3] p. 49, section 19, or [4], after a sequence of frequentistic arguments, he notes that

> On trouve facilement que, si L et Λ sont les *frèquences* d'une certaine condition dans les séries
>
> $$\begin{array}{c} f\left(\frac{n}{1}\right), \quad f\left(\frac{n}{2}\right), \quad f\left(\frac{n}{3}\right), \dots \quad f\left(\frac{n}{n}\right) \\ f\left(\frac{1}{n}\right), \quad f\left(\frac{2}{n}\right), \quad f\left(\frac{3}{n}\right), \dots \quad f\left(\frac{n}{n}\right) \end{array} \tag{1}$$
>
> la probabilité que, en prenant u et v au hasard, $f\left(\frac{u}{v}\right)$ satisfasse à la même condition, est ègale à la moyenne arithmétique des deux fréquences. Ainsi:
>
> $$P = \frac{1}{2}\left(L + \Lambda\right) \tag{2}$$

At this point, taking $f(x) = x$ in the first of two series (1) he states that

> D'autre part, dans la seconde des séries (1), chaque chiffre est également fréquent. Donc, d'après (2), on peut affirmer que: Dans une division quelconque, la probabilité que le premier chiffre décimal soit r, est:
>
> $$\frac{1}{20} + 5\int_0^1 \frac{1-\varphi}{1-\varphi^{10}}\varphi^{9+r}\mathrm{d}\varphi. \tag{3}$$

Next he states that, performing an arbitrary division of two random integers, the ith decimal digit has a probability to be r given by

$$\frac{1}{20} + \frac{10^i}{2}\int_0^1 \frac{1-\varphi}{1-\varphi^{10}}\,\varphi^{10^i-1+r}\mathrm{d}\varphi. \tag{4}$$

Our contribution starts by evaluating the integrals (3) and (4). Such integrals, concerning a ratio of rational functions,

$$\frac{1-\varphi}{1-\varphi^{10}}\varphi^{9+r} = \frac{\varphi^{9+r}}{(\varphi+1)P(\varphi)},$$

where

$$\begin{aligned} P(\varphi) = &\left(\varphi^2 - \frac{1}{2}\left(\sqrt{5}-1\right)\varphi + 1\right)\left(\varphi^2 + \frac{1}{2}\left(\sqrt{5}-1\right)\varphi + 1\right) \times \\ &\times \left(\varphi^2 - \frac{1}{2}\left(1+\sqrt{5}\right)\varphi + 1\right)\left(\varphi^2 + \frac{1}{2}\left(1+\sqrt{5}\right)\varphi + 1\right), \end{aligned}$$

are not easy. It is then better to compute them by standing on a higher viewpoint, i.e., introducing the Euler ψ function. Such a computation was not performed by Cesàro, who probably judged it too involved. But through an alternative representations of ψ, we realized that those integrals could be expressed by means of infinite series. This inspired an elementary proof of the same statement of Cesàro, followed by some generalizations to be found in our Sections 3 and 4.

A theorem of Gauss on ψ of rational arguments simplifies the process, avoiding decomposition into simple fractions.

1.2 Formal probability definition

In his arithmetic papers Cesàro freely mentions the probability $p_r^{(i)}$ that the ith decimal digit of the ratio of two integers to be a given digit, say $r \in \{0, \dots, 9\}$, but quite informally. For better clarity and chiefly to highlight the difference between the definition in our paper (two real positive numbers randomly chosen) and that implicitly followed by Cesàro (two integer numbers randomly chosen), we define this discrete version: if $d_i(v)$ is the ith decimal digit of the number v, we put:

$$p_r^{(i)} := \lim_{T\to\infty} \frac{\#\{(n_1, n_2) \in \mathbb{N}^2 \mid n_j \le T \text{ for } j = 1, 2,\ d_i(n_1/n_2) = r\}}{\#\{(n_1, n_2) \in \mathbb{N}^2 \mid n_j \le T \text{ for } j = 1, 2\}}$$

whenever such a limit exists. Here $\#A$ stands for the cardinality of A. When $i = 1$ we write p_r instead of $p_r^{(1)}$.

2 Cesàro's integrals evaluation through the ψ function

The main properties of Euler's ψ function will be recalled, other details being available in [1] and [6]. We have

$$\psi(x) := \frac{\Gamma'(x)}{\Gamma(x)} = -\gamma - \frac{1}{x} + \sum_{n=1}^{\infty} \frac{x}{n(x+n)}, \tag{5}$$

where γ is the Euler-Mascheroni constant,

$$\gamma = \lim_{n\to\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n\right) = \int_1^{\infty} \left(\frac{1}{\lfloor x \rfloor} - \frac{1}{x}\right) \mathrm{d}x.$$

where $\lfloor x \rfloor$ is the *floor of* x. From (5) follows the recursive relation

$$\psi(x+1) = \psi(x) + \frac{1}{x}. \tag{6}$$

Furthermore these integral representations for $x > 0$ will be useful:

$$\psi(x) = -\gamma + \int_0^{\infty} \frac{e^{-t} - e^{-xt}}{1 - e^{-t}}\, \mathrm{d}t \tag{7}$$

$$\psi(x) = -\gamma + \int_0^{1} \frac{1 - u^{x-1}}{1 - u}\, \mathrm{d}u \tag{8}$$

Of course (8) is obtained from (7) by putting $e^{-t} = u$. From the second integral formula, taking $a, b > 0$, one obtains:

$$\int_0^1 \frac{y^a - y^b}{1 - y}\, \mathrm{d}y = \psi(b+1) - \psi(a+1)$$

of which the Cesàro integral in (3) is a special case:

$$\begin{aligned} \int_0^1 \frac{1-\varphi}{1-\varphi^{10}}\, \varphi^{9+r} \mathrm{d}\varphi &= \frac{1}{10} \int_0^1 \frac{y^{\frac{r}{10}} - y^{\frac{r+1}{10}}}{1-y}\, \mathrm{d}y \\ &= \frac{1}{10} \left\{ \psi\left(\frac{r+11}{10}\right) - \psi\left(\frac{r+10}{10}\right) \right\} \end{aligned} \tag{9}$$

In such a way the probability in (3) is given by

$$p_r = \frac{1}{20} + \frac{1}{2} \left\{ \psi\left(\frac{r}{10} + \frac{11}{10}\right) - \psi\left(\frac{r}{10} + 1\right) \right\}. \tag{10}$$

Gauss, see his *Gesammelte Werke*, Bd. 3, pp. 155–156, (Göttingen, 1866), found for ψ of rational arguments an expression involving elementary functions only. In fact, for any $p, q \in \mathbb{N}$ with $1 \le p < q$,

$$\psi\left(\frac{p}{q}\right) = -\gamma - \frac{1}{2}\pi \cot\left(\frac{p\pi}{q}\right) - \ln q + \sum_{k=1}^{q-1} \cos\left(\frac{2kp\pi}{q}\right) \ln\left(2 \sin\left(\frac{k\pi}{q}\right)\right). \tag{11}$$

A relevant proof, more elementary than the original one, can be found in [6]. Joining (6), (10), and (11) we have a closed form expression of all the required probabilities:

$$
\begin{aligned}
p_0 &= \tfrac{101}{20} - \tfrac{\pi}{4}\sqrt{5+2\sqrt{5}} - \tfrac{1}{8}\left[\ln(800\,000) + \sqrt{5}\ln\left(9+4\sqrt{5}\right)\right] \\
p_1 &= \tfrac{1}{20}\left[-49 + \pi\sqrt{10\left(5+\sqrt{5}\right)} + 5\left(\ln 16 + \sqrt{5}\,\ln\tfrac{3+\sqrt{5}}{2}\right)\right] \\
p_2 &= \tfrac{1}{60}\left[-47 + 6\pi\sqrt{25-10\sqrt{5}} + 30\left(\sqrt{5}\,\ln\tfrac{3+\sqrt{5}}{2} - \ln 4\right)\right] \\
p_3 &= \tfrac{1}{60}\left[-22 + 3\pi\left(5-\sqrt{5}\right)\sqrt{5-2\sqrt{5}} - 15\left(\sqrt{5}\,\ln\tfrac{3+\sqrt{5}}{2} - \ln 16\right)\right] \\
p_4 &= \tfrac{1}{40}\Big[-8 + 2\pi\sqrt{25-10\sqrt{5}} - \left(40+\sqrt{5}\right)\ln 2 + \\
&\quad + 25\ln 5 + \sqrt{5}\ln\left(123-55\sqrt{5}\right)\Big] \\
p_5 &= \tfrac{1}{120}\left[-14 + 6\pi\sqrt{25-10\sqrt{5}} - 15\left(\ln\tfrac{3125}{256} - 2\sqrt{5}\ln\tfrac{1+\sqrt{5}}{2}\right)\right] \\
p_6 &= -\tfrac{29}{420} + \tfrac{\pi}{2}\sqrt{\tfrac{25-11\sqrt{5}}{10}} + \ln\tfrac{\sqrt{1+\sqrt{5}}}{4} + \tfrac{1}{2}\left[\ln\left(\sqrt{5}-1\right) + \tfrac{\sqrt{5}}{2}\,\ln\tfrac{3+\sqrt{5}}{2}\right] \\
p_7 &= -\tfrac{11}{280} + \tfrac{\pi}{2}\sqrt{1-\tfrac{2}{\sqrt{5}}} + \ln 2 + \tfrac{\sqrt{5}}{2}\,\ln\tfrac{3-\sqrt{5}}{2} \\
p_8 &= \tfrac{1}{360}\left(-7 + 18\pi\sqrt{10\left(5+\sqrt{5}\right)} - 360\ln 2 - 90\sqrt{5}\ln\tfrac{3+\sqrt{5}}{2}\right) \\
p_9 &= \tfrac{1}{360}\left\{-2 - 90\pi\sqrt{5+2\sqrt{5}} + 45\left[\ln(800\,000) + \sqrt{5}\ln\left(9+4\sqrt{5}\right)\right]\right\}.
\end{aligned}
$$

With the help of Mathematica® it is immediate to verify that

$$\sum_{k=0}^{9} p_k = 1.$$

The correspondent numerical values are

$$
\begin{aligned}
&p_0 = 0.126730362245; \quad p_1 = 0.117357521909; \quad p_2 = 0.109924503863; \\
&p_3 = 0.103903172141; \quad p_4 = 0.098937259282; \quad p_5 = 0.094778739397; \\
&p_6 = 0.091250211050; \quad p_7 = 0.088221779210; \quad p_8 = 0.085596363935; \\
&p_9 = 0.083300086967.
\end{aligned}
$$

The computed first digit probability is then decreasing with increasing digits, so that the highest is 'zero', and the lowest is 'nine'.

Let us quote the recent paper of Qiu and Vuorinen, [11], where, formula (2.3) p. 727, it is proved that for any $x > 1$,

$$\ell(x) := \ln x - \frac{1}{2x} - \frac{1}{12x^2} < \psi(x) < \ln x - \frac{1}{2x} - \frac{2\gamma - 1}{2x^2} := u(x).$$

So, we see that

$$\sup_{x>1}\left(u(x) - \ell(x)\right) = \frac{7}{12} - \gamma \simeq 0.00611767$$

and then the use of either $u(x)$ or $\ell(x)$ instead of $\psi(x)$ produces an error less than $1/100$.

2.1 An experimental check of the Cesàro's law

Before going ahead with the generalizations, the reader should know a possible experimental check on the above law's reliability. In order to do it, following the frequentist approach, after a very high number of divisions of two positive integers, chosen randomly, we will extract the first digit, recording the occurrence of each of ten digits. In such a sense we will follow the approach of the empirical checks, performed via Mathematica® which is the same as our tool, on the Weisstein web site [12] about Benford's law.

It will be recalled that the leading digit in tables and physical data is not evenly distributed among the digits 1 and 9. The first known written reference is a two-pages article by Newcomb, [10] 1881, who stated[2] that

$$\text{Prob}(\text{first significant digit} = r) = \log_{10}\left(1+\frac{1}{r}\right), \quad r = 1, \ldots, 9.$$

In such a way he conjectured that the digit '9' occurs about 4.6% of the time. In 1938, Benford published a paper [2] describing how well his 20,229 observations were fit by the logarithmic law, nowadays called the *Benford Law* stating that for a dimensionless data sequence, the digit '1' tends to occur with probability $\simeq 30\%$, almost three times greater than the naive value[3], $\simeq 11.\bar{1}\%$. There is also a general significant digit Benford law including not only the first digit, but also the second, which may be zero, and all higher significant digits. For instance in [5], the probability that the first three significant digits are 3, 1, 4 in that order, is given by

$$\log_{10}\left(1+\frac{1}{314}\right) \simeq 0.14\%$$

and similarly for other significant digit patterns.

Cesàro's law has a different object than Benford's: in fact Cesàro's argument is theoretical, founded upon number theory; on the contrary *in the beginning* the Benford law was nothing but a empirical statement about real-world datasets, applicable to some datasets but not all, and only approximate. Things are quite different *nowadays*, the law having been embodied as a theoretical mathematical result (see for instance [9] and [7] and the references therein). Accordingly, a sequence of *positive* numbers x_n is defined as a *Benford (base b)* if the probability of observing the first digit of x_n in base b is $j \in \{j = 1,\, 2,\, \ldots b-1\}$ is given by:

$$\lim_{N\to\infty} \frac{\#\{n < N : \text{first digit of } x_n \text{ in base } b \text{ is} j\}}{N} = \log_b\left(1+\frac{1}{j}\right)$$

From this definition, one can, for instance, see that the first digit of 2^n is Benford in base 10, but not Benford in base 2 because the first digit is always 1 in this second case. In [9] (Theorem 9.3.1 p. 220) some conditions are given in order that a recurrence relation of assigned length meets Benford; it is also shown why the first digit of a geometric Brownian motion is Benford.

But any sequence of digit obtained with the Cesàro procedure takes into account the 'zero' digit's occurrences, so this phenomenon is not relevant for Benford's, dealing with sequences of positive numbers.

Anyway: we did an experimental check by means of Mathematica®, randomly taking couples of positive integers between 1 and 10^7, taking the quotient, repeating 10^7 times, and recording the relative frequency of each digit. Comparing such values to the computed ones, we found for each digit a difference on the order of magnitude of $\pm 10^{-6}$, which has been successively confirmed by a similar test on the second place also.

2.2 Further digits

The integral in (4) can be again expressed through ψ:

$$p_r^{(i)} = \frac{1}{20} + \frac{10^i}{20}\left\{\psi\left(\frac{r+10^i+1}{10}\right) - \psi\left(\frac{r+10^i}{10}\right)\right\}. \tag{12}$$

Like in the previous analysis of the first digit, one can compute via elementary functions the probability that, dividing two randomly chosen integers, the ith digit will be $r = 0,\, 1,\, \ldots,\, 9$. But, due to the recursive relationship on ψ,

$$\psi(x+n) = \psi(x) + \frac{1}{x} + \frac{1}{x+1} + \cdots + \frac{1}{x+n-1},\ n \in \mathbb{N},$$

[2] Assuming that a universal probability distribution exists over such numbers, it will be invariant under a change of scale. Normalizing and differentiating such a invariance one will find a hyperbolic probability distribution, so that by integration the logarithmic phenomenon appears known as Benford's law.

[3] Namely the percentage one would get if the digits were equidistributed.

such formulae quickly become intractable, having to be evaluated with $n = 10^{i-1}$. For instance the probability of the second decimal digit being zero is

$$\begin{aligned} p_0^{(2)} &= \frac{2\,991\,620\,812\,234\,909}{303\,502\,130\,011\,080} - \frac{\pi}{2}\sqrt{5+2\sqrt{5}} + \frac{\sqrt{5}}{4}\ln 5 \\ &- \frac{5\ln 20}{4} - \frac{\sqrt{5}-1}{2}\ln\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right) \\ &+ \frac{1}{2}\left[\sqrt{5}\ln\left(7-3\sqrt{5}\right) + \ln\left(\sqrt{5}-1\right) - \sqrt{5}\ln\left(3\sqrt{5}-5\right)\right]. \end{aligned}$$

As $i \to \infty$, the events become equiprobable. Roughly speaking, if, instead of taking the first, we take for instance the millionth digit, then the probability distribution of the ten occurrences $(0, \ldots, 9)$ is uniform.

In fact, see [6] p. 165, the following series expansion holds:

$$\psi(s) - \psi(s-r) = \sum_{n=1}^{\infty} \frac{(r)_n}{n(s)_n},$$

where $(s)_n = s(s+1)\cdots(s+n-1)$ is the Pochhammer symbol, so that for k, α, $\beta > 0$,

$$k\left[\psi(\alpha+k) - \psi(\beta+k)\right] = k\left[\frac{\alpha-\beta}{\alpha+k} + \frac{(\alpha-\beta)(\alpha-\beta+1)}{2(\alpha+k)(\alpha+k+1)} + \cdots\right],$$

and then

$$\lim_{i\to\infty} p_r^{(i)} = \frac{1}{20} + \lim_{i\to\infty}\frac{10^i}{20}\left\{\psi\left(\frac{r+10^i+1}{10}\right) - \psi\left(\frac{r+10^i}{10}\right)\right\} = \frac{1}{20} + \frac{1}{20} = \frac{1}{10}.$$

3 An elementary proof of both $\frac{n}{d}$ theorems

In this section Cesàro's statements will be proved by methods of elementary probability theory, and then they will be extended to the case of the quotient of positive real numbers with respect to an arbitrary base $b \geq 2$. Choose at random two real positive numbers n, d. We provide an alternative definition of the probability that the ith digit of the ratio n/d will be $r \in \{0, \ldots, b-1\}$. There being a choice made in the whole continuum of possible outcomes, we cannot make use, as for the discrete case, of the ratio of two cardinal numbers. In such a way the definition can be applied to an arbitrary base b and not only to a simple ratio but also to a power $\alpha > 0$ of such a ratio. Given two numbers $(n, d) \in {\mathbb{R}_+}^2$, we need to define the probability ${}^{b}_{\alpha}p_r^{(i)}$ that the ratio $\left(\frac{n}{d}\right)^{\alpha}$ has, to base $b \geq 2$, the ith digit r. To do this first we point out formally how to extract the ith digit in base b of a given real number. Given an integer $b \geq 2$ called the base and using the b digits $c_i \in \{0, 1, \ldots, b-1\}$, we have a representation of a positive real number ν in base b as

$$\nu = \sum_{i=-\infty}^{\infty} c_i b^i,$$

Observe that there exists $N \in \mathbb{N}$ such that $i > N \implies c_i = 0$ thus

$$\sum_{i=0}^{N} c_i b^i \quad \text{and} \quad \sum_{i=-\infty}^{-1} c_i b^i$$

are respectively the integer part and the fractional part of ν. The coefficient c_{-i} in front of b^{-i}, $i = 1, 2, \ldots$ is called the ith digit in the base b representation and will be denoted $d_i^b(\nu)$. Thus

$$d_i^b(\nu) = r \in \{0, 1, \ldots, b-1\}$$

if and only if there exists $k = 0, 1, \ldots$ such that $b^{i-1}\nu \in \left[k + \frac{r}{b}, k + \frac{r+1}{b}\right[$.

Definition 1. Let $d_i^b(\nu)$ be the ith decimal digit for the real number ν; then, given two real positive numbers n, d and $\alpha > 0$ we define the probability ${}_{\alpha}^{b}p_r^{(i)}$ that the ith decimal digit to base b of the ratio $(n/d)^\alpha$ to be $r \in \{0, \ldots, b-1\}$ as

$${}_{\alpha}^{b}p_r^{(i)} = \lim_{T\to\infty} \frac{\mu\{(n,d) \in \mathbb{R}_+^2 \,|\, n,\, d \le T,\, d_i^b\left(\frac{n^\alpha}{d^\alpha}\right) = r\}}{\mu\{(n,d) \in \mathbb{R}_+^2 \,|n,\, d \le T\}}$$

if the limit exists. Here μ is the Lebesgue measure on $\mathbb{R}_+^2$.

We will keep the notation $p_r^{(i)}$ whenever $\alpha = 1$ and $b = 10$ and the notation p_r if $i = 1$, $\alpha = 1$, $b = 10$.

As a first step we will, by means of the series expansion (5) of ψ, rearrange (10). In fact from (5) we deduce the following relation, which holds for b, $x > 0$:

$$\psi\left(\frac{x+1}{b}\right) - \psi\left(\frac{x}{b}\right) = b\sum_{n=0}^{\infty} \frac{1}{(x+nb)(x+1+nb)} \tag{13}$$

In such a way, using (13) we see that (10) can be written using a numerical series:

$$p_r = \frac{1}{20} + 5\sum_{k=0}^{\infty} \frac{1}{(10+10k+r)(11+10k+r)}. \tag{14}$$

This fact inspired us in looking for an alternative proof to the Cesàro statement, we are going to present in the following theorem.

Theorem 1. *For a couple of real positive numbers n, d taken at random and represented in base b, the probability ${}_bp_r^{(i)}$ that the ith decimal digit of n/d will be $r \in \{0, 1, \ldots, b-1\}$ can be computed by the formula*

$$\begin{aligned} {}_bp_r^{(i)} &= \frac{1}{2b} + \frac{1}{2}\sum_{k=0}^{\infty} \frac{b^i}{(b^i+bk+r)(b^i+bk+r+1)} \\ &= \frac{1}{2b} + \frac{b^{i-1}}{2}\left\{\psi\left(\frac{b^i+1+r}{b}\right) - \psi\left(\frac{b^i+r}{b}\right)\right\}. \end{aligned} \tag{15}$$

Proof. We have

$$\left\{(n,d) \in]0,T]^2 |\, d_i^b(n/d) = r\right\} = \bigcup_{k=0}^{\infty}\left\{(n,d) \in]0,T]^2 |\, \frac{kb+r}{b^i} \le \frac{n}{d} < \frac{kb+r+1}{b^i}\right\}.$$

This union is split as $U \cup L$, where

$$U = \bigcup_{k=0}^{b^{i-1}-1}\left\{(n,d) \in]0,T]^2 |\, \frac{kb+r}{b^i} \le \frac{n}{d} < \frac{kb+r+1}{b^i}\right\}$$

and

$$L = \bigcup_{k=b^{i-1}}^{\infty}\left\{(n,d) \in]0,T]^2 |\, \frac{kb+r}{b^i} \le \frac{n}{d} < \frac{kb+r+1}{b^i}\right\}$$

Let us explain the split. Reasoning in the plane (n,d) for $0 \le k \le b^{i-1} - 1$, each of the subsets forming U succeeds in being a triangle with the base on the horizontal straight line of equation $d = T$ bounded by the straight lines with slopes ≥ 1. If, on the contrary, $k \ge b^{i-1}$, each of the subsets forming V is a triangle based on the vertical straight line $n = T$ and sided by the straight lines with slopes < 1, as shown in the figure below. For $k = 0, \ldots, b^{i-1} - 1$ the kth set in U is the triangle with vertices in the (n,d)-plane

$$(0,0), \quad \left(\frac{kb+r}{b^i}T, T\right), \quad \left(\frac{kb+r+1}{b^i}T, T\right),$$

hence its Lebesgue measure is $T^2/(2b^i)$.

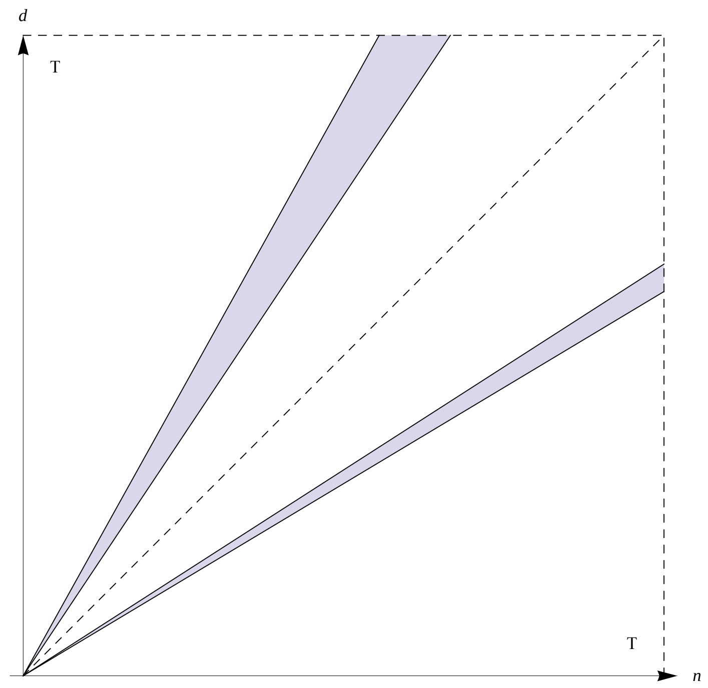

Figure 1: Reference (n,d): the shaded areas sketch the standard subsets making up the set U (on the left) and the set L (on the right).

For $k = b^{i-1}, \dots$ the kth set in L is the triangle with vertices

$$(0,0), \quad \left(T, \frac{b^i}{kb+r}T\right), \quad \left(T, \frac{b^i}{kb+r+1}T\right),$$

with Lebesgue measure

$$\frac{b^i T^2}{2(kb+r)(kb+r+1)}.$$

Therefore,

$$\begin{aligned} {}_b p_r^{(i)} &= \sum_{k=0}^{b^{i-1}-1} \frac{1}{2b^i} + \sum_{k=b^{i-1}}^{\infty} \frac{b^i}{2(kb+r)(kb+r+1)} \\ &= \frac{1}{2b} + \frac{b^i}{2} \sum_{n=0}^{\infty} \frac{1}{(b^i+bn+r)(b^i+bn+r+1)} \end{aligned}$$

This shows the first of (15), while the second relation follows from (13). □

Remark 1.1. Theorem 1 given above does not prove the original statement of Cesàro, which concerns integer variables (the discrete case) while Theorem 1 concerns all the real ones (the continuous case). It is however not difficult to reprove also Cesàro's statement, by combining the above ideas with Gauss's lattice point technique, see e.g. [8] section 1.1, theorems 1.1 and 1.4. In fact, considering pairs of *integers* inside the triangles described by the sets U and L, the number of such integer points inside a triangle is approximated by the area of the triangle with an error which is growing as the perimeter of the triangle. In Theorem 1 the number of integer pairs in U is equal to

$$\sum_{k=0}^{b^{i-1}-1} \left(\frac{T^2}{2b^i} + O(T)\right)$$

which, when divided by T^2 converges to $1/2b$ as $T \to \infty$.

The sum of integers pairs in L requires further attention, since L contains infinitely many triangles. We note that the number of integer points in L approximated by the number of integer points in the triangle with $k \le \lfloor\sqrt{T}\rfloor$. The integer points not counted by such restriction all lie in the triangle with vertices

$$(0,0), \quad (T,0), \quad (T, C\lfloor\sqrt{T}\rfloor)$$

for some constant $C > 0$. It follows that the number of integer points in L equals

$$\sum_{k=b^{i-1}}^{\lfloor\sqrt{T}\rfloor}\left(\frac{b^i T^2}{(kb+r)(kb+r+1)} + O(T)\right) + O(T^{3/2}).$$

Dividing by T^2 and letting $T \to \infty$ gives the desired result.

4 The $\left(\frac{n}{d}\right)^\alpha$ theorems

In this section we can see how the same approach can easily lead to the any decimal digit of n^α/d^α for α real and positive.

Let us state in advance something about the *Hurwitz zeta* function. It will be recalled that the *Riemann zeta* function $\zeta(s)$ is defined by

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}, \quad \operatorname{Re}(s) > 1$$

where s is a complex variable.

The Hurwitz zeta function $\zeta(s,x)$, generalizing Riemann's, is defined by

$$\zeta(s,x) := \sum_{n=0}^{\infty}\frac{1}{(n+x)^s} \tag{16}$$

where $\operatorname{Re} s > 1$ and $\operatorname{Re} x > 0$. We have $\zeta(s) = \zeta(s,1)$. Recall the Hurwitz ζ integral representation

$$\zeta(s,x) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{e^{-xt}}{1-e^{-t}}t^{s-1}\mathrm{d}t, \tag{17}$$

which holds for $\operatorname{Re} s > 1$ and $\operatorname{Re} x > 0$. It is known that the Hurwitz zeta function can be meromorphically continued to be defined for all complex numbers s with $s \neq 1$. At $s = 1$ it has a simple pole with residue 1. Its meromorphic continuation becomes explicit for $s \neq 1$ and $x > 0$ by means of Hermite's integral representation theorem:

$$\zeta(s,x) = \frac{1}{2}x^{-s} + \frac{1}{s-1}x^{1-s} + 2\int_0^\infty \frac{\sin(s\arctan\frac{t}{x})}{(x^2+t^2)^{s/2}(e^{2\pi t}-1)}\mathrm{d}t \tag{18}$$

More information about $\zeta(s,x)$ can be found in [13] chapter 13 or in [1].

The Hurwitz zeta function is linked to the polygamma function if its first variable is a positive integer, i.e., $n \in \mathbb{N}$:

$$\psi_n(z) := \frac{d^n}{dz^n}\psi(z) = (-1)^{n+1}n!\,\zeta(n+1,z). \tag{19}$$

In what follows, we will need the integration formula

Lemma 1. *Let $s \neq 1$ and $a \in \mathbb{R}^+$, $b \in \mathbb{R}$ be such that $a+b \in \mathbb{R}^+$. Then*

$$\int_0^1 \zeta(s+1, a+bx)\,\mathrm{d}x = \frac{1}{sb}\left\{\zeta(s,a) - \zeta(s,a+b)\right\}. \tag{20}$$

Proof. Formula (20) follows by:

$$\int \zeta(s,x)\mathrm{d}x = \frac{1}{1-s}\zeta(s-1,x)$$

which, in turn, follows from the relationship:

$$\frac{\partial\zeta}{\partial x}(s-1,x) = (1-s)\zeta(s,x),$$

which is a consequence of (16) □

Finally for $b>0$, $\mathrm{Re}(s)>1$, $\mathrm{Re}(x)>0$ we will employ this relation which is an immediate consequence of the definition of $\zeta(s,x)$

$$\sum_{n=0}^{\infty}\frac{1}{(bn+x)^s}=\frac{1}{b^s}\zeta(s,\frac{x}{b}) \tag{21}$$

We then can present our generalization of Cesàro's results.

Theorem 2. *For a couple of positive real numbers n, d taken at random and represented in base b, the probability ${}^{b}_{\alpha}p_r^{(i)}$, with $\alpha>0$, that the ith decimal digit of n^α/d^α will be $r\in\{0,1,\ldots,b-1\}$ is given by*

$$\begin{aligned}{}^{\alpha}_{b}p_r^{(i)} &= \frac{1}{2\sqrt[\alpha]{b^i}}\sum_{k=0}^{b^{i-1}-1}\left[(kb+r+1)^{1/\alpha}-(kb+r)^{1/\alpha})\right]\\ &+\frac{1}{2}\sqrt[\alpha]{b^i}\sum_{k=0}^{\infty}\left(\frac{1}{\sqrt[\alpha]{b^i+bk+r}}-\frac{1}{\sqrt[\alpha]{b^i+bk+r+1}}\right)\end{aligned} \tag{22}$$

Moreover when $\alpha<1$ in equation (22) *the infinite series can be written in terms of Hurwitz zeta function:*

$$\begin{aligned}{}^{\alpha}_{b}p_r^{(i)} &= \frac{1}{2\sqrt[\alpha]{b^i}}\sum_{k=0}^{b^{i-1}-1}\left[(kb+r+1)^{1/\alpha}-(kb+r)^{1/\alpha})\right]\\ &+\frac{\sqrt[\alpha]{b^{i-1}}}{2}\left\{\zeta\left(\frac{1}{\alpha},\frac{b^i+r}{b}\right)-\zeta\left(\frac{1}{\alpha},\frac{b^i+r+1}{b}\right)\right\}\end{aligned} \tag{22b}$$

Proof. The proof is similar to Theorem 1. In this case we have

$$\begin{aligned}&\left\{(n,d)\in]0,T]^2|\, d_i^b(n^\alpha/d^\alpha)=r\right\}\\ &=\bigcup_{k=0}^{\infty}\left\{(n,d)\in]0,T]^2|\,\frac{kb+r}{b^i}\le\frac{n^\alpha}{d^\alpha}<\frac{kb+r+1}{b^i}\right\}\\ &=\bigcup_{k=0}^{\infty}\left\{(n,d)\in]0,T]^2|\,\sqrt[\alpha]{\frac{kb+r}{b^i}}\le\frac{n}{d}<\sqrt[\alpha]{\frac{kb+r+1}{b^i}}\right\}.\end{aligned}$$

As for the previous case in Theorem 1, this union can be split as $U\cup L$, where

$$U=\bigcup_{k=0}^{b^{i-1}-1}\left\{(n,d)\in]0,T]^2|\sqrt[\alpha]{\frac{kb+r}{b^i}}\le\frac{n}{d}<\sqrt[\alpha]{\frac{kb+r+1}{b^i}}\right\}$$

and

$$L=\bigcup_{k=b^{i-1}}^{\infty}\left\{(n,d)\in]0,T]^2|\sqrt[\alpha]{\frac{kb+r}{b^i}}\le\frac{n}{d}<\sqrt[\alpha]{\frac{kb+r+1}{b^i}}\right\}$$

For $k=0,\ldots,b^{i-1}-1$ the kth set in U is the triangle with vertices in the (n,d)-plane

$$(0,0),\quad\left(\sqrt[\alpha]{\frac{kb+r}{b^i}}T,T\right),\quad\left(\sqrt[\alpha]{\frac{kb+r+1}{b^i}}T,T\right),$$

hence its Lebesgue measure is

$$\frac{T^2(\sqrt[\alpha]{kb+r+1}-\sqrt[\alpha]{kb+r})}{2\sqrt[\alpha]{b^i}}.$$

For $k=b^{i-1},\ldots$ the kth set in L is the triangle with vertices

$$(0,0),\quad\left(T,\sqrt[\alpha]{\frac{b^i}{kb+r}}\,T\right),\quad\left(T,\sqrt[\alpha]{\frac{b^i}{kb+r+1}}\,T\right),$$

with Lebesgue measure

$$\frac{1}{2}\left(\sqrt[\alpha]{\frac{b^i}{kb+r}}-\sqrt[\alpha]{\frac{b^i}{kb+r+1}}\right)T^2$$

Therefore summing the measures of the disjointed events in order to detect the required probability we find the first formula for ${}_{\alpha}^{b}p_r^{(i)}$ given in (22). To end the proof and to get the formula (**??**b) for $\alpha < 1$, first notice that the series (22) converges also for this set of α's, then use the following identity $a, k, \beta > 0$

$$\frac{1}{(a+k)^\beta}-\frac{1}{(1+a+k)^\beta}=\int_0^1\frac{\beta}{(1+a+k-x)^{\beta+1}}\mathrm{d}x$$

then summing on k and interchanging summation and integration, we see that

$$\begin{aligned}&\sum_{k=0}^{\infty}\left(\frac{1}{\sqrt[\alpha]{b^i+bk+r}}-\frac{1}{\sqrt[\alpha]{b^i+bk+r+1}}\right)\\&=\frac{1}{\alpha}\int_0^1\sum_{k=0}^{\infty}\frac{1}{(b^i+r+1+kb-x)^{\frac{1}{\alpha}+1}}\mathrm{d}x\\&=\frac{1}{\alpha b^{\frac{1}{\alpha}+1}}\int_0^1\zeta\left(1+\frac{1}{\alpha},\frac{b^i+r-x+1}{b}\right)\mathrm{d}x\end{aligned}$$

Formula (**??**b) now follows using (21) and (20). □

Remark 2.1. We can also prove a discrete version of Theorem 2, by using Gauss's technique as in Remark 1.1.

Let us close with some points to note.

If in (22) we take the limit $\alpha \to 1$, (15) shall by necessity be obtained, and this is done easily using the expressions containing infinite series. If one instead wants to use the special functions representations, the pole at $s = 1$ of $\zeta(s, x)$ can be bypassed: in fact if x, y are two positive real numbers, then

$$A_{xy}(s) := \zeta(s,x)-\zeta(s,y)$$

is continuous in $s = 1$ attaining there the value

$$A_{xy}(1)=\psi(y)-\psi(x). \tag{23}$$

In fact, expressing A_{xy} by means of the integral representation (17) of $\zeta(s,\cdot)$, we have

$$A_{xy}(s)=\zeta(s,x)-\zeta(s,y)=\frac{1}{\Gamma(s)}\int_0^\infty\frac{e^{-xt}-e^{-yt}}{1-e^{-t}}t^{s-1}\mathrm{d}t$$

the limit under the integral for $s \to 1$ is allowed due to the integrand's summability (including the origin). So that, by the first part of (7):

$$A_{xy}(1)=\int_0^\infty\frac{e^{-xt}-e^{-yt}}{1-e^{-t}}\mathrm{d}t=\int_0^\infty\frac{e^{-xt}-e^{-t}+e^{-t}-e^{-yt}}{1-e^{-t}}\mathrm{d}t=\psi(y)-\psi(x).$$

Then, by means of (23), we can put $\alpha = 1$ in (22), finding (15) again.

Whilst in Cesàro's formulae the digamma appears, in ours we use the Hurwitz ζ function: this depends on the non-integrality of the first argument of the generalized zeta function. Therefore if in (**??**b) one puts $\alpha = 1/n$ with $n \in \mathbb{N}$, then thanks to (19) it would be possible to express (**??**b) by a digamma.

Conclusions

In 1885 E. Cesàro considered in [4] the probability that a whichever digit r, *zero included*, could occur as the first when dividing two integers taken at random, say n and d. Such a probability was stated without proof as depending on the integral

$$\int_0^1\frac{1-\varphi}{1-\varphi^{10}}\varphi^{9+r}\mathrm{d}\varphi,\quad r=0,1,\ldots,9$$

which is a non-trivial function of r. In our article, an elementary proof is given of such a formula in the case of two arbitrary chosen positive real numbers; next, the integral is computed by reducing it to the Euler ψ function, namely the logarithmic derivative of the Γ function. Such a law has been tested by computer, taking random couples of positive integers between 1 and 10^7, performing the division, repeating 10^7 times, recording the relative frequency of each digit. By comparison of such values to the computed ones, we found a difference for each digit of the order of $\pm 10^{-6}$, which has been successively confirmed when arranging a similar test of the second digit. In fact further theorems have been obtained generalizing such a result to the ith decimal digit of n/d and to the case of numbers in a whichever base b. Further generalizations concerning the probabilistic occurrence to the second power and finally to the αth power of n/d are proved by using the Hurwitz ζ function by means of theorems about that function whose role was long ago established through number theory.

Acknowledgement

We wish to thank the anonymous referee for his/her fruitful suggestions which improved our article.

References

[1] G. E. Andrews, R. Askey, R. Roy, Special Functions, Encyclopedia of Mathematics and its Applications, 71. Cambridge University Press, Cambridge, 1999.

[2] F. Benford, The law of anomalous numbers, Proc. Amer. Phil. Society 78 (1938) 551–572.

[3] E. Cesàro, Excursions arithmetiques a l'infini, Hermann, Paris, 1885.

[4] E. Cesàro, Eventualités de la division arithmétique, Ann. Mat. Pura Appl. 13 (1885) 269–290.

[5] T. P. Hill, A statistical derivation of the significant-digit law, Statistical Science 10 (1995) 354–363.

[6] J. L. W. V. Jensen, An elementary exposition of the theory of the gamma function, Ann. Math. 17 (1915) 124–166.

[7] A. V. Kontorovich, S. J. Miller, Benford's law, values of L-functions and the $3x + 1$ problem, Acta Arith. 120 (2005) 269–297.

[8] E. Krätzel, Lattice points. Kluwer Academic Publishers Group, Dordrecht, 1988.

[9] S. J. Miller, R. Takloo-Bighash, An invitation to modern number theory. Princeton University Press, Princeton, 2006.

[10] S. Newcomb, Note on the frequency of use of the different digits in natural numbers, Proc. Amer. Journal of Math. 4 (1881) 39–40.

[11] S. L. Qiu, M. Vuorinen, Some properties of the Gamma and Psi Functions, with Applications. Math. Comp. 74 (2004) 724–742.

[12] E. W. Weisstein, Benford's Law. In: MathWorld A Wolfram Web Resource mathworld.wolfram.com/BenfordsLaw.html, 2011.

[13] E. T. Whittaker, G. N. Watson, A Course of Modern Analysis. Cambridge University Press, Cambridge, 1927.